CALC 1000 ACE Booklet Solutions (Winter 2026)

A. Solving Trig Equations

Example 1.

a) $\cos(x - y)$

$b)$ $\frac{\sin 2x}{2}$

$c)$ $\sin(x - y)$

Example 2. $\cos x + \sin 2x = 0$

$\cos x + 2\sin x \cos x = 0$

$\cos x(1 + 2\sin x) = 0$

$\cos x = 0$ $\quad$ $1 + 2\sin x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $\sin x = \frac{-1}{2}$

$x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

$x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

Example 3.

$a)\ sin\theta cot\theta = cos\theta$

$LS = sin\theta cot\theta$

$= sin\theta\left(\frac{cos\theta}{sin\theta}\right)$

$= cos\theta$

$RS = cos\theta$

LS=RS

b) $(sinx + cosx)^2 = 1 + sin2x$

$LS = (sinx + cosx)(sinx + cosx)$

$= sin^2x + 2sinxcosx + cos^2x$

$= 1 + 2sinxcosx$

$= 1 + sin2x$

LS=RS

A1. $\cos(2x) = -\sin^2 x$

$\cos^2 x - \sin^2 x = -\sin^2 x$

$\cos^2 x - \sin^2 x + \sin^2 x = 0$

$\cos^2 x = 0$

$\cos x = 0 \qquad x = \frac{\pi}{2}, \frac{3\pi}{2}$

A2. $2\cos^2 x - \cos x - 1 = 0$

$(\cos x - 1)(2\cos x + 1) = 0$

$\cos x = 1 \qquad or \;\; 2\cos x + 1 = 0$

$x = 0,2\pi \qquad\qquad 2\cos x = -1$

$\cos x = \frac{-1}{2}$

$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \qquad x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

$A3.\quad \tan x = \sin 2x \quad [0, \pi]$

$\tan x = 2\sin x \cos x$

$\frac{\sin x}{\cos x} = 2\sin x \cos x$

$\sin x = 2\sin x \cos^2 x$

$0 = 2\sin x \cos^2 x - \sin x$

$0 = \sin x(2\cos^2 x - 1)$

$\sin x = 0 \qquad\qquad 2\cos^2 x - 1 = 0$

$x = 0, \pi \qquad\qquad \cos^2 x = \frac{1}{2}$

$\cos x = \frac{1}{\sqrt{2}} \quad or \quad \cos x = \frac{-1}{\sqrt{2}}$

$x = \frac{\pi}{4} \qquad\qquad x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$all\ other\ solutions\ from\ the\ CAST\ rule\ would\ be$

$greater\ than\ \pi, so\ we\ don't\ include\ them$

$A4.\quad \tan x = \sin x \qquad [0, \pi]$

$\frac{\sin x}{\cos x} = \sin x$

$\sin x = \sin x \cos x$

$\sin x - \sin x \cos x = 0 \qquad [0, \pi]$

$\sin x(1 - \cos x) = 0$

$\sin x = 0 \qquad\qquad 1 = \cos x$

$x = 0, \pi, 2\pi \qquad\qquad x = 0, 2\pi$ ... 2π *is not in our interval*

$\therefore\ solution\ is\ x = 0, \pi$

A5.

a) $(1 - sin\theta)(1 + sin\theta) = cos^2\theta$

$LS = (1 - sin\theta)(1 + sin\theta)$

$= 1 + sin\theta - sin\theta - sin^2\theta$

$= 1 - sin^2\theta$

$= cos^2\theta$

LS=RS

b) $sinxsin2x + cosxcos2x = cosx$

$LS = sinxsin2x + cosxcos2x$

LS=$sinx(2sinxcosx) + cosx(1 - 2sin^2x)$

$= 2sin^2xcosx + cosx(1 - 2sin^2x)$

=$2sin^2xcosx + cosx - 2sin^2xcosx$

=cosx

LS=RS

c) $\frac{1+tan^2x}{1+cot^2x} = tan^2x$

LS=$\frac{1+tan^2x}{1+cot^2x}$

$$= \frac{1+\frac{sin^2x}{cos^2x}}{1+\frac{cos^2x}{sin^2x}}$$

$$= \frac{\frac{cos^2x+sin^2x}{cos^2x}}{\frac{sin^2x+cos^2x}{sin^2x}}$$

$$= \frac{\frac{1}{cos^2x}}{\frac{1}{sin^2x}}$$

$$= \frac{1}{cos^2x}\left(\frac{sin^2x}{1}\right)$$

=tan^2x

LS=RS

d) $tan^2\vartheta - sin^2\vartheta = tan^2\vartheta sin^2\vartheta$

$$LS = \frac{sin^2\vartheta}{cos^2\vartheta} - \frac{sin^2\vartheta}{1}$$

$$= \frac{sin^2\vartheta}{cos^2\vartheta} - \frac{sin^2\vartheta cos^2\vartheta}{cos^2\vartheta}$$

$$= \frac{sin^2\vartheta(1-cos^2\vartheta)}{cos^2\vartheta}$$

$$= tan^2\vartheta sin^2\vartheta$$

LS=RS

e) $2csc2\vartheta = sec\vartheta csc\vartheta$

LS=$2csc2\vartheta$

LS=$2\left(\frac{1}{sin2\vartheta}\right)$

$$= \frac{2}{2sin\vartheta cos\vartheta}$$

$$= \frac{1}{sin\vartheta}\left(\frac{1}{cos\vartheta}\right)$$

$$=csc\vartheta sec\vartheta$$

LS=RS

B. Graphing Exponential and Logarithmic Functions

Example 1.

$y = 2^{x+1}$

$D \; x \in R \quad or \quad (-\infty, \infty)$

$R \quad y \in R/y > 0 \quad or \quad (0, \infty)$

$HA \quad y = 0$

$x = int \quad none$

$y - int \; (0,2)$

Example 2.

$y = 3^x$

$D \; x \in R \quad or \quad (-\infty, \infty)$

$R \; y \in R/y > 0 \quad or \; (0, \infty)$

$HA \quad y = 0$

$x - int \quad none$

$y - int \; (0,1)$

Example 3.

$y = -2^x - 4$

$D \quad x \in R \quad or \quad (-\infty, \infty)$

$R \quad y \in R/y < -4 \quad or \quad (-\infty, -4)$

$HA \quad y = -4$

$x - int = none \qquad y - int = (0, -5)$

Example 4.

$y = 3(2^x) - 2$

$D \quad x \in R \quad (-\infty, \infty)$

$R \quad y \in R/y > -2 \qquad (-2, \infty)$

$HA \ \ y = -2$

$x - int = \dfrac{\ln\frac{2}{3}}{\ln 2}$

$y - int = (0,1)$

Work for how to get the intercepts...

$x - int \quad let\ y = 0 \qquad 0 = 3(2^x) - 2$

$3(2^x) = 2$

$2^x = \frac{2}{3}$

$\ln 2^x = \ln\frac{2}{3} \qquad x = \dfrac{\ln\frac{2}{3}}{\ln 2}$

$y - int \quad let\ x = 0 \qquad y = 3(2^0) - 2 = 3 - 2 = 1$

Example 5.

$y = (0.5)^x$

$D = x \in R \quad or \ \ (-\infty, \infty)$

$R = \ \ y \in R/y > 0 \quad or \ \ (0, \infty)$

$HA \quad y = 0$

$x - int = none\ \ DNE \qquad y - int = (0,1)$

Example 6.

$y = e^x$

$HA \quad y = 0$

$x - int = none \quad y - int = (0,1)$

$y = \ln x \quad HA \quad none$

VA x=0

$x - int \quad (1,0) \quad y - int \quad (none)$

Example 7.

$y = 3\log_2(x-1) + 3$

↑ ↑ ↑

Vertical stretch *right 1* *up 3*

(multiply y by 3)

Example 8. Sub. (1,7) into the equation $y=b^x+c$

$7 = b^1 + c$

c=7-b

Subst. (2,13) into the equation and get:

$13 = b^2 + c$ subst. c=b-7 into this equation

$13 = b^2 + (7 - b)$

Rearrange and we get: $b^2 - b - 6 = 0$

Factor and get (b-3)(b+2)=0

And b=3 or b= -2

If b=3, we get c=7-b=7-3=4 and the equation is $y=3^x+4$

If b= -2, we get c=7-b=7-(-2)=9 and the equation is $y=(-2)^x+9$

C. Exponential Equations

Example 1

a) $\frac{1}{4}b^5$

b) $9x^4y^2z^8$

c) $\frac{(4abc^2)(9a^2b^4)}{-6abc^2} = \frac{36a^3b^5c^2}{-6abc^2} = -6a^2b^4$

d) $(-2x^{-1}y^3)^2 = 4x^{-2}y^6 = \frac{4y^6}{x^2}$

Example 2. Solve each of the following:

a) $7^{5-2x} = 7^{3x+8}$ $\therefore 5 - 2x = 3x + 8$

$$5 - 8 = 5x$$

$$-3 = 5x$$

$$x = -\frac{3}{5}$$

b) $6^{3x} = \left(\frac{1}{216}\right)^{3x+8}$

$$6^{3x} = (6^{-3})^{3x+8}$$

$$3x = -3(3x + 8)$$

$$3x = -9x - 24$$

$$12x = -24 \qquad x = -2$$

c) $4^{3\mathrm{x}-5} = \left(\frac{1}{64}\right)^3$ $=(4^{-3})^3$

$4^{3x-5} = 4^{-9}$

$3x - 5 = -9$

$3x = -9 + 5$

$3x = -4$

$x = \frac{-4}{3}$

*d) $If\ 3^{-2t} = e^{ct}, find\ the\ value\ of\ c\ and\ simplify$.

Take the ln of both sides $\ln 3^{-2t} = \ln e^{ct}$

$$-2t \ln 3 = ct \ln e$$

$$-2 \ln 3 = c \quad since\ \ln e = 1$$

$$c = ln3^{-2} = \ln\left(\frac{1}{9}\right) = ln1 - ln9 = -ln9$$

e) (3x-1)(2x-1)=3(4x-1)-1

$6x^2 - 3x - 2x + 1 = 12x - 4$

$6x^2 - 17x + 5 = 0$

(2x-5)(3x-1)=0 so x=5/2 or x= 1/3

Example 3.

Co $= 30$

Tripling time = 3 hr

$$\rightarrow \therefore \begin{cases} 3 = 1e^{r(3)} \\ 3 = e^{3r} \\ \ln 3 = \ln e^{3r} \\ \ln 3 = 3r \overbrace{\cancel{\ln e}}^{1} \\ r = \frac{\ln 3}{3} \end{cases}$$

Let t=9hr

a) $C(t) = \text{Co}\, e^{rt}$

$$C(9) = 20e^{\frac{\ln 3}{3}(9)}$$
$$= 20e^{3\ln 3}$$
$$= 20e^{\ln 3^3}$$
$$= 20e^{\ln 27}$$
$$= 20(27) = 540$$

b) $C(30) = 20e^{\frac{\ln 3}{3}(30)}$

$$= 20e^{10\ln 3}$$
$$= 20e^{\ln 3^{10}}$$
$$= 20(3^{10})$$

C1. $2^{4x-1} = 2^6$

4x-1=6

4x=7

x=7/4

Therefore, answer is A) .

C2. $3^{3x-4} = 3^4$

3x-4=4

3x=8

x=8/3. Therefore, answer is B) .

C3. $5^{6x} = 5^4$

6x=4

x=2/3

Therefore, answer is C) .

C4. $(2^{-6})^2 = 4^{3x}$

$2^{-12} = 2^{6x}$

-12=6x

x=-2

$C5.\quad e^{-x} = \ln 2$

$\ln e^{-x} = \ln(\ln 2)$

$-x \ln e = \ln(\ln 2)$

$x = -\ln(\ln 2)$

C6.(3x-1)(2x-1)=3(4x-1)-1

$6x^2 - 3x - 2x + 1 = 12x - 4$

$6x^2 - 17x + 5 = 0$

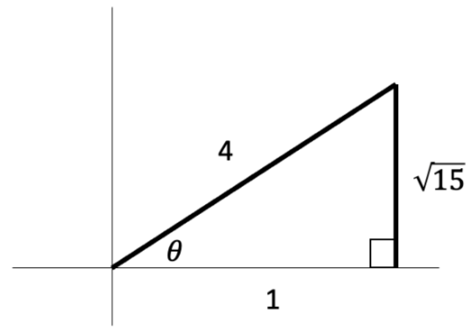
(2x-5)(3x-1)=0 so x=5/2 or x= 1/3

D. Cancellation Laws

Example 1.

$\cos(\cos^{-1}(\frac{-1}{2})) = \frac{-1}{2}$ *since* $x \in [-1,1]$

Example 2.



$\cos\left[2\arccos\left(\frac{1}{4}\right)\right]$

Let $\arccos\left(\frac{1}{4}\right) = \theta$, *then we want* $\cos 2\theta = 2\cos^2\theta - 1$

Since $\arccos\left(\frac{1}{4}\right) = \theta$, *we have* $\cos\theta = \frac{1}{4}$

So, $2\cos^2\theta - 1 = 2\left(\frac{1}{4}\right)^2 - 1 = \frac{2}{16} - \frac{16}{16} = -\frac{14}{16} = -\frac{7}{8}$

$\sin\left[2\arccos\left(\frac{1}{4}\right)\right]$ =we need to find $\sin 2\theta = 2\sin\theta\cos\theta$

$$\sin\theta = \frac{opp}{hyp} = \frac{\sqrt{15}}{4}$$

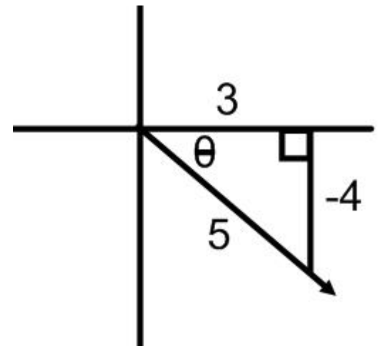
$\sin\left[2\arccos\left(\frac{1}{4}\right)\right] = 2\left(\frac{\sqrt{15}}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{8}\sqrt{15}$

***note: for the first part, you can use any of the identities for $\boldsymbol{\cos 2\theta}$ and you will get the same answer**

Example 3.

$\cos\left[arctan\left(-\frac{4}{3}\right)\right]$

Let $arctan\left(-\frac{4}{3}\right)=\theta, so\ tan\theta=-\frac{4}{3}=\frac{opp}{adj}$ and since it was arctanx we know it is in the fourth quadrant since it was negative (p.32 of booklet)



Find the other side using Pythagorean theorem and we get 5.

Now, we want $\cos\theta=\frac{adj}{hyp}=\frac{3}{5}$.

Example 4.

$\cos\left[2arcsin\left(\frac{1}{5}\right)\right]$

Let $arcsin\left(\frac{1}{5}\right)=\theta, we\ want\ cos2\theta=2cos^2\theta-1$

From $arcsin\left(\frac{1}{5}\right)=\theta, we\ know\ sin\theta=\frac{1}{5}=\frac{opp}{hyp}$ Using Pythagorean theorem, the other side would be $\sqrt{24}$ and then $\cos\theta=\frac{\sqrt{24}}{5}$. (In first quadrant, since it was arcsinx of a positive x)

So, $cos2\theta=2cos^2\theta-1$=$2(\frac{\sqrt{24}}{5})^2-1=2\left(\frac{24}{25}\right)-1=\frac{48}{25}-\frac{25}{25}=\frac{23}{25}$

Example 5.

$\sin(2\cos^{-1}(x))$

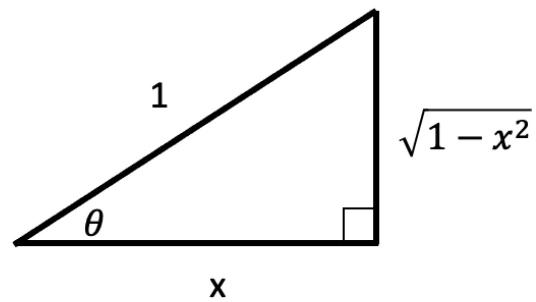
Let $\theta = \cos^{-1}(x)$

$\therefore \cos\theta = \frac{\mathrm{x}}{1} = \frac{\mathrm{adj}}{\mathrm{hyp}}$

$\sin\theta = \frac{\mathrm{opp}}{\mathrm{hyp}} = \sqrt{\frac{1-x^2}{1}}$

$\therefore \sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(\sqrt{\frac{1-x^2}{1}}\right)\left(\frac{x}{1}\right) = 2x\sqrt{1-x^2}$



D1. $\sin(\sin^{-1}(\frac{-1}{2})) = \frac{-1}{2}$ $\quad since\ x \in [-1,1]$

D2.

$\cos^{-1} x$ $\quad defined\ in\ 1st\ and\ 2nd\ quadrants\ since\ we\ want$ $\arccos\left(\frac{-1}{2}\right)$

$negative\ number$

$\therefore\ looking\ at\ 2nd\ quadrant\ by\ CAST\ rule$

$arc\cos\left(\frac{-1}{2}\right) = \theta$ $\qquad \cos\theta = \frac{-1}{2}$ $\quad adj\ hyp$

$\therefore \sin(arc\cos(\frac{-1}{2})) = \sin\theta = \frac{opp}{hyp} = \frac{\sqrt{3}}{2}$

$D3.\quad \sin(\tan^{-1}\left(\frac{-1}{2}\right) = \sin\theta \qquad \tan^{-1}\left(\frac{-1}{2}\right) = \theta \quad negative$

$$\therefore we\ are\ in\ 4th\ quadrant$$

$2^2 + (-1)^2 = c^2 \qquad \tan\theta = \frac{-1}{2}\frac{opp}{adj}$

$5 = c^2$

$c = \sqrt{5}$

$$\therefore \sin(\tan^{-1}\left(\frac{-1}{2}\right) = \sin\theta = \frac{-1}{\sqrt{5}}$$

D4. $\tan(arc\tan 1) = 1 \quad since\ arc\tan x\ is\ defined\ for\ all\ x \in R$

D5. $arc\cos(\cos\frac{\pi}{3}) = arc\cos\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$*recall\ arc\cos(\cos x) = x\ \ for\ all\ \ x \in\ [0, \pi]$

D6. $\sin(arc\cos\frac{1}{2}) = \sin\theta$

$arc\cos\frac{1}{2} = \theta$

$\cos\theta = \frac{1}{2}\frac{adj}{hyp} \quad arc\cos x\ is\ defined\ in\ 1st\ and\ 2nd\ qaudrant.$

$Since\ \frac{1}{2}\ is\ positive, we\ are\ in\ the\ 1st\ quadrant.$

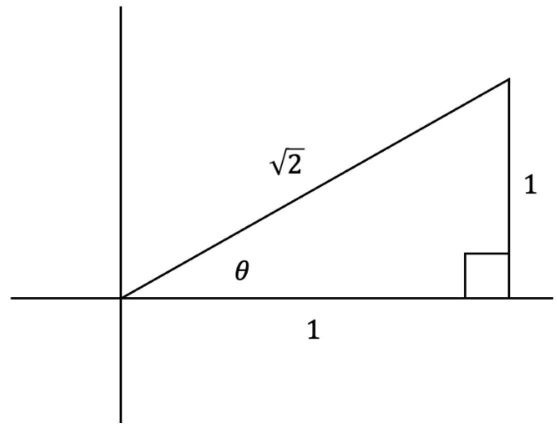
$$\sin(arc\cos\frac{1}{2}) = \sin\theta = \frac{opp}{hyp} = \frac{\sqrt{3}}{2}$$

D7. $\tan(arc\cos\frac{1}{\sqrt{2}})$ $\quad positive \quad \therefore in\ 1st\ quadrant$

$arc\cos\frac{1}{\sqrt{2}} = \theta$

$\cos\theta = \frac{1}{\sqrt{2}}\ \frac{adj}{hyp}$

$\therefore \tan(arc\cos\frac{1}{\sqrt{2}}) = \tan\theta = \frac{1}{1} = 1$



D8. $\cos(arc\sin\frac{1}{2})$ $\quad arc\ sinx\ \ is\ defined\ in\ 1st\ and\ 4th$

$Let\ arc\sin\frac{1}{2} = \theta \qquad \sin\theta = \frac{1}{2}\quad \frac{opp}{hyp}$

Since 1/2 is a positive number, we are in the 1st quadrant

$\cos(arc\sin\frac{1}{2}) = \cos(\theta) = \frac{\sqrt{3}}{2}$

D9. $\cos(arc\cos\sqrt{3}/\sqrt{2}\,) \ = \frac{\sqrt{3}}{2} \quad since\ \frac{\sqrt{3}}{2} \in [-1,1]$

D10. $\cos^{-1}(\tan\frac{\pi}{4}) = \cos^{-1}(1) = 0$

D11. $\cos(\sin^{-1}\left(\frac{-3}{5}\right)$ $\sin^{-1}\left(\frac{-3}{5}\right) = \theta$

$\sin\theta = \frac{-3}{5}$ $negative\ \therefore$

$\sin^{-1}x\ is\ in\ the\ 4th\ quadrant$

$(-3)^2 + x^2 = 5^2$

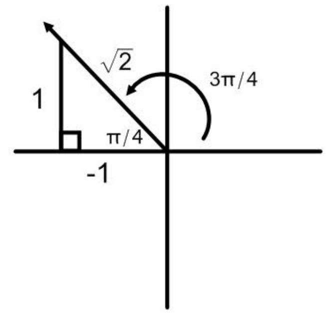
$x^2 = 25 - 9$

$x^2 = 16$

$x = 4$

$\cos(\sin^{-1}(\frac{-3}{5})) = \cos\theta = \frac{4}{5}$

D12. arctan $\left[tan\left(\frac{3\pi}{4}\right)\right]$



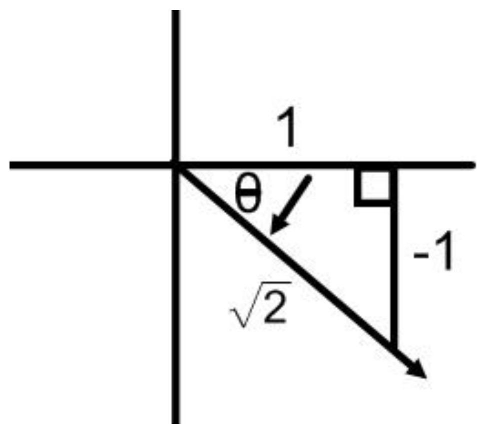
$tan\left(\frac{3\pi}{4}\right) = \frac{opp}{adj} = \frac{1}{-1} = -1$ using special triangles

$$\text{arctan}\left[tan\left(\frac{3\pi}{4}\right)\right] = \text{arctan}(-1)$$

$= \theta$ *so we draw the angle in the* 4*th quadrant* (*see p.* 36)

and $tan\theta = -1$

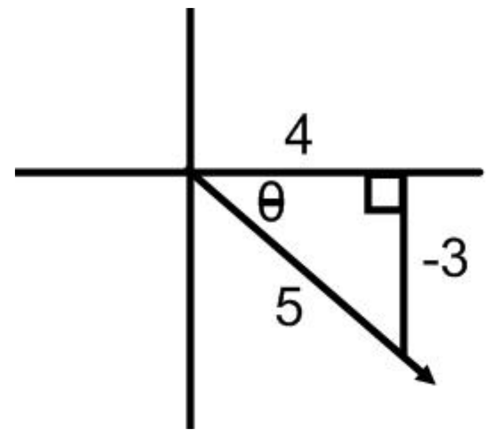
$\theta = -\frac{\pi}{4}$.



D13. $\cos\left[arcsin\left(-\frac{3}{5}\right)\right]$

Let $arcsin\left(-\frac{3}{5}\right) = \theta, then\ sin\theta = -\frac{3}{5}\ and\ it\ is\ in\ the\ 4th\ quadrant$. Using Pythagorean Theorem, we get the other side is 4.

We want to find $\cos\left[arcsin\left(-\frac{3}{5}\right)\right] = cos\ \theta = \frac{adj}{hyp} = \frac{4}{5}$ from our diagram.



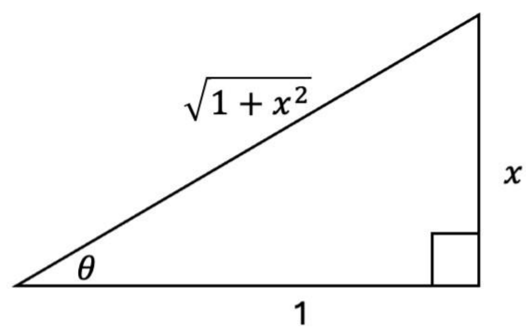
D14. $\cos(2\tan^{-1}(x))$

Let $\theta = \tan^{-1} x$

$\tan\theta = \frac{x}{1}$

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{1+x^2}}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1+x^2}}$



$\cos 2\theta = cos^2\,\theta - sin^2\theta$

$= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 - \left(\frac{x}{\sqrt{1+x^2}}\right)^2$

$= \frac{1}{1+x^2} - \frac{x^2}{1+x^2}$

$= \frac{1-x^2}{1+x^2}$

E. Inverse

Example 1.

1. no
2. no
3. yes
4. yes

Visually Example

$f(4) = 5$ means $f^{-1}(5) = 4$

$f^{-1}(10) = 12 \therefore f(12) = 10$

Example 2

$g^{-1}(4) = 6$

$\therefore g(6) = 4$

$f(x) = \sqrt[3]{2x - 2}$

$f(-1) = \sqrt[3]{2(-1) - 2}$

But

$f^{-1}(-1)?$

$(-1)^3 = \left(\sqrt[3]{2(-1) - 2}\right)^3$

$-1 = 2x - 2$

$-1 + 2 = 2x$

$2x = 1$

$x = \frac{1}{2}$

Point (7,2) is on g

$\therefore g(7) = 2$

$g^{-1}(2) = 7$

a) $f^{-1}(-1) + g(6)$

$= \frac{1}{2} + \frac{4}{1}$

$= \frac{1}{2} + \frac{8}{2} = \frac{9}{2}$

b) $g^{-1}(f(5))$

$f(x) = \sqrt[3]{2x-2}$

$f(5) = \sqrt[3]{2(5)-2}$

$= \sqrt[3]{8} = 2$

$\therefore g^{-1}(f(5)) = g^{-1}(2) = 7$

Example 3.

$y = \frac{4}{x+2}$ X←→ Y

$$x = \frac{4}{y+2}$$

$xy + 2x = 4$

$xy = 4 - 2x$

$y = \frac{4-2x}{x}$ $\therefore f^{-1}(x) = \frac{4-2x}{x}$

Example 4. $y = \frac{3x+1}{x-1}$ X←→ Y

$$x = \frac{3y+1}{y-1}$$

$xy - x = 3y + 1$

$xy - 3y = 1 + x$

$y(x-3) = 1 + x$

$$y = \frac{1+x}{x-3} \qquad \therefore f^{-1}(x) = \frac{1+x}{x-3}$$

Example 5.

$y = \sqrt{1-x^2},\ 0 \leq x \leq 1$ X←→ Y

$x = \sqrt{1-y^2}$ square both sides

$x^2 = 1 - y^2$

$y^2 = -x^2 + 1$

$y = \sqrt{1-x^2}, 0 \leq y \leq 1$

$\therefore f^{-1}(x)=\sqrt{1-x^2}, 0 \leq y \leq 1$

Example 6. $y = tan^{-1}(lnx)$ X←→ Y

$x = tan^{-1}(lny)$

$tanx = lny$ exponential form gives: $y = e^{tanx}$ $\therefore f^{-1}(x)=e^{tanx}$

Example 7.

$y = 20(1 - e^{-4x})$

X←→ Y

$\frac{x}{20} = \frac{20}{20}(1 - e^{-4y})$

$\frac{x}{20} = (1 - e^{-4y})$

$\frac{x}{20} - 1 = -e^{-4y}$

$-\frac{x}{20} + 1 = e^{-4y}$

$\ln\left|-\frac{x}{20} + 1\right| = \ln e^{-4y}$

$\ln\left|-\frac{x}{20} + 1\right| = -4y \sout{\ln e}$

$\mathrm{y} = -\frac{1}{4}\ln\left|-\frac{x}{20} + 1\right|$

E1. $y = 9e^{x+5} - 2$

$x = 9e^{y+5} - 2$

$x + 2 = 9e^{y+5}$

$\frac{x+2}{9} = e^{y+5}$

$ln\left(\frac{x+2}{9}\right) = ln e^{y+5}$

$y + 5 = ln\left(\frac{x+2}{9}\right)$

$y = ln\left(\frac{x+2}{9}\right) - 5$

$\therefore f^{-1}(x) = ln\left(\frac{x+2}{9}\right) - 5$

E2. $y = ln(7x + 2)$

$x = ln(7y + 2)$

$e^x = e^{ln(7y+2)}$

$e^x = 7y + 2$

$7y = e^x - 2$

$y = \frac{e^x - 2}{7}$

$f^{-1}(x) = \frac{e^x - 2}{7}$

E3. $y = e^{3x+5}$

$x = e^{3y+5}$

$\ln x = \ln e^{3y+5}$

$\ln x = (3y+5)\ln e$

$\dfrac{\ln x - 5}{3} = y \qquad \therefore f^{-1}(x) = \dfrac{\ln x - 5}{3}$

E4. $y = \frac{x}{3+x}$

$x = \dfrac{y}{3+y}$

$3x + xy = y$

$3x = y - xy$

$3x = y(1-x)$

$y = \dfrac{3x}{1-x} \qquad \therefore f^{-1}(x) = \dfrac{3x}{1-x}$

E5. $y = \ln(4x-2)$

$x = \ln(4y-2)$

$e^x = e^{\ln(4y-2)}$

$e^x = 4y - 2$

$4y = e^x + 2 \qquad \therefore f^{-1}(x) = \dfrac{e^x + 2}{4}$

E6. $g(x) = \frac{1}{\sqrt{4-x^2}}$ $\quad -2 < x \leq 0$

$$y = \frac{1}{\sqrt{4-x^2}}$$

x←→ y $\quad x = \frac{1}{\sqrt{4-y^2}}$

$$\left(x\sqrt{4-y^2}\right)^2 = (1)^2$$

$$x^2(4-y^2) = 1$$

$$4x^2 - x^2y^2 = 1$$

$$4x^2 = 1 + x^2y^2$$

$$4x^2 - 1 = x^2y^2$$

$$y^2 = \frac{4x^2-1}{x^2}$$

$$y = -\sqrt{\frac{4x^2-1}{x^2}} \qquad (-2 < y \leq 0)$$

$$f^{-1}(x) = -\sqrt{\frac{4x^2-1}{x^2}}$$

F. Logarithms

Example 1. Solve each of the following:

a) $log_3 27$ =3 since $3^3 = 27$

b) $log_{25} 5$ =$\frac{1}{\log_5 25} = \frac{1}{2}$

c) $log_2 32$ =5 since $2^5 = 32$

Example 2. Simplify each of the following using logarithmic rules.

a) $6^{3\log_6 2}$=$6^{\log_6 2^3}$

$= 2^3$

=8

b) $\log_4 4^3$ =3

c) $e^{ln3+ln6}$

=$e^{\ln 18} = 18$

d) $e^{ln12-ln4}$

$= e^{\ln(\frac{12}{4})} = e^{\ln 3} = 3$

e) $e^{2ln4+ln5}$

$=e^{\ln 4^2+\ln 5}$

$=e^{\ln 16+\ln 5}$

$=e^{\ln(16\times 5)} = 80$

f) $\log_3 27 - \log_2 32$

=3-5=-2

Example 3. Solve each of the following:

b) $\mathrm{e}^{3\mathrm{t}} = 14$ take ln of both sides $\ln e^{3t} = \ln 14$

$3t \ln e = \ln 14$

$3t = \ln 14$

$t = \frac{\ln 14}{3}$

c) $\log_3(4\mathrm{x}+7) - \log_3(\mathrm{x}+1) = 2$

$\log_3\left(\frac{4x+7}{x+1}\right) = 2$

$3^2 = \frac{4x+7}{x+1}$

$9x + 9 = 4x + 7$

$5x = -2$

$x = -\frac{2}{5}$ substitute into both original logarithms and both are >0, so it is a solution

d) $2(3^{x-5}) = 28$ divide both sides by 2

$3^{x-5} = 14$ take ln of both sides

$\ln 3^{x-5} = \ln 14$

$(x-5)\ln 3 = ln14$

$x \ln 3 - 5\ln 3 = ln14$

$x \ln 3 = \ln 14 + \ln 3^5$

$x = \frac{\ln 14 + \ln 3^5}{\ln 3}$

*f) $\frac{\log_3 5 + \log_3 25}{\log_3 125 - \log_3 5} = \frac{\log_3 125}{\log_3\left(\frac{125}{5}\right)} = \frac{\log_3 125}{\log_3 25} = \frac{\log_5 125}{\log_5 25} = \frac{3}{2}$

*g) $\log_{25} 625 + \log_6 36 + \log_4 32 = \frac{\log 625}{\log 25} + 2 + \frac{\log 32}{\log 4}$

$= \frac{\log_5 625}{\log_5 25} + 2 + \frac{\log_2 32}{\log_2 4}$

$= \frac{4}{2} + \frac{4}{2} + \frac{5}{2}$

$= \frac{13}{2}$

*h) $\frac{\log_4 x + \log_4 y}{\log_4 x^2 + \log_4 y^2} = \frac{\log_4 x + \log_4 y}{2\log_4 x + 2\log_4 y} = \frac{\log_4 x + \log_4 y}{2(\log_4 x + \log_4 y)}$

$= \frac{1}{2}$

*i) $\frac{\log_4 32}{\log_{27} 9} = \frac{\frac{log 32}{\log 4}}{\frac{\log 9}{\log 27}} = \frac{\frac{\log_2 32}{\log_2 4}}{\frac{\log_3 9}{\log_3 27}} = \frac{\frac{5}{2}}{\frac{2}{3}} = \frac{5}{2} \times \frac{3}{2} = \frac{15}{4}$

J) $\log_9 x + \log_3 x = \frac{1}{2}$

$\frac{\log x}{\log 9} + \log_3 x = \frac{1}{2}$

$\frac{\log_3 x}{\log_3 9} + \log_3 x = \frac{1}{2}$

$\frac{\log_3 x}{2} + \log_3 x = \frac{1}{2}$

$\frac{1}{2}\log_3 x + 1 \log_3 x = \frac{1}{2}$

$\frac{3}{2}\log_3 x = \frac{1}{2}$

$\log_3 x = \frac{1}{2} \div \frac{3}{2}$

$\log_3 x = \frac{1}{2} \times \frac{2}{3}$

$\log_3 x = \frac{1}{3}$

$x = 3^{\frac{1}{3}} = \sqrt[3]{3}$

Example 4.

Substitute $\binom{1,2}{x,y}$ $\quad 2 = \log_a 1 + b$

$$2 = 0 + b$$

$$b = 2$$

Substitute $B\binom{4,3}{x,y}$ $\quad y = \log_a x + 2$

$$3 = \log_a 4 + 2$$

$$1 = \log_a 4$$

$$a^1 = 4$$

$$a = 4$$

$$\therefore \textit{the equation is } \; y = \log_4 x + 2$$

F1. $e^{3x+5} = 6$

$\ln e^{3x+5} = \ln 6$

$(3x+5)\ln e = \ln 6$

$\frac{3x}{3} = \frac{\ln 6 - 5}{3}$ $\qquad x = \frac{\ln 6 - 5}{3}$

F2. $\ln(x^2 + x) = 2 + \ln x$

$\ln(x^2 + x) - \ln x = 2$

$\ln\left(\frac{x^2 + x}{x}\right) = 2$ $\quad$ *put into exp. form*

$e^2 = \frac{x^2 + x}{x}$

$e^2 x = x^2 + x$

$x^2 + x - e^2 x = 0$

$x^2 + x(1 - e^2) = 0$

$x[x + (1 - e^2)] = 0$

$x = 0 \quad OR \quad x + 1 - e^2 = 0 \qquad x = e^2 - 1$

$x \neq 0$ *since* $\ln 0$ *is undefined* $\quad \therefore x = e^2 - 1$

F3. $\log_8 2 + \log_2 8 \;=\; \frac{1}{\log_2 8} + \log_2 8 \;=\; \frac{1}{3} + 3$

$= 3\frac{1}{3}$ OR $\frac{10}{3}$

F4. $\log_2(x+2) - \log_2(2x+1) = 4$

$$\log_2\left(\frac{x+2}{2x+1}\right) = 4$$

$$2^4 = \frac{x+2}{2x+1}$$

$$16 = \frac{x+2}{2x+1}$$

$$32x + 16 = x + 2$$

$$31x = 2 - 16$$

$$31x = -14$$

$$x = -\frac{14}{31} \quad \mathit{Check} \quad \log_2\left(2\left(-\frac{14}{31}\right) + 1\right)$$

$$= \log_2\left(-\frac{28}{31} + \frac{31}{31}\right)$$

$$= \log_2\left(\frac{3}{31}\right) \ \mathit{is\ defined} \quad \therefore x = -\frac{14}{31}$$

F5. $\log_2 16 - \log_2 32 = \log_2(\frac{16}{32}) = \log_2 \frac{1}{2} = -1$

Therefore, answer is E).

F6. $2\log_2 4 + \log_2 8 = \log_2 4^2 + 3 = \log_2 16 + 3 = 4 + 3 = 7$

Therefore, answer is A).

F7. $\log_4 32 - \log_4 16 = \log_4(\frac{32}{16}) = \log_4 2 = \frac{1}{\log_2 4} = \frac{1}{2}$

Therefore, answer is C).

F8. $e^{ln3+ln9} = e^{\ln 27} = 27$

Therefore, answer is B).

F9. $\frac{\log_5 3+\log_5 81}{\log_5 81} = \frac{\log_5 243}{\log_5 81} = \frac{\log_3 243}{\log_3 81} = \frac{5}{4}$

$$Change\ both\ bases, so\ you\ can\ evaluate$$

F10. $\frac{\log_4 25}{\log_4 125} = \frac{\log_5 25}{\log_5 125} = \frac{2}{3}$ Change of base

Therefore, answer is C).

F11. Simplify: $\frac{\log_9 3-\log_2 16}{\log_4 64} = \frac{\frac{1}{\log_3 9}-4}{3} = \frac{\frac{1}{2}-4}{3} = \frac{\frac{1}{2}-\frac{8}{2}}{\frac{3}{1}} = \frac{-7}{2} \times \frac{1}{3} = -\frac{7}{6}$

F12. $\log_8 32 + \log_9 27 = \frac{log32}{\log 8} + \frac{\log 27}{\log 9} = \frac{\log_2 32}{\log_2 8} + \frac{\log_3 27}{\log_3 9}$

$$= \frac{5}{3} + \frac{3}{2} = \frac{10}{6} + \frac{9}{6} = \frac{19}{6}$$

F13. $\frac{lnx^8+lny^8}{ln(x^4y^4)} = \frac{\ln(x^8y^8)}{ln(x^4y^4)} = \frac{ln(x^4y^4)^2}{ln(x^4y^4)} = \frac{2ln(x^4y^4)}{ln(x^4y^4)}$

$= 2$

F14. Solve: $log_3 x + \ log_3(x+2) = log_3(x+20)$

$log_3\big(x(x+2)\big) = log_3(x+20)$

$x(x+2) = x + 20$

$x^2 + x - 20 = 0\ factor$

(x+5)(x-4)=0

x= -5,4 but we can't take the log of a negative number (or 0), so we exclude x=-5 and the only solution is x=4.

G. Limits

Example 1. $a)\ \lim_{x\to -4-} f(x) = 3 \qquad \lim_{x\to -4^+} f(x) = 0$

$\therefore \lim_{x\to -4} f(x) = DNE$

$b)\ \lim_{x\to -2} f(x) = -4$

$c)\ \lim_{x\to 4+} f(x) = 5$

$d)\ \lim_{x\to 4-} f(x) = 5$

$e)\ \lim_{x\to 4} f(x) = 5$

$f)\ f(4) = 5$

$g)\ \lim_{x\to 0} f(x) = DNE \quad since \ \lim_{x\to 0^-} f(x) = 0 \quad and \ \lim_{x\to 0^+} f(x) = 1$

$h)\ f(0) = 1$

H. Properties of Limits

Example 1. $\lim_{x\to4}\sqrt{x}=\sqrt{4}=2$

Example 2. $\lim_{x\to1}(x^2+3x)=1^2+3(1)=4$

Example 3. $\lim_{x\to2}\frac{x^2-4}{5x}=\frac{2^2-4}{5(2)}=\frac{0}{10}=0$

Example 5. $=\lim_{x\to2}\frac{(x-2)(x-3)}{(x-2)}$

$=\lim_{x\to2}(x-3)$

$=2-3$

$=-1$

Example 6. $=\lim_{x\to3}\frac{(x-3)(\sqrt{x-2}+1)}{(\sqrt{x-2}-1)(\sqrt{x-2}+1)}$

$=\lim_{x\to3}\frac{(x-3)(\sqrt{x-2}+1)}{x-2-1}$

$=\lim_{x\to3}\frac{(x-3)(\sqrt{x-2}+1}{(x-3)}$

$=\lim_{x\to3}(\sqrt{x-2}+1)$

$=1+1$

$=2$

Example 7. Since $x \to 3$ makes $|x-3| = 0$ $\therefore$ we must do left and right hand limits

$\therefore \lim_{x\to 3^-} \frac{(x-2)(x-3)}{-(x-3)}$

$\lim_{x\to 3^-} \frac{x-2}{-1}$

$= \frac{3-2}{-1}$

$= -1$

$\lim_{x\to 3^+} \frac{(x-2)(x-3)}{(x-3)}$

$\lim_{x\to 3^+} (x-2)$

$= 3-2$

$= 1$

$\therefore \lim_{x\to 3} \frac{x^2-5x+6}{|x-3|} = DNE$

Since $\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$

Example 8. $\lim_{x\to 0} \frac{sin5x}{sin2x}$

$= \lim_{x\to 0} \frac{5xsin5x/5x}{2xsin2x/2x}$

$= \frac{5}{2} \lim_{x\to 0} \frac{sin5x/5x}{sin2x/2x}$

$= \frac{5}{2} \frac{\lim_{x\to 0} sin5x/5x}{\lim_{x\to 0} sin2x/2x}$

$=\frac{5}{2}$ since $\lim_{x\to 0} \frac{sinx}{x} = 1$ $(p.79\ of\ booklet)$

Example 9. The limit $lim_{x\to\pi}\ \frac{f(x)-4}{x-\pi}$ can only equal a number if f(x)−4 → 0

a) So, therefore, f(x) → 4 and the limit is 4 as well. Since the answer to the limit is just a number, it would have to be 0/0 and you did something to make it a constant answer. Otherwise, since the bottom is 0, it would be #/0 which would either be plus or minus infinity.

b) If $\lim_{x\to 3^+} \frac{f(x)-7}{x-3} = 2, find \lim_{x\to 3^+} f(x)$. Here again, in order to get a numerical answer of 2, it must be 0/0 as otherwise it would be #/0 which is +/- infinity. So, f(x) – 7 → 0 and $\lim_{x\to 3^+} f(x) = 7$

<u>Example 10.</u>

$$\lim_{x\to 4}\frac{\sqrt{8-x}-2}{\sqrt{5-x}-1} \qquad (0/0)$$

$$=\lim_{x\to 4}\frac{\left(\sqrt{8-x}-2\right)\left(\sqrt{5-x}+1\right)}{\left(\sqrt{5-x}-1\right)\left(\sqrt{5-x}+1\right)}$$

$$=\lim_{x\to 4}\frac{\left(\sqrt{8-x}-2\right)\left(\sqrt{5-x}+1\right)}{(5-x-1)}$$

$$=\lim_{x\to 4}\frac{\left(\sqrt{5-x}+1\right)\left(\sqrt{8-x}-2\right)\left(\sqrt{8-x}+2\right)}{4-x\left(\sqrt{8-x}+2\right)}$$

$$=\lim_{x\to 4}\frac{\left(\sqrt{5-x}+1\right)(8-x-4)}{(4-x)\left(\sqrt{8-x}+2\right)}$$

$$=\lim_{x\to 4}\frac{\left(\sqrt{5-x}+1\right)(\cancel{x-4})}{\cancel{(4-x)}\left(\sqrt{8-x}+2\right)}$$

$$\lim_{x\to 4}\frac{\left(\sqrt{5-x}+1\right)}{\left(\sqrt{8-x}+2\right)}$$

$$=\frac{\sqrt{5-4}+1}{\sqrt{8-4}+2}$$

$$=\frac{2}{4}=\frac{1}{2}$$

Example 11 $= \lim_{t\to 0}(\frac{1}{t} + \frac{1}{t(t-1)})$

$$= \lim_{t\to 0}(\frac{1(t-1)}{t(t-1)} + \frac{1}{t(t-1)})$$

$$= \lim_{t\to 0}\frac{t-1+1}{t(t-1)}$$

$$= \lim_{t\to 0}\frac{t}{t(t-1)}$$

$$= \lim_{t\to 0} t = \frac{1}{0-1} \quad = -1$$

H1. (%)

$$= \lim_{h \to 0} \frac{h(4h+1)}{h(1+h)}$$

$$= \lim_{h \to 0} \frac{4h+1}{1+h}$$

$$= \frac{1}{1}$$

$$= 1$$

H2. $= \lim_{x \to 5} \frac{\sqrt{5-2}}{5-5} = \frac{\sqrt{3}}{0} = \infty$

H3. (%)

$$= \lim_{x \to 4} \frac{(x-4)(x+4)}{(x-4)}$$

$$= \lim_{x \to 4} \frac{(x+4)}{1} = 4 + 4 = 8$$

H4. $= \frac{0}{1+2(4)} = \frac{0}{9} = 0$

H5. (%)

$$= \lim_{h \to 0} \frac{h(6+h)}{h(1+h^2)}$$

$$= \lim_{h \to 0} \frac{6+h}{1+h^2} = \frac{6}{1} = 6$$

H6. $|x| = x \quad since\ x \to 0^+$

$$\therefore \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

H7. $= \lim_{x \to 7} \frac{(x-7)(x+7)}{x-7}$

$= \lim_{x \to 7} (x+7) = 7 + 7 = 14$

H8. (%)

$= \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)(x-2)}$

$= \lim_{x \to 2} \frac{x+2}{x-2} = \frac{2+2}{2-2} = \frac{4}{0} = \infty$

H9. $= \lim_{h \to 0} (\frac{1}{h} + \frac{1}{h(h-1)})$

$= \lim_{h \to 0} (\frac{1(h-1)}{h(h-1)} + \frac{1}{h(h-1)})$

$= \lim_{h \to 0} \frac{h-1+1}{h(h-1)}$

$= \lim_{h \to 0} \frac{h}{h(h-1)}$

$= \lim_{h \to 0} \frac{1}{h-1} = \frac{1}{0-1} = -1$

H10. (%)

$= \lim_{x \to 3} \frac{(x-3)(\sqrt{x-2}+1)}{(\sqrt{x-2}-1)(\sqrt{x-2}+1)}$

$= \lim_{x \to 3} \frac{(x-3)(\sqrt{x-2}+1)}{x-2-1}$

$= \lim_{x \to 3} (\sqrt{x-2} + 1)$

$= \sqrt{3-2} + 1 = 2$

H11. (%)

$$\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}$$

$$=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{x-1}$$

$$\lim_{x\to 1}(\sqrt{x}+1)=1+1=2$$

H12. $$=\lim_{x\to 0}\frac{-(x-3)-(6x+3)}{x}$$

$$=\lim_{x\to 0}\frac{-x+3-6x-3}{x}$$

$$=\lim_{x\to 0}\frac{-7x}{x}=-7$$

H13. $$=\lim_{x\to 0}\frac{-(x-5)-(x+5)}{x}$$

$$=\lim_{x\to 0}\frac{-x+5-x-5}{x}$$

$$=\lim_{x\to 0}\frac{-2x}{x}=-2$$

H14. $$\lim_{h\to 0}\frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)}$$

$$=\lim_{h\to 0}\frac{9+h-9}{h(\sqrt{9+h}+3)}$$

$$=\lim_{h\to 0}\frac{h}{h(\sqrt{9+h}+3)}$$

$$=\lim_{h\to 0}\frac{1}{(\sqrt{9+h}+3)}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}$$

H15. $\lim_{x\to 0}\frac{sin3x}{sin4x}$

$$=\lim_{x\to 0}\frac{3xsin3x/3x}{4xsin4x/4x}$$

$$=\frac{3}{4}\lim_{x\to 0}\frac{sin3x/3x}{sin4x/4x}$$

$$=\frac{3}{4}\frac{\lim_{x\to 0}}{\lim_{x\to 0}}\frac{sin3x/3x}{sin4x/4x}$$

$=\frac{3}{4}$ since $\lim_{x\to 0}\frac{sinx}{x}=1$ $\quad(p. 70\ of\ booklet)$

H16. The limit $\lim_{x\to 5^+}\frac{f(x)+5}{x-5}$ can only equal a number if f(x)+5 $\to 0$

So, therefore, f(x) $\to -5$

H17. $\lim_{x\to-\infty} \frac{5x-2}{\sqrt{5x^2+x^3-5x+1}}$

$\lim_{x\to-\infty} \frac{5x-2}{\sqrt{x^3\left(\frac{5}{x}+1-\frac{5}{x^2}+\frac{1}{x^3}\right)}}$ $\sqrt{x^3} = x^{\frac{3}{2}}$

$$\lim_{x\to-\infty} \frac{5x-2}{x^{\frac{3}{2}}\sqrt{\left(\frac{5}{x}+1-\frac{5}{x^2}+\frac{1}{x^3}\right)}}$$

$$= \lim_{x\to-\infty} \frac{\frac{5x}{x^{\frac{3}{2}}}-\frac{2}{x^{\frac{3}{2}}}}{\frac{x^{\frac{3}{2}}}{x^{\frac{3}{2}}}\sqrt{\left(\frac{5}{x}+1-\frac{5}{x^2}+\frac{1}{x^3}\right)}}$$

$$= \lim_{x\to-\infty} \frac{\frac{5x^1}{x^{\frac{3}{2}}}-\frac{2}{x^{\frac{3}{2}}}}{\sqrt{\left(\frac{5}{x}+1-\frac{5}{x^2}+\frac{1}{x^3}\right)}} = \lim_{x\to-\infty} \frac{\frac{5}{x^{\frac{1}{2}}}-\frac{2}{x^{\frac{3}{2}}}}{\sqrt{\left(\frac{5}{x}+1-\frac{5}{x^2}+\frac{1}{x^3}\right)}} = \frac{0\ -0}{\sqrt{(0+1-0+0)}}$$

$= \frac{0}{\sqrt{1}} = 0$

I. Continuity

Example 1. $x^2 - 25 = 0$

$x^2 = 25$

$x = 5, -5$

$\therefore f(x) \text{ is discontinuous at } x = 5, -5$

Example 2. a) $= 1^2 - 9 = -8$

b) $= \lim_{x \to 1^-} (x^2 - 9) = 1^2 - 9 = -8$

c) $= \lim_{x \to 1^+} (2x + 4) = 2(1) + 4 = 6$

d) No, since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

Example 3. Since the function is continuous everywhere,
$f(x) \text{is continuous at } x = 3$

$\therefore k(3^2 - 3) = 1 + 3k^2(3)$

$6k = 9k^2 + 1$

$9k^2 - 6k + 1 = 0$

$(3k - 1)(3k - 1) = 0$

$k = \frac{1}{3}$

Example 4.

$$\lim_{x\to 0^+} f(x) = f(0) \quad \therefore \lim_{x\to 0^+} \frac{\sin(3x)}{x} = A$$

$$H = \lim_{x\to 0^+} \frac{3\cos(3x)}{1} = A$$

$$\frac{3(1)}{1} = A \qquad \therefore A = 3$$

Or use the limit $lim_{x\to 0^+} \frac{sin\ (3x)}{x} = A$

$$lim_{x\to 0^+} \frac{3sin\ (3x)}{3x} = A$$

$$3lim_{x\to 0^+} \frac{sin\ (3x)}{3x} = A$$

A=3 since the limit is 1. (see p. 79 booklet)

$$\lim_{x\to 0^-} f(x) = f(0) = A = 3$$

$$\lim_{x\to 0^-} \frac{(3-\sqrt{9-x})(3+\sqrt{9-x})}{Bx(3+\sqrt{9-x})} = 3$$

$$\lim_{x\to 0^-} \frac{(9-(9-x))}{Bx(3+\sqrt{9-x})} = 3$$

$$\lim_{x\to 0^-} \frac{x}{Bx(3+\sqrt{9-x})} = 3$$

$$\frac{1}{B(3+\sqrt{9})} = 3$$

$$\frac{1}{B(6)} = 3$$

$$18B = 1$$

$$B = \frac{1}{18}$$

I1. a) $= 1^2 + 4 = 5$

b) $= \lim_{x\to 1^-}(x^2+4) = 1^2 + 4 = 5$

c) $= \lim_{x\to 1^+}(2x+3) = 2(1) + 3 = 5$

d) $\therefore Yes, since \;\; f(1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

I2. $x^2 - 100 = 0$

$x^2 = 100$

$x = 10, -10$

$\therefore f(x) is\ discontinuous\ at\ x = 10, -10$

I3. a) $f(2) = 2^2 - 4 = 0$

b) $\lim_{x\to 2^-}(x^2-4) = 2^2 - 4 = 0$

c) $\lim_{x\to 2^+}(4x+3) = 4(2) + 3 = 11$

d) $\therefore No, since \;\; \lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$

I4. $f(x)\ is\ continuous\ at\ x = 3$

$\therefore k(3)^2 + 1 = 4(3) + 3$

$9k + 1 = 12 + 3$

$9k = 14$

$k = \frac{14}{9}$

I5. $\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(\ln x) = \ln 1 = 0$

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} e^x = e^1$

$\therefore No, since\ e \neq 0$

I6. $f(x) is\ continuous\ at\ x = 2$

$\therefore 4(2)^2 + k(2) = 2(2)^3 - 5k$

$16 + 2k = 16 - 5k$

$7k = 0$

$k = 0$

I7. $\lim_{x\to0^+} f(x) = f(0) \quad \therefore \lim_{x\to0^+} \frac{\sin(2x)}{x} = A$

$H = \lim_{x\to0^+} \frac{2\cos(2x)}{1} = A$

$\frac{2(1)}{1} = A \qquad \therefore A = 2$

$\lim_{x\to0^-} f(x) = f(0) = A = 2$

$\lim_{x\to0^-} \frac{(4-\sqrt{16-x})(4+\sqrt{16-x})}{Bx(4+\sqrt{16-x})} = 2$

$\lim_{x\to0^-} \frac{(16-(16-x))}{Bx(4+\sqrt{16-x})} = 2$

$\lim_{x\to0^-} \frac{x}{Bx(4+\sqrt{16-x})} = 2$

$\frac{1}{B(4+\sqrt{16})} = 2$

$\frac{1}{B(8)} = 2$

$16B = 1 \quad B = \frac{1}{16}$

J. Limits at Infinity

Example 2. $\lim_{x\to\infty} \frac{x^2(3+\frac{4}{x}-\frac{5}{x^2})}{x^2(6+\frac{2}{x}-\frac{1}{x^2})}$

$$= \lim_{x\to\infty} \frac{3+\frac{4}{x}-\frac{5}{x^2}}{6+\frac{2}{x}-\frac{1}{x^2}} = \frac{3}{6} = \frac{1}{2}$$

Example 3. $\lim_{t\to-\infty} \frac{\sqrt{16t^2+t-4}}{t-4}$

$$= \lim_{t\to-\infty} \frac{\frac{\sqrt{16t^2+t-4}}{t}}{\frac{t-4}{t}}$$

$$= \lim_{t\to-\infty} \frac{-\sqrt{\frac{16t^2+t-4}{t^2}}}{1-\frac{4}{t}}$$

$$= \lim_{t\to-\infty} \frac{-\sqrt{16+\frac{1}{t}-\frac{4}{t^2}}}{1-\frac{4}{t}}$$

$$= -\frac{\sqrt{16+0+0}}{1-0}$$

$$= -4$$

Factor out the highest power in the denominator

J1. $= \lim_{x\to\infty} \frac{x(3x-4-\frac{6}{x})}{x(1-\frac{6}{x})}$

$= \lim_{x\to\infty} \frac{3x-4-\frac{6}{x}}{1-\frac{6}{x}} = \frac{\infty-4-0}{1-0} = \infty$

J2. $= \lim_{x\to\infty} \frac{3-\frac{6}{x}}{1+\frac{2}{x}-\frac{1}{x^2}} = \frac{3}{1} = 3$

J3. $= \lim_{x\to\infty} \frac{x^2(1)}{x^2(6+\frac{1}{x^2})}$

$= \lim_{x\to\infty} \frac{1}{6+\frac{1}{x^2}} = \frac{1}{6+0} = \frac{1}{6}$

J4. $\lim_{t\to-\infty} \frac{t(t^2-\frac{2}{t})}{t(2-\frac{4}{t})}$

$= \lim_{t\to-\infty} \frac{t^2-\frac{2}{t}}{2-\frac{4}{t}} = \frac{(-\infty)^2-0}{2-0} = \infty$

J5.

$$= \lim_{t\to-\infty} \frac{\frac{\sqrt{9t^2+t-3}}{t}}{\frac{t-4}{t}}$$

$$= \lim_{t\to-\infty} \frac{-\sqrt{\frac{9t^2+t-3}{t^2}}}{1-\frac{4}{t}}$$

$$= \lim_{t\to-\infty} \frac{-\sqrt{9+\frac{1}{t}-\frac{3}{t^2}}}{1-\frac{4}{t}}$$

$$= \frac{-\sqrt{9+0+0}}{1-0}$$

$$= -3$$

$J6.$ $\lim_{x\to 0} \sin\left(\frac{1}{x}\right) = DNE\ since \quad -1 \le \sin(\frac{1}{x}) \le 1$

$J7.$ $\lim_{x\to\infty} \frac{4x^4+3x^2+2x}{x^4+2x+1} = \lim_{x\to\infty} \frac{x^4(4+\frac{3}{x^2}+\frac{2}{x^3})}{x^4(1+\frac{2}{x^3}+\frac{1}{x^4})} = \lim_{x\to\infty} \frac{(4+\frac{3}{x^2}+\frac{2}{x^3})}{(1+\frac{2}{x^3}+\frac{1}{x^4})} = \frac{4}{1} = 4$

K. Vertical and Horizontal Asymptotes

Example 1.

a) For VA, if you have a rational function

I.e. $y = \frac{p(x)}{q(x)}$, we check the values of $x = a$ where the denominator $q(a) = 0$ and

a) if numerator $p(a) \neq 0$

→ we have a VA

$$f(x) = \frac{p(x)}{q(x)} = \frac{x+8}{x^2-9}$$

$x^2 - 9 = 0$

$x^2 = 9$

$x = 3, -3$

$p(3) = 3 + 8 = 11 \neq 0$

$p(-3) = -3 + 8 = 5 \neq 0$

∴ we have 2 VA at $x = 3, -3$

$$HA \qquad \lim_{x\to\infty} \frac{x+3}{x^2-9} = \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{3}{x^2}}{1 - \frac{9}{x^2}} = \frac{0+0}{1-0} = \frac{0}{1} = 0$$

$$\therefore y = 0$$

NOTE: if we do the limit as x approaches -∞, it is the same!

b) $f(x) = \frac{x^2-x-12}{x-4}$

if $p(a) = 0$, then, we have to find $\lim_{x \to a} f(x)$ to see if it is $\pm\infty$

if it is $\pm\infty$, there is a VA

VA

$$f(x) = \frac{p(x)}{q(x)}$$

$q(x) = 0 \quad x - 4 = 0 \quad x = 4$

if $p(4) \neq 0$ we have a VA

$$p(x) = (x-4)(x+3)$$

but here $p(4) = (4-4)(4+3) = 0$

Since $p(4) = 0$ we have to do the limit

$$\lim_{x \to 4} \frac{\cancel{(x-4)}(x+3)}{\cancel{(x-4)}} = 4 + 3 = 7 \neq \infty$$

No VA at $x = 4$

$factor \quad f(x) = \frac{(x-4)(x+3)}{(x-4)} = x + 3$ (cancel) This is really just a line with a hole at x=4!

$$HA \quad \lim_{x \to \pm\infty} (x+3) = \pm\infty \quad \therefore none$$

Example 2.

a) $\lim_{x\to\infty} \frac{\frac{3x}{x}+\frac{1}{x}}{\frac{x}{x}+\frac{\sqrt{4x^2+5}}{x}}$ $\qquad$ $\lim_{x\to-\infty} \frac{\frac{3x}{x}+\frac{1}{x}}{\frac{x}{x}-\sqrt{\frac{4x^2+5x}{x^2}}}$

$= \lim_{x\to\infty} \frac{3+\frac{1}{x}}{1+\sqrt{\frac{4x^2+5}{x^2}}}$ $\qquad$ $= \lim_{x\to-\infty} \frac{3+\frac{1}{x}}{1-\sqrt{4+\frac{5}{x^2}}}$

$= \lim_{x\to\infty} \frac{3+\frac{1}{x}}{1+\sqrt{4+\frac{5}{x^2}}}$ $\qquad$ $= \frac{3+0}{1-\sqrt{4+0}}$

$= \frac{3+0}{1+\sqrt{4+0}}$ $\qquad$ $= \frac{3}{1-2}$

$= \frac{3}{1+2} = \frac{3}{3}$ $\qquad$ $= \frac{3}{-1}$

$= 1$ $\qquad$ $= -3$

$y = 1$ *and* $y = -3$ *are the horizontal asymptotes.*

b) $\lim_{x\to\infty} \frac{2^{2x}-3^x}{3^x+4^{x+1}}$

$$= \lim_{x\to\infty} \frac{(2^2)^x - 3^x}{3^x + 4^{x+1}}$$

$$= \lim_{x\to\infty} \frac{4^x - 3^x}{3^x + 4^1 \cdot 4^x}$$

$$= \lim_{x\to\infty} \frac{\frac{4^x}{4^x} - \frac{3^x}{4^x}}{\frac{3^x}{4^x} + \frac{4\cdot 4^x}{4^x}}$$

$$= \lim_{x\to\infty} \frac{1 - \left(\frac{3}{4}\right)^x}{\left(\frac{3}{4}\right)^x + 4}$$

$\left(\frac{3}{4}\right)^x \to 0 \; as \; x \to \infty \; (fraction \; less \; than \; 1)$

Final answer

$= \frac{1+0}{0+4}$ $=1/4$

****NOTE: If it is a long answer, do the full solution we went through in prep!!!**

K1. $f(x) = \frac{x-4}{x^2-5x-6} = \frac{x-4}{(x-6)(x+1)}$

$VA \quad x = 6, -1$

$$HA \quad \lim_{x\to\infty} \frac{x-4}{x^2-5x-6} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{4}{x^2}}{1-\frac{5}{x}-\frac{6}{x^2}} = \frac{0-0}{1-0-0} = 0$$

$$y = 0$$

K2. $f(x) = \frac{4-x}{2x+3}$

$VA \quad 2x + 3 = 0$

$$x = -\frac{3}{2}$$

$$HA \quad \lim_{x\to\infty} \frac{4-x}{2x+3} = \lim_{x\to\infty} \frac{\frac{4}{x}-1}{2+\frac{3}{x}} = \frac{0-1}{2+0} = -\frac{1}{2} \qquad \therefore y = -\frac{1}{2}$$

K3. VA $2e^x - 4 = 0$

$$2e^x = 4$$

$$e^x = 2$$

$$\ln e^x = \ln 2$$

$$x \ln e = \ln 2$$

$$x = \ln 2$$

HA $\lim\limits_{x\to\infty} \frac{\sqrt{4e^{2x}+1}}{2e^x-4}$

$$= \lim_{x\to\infty} \frac{\sqrt{e^{2x}(4+\frac{1}{e^{2x}})}}{e^x(2-\frac{4}{e^x})}$$

$$= \lim_{x\to\infty} \frac{|e^x|\sqrt{4+\frac{1}{e^{2x}}}}{e^x(2-\frac{4}{e^x})}$$

NOTE: $|e^x| = e^x \ \ since \ \ x \to \infty$

$$\therefore e^x > 0$$

$$= \lim_{x\to\infty} \frac{\sqrt{4+e^{-2x}}}{2-4e^{-x}}$$

$$= \frac{\sqrt{4+0}}{2-0}$$

$$= \frac{\sqrt{4}}{2}$$

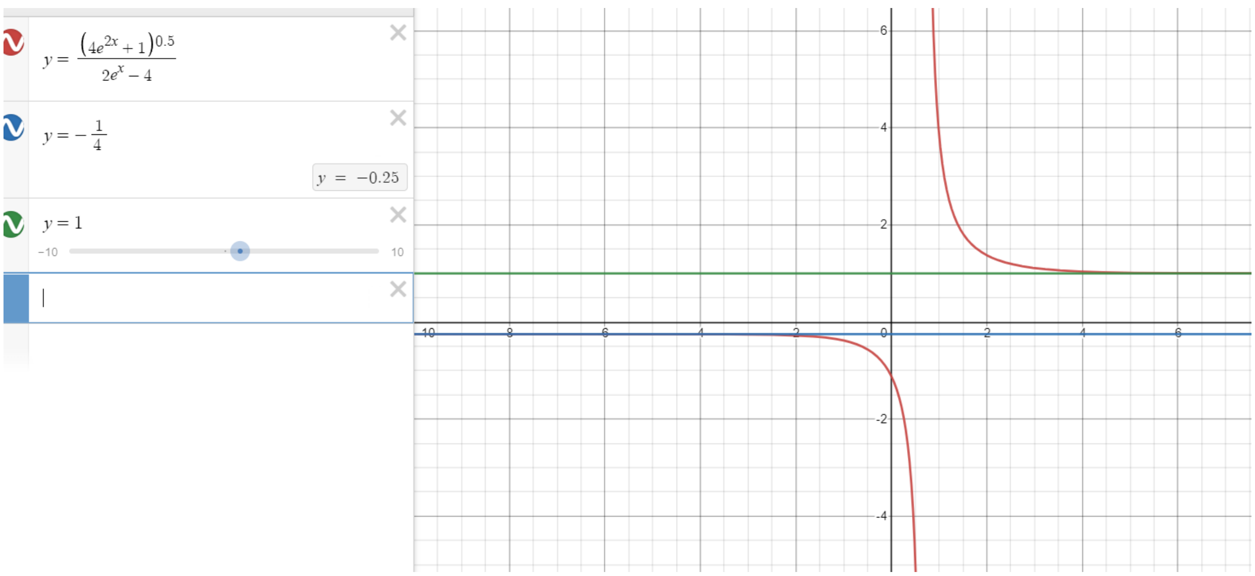
$$= 1$$

$$\therefore y = 1$$

$$HA \lim_{x\to-\infty} \frac{\sqrt{4e^{2x}+1}}{2e^x-4}$$

$$= \frac{\sqrt{4\lim\limits_{x\to-\infty} e^{2x}+1}}{2\lim\limits_{x\to-\infty} e^x - 4}$$

$$= \frac{\sqrt{0+1}}{0-4} = -\frac{1}{4} \quad \therefore y = -\frac{1}{4}$$



K4. a. $f(x) = \frac{e^{2x}+5e^{x}+2}{e^{2x}-1}$

VA

$e^{2x} - 1 = 0$

$e^{2x} = 1$

$\ln e^{2x} = \ln 1$

$2x \ln e = \ln 1$

$2x = 0$

$x = 0$

HA

$$\lim_{x\to\infty} \frac{e^{2x}+5e^{x}+2}{e^{2x}-1}$$

$$= \lim_{x\to\infty} \frac{e^{2x}(1+\frac{5}{e^{x}}+\frac{2}{e^{2x}})}{e^{2x}(1-\frac{1}{e^{2x}})}$$

$$= \lim_{x\to\infty} \frac{1+\frac{5}{e^{x}}+\frac{2}{e^{2x}}}{1-\frac{1}{e^{2x}}}$$

$$= \frac{1+0+0}{1-0} = 1 \ \therefore y = 1$$

$$\lim_{x\to-\infty} \frac{e^{2x}+5e^{x}+2}{e^{2x}-1}$$

$$= \frac{\lim_{x\to-\infty} e^{2x} + \lim_{x\to-\infty} e^{x} + \lim_{x\to-\infty} 2}{\lim_{x\to-\infty} e^{2x} - \lim_{x\to-\infty} 1}$$

$$= \frac{0+0+2}{0-1} = \frac{2}{-1} = -2$$

$\therefore y = -2$

$\therefore HA\ at\ y = 1, y = -2$

b) $g(x) = \frac{2x}{\sqrt[4]{2x^4+4}}$

VA None since $2x^4 > 0$ for all x

HA

$\lim_{x\to\infty} \frac{2x}{\sqrt[4]{2x^4+4}}$

$= \lim_{x\to\infty} \frac{2x}{\sqrt[4]{x^4\left(2+\frac{4}{x^4}\right)}}$

$= \lim_{x\to\infty} \frac{2x}{x\sqrt[4]{2+\frac{4}{x^4}}}$ since $x \to +\infty$ $\sqrt[4]{x^4} = |x| = x$

$= \lim_{x\to\infty} \frac{2}{\sqrt[4]{2+\frac{4}{x^4}}} = \frac{2}{\sqrt[4]{2+0}} = \frac{2}{\sqrt[4]{2}}$

$\therefore y = \frac{2}{\sqrt[4]{2}}\ is\ a\ HA$

$\lim_{x\to-\infty} \frac{2x}{\sqrt[4]{2x^4+4}}$

$= \lim_{x\to-\infty} \frac{2x}{\sqrt[4]{x^4\left(2+\frac{4}{x^4}\right)}}$

$= \lim_{x\to-\infty} \frac{2x}{(-x)\sqrt[4]{2+\frac{4}{x^4}}}$ since $|x| = -x\ as\ x \to -\infty$

$$= \lim_{x \to -\infty} \frac{-2}{\sqrt[4]{2 + \frac{4}{x^4}}}$$

$$= \lim_{x \to -\infty} \frac{-2}{\sqrt[4]{2+0}} = -\frac{2}{\sqrt[4]{2}}$$

$$y = -\frac{2}{\sqrt[4]{2}}$$

$$\therefore HA\ y = \pm\frac{2}{\sqrt[4]{2}}$$

L. Squeeze Theorem

Example 2.

$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$ $\qquad -1 \le \cos(\frac{1}{x}) \le 1$

By squeeze theorem, $\qquad x^2(-1) \le x^2\cos(\frac{1}{x}) \le 1(x^2)$

$\lim_{x\to 0} -x^2 = 0$ & $\lim_{x\to 0} x^2 = 0$ $\qquad \uparrow \qquad \uparrow$

$as\ x \to 0 \qquad as\ x \to 0$

$lim \to 0 \qquad lim \to 0$

$\therefore \lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$

Example 3.

$\lim\limits_{x\to\infty} \frac{3-\cos x}{x+3}$ $\quad -1 \leq \cos x \leq 1 \quad \textit{multiply by} -1$

$1 \geq -\cos x \geq -1 \quad \textit{reverse inequalities}$

$\therefore -1 \leq -\cos x \leq 1$

Add 3 to each component

$-1+3 \leq 3-\cos x \leq 1+3$

$\therefore 2 \leq 3-\cos x \leq 4$

Divide by $x+3 \quad since \quad x \to \infty, x+3 > 0$

$\frac{2}{x+3} \leq \frac{3-\cos x}{x+3} \leq \frac{4}{x+3}$

$\lim\limits_{x\to\infty} \frac{2}{x+3} = 0 = \lim\limits_{x\to\infty} \frac{4}{x+3}$

$\therefore \textit{from squeeze theorem}$

$\lim\limits_{x\to\infty} \frac{3-\cos x}{x+3} = 0$

Example 4.

$-1 \leq \sin\left(\frac{2020}{x}\right) \leq 1 \qquad multiply\ by\ e^{\frac{1}{x}}$

$-e^{\frac{1}{x}} \leq e^{\frac{1}{x}}\sin(\frac{2020}{x}) \leq e^{\frac{1}{x}}$

$\lim_{x\to 0^-}\left(-e^{\frac{1}{x}}\right) = -e^{-\infty} = -\frac{1}{e^{\infty}} = 0$

And $\lim_{x\to 0^-}\left(e^{\frac{1}{x}}\right) = e^{-\infty} = \frac{1}{e^{\infty}} = 0$

$\downarrow \quad below\ 0 \ \therefore a\ negative\ number$

$\therefore \lim_{x\to 0^-}\sin(\frac{2020}{x})e^{\frac{1}{x}} = 0 \quad by\ Squeeze\ Theorem$

Example 5.

$\lim_{x \to -\infty} g(x)$

$= \lim_{x \to -\infty} \frac{5x^2+10x+15}{x^2+1}$

$= \lim_{x \to -\infty} \frac{5+\frac{10}{x}+\frac{15}{x^2}}{1+\frac{1}{x^2}}$

$= \frac{5}{1} = 5$

$\lim_{x \to -\infty} h(x)$

$= \lim_{x \to -\infty} (5 - 3^x)$

$= 5 - 3^{-\infty}$

$= 5 - \frac{1}{\infty}$

$= 5 - 0 = 5$

$\therefore$ since $g(x) \le f(x) \le h(x)$

And $\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} h(x) = 5$

$\therefore \lim_{x \to -\infty} f(x) = 5$ by the squeeze theorem

L1. $\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})}$

$-1 \le \sin(\frac{1}{x}) \le 1$

$e^{-1} \le e^{\sin(\frac{1}{x})} \le e^1$

$\frac{1}{e} \le e^{\sin(\frac{1}{x})} \le e$

$\frac{x^2}{e} \le x^2 e^{\sin(\frac{1}{x})} \le ex^2$

$\lim_{x\to 0} \frac{x^2}{e} = 0$ & $\lim_{x\to 0} ex^2 = 0$

$\therefore$ *By squeeze Theorem,* $\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})} = 0$

L2. $\lim_{x\to\infty} \frac{\cos^2(2x)}{5-2x}$

$-1 \le \cos(2x) \le 1$ ∴think about squaring this graph, it would only have values from 0 to 1

divide by $5-2x$ *and* $x \to \infty$, *assume* $5-2x < 0$

$\therefore \frac{0}{5-2x} \ge \frac{\cos^2(2x)}{3-2x} \ge \frac{1}{5-2x}$

(inequalities switch because we're dividing by a negative value)

$\therefore \frac{1}{5-2x} \le \frac{\cos^2(2x)}{5-2x} \le 0$

Now, $\lim_{x\to\infty} \frac{1}{5-2x} = 0$ *and* $\lim_{x\to\infty} 0 = 0$

$\therefore$ *by squeeze theorem* $\lim_{x\to\infty} \frac{\cos^2(2x)}{5-2x} = 0$

L3. $\lim_{x\to\infty} \frac{6x^2-\sin(2x)}{x^2+10}$

$-1 \le \sin(2x) \le 1$ since the sine curve oscillates between -1 and 1.

multiply by -1

$1 \ge -\sin(2x) \ge -1$ same as

$-1 \le -\sin(2x) \le 1$

add $6x^2$ *to each term*

$$6x^2 - 1 \le 6x^2 - \sin(2x) \le 6x^2 + 1$$

$$\frac{6x^2-1}{x^2+10} \le \frac{6x^2-\sin(2x)}{x^2+10} \le \frac{6x^2+1}{x^2+10}$$

$$\lim_{x\to-\infty} \frac{6x^2-1}{x^2+10} = \lim_{x\to-\infty} \frac{x^2\left(6-\frac{1}{x^2}\right)}{x^2(1+\frac{10}{x^2})} =$$

$$= \frac{6-0}{1+0} = 6$$

$$\lim_{x\to-\infty} \frac{6x^2+1}{x^2+10} = 6$$

$\lim_{x\to-\infty} \frac{6x^2-\sin(2x)}{x^2+10} = 6$ by the Squeeze Theorem

M. Intermediate Value Theorem

Example 2. $f(x) = 2x^3 - 3x^2 - 3$

First, always state that f(x) is continuous on [a,b] or they take off marks:(

$f(1) = 2 - 3 - 3 = -4 < 0$

$f(3) = 2(3)^3 - 3(3)^2 - 3$

$= 54 - 27 - 3 \;\; > 0$ $\qquad f(1) < f(c) < f(3)$

$= 24 > 0$

There is at least one c in $(1,3)$ *such that* $f(c) = 0$ *by IVT and* c *is a root of*

$2c^3 - 3c^2 - 3 = 0$

Example 3.

$f(x) = \frac{1}{3}e^{\cos(2\pi x)} - 2\sin(\frac{\pi}{6}x)$

$f(x)$ *is continuous on* $(-\infty, \infty)$

$f(0) = \frac{1}{3}e^{\cos 0} - 2\sin 0 = \frac{1}{3}e^1 - 0 = \frac{1}{3}e > 0$

$f(1) = \frac{1}{3}e^{\cos 2\pi} - 2\sin\frac{\pi}{6} = \frac{1}{3}e^1 - 1 < 0$

$\therefore$ *by IVT there is at least one c in* $[0, 1]$ *such that* $f(c) = 0$

$\therefore$ *C is a root of* $\frac{1}{3}e^{\cos(2\pi c)} = 2\sin(\frac{\pi}{6}c)$

Example 4. IVT

$f(x) = \sqrt[n]{x} - 1 + 2x$

Check $f(0) = \sqrt[n]{0} - 1 + 2(0)$

$= 0 - 1 + 0 = -1 < 0$

$f(1) = \sqrt[n]{1} - 1 + 2(1) \qquad n \geq 4$

$= 1^{\frac{1}{n}} - 1 + 2$

$= 1^{\frac{1}{n}} + 1$

$= 1 + 1 = 2 > 0$ but $1^{\frac{1}{n}} = 1$ for $n \geq 4$

$\therefore f(0) < f(c) < f(1)$

∴ is at least one c in (0,2) such that f (c) = 0.

∴ there is at least one root in (0,2). A root means there is at least one solution

∴ c is a root of $f(c) = \sqrt[n]{c} - 1 + 2c$ by IVT

M1. $f(x) = x^4 + x^3 - x - 2$

First, always state that f(x) is continuous on [a,b] or they take off marks:(

$f(0) = 0 + 0 - 0 - 2 = -2 < 0$

$f(2) = 16 + 8 - 2 - 2 = 24 - 4 = 20 > 0$

$f(0) < f(c) < f(2)$

$\therefore$ *there is at least one c in* $(0,2)$ *such that* $f(c) = 0$ *and by IVT* c *is a root of*

$c^4 + c^3 - c - 2 = 0$

*M*2. $f(x) = \cos x$ $[0,2\pi]$ *First, always state that f(x) is continuous on [a,b] or they take off marks:(Note: if you used the endpoints, you would get both are positive*

$f(0) = \cos 0 = 1 > 0$

$f(\pi) = \cos \pi = -1 < 0$

$f(\pi) < f(c) < f(0)$

$\therefore$ *there is at least one c in* $(0, \pi)$ *such that* $f(c) = 0$ *and by IVT, c is a root of*

$\cos c = 0$

M3. $f(x) = \sin^2 x - x^3 + 2 \qquad (0, \frac{3\pi}{2})$

$f(0) = (\sin 0)^2 - 0^3 + 2 = 2 > 0$

$$f\left(\frac{3\pi}{2}\right) = \sin^2\left(\frac{3\pi}{2}\right) - (\frac{3\pi}{2})^3 + 1 = (-1)^2 - \left(\frac{27\pi^3}{8}\right) + 2$$

$$= pretty\ difficult\ without\ a\ calculator$$

$try\ f(\pi) = \sin^2 \pi - \pi^3 + 2$

$$= 0 - \pi^3 + 2 = 2 - (3.14)^3 < 0$$

$f(\pi) < f(c) < f(0)$

$\therefore\ there\ is\ at\ least\ one\ c\ in\ \left(0, \frac{3\pi}{2}\right) such\ that\ f(c) = 0$

$and\ by\ IVT\ c\ is\ a\ root\ of\ the\ equation$

$$\sin^2 c - c^3 + 2 = 0$$

N. Definition of the Derivative

Example 1.

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$$f'(3) = \lim_{x\to 3}\frac{f(x)-f(3)}{x-3} = \lim_{x\to 3}\frac{x^3-3^3}{x-3} = \lim_{x\to 3}\frac{x^3-27}{x-3} = H =$$

$$= \lim_{x\to 3}\frac{3x^2}{1} = 3(3)^2 = 27$$

*here you can factor as a difference of cubes, or use L'Hospitals Rule

$$\ldots\lim_{x\to 3}\frac{x^3-27}{x-3} = \lim_{x\to 3}\frac{(x-3)(x^2+3x+9)}{x-3} = \lim_{x\to 3}(x^2+3x+9) =$$

$$= 3^2 + 3(3) + 9 = 27$$

Example 2.

$$f(x) = \sqrt{x+1}$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(\sqrt{x+h+1}-\sqrt{x+1})}{h}\frac{(\sqrt{x+h+1}+\sqrt{x+1})}{(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \lim_{h\to 0}\frac{x+h+1-(x+1)}{h(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \lim_{h\to 0}\frac{x+h+1-x-1}{h(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \lim_{h\to 0}\frac{1}{(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \frac{1}{\sqrt{x+1}+\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

Example 3.

f(x)=sinx f(x+h)=sin(x+h)

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Now, from identities sin(x+h)=sinxcosh + cosxsinh

$$f'(x) = \lim_{h\to 0}\frac{sinxcosh+cosxsinh-sinx}{h}$$ rearrange

$$=\lim_{h\to 0}\frac{sinxcosh-sinx+cosxsinh}{h}$$ common factor of sinx from first two terms

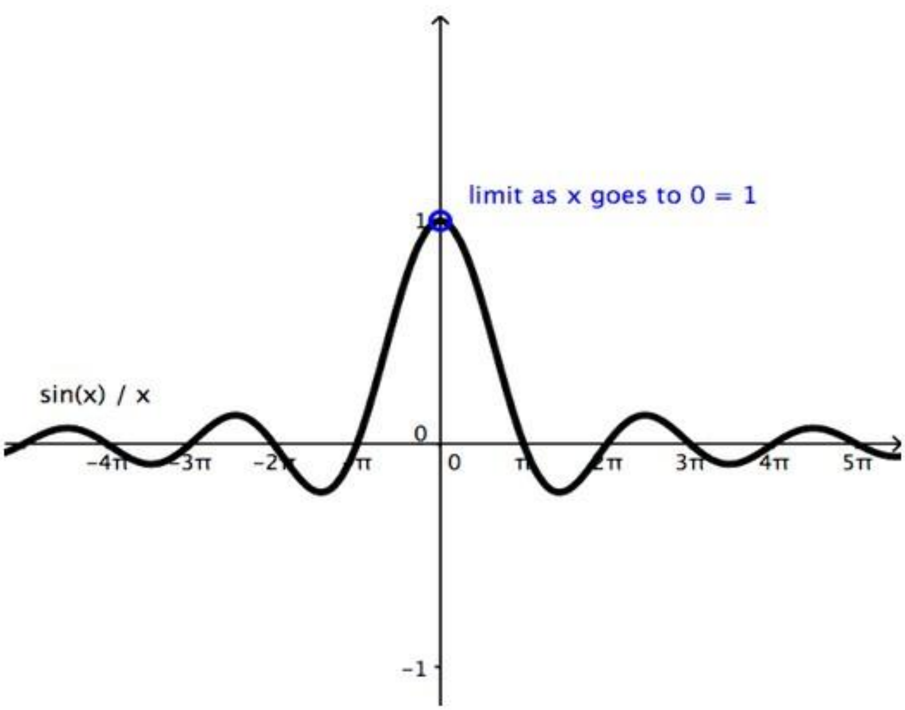
$$=\lim_{h\to 0}\frac{sinx(cosh-1)+cosxsinh}{h}$$ split up limits

$$=\lim_{h\to 0} sinx(\frac{cosh-1}{h}) + \lim_{h\to 0} cosx(\frac{sinh}{h})$$

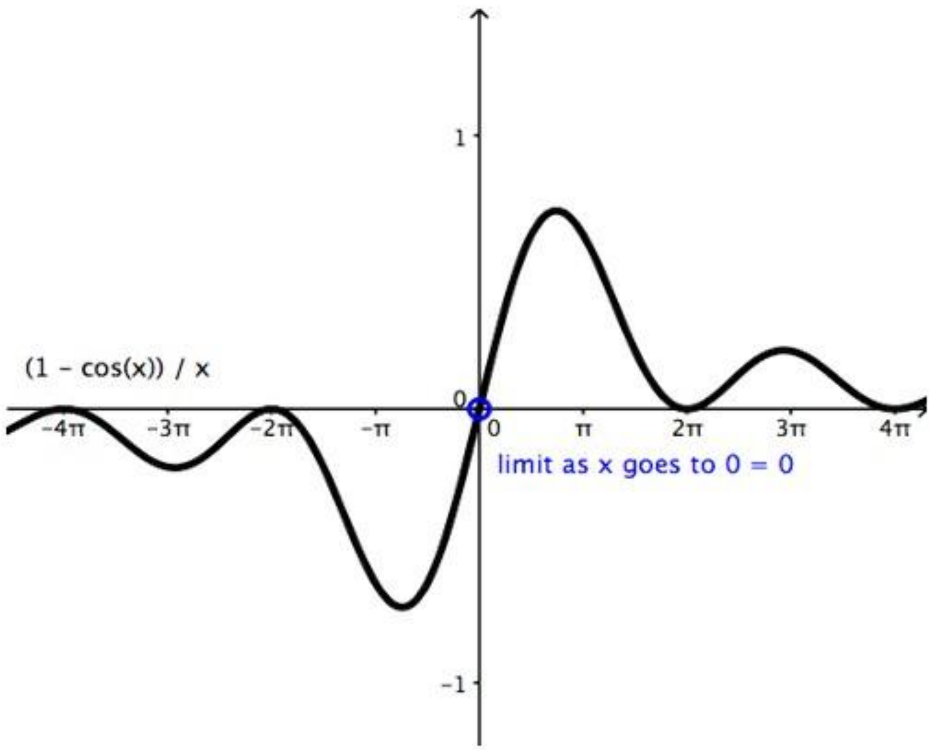
=sinx (0)+cosx(1) from limits you've memorized! (I hope!!) p.56 booklet

=cosx

We can see that the first limit converges to 1



and the second limit converges to 0.



We can plug in 1 and 0 for the limits and get cos(x)

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)(1) - sin(x)(0)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$$

Example 4.

$f(x) = (x+2)^2 \; at \; x = 2.$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$f'(2) = \lim_{h\to 0} \frac{(2+h+2)^2 - (2+2)^2}{h}$$

$$f'(2) = \lim_{h\to 0} \frac{(4+h)^2 - 16}{h}$$

$$f'(2) = \lim_{h\to 0} \frac{16 + 8h + h^2 - 16}{h}$$

$$f'(2) = \lim_{h\to 0} \frac{h^2 + 8h}{h}$$

$$f'(2) = \lim_{h\to 0} \frac{h(h+8)}{h}$$

m=8
$f(x) = (x+2)^2 \; at \; x = 2.$
Subst. x=2 into f(x) to find y
$f(2) = (2+2)^2 = 16$
Point (2,16)
$y - 16 = 8(x-2)$
$y - 16 = 8x - 16$
y=8x

Example 5.

Subst. x=3 into f(x) and get $f(3) = \sqrt{3+1} = 2$ Point (3,2)

$$m = f'(3) = \lim_{h\to 0} \frac{(\sqrt{3+h+1}-2)(\sqrt{3+h+1}+2)}{h(\sqrt{3+h+1}+2)}$$

$$m = \lim_{h\to 0} \frac{(3+h+1-4)}{h(\sqrt{3+h+1}+2)}$$

$$m = \lim_{h\to 0} \frac{h}{h(\sqrt{3+h+1}+2)}$$

$$m = \lim_{h\to 0} \frac{1}{(\sqrt{3+h+1}+2)}$$

$$m = \frac{1}{4}$$

$$y - 2 = \frac{1}{4}(x-3) or\ y = \frac{1}{4}x - \frac{3}{4} + 2$$

$$y = \frac{1}{4}x + \frac{5}{4}$$

Example 6.

a) Yes, $f(x)$ is defined at $x = 0$
$f(0) = 4(0) = 0$

b) Check differentiability

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - a}$$

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{4x - 0}{x - 0} = \lim_{x\to 0^+} \frac{4x}{x} = 4$$

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{2\mathrm{x}\sin\left(\frac{1}{x^2}\right) - 0}{x - 0}$$

$$= \lim_{x\to 0^-} \frac{2\sin\left(\frac{1}{x^2}\right)}{1} = DNE$$

$\therefore f'(0)$ is undefined

c)

$$\lim_{x\to 0^+} 4x = 4(0) = 0$$

$\lim_{x\to 0^+} 2\sin\left(\frac{1}{x^2}\right) = 0$ by Squeeze Theorem

$$\therefore \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$$

$\therefore f(x)$ is continuous at $x = 0$

$N1.\ f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \quad \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$$

N2. $f(x) = (x+1)^2 = x^2 + 2x + 1$

$$f(x+h) = (x+h+1)^2 = (x+h+1)(x+h+1)$$

$$= x^2 + xh + x + xh + h^2 + h + x + h + 1$$

$$= x^2 + 2xh + 2x + 2h + h^2 + 1$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{x^2+2xh+2x+2h+h^2+1-(x^2+2x+1)}{h}$$

$$= \lim_{h\to 0} \frac{2xh+2h+h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+2+h)}{h}$$

$$= \lim_{h\to 0}(2x + 2 + h)$$

$$= 2x + 2$$

$N3.\ f(x) = x^2 - x$

$f(x+h) = (x+h)^2 - (x+h) = x^2 + 2xh + h^2 - x - h$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{x^2+2xh+h^2-x-h-(x^2-x)}{h}$

$= \lim_{h\to 0} \frac{2xh+h^2-h}{h}$

$= \lim_{h\to 0} \frac{h(2x+h-1)}{h}$

$= \lim_{h\to 0} \frac{(2x+h-1)}{1}$

$= 2x + 0 - 1 \quad = 2x - 1$

N4. $f(x) = \frac{1}{x+3}$ $\qquad f(x+h) = \frac{1}{x+h+3}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h}$ $\quad$ *get a common denominator*

$= \lim_{h\to 0} \frac{\frac{1(x+3)}{(x+3)(x+h+3)} - \frac{1(x+h+3)}{(x+3)(x+h+3)}}{h}$

$= \lim_{h\to 0} \frac{\frac{x+3-x-h-3}{(x+3)(x+h+3)}}{h}$

$\lim_{h\to 0} \frac{-h}{(x+3)(x+h+3)}\left(\frac{1}{h}\right) = \frac{-1}{(x+3)(x+0+\#)} = \frac{-1}{(x+3)^2}$

$N5.\ \ f(x) = \dfrac{1}{\sqrt{x}} \qquad\qquad f(x+h) = \dfrac{1}{\sqrt{x+h}}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{\sqrt{x}}{\sqrt{x}\sqrt{x+h}}-\frac{\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{(\sqrt{x}-\sqrt{x+h})}{\sqrt{x}\sqrt{x+h}}\frac{(\sqrt{x}+\sqrt{x+h})}{(\sqrt{x}+\sqrt{x+h})}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{x-(x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h}}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{x-x-h}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}}{h}$$

$$= \lim_{h\to 0}\frac{-h}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}\left(\frac{1}{h}\right) = \frac{-1}{\sqrt{x}\sqrt{x+0}(\sqrt{x}+\sqrt{x+0}} = \frac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x}+\sqrt{x})} =$$

$$\frac{-1}{x(2\sqrt{x})} = \frac{-1}{2x^1x^{\frac{1}{2}}}$$

$$= \frac{-1}{2x^{\frac{3}{2}}} = \frac{-1x^{\frac{-3}{2}}}{2}$$

N6. A) Find $f'(x)$ using the definition of the derivative if

$f(x) = (2x-1)^2$.

$f(x+h) = [2(x+h)-1]^2 = (2x+2h-1)(2x+2h-1)$

$f(x+h) = 4x^2 + 4xh - 2x + 4xh + 4h^2 - 2h - 2x - 2h + 1$

$f(x+h) = 4x^2 + 8xh - 4x - 4h + 4h^2 + 1$

And

$f(x) = (2x-1)^2 = (2x-1)(2x-1) = 4x^2 - 4x + 1$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{4x^2 + 8xh - 4x - 4h + 4h^2 + 1 - (4x^2 - 4x + 1)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{4x^2 + 8xh - 4x - 4h + 4h^2 + 1 - 4x^2 + 4x - 1}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{8xh - 4h + 4h^2}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{h(8x - 4 + 4h)}{h}$$

$$f'(x) = \lim_{h\to 0}(8x - 4 + 4h)$$

$f'(x) = 8x - 4$

B) Find the equation of the tangent to f(x) at x=3.

$f'(x) = 8x - 4$

$f'(3) = 8(3) - 4 = 20$

Subst. x=3 into $f(x) = (2x - 3)^2$ to find y

f(2)=9, so y=9. The point is (3,9)

So, the equation is

$y - 9 = 20(x - 3)$ *or* $y = 20x - 60 + 9$ *or* $y = 20x - 51$

*N7. A) Find $f'(x)$ using the definition of the derivative if $\mathrm{f}(x) = \sqrt{2x-4}$.

A) Find $f'(x)$ using the definition of the derivative if $\mathrm{f}(x) = \sqrt{2x-4}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{2x+2h-4} - \sqrt{2x-4}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(\sqrt{2x+2h-4} - \sqrt{2x-4})(\sqrt{2x+2h-4} + \sqrt{2x-4})}{h(\sqrt{2x+2h-4} + \sqrt{2x-4})}$$

$$f'(x) = \lim_{h\to 0} \frac{2x+2h-4-(2x-4)}{h(\sqrt{2x+2h-4} + \sqrt{2x-4})}$$

$$f'(x) = \lim_{h\to 0} \frac{2x+2h-4-2x+4}{h(\sqrt{2x+2h-4} + \sqrt{2x-4})}$$

$$f'(x) = \lim_{h\to 0} \frac{2h}{h(\sqrt{2x+2h-4} + \sqrt{2x-4})}$$

$$f'(x) = \lim_{h\to 0} \frac{2}{(\sqrt{2x+2h-4} + \sqrt{2x-4})}$$

$$f'(x) = \frac{2}{\sqrt{2x-4}) + \sqrt{2x-4})}$$

$$f'(x) = \frac{2}{2\sqrt{2x-4})}$$

$$f'(x) = \frac{1}{\sqrt{2x-4}}$$

The slope, m at x=7 would be m=$\frac{1}{\sqrt{2(7)-4}} = \frac{1}{\sqrt{10}}$

B). Find the equation of the tangent to f(x) at x=7.

At x=7, $f(7) = \sqrt{2(7)-5} = \sqrt{9}$=3

The point is (7,3)

So, the equation is

$$y - 3 = \frac{1}{3}(x-7) \; or \; y = \frac{1}{3}x - \frac{7}{3} + 3$$

$$or \; y = \frac{1}{3}x + \frac{2}{3}$$

N8. See example 3. Repeat for you to try it again!

N9. Find the derivative of y=cosx using the definition of the derivative.

Using the definition of a derivative:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \text{ where } h = \delta x$$

We substitute in our function to get:

$$\lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

Using the Trig identity:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b,$$

we get:

$$\lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$$

Factoring out the $\cos x$ term, we get:

$$\lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

This can be split into 2 fractions:

$$\lim_{h \to 0} \frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \sin h}{h}$$

Now comes the more difficult part: recognizing known formulas.

The 2 which will be useful here are:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \text{ and } \lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

Since those identities rely on the variable inside the functions being the same as the one used in the $\lim$ portion, we can only use these identities on terms using h, since that's what our $\lim$ uses. To work these into our equation, we first need to split our function up a bit more:

$$\lim_{h \to 0} \frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \sin h}{h}$$

becomes:

$$\lim_{h \to 0} \cos x \left(\frac{\cos h - 1}{h} \right) - \sin x \left(\frac{\sin h}{h} \right)$$

Using the previously recognized formulas, we now have:

$$\lim_{h \to 0} \cos x(0) - \sin x(1)$$

which equals:

$$\lim_{h \to 0} (-\sin x)$$

Since there are no more h variables, we can just drop the $\lim_{h \to 0}$, giving us a final answer of: $-\sin x$.

O. Evaluating Limits by recognizing them as derivatives

Example 2.

a=x=4 to find f(x) replace 4+h with an x

f(x)=lnx

$$f'(x) = \frac{1}{x}$$

$$f'(x) = \frac{1}{4}$$

Example 3.

a=x=3 to find f(x) look at the first part of the limit

$$f(x) = e^{3x}$$

$$f'(x) = 3e^{3x}$$

$$f'(x) = 3e^{9}$$

Example 4.

$$\lim_{h\to 0}\frac{(1+h)^3-2(1+h)^2+3(1+h)-2}{h} \qquad (\%)$$

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \ \therefore a=x=1$$

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

Replace $1+h\to x$

$\therefore f(x)=x^3-2x^2+3x-2$ note that f(1)=0

$f'(x)=3x^2-4x+3$

$f'(1)=3(1)^2-4(1)+3$

$=2$

O1. $a+h=4+h \quad \therefore a=4$

To find $f(x)$ $replace$ $4+h$ $with$ an x

$\therefore f(x)=2^x$

$\therefore f'(x)=2^x\ln 2$

$f'(4)=2^4\ln 2=16\ln 2$

O2. $x \to a \quad \therefore a = \frac{\pi}{3}$

$$f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$f'\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} \quad = -\frac{\sqrt{3}}{2}$$

O3. $a + h = 5 + h \quad \therefore a = 5$

To find $f(x)$ *replace* $5 + h$ *with an* $x \quad \therefore f(x) = x^3$

$f'(x) = 3x^2$

$f'(5) = 3(5)^2 = 3(25) = 75$

$O4. \quad \lim\limits_{x\to\pi/3} \dfrac{e^{\sin x} - \frac{1}{2}e^{\sqrt{3}/2}}{x - \pi/3}$

Definition $\quad f'(a) = \lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}$

$\therefore a = \frac{\pi}{3} \quad f(a) = f(\frac{\pi}{3}) = \frac{1}{2}e^{\sqrt{3}/2} \quad f(x) = e^{\sin x}$

$f'(x) = e^{\sin x}(\cos x)$

$f'(\pi/3) = e^{\sin(\frac{\pi}{3})}(\cos(\frac{\pi}{3})) = 1/2\, e^{\sqrt{3}/2}$

O5.

$$\lim_{x\to\frac{1}{2}} \frac{arc\sin x - \frac{\pi}{6}}{x - \frac{1}{2}} \qquad Definition \quad f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

$$a = \frac{1}{2} \qquad f\left(\frac{1}{2}\right) = arc\ sin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$f(x) = arc\sin x$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f'\left(\frac{1}{2}\right) = \frac{1}{\sqrt{1-(\frac{1}{2})^2}} = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{1}{\sqrt{\frac{3}{4}}}$$

$$\therefore f'\left(\frac{1}{2}\right) = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

O6. $\lim_{h\to 0} \frac{3^{2h}-1}{h}$ $\qquad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$a = 0 \qquad f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$f(x) = 3^{2x}$$

$$f'(x) = 3^{2x}(\ln 3)(2)$$

$$f'(0) = 3^0(\ln 3)(2) = 2\ln 3 = \ln 3^2 = \ln 9$$

O7. $\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2}+h\right)-1}{h}$ $\qquad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$a=\pi/2$ $\qquad f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$f(\pi/2)=\sin\left(\frac{\pi}{2}\right)=1$

$f(x)=sinx$

$f'(x)=cosx$

$f'(\pi/2\,)=cos(\frac{\pi}{2})=0$

P. Average vs. Instantaneous Velocity and Acceleration

P1. a) $s(1) = 1^3 + 2(1)^2 + 1$

$= 1 + 2 + 1 = 4m$

b) $\frac{\Delta s}{\Delta t} = \frac{(2^3+2(2)^2+1)-4}{2-1} = \frac{8+8+1-4}{1} = 13\text{m/s}$

c) $s'(t) = 0 \qquad s'(t) = 3t^2 + 4t$

d) $s'(3) = 3(3)^2 + 4(3) = 3(9) + 12$

$= 27 + 12 = 39\text{ m/s}$

e) $s''(t) = 6t + 4$

$s''(3) = 6(3) + 4 = 22\text{ m/}s^2$

P2. a) $\frac{\Delta s}{\Delta t} = \frac{(50-2(2)^2)-(50-2(1)^2)}{2-1}$

$= \frac{(50-8)-(50-2)}{1} = \frac{42-48}{1} = -6\text{ m/s}$

b) $s'(t) = 0$

$s'(2) = -4(2) = -8\text{ m/s}$

c) $s(t) = 0 \qquad 50 - 2t^2 = 0$

$50 = 2t^2$

$25 = t^2$

$t = 5s$

d) $s''(t) = -4\text{ m/}s^2$

$s''(3) = -4\, m/s^2$ (constant acceleration)

Q. Derivative Rules

Example 1. (a) $f(x) = x^{\frac{1}{2}}(x^4 + 3x)$

$$f(x) = x^{\frac{9}{2}} + 3x^{\frac{3}{2}}$$

$$f'(x) = \frac{9}{2}x^{\frac{7}{2}} + \frac{9}{2}x^{\frac{1}{2}}$$

(b) $g'(x) = \frac{3(2-x)-(-1)(1+3x)}{(2-x)^2}$

$$= \frac{6-3x+1+3x}{(2-x)^2}$$

$$= \frac{7}{(2-x)^2}$$

(c) $f'(x) = \frac{(2x+2)(x^3+4)-3x^2(x^2+2x+2)}{(x^3+4)^2}$

$$= \frac{2x^4+8x+2x^3+8-3x^4-6x^3-6x^2}{(x^3+4)^2}$$

$$= \frac{-x^4-4x^3-6x^2+8x+8}{(x^3+4)^2}$$

(d) $f(x) = x^2 - \sqrt{x}\,x + \sqrt{x} \cdot x - x$

$$= x^2 - x$$

$$f'(x) = 2x - 1$$

Example 2. Find the derivative of each of the following:

a) $f(x) = 3^x$

$f'(x) = 3^x \ln 3$

b) $f(x) = 7^x + 4x^7$

$f'(x) = 7^x ln7 + 28x^6$

c) $f(x) = 4e^x + 3x^9 + 6$

$f'(x) = 4e^x + 27x^8$

(d) $f'(x) = 5^x ln5(3e^x + 5x^2 + 6x + 12) + 5^x(3e^x + 10x + 6)$

$= 5^x[\ln 5(3e^x + 5x^2 + 6x + 12) + (3e^x + 10x + 6)]$

(e) $f'(x) = \frac{3^x ln3(x^3+4x^2) - 3^x(3x^2+8x)}{(x^3+4x^2)^2}$

Example 3. Find the derivative of each of the following using the Product Rule.

a) $f(x) = 6x^7e^x$ $\quad f'(x) = 42x^6e^x + e^x(6x^7)$

$f'(x) = 6x^6e^x(7 + x)$

b) $y = 8^x log_5 x$ $\quad y' = 8^x\ ln\ 8(x) + \frac{1}{xln5}(8^x)$

Example 4. Find the derivative of each of the following using the Quotient Rule.

$f(x) = \frac{lnx}{6x^5}$ $\quad f'(x) = \frac{\frac{1}{x}(6x^5) - 30x^4(lnln\ x)}{(6x^5)^2} = \frac{6x^4 - 30x^4 lnx}{36x^{10}} = \frac{6x^4(1-5lnx)}{36x^{10}}$

$= \frac{(1 - 5lnx)}{6x^6}$

Example 6. Find the derivative of each of the following using the Chain Rule.

a) $y = (3x + 6)^4$ $\quad y' = 4(3x + 6)^3(3) = 12(3x + 6)^3$

b) $f(x) = 6^{8x-1}$ $\quad f'(x) = 6^{8x-1}(ln\ 6)(8)$
$= (8\ ln\ 6)6^{8x-1}$

c) $y = e^{9x-2}$ $\quad y' = 9e^{9x-2}$

d) $f(x) = 4ln\ (8x-7)$ $\quad f'(x) = 4\left(\frac{8}{8x-7}\right) = \frac{32}{8x-7}$

e) $y = log_3(2x^2 - x + 5)$ $\quad y' = \frac{(4x-1)}{(2x^2-x+5)ln3}$

f) $f(x) = [ln\ (2x-5)]^4$ $\quad f'(x) = 4[ln\ (2x-5)\ ]^3\left(\frac{2}{2x-5}\right)$

$= \frac{8}{2x-5}[ln\ (2x-5)\ ]^3$

g) $f(x) = ln\ (2x-5)^4$ $\quad f(x) = 4\ ln\ (2x-5)$ log rules

$f'(x) = 4\left(\frac{2}{2x-5}\right) = \frac{8}{2x-5}$

(h) $(x) = \sqrt{4e^x - 9x^3} = (4e^x - 9x^3)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}(4e^x - 9x^3)^{\frac{-1}{2}}(4e^x - 27x^2)$$

$$f'(x) = \frac{4e^x - 27x^2}{2(4e^x - 9x^3)^{\frac{1}{2}}} = \frac{4e^x - 27x^2}{2\sqrt{4e^x - 9x^3}}$$

***Example 7.** Find the derivative of each of the following:

a) $f(x) = ln\ (lnx)\ at\ x = e^2$

$f'(x) = \frac{\frac{1}{x}}{lnx} = \frac{1}{xlnx}$

$f'(e^2) = \frac{1}{e^2lne^2} = \frac{1}{e^2(2lne)} = \frac{1}{2e^2}$

b) $y = e^x lnx\ at\ x = e$ $\quad y' = e^x\ ln\ x + \frac{1}{x}e^x = e^x(ln\ x\ + \frac{1}{x})$

$y'(e) = e^e\left(ln\ e\ + \frac{1}{e}\right) = e^e(1 + \frac{1}{e})$

***Example 8.** Find the derivative of $y = ln\,(2x^2 + 5e)^{3e}$

$$y = 3e\,ln\,(2x^2 + 5e)$$

$$y' = 3e\left(\frac{4x}{2x^2+5e}\right) = \frac{12ex}{2x^2+5e}$$

***Example 9.** Find the derivative of $y = [ln(2x^2 + 5e)]^{3e}$

$$y' = 3e[ln\,(2x^2 + 5e\,]^{3e-1}\left(\frac{4x}{2x^2+5e}\right)$$

$$= \frac{12ex}{2x^2+5e}[ln\,(2x^2 + 5e\,]^{3e-1}$$

Q1. a) $f'(x)$=$4e^{4x}$

b) $f'(x)$=$3^{-4x}(ln\,3\,)(-4)$

c) $f'(x)$=$3x^2(e^{7x}) + 7e^{7x}(x^3)$=$x^2(e^{7x})(3 + x)$

d) $f'(x)$=$\frac{1}{(x-5)ln3}$

e) $f'(x)$=$12x^2 + 4e^{3x}(3) = 12x^2 + 12e^{3x}$

f) $f'(x)$=$\frac{4e^{4x}(x^2)-2x(e^{4x})}{(x^2)^2} = \frac{2xe^{4x}(2x-1)}{x^4} = \frac{2e^{4x}(2x-1)}{x^3}$

g) $f'(x)$=$e^{\sqrt{x}}\left(\frac{1}{2}x^{\frac{-1}{2}}\right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

h) $f'(x)$=$6^{x^2-7x}(ln6)(2x - 7)$

i) $f'(x)$=$\frac{6}{6x-3} = \frac{2}{2x-1}$

j) $f'(x)$=$\frac{e^{x+5}(x+e^x)-(1+e^x)(e^{x+5})}{(x+e^x)^2}$

k) $f(x)=10^x(x^2+3x)^{\frac{1}{2}}$

$f'(x)=10^x \ln 10\,\sqrt{x^2+3x}+\frac{1}{2}(x^2+3x)^{\frac{-1}{2}}(2x+3)(10^x)$

l) $\mathrm{f}(x)=\ln(x-5)-\ln(x-2)$ ...using log rules

$f'(x)=\frac{1}{x-5}-\frac{1}{x-2}$

m) USE LOG RULES y=$3\ln(x^2-2x)$

y'= $\frac{3(2x-2)}{x^2-2x}$

Q2. $f'(x)e^{x^4+2x}(4x^3+2)$

$f'(1)=e^3(4+2)=6e^3$

Therefore, answer is B).

Q3. $f'(x)=\frac{1}{x\ln 5}$ $f'(2)=\frac{1}{2\ln 5}$

Therefore, answer is B).

Q4.

$$\frac{d}{dx}(4^{2\log_4 x})=\frac{d}{dx}(4^{\log_4 x^2})=\frac{d}{dx}(x^2)=2x$$

Q5. $f'(x)=\frac{2x}{(x^2+2)\ln 3}$

$f'(1)=\frac{2(1)}{3\ln 3}=\frac{2}{3\ln 3}$

Therefore, answer is E).

Q6. $f'(x)$=2x $ln\ x\ +\frac{1}{x}(x^2) = 2x\ ln\ x + x$

$f'(e^2)$=$2e^2\ ln\ e^2 + e^2 = 2e^2(2) + e^2 = 5e^2$

Therefore, answer is A).

Q7. $f'(x)$=$\frac{\frac{1}{x}(x^2)-2xlnx}{x^4} = \frac{x-2elnx}{x^4}$

f'(e)=$\frac{e-2elne}{e^4} = \frac{-e}{e^4} = -\frac{1}{e^3}$

Therefore, answer is C).

Q8. $f(x)$=$6e^x - 4\ ln\ x\ +\frac{1}{4}x^{-1}$

$f'(x)$=$6e^x - 4\left(\frac{1}{x}\right) + \frac{1}{4}(-x^{-2}) = 6e^x - \frac{4}{x} - \frac{1}{4x^2}$

Q9. $f'(x)$=4e$[ln\ (3x+e)\ ]^{4e-1}\left(\frac{3}{3x+e}\right)$

Q10. $f'(x)$=$6^{2x-1}(ln\ 6\ )(2)$

Q11. $f(x) = \frac{e^x}{3} - x^2$

$f'(x) = \frac{1}{3}e^x - 2x$

$f'(ln2) = \frac{1}{3}e^{ln2} - 2ln2 = \frac{2}{3} - ln4$

Q12. $\frac{d}{dx}(\log_4 x^8) = \frac{8x^7}{x^8 ln4} = \frac{8}{xln4}$

Q13.a) $f(x) = 2e^{x^{\frac{1}{2}}}$ at x=16

$$f'(x) = 2e^{x^{\frac{1}{2}}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{2e^{\sqrt{x}}}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{\sqrt{x}}$$

$$f'(16) = \frac{e^{\sqrt{16}}}{\sqrt{16}} = \frac{e^4}{4}$$

b) $y = \log_3 x \; at \; x = 27.$

$$y' = \frac{1}{xln3}$$

$$y'(27) = \frac{1}{27ln3}$$

c) $f(x) = xe^{2x}$

$$f'(x) = (1)e^{2x} + x(2e^{2x}) = e^{2x}(1 + 2x)$$

$$f'(1) = e^2\big(1 + 2(1)\big) = 3e^2$$

d) $f(x) = ln\,(lnx)$

$$f'(x) = \frac{1}{lnx}\left(\frac{1}{x}\right)$$

$$f'(e^2) = \frac{1}{e^2\,lne^2} = \frac{1}{2e^2}$$

Q14. $F'(x) = f'\big(f(x)\big) \cdot f'(x)$

$$F'(4) = f'\big(f(4)\big) \cdot f'(4)$$

$$= f'(3) \cdot 12$$

$$= 5 \cdot 12 = 60$$

Q15. $g'(x) = 5[f(x)]^4 \cdot f'(x)$

$\therefore g'(4) = 5[f(4)]^4 \cdot f'(4)$

$= 5\,(3)^4 \cdot (2)$

$= 5\,(81)(2)$

$= 5\,(162)$

$= 810$

Q16. a) $f'(x) = \frac{2x(2x+3)-2(x^2-4)}{(2x+3)^2}$

$= \frac{4x^2+6x-2x^2+8}{(2x+3)^2}$

$= \frac{2x^2+6x+8}{(2x+3)^2}$

b) $f'(x) = \frac{1(x+3)-1(x+2)}{(x+3)^2}$

$= \frac{x+3-x-2}{(x+3)^2}$

$= \frac{1}{(x+3)^2}$

c) $(x) = \frac{1}{2}e^x + 18x + 6$

$f'(x) = \frac{1}{2}e^x + 18$

d) $f(x) = 3x^{2ln3}$

$f(x) = 3x^{ln9}$

$f'(x) = 3[(ln\,9\,)x^{ln9-1}]$

e) $f(x) = x^3 lnx, find\ f'(e^2)$.

$$f'(x) = 3x^2\ ln\ x\ + \frac{1}{x}(x^3) = 3x^2\ ln\ x\ + x^2$$
$$f'(x) = x^2(3\ ln\ x + 1)$$
$$f'(e^2) = (e^2)^2(3\ ln\ e^2 + 1) = e^4(6lne + 1) = 7e^4$$

f) $f(x) = 3^{2x} \quad f'(x) = 3^{2x}(ln\ 3)(2)\ OR\ \ 2\ ln\ 3(3)^{2x}$

g) find $\frac{d}{dx} ln\ (x + 3 + \frac{1}{x}) \quad = \frac{1+0-1x^{-2}}{x+3+\frac{1}{x}} = \frac{1-\frac{1}{x^2}}{x+3+\frac{1}{x}}$

Q17. a) $f(x) = x^4 e^x \quad f'(x) = 4x^3(e^x) + e^x(x^4)$

$$= x^3 e^x(4 + x)$$

b) $y = (3x^3 - 5)lnx \quad y' = 9x^2\ ln\ x\ + \frac{1}{x}(3x^3 - 5)$

Q18.

a) $f(x) = \frac{e^x}{x^8} \quad f'(x) = \frac{e^x(x^8)-8x^7(e^x)}{(x^8)^2} = \frac{x^7 e^x(x-7)}{x^{16}} = \frac{e^x(x-8)}{x^9}$

b) $y = \frac{(4x+5)}{x^4} \quad y' = \frac{4(x^4)-4x^3(4x+5)}{(x^4)^2} = \frac{4x^4-16x^4-20x^3}{x^8} = \frac{-12x^4-20x^3}{x^8}$

$$= \frac{-4x^3(3x+5)}{x^8} = \frac{-4(3x+5)}{x^5}$$

Q19. Find $\frac{d}{dx}(ln\ (e^{-3x} + 2x)\ at\ x = 1$

$\frac{dy}{dx} = \frac{(-3e^{-3x}+2)}{e^{-3x}+2x} \quad at\ x = 1, \quad \frac{dy}{dx} = \frac{-3e^{-3}+2}{e^{-3}+2}$

R. Trig Derivatives

Example 1. Find the derivative of each of the following. Don't forget to use the Product Rule, the Quotient Rule and the Chain Rule!!

a) $y = 2cosx \quad y' = -2\sin x$

b) $f(x) = 2sin8x \quad f'(x) = 2(8\cos 8x) = 16\cos(8x)$

c) $f(x) = sin^2x + cosx + e^{8x} \quad f(x) = (\sin x)^2 + \cos x + e^{8x}$

$$f'(x) = 2(\sin x)^1(\cos x) + (-\sin x) + 8e^{8x}$$

$$f'(x) = 2\sin x\cos x - \sin x + 8e^{8x}$$

$f'(x) = \sin 2x - \sin x + 8e^{8x}$ trig identity

d) $f(x) = \cot(3x)$ chain rule

$$f'(x) = -3\csc^2(3x)$$

e) $f(x) = \text{sin3x} + e^{\sqrt{x}}$ chain rule

$$f'(x) = 3\cos 3x + e^{\sqrt{x}} \quad (\tfrac{1}{2}x^{-\frac{1}{2}}) = 3\cos 3x + \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

f) $f(x) = (cosx + tanx)^6$ chain rule

$$f'(x) = 6(\cos x + \tan x)^5(-\sin x + \sec^2 x)$$

g) $f(x) = \frac{tanx}{(1+\cos x)^5}$ quotient rule

$$f'(x) = \frac{(\sec^2 x)(1+\cos x)^5 - 5(1+\cos x)^4(-sinx)(tanx)}{[(1+\cos x)^5]^2}$$

$$f'(x) = \frac{(1+\cos x)^4\,[\sec^2 x(1+cosx) + 5sinxtanx]}{(1+cosx)^{10}}$$

$$f'(x) = \frac{[\sec^2 x(1+cosx) + 5sinxtanx]}{(1+cosx)^6}$$

Example 2. If $f(x) = sin^2x, find\ f''\left(\frac{\pi}{4}\right)$. chain rule

$$f(x) = (\sin x)^2$$

$f'(x) = 2(\sin x)(\cos x)$=sin2x (easier than doing a product rule for the second derivative)

$$f''(x) = 2cos2x$$

$$f''\left(\frac{\pi}{4}\right) = 2\cos\left(\frac{2\pi}{4}\right) = 2cos\frac{\pi}{2} = 0$$

Example 3. Find the equation of the tangent of $y = x^3e^{sinx} at\ x = \pi$.

$$y' = 3x^2e^{\sin x} + e^{\sin x}(\cos x)(x^3)$$

$$y'(\pi) = 3\pi^2e^{\sin\pi} + e^{\sin\pi}(\cos\pi)(\pi)^3$$

$$= 3\pi^2(e)^0 + e^0(-1)(\pi)^3$$

$$= 3\pi^2 - \pi^3 = \pi^2(3-\pi)$$

at x= $\pi, y = x^3e^{sinx} = \pi^3e^0 = \pi^3$

$y - y_1 = m(x - x_1)$ becomes $y - \pi^3 = \pi^2(3-\pi)(x-\pi)$

Practice Exam Question on Trigonometric Derivatives

R1.a) y=2secx
$y' = 2secxtanx$

b) $f(x) = 3cos7x$
$f'(x) = 3(-4sin7x) = -12sin7x$

c) $f(x) = (sinx)^2$
$f'(x) = 2(sinx)(cosx)$

d) $f(x) = sec\ (2x)$
$f'(x) = (2x)\ tan\ (2x)$

e) $f(x) = 3e^x sinx$
$f(x) = (3e^x)(sinx) + (cosx)(3e^x) = 3e^x(sinx + cosx)$

f) $f(x) = ln\ (sinx)$
$$f'(x) = \frac{1}{sinx}(cosx) = cotx$$

g) $f(x) = (x + sinx)^4$
$f'(x) = 4(x + sinx)^3(1 + cosx)$

h) $f(x) = \frac{sinx}{secx} = \frac{sinx}{1/cosx} = sinx(cosx)$

$f'(x) = (cosx)(cosx) + (-sinx)(sinx)$
$= cos^2x - sin^2x = cos^2x - (1 - cos^2x)$
$= -1 + 2cos^2x$

i) $f(x) = sin(x^{\frac{1}{2}})$
$$f'(x) = cos(x^{\frac{1}{2}})\frac{1}{2}x^{-1/2}$$

$f'(x) = \frac{cos\sqrt{x}}{2\sqrt{x}}$

j) $f(x) = sin\,(e^x)$
$f'(x) = cos(e^x) \cdot e^x = e^x cos(e^x)$

k) $y = cotxsinx = \frac{cosx}{sinx} sinx = cosx$
$y' = -sinx$

l) $y = csc\,(2x)$
$y' = -\,csc\,(2x)\,cot\,(2x)\ \cdot 2 = -2\,csc\,(2x)\,cot\,(2x)$

R2. $f(x) = x^2 sinx$
$f'(x) = 2xsinx + x^2 cosx$
$f'(\pi) = 2\pi sin\pi + \pi^2 cos\pi = -\pi^2$

R3. f(x)=$(sinx)^2$
$f'(x) = 2sinx(cosx)$
$f''(x) = (2cosx)(cosx) + (-sinx)(2sinx) = 2cos^2x - 2sin^2x$
$f''\left(\frac{\pi}{6}\right) = 2(cos\frac{\pi}{6})^2 - 2(sin\frac{\pi}{6})^2$
$f''\left(\frac{\pi}{6}\right) = 2(\frac{\sqrt{3}}{2})^2 - 2(\frac{1}{2})^2 = 2\left(\frac{3}{4}\right) - 2\left(\frac{1}{4}\right) = \frac{4}{4} = 1$

$R4. a)$
$f(x) = sin\,(5x)$
$f'(x) = 5\,cos\,(5x)$

$b)$
$f(x) = cos\,\sqrt{x} = cos\,(x^{\frac{1}{2}})$

$f'(x) = -\,sin\left(x^{\frac{1}{2}}\right)(\frac{1}{2}x^{\frac{-1}{2}})$

$f'(x) = \frac{-1\ sin\sqrt{x}}{2\sqrt{x}}$

$c)$
$f(x) = e^{sinx}$
$f'(x) = e^{sinx}\,(cos\ x\ )$

d)
$f(x) = ln\ x\ sec\ x$
$f'(x) = \frac{1}{x} sec\ x + (sec\ x\ tan\ x)\ ln\ x = sec\ x(\frac{1}{x} + tan\ x\ ln\ x)$

$e)$
$f(x) = sec^2(2x+1) = [sec\ (2x+1)\ ]^2$
$f'(x) = 2[sec\ (2x+1)\ ]^1\ sec\ (2x+1)\ tan\ (2x+1)(2)$
$= 4sec^2(2x+1)\ tan\ (2x+1)$

$f)\quad f(x) = e^{tanx}\ ln\ (cosx\ )$
$f'(x) = e^{tanx}\,(x)\ ln\ (cos\ x) + \frac{1(sinx)}{cosx} e^{tanx}$
$= e^{tanx}\,(x)\ ln\ (sin\ x) + e^{tanx}$
$= e^{tanx}\,[(x)\ ln\ (sin\ x) + tanx]$

g) $f(x) = \frac{sinx}{1+cosx}$

$f'(x) = \frac{cosx(1+cosx)-(-sinx)(sinx)}{(1+cosx)^2} = \frac{cosx+cos^2x+sinx}{(1+cosx)^2}$

$= \frac{cosx+1}{(1+cosx)^2} \qquad since\ x + x = 1$

$= \frac{1}{1+cosx}$

R5. f(x)=$(cos\ x)^2$
$f'(x)$=2$cos\ x\ (-\ sin\ x\ ) = -2\ sin\ x\ cos\ x$

Therefore, answer is C).

R6. $f'(x)$= $cos\ (4x)(4)$
$f'(\frac{\pi}{4}) = 4\ cos\left(\frac{4\pi}{4}\right) = -4$

Therefore, answer is C).

R7. $f'(x)$=(1)$sin\ x\ + (cos\ x\ )(x)$
$f'\ (\pi)$ =$sin\ \pi\ + (cos\ \pi\ )(\pi) = 0 - \pi = -\pi$
At x=0, y=0, so the equation is y=0= - $\pi(x-0) or\ y = -\pi$

R8. $f'(x)$=(2$sec2xtan2x)(tan2\ x) + 2sec^2(2x)(sec\ 2x)$
$= 2\ sec\ (2x)\ tan^2(2x) + 2sec^3(2x)$ Product rule

R9. f(x)=$[csc\ (2x)\ ]^2$
$f'(x)$=2$csc\ (2x)(-\ csc\ (2x)\ cot\ (2x)(2))$
$= -4\ cot\ (2x)csc^2\ (2x)$ *chain rule*

R10. $f'(x)$=5$cos\ x(cos\ (2x)) + (-\ sin\ (2x)(2)(5\ sin\ x)$ product rule
$f'(0)$= $5(1)(1) - 0 = 5$

R11. Find the slope of the tangent of $f(x) = sin\ (2x)\ at\ x = \frac{\pi}{4}$.

$$f'(x) = 2\ cos\ (2x)$$

$$f'\left(\frac{\pi}{4}\right) = 2\ cos\ \frac{2\pi}{4} = 2(-\ cos\ \frac{\pi}{2}) = 0$$

R12.

a) $f(x) = 3e^{2x}\cos\ (6x)$ product rule

$$f'(x) = 3e^{2x}(2)\cos(6x) + (-6\sin(6x)(3e^{2x})$$

$$= 6e^{2x}\cos(6x) - 18e^{2x}\sin(6x)$$

$=6e^{2x}(\cos(6x) - 3\sin(6x))$

b) $f(x) = sin\sqrt{cosx}$ chain rule

$$f'(x) = \cos(cosx)^{\frac{1}{2}} \cdot \frac{1}{2}(\cos x)^{\frac{-1}{2}}(-sinx)$$

$$f'(x) = \frac{-sinx\cos\sqrt{\cos x}}{2\sqrt{\cos x}}$$

c) $y = cos(e^x)$ chain rule

$$y' = -\sin(e^x)\cdot e^x = -e^x\sin(e^x)$$

d) $y = cotxcscx$ product rule

$$y' = -csc^2x(cscx) + cotx(-cscxcotx) = -csc^3x - cot^2xcscx$$

If the answers involve cosx/sinx:

$$y = \frac{\cos x}{\sin x}\left(\frac{1}{\sin x}\right) = \frac{\cos x}{(\sin x)^2}$$

$$y' = \frac{-\sin x(\sin^2 x) - 2\sin x(\cos x)(\cos x)}{\sin^4 x}$$

$$= \frac{-\sin^3 x - 2\sin x\cos^2 x}{\sin^4 x}$$

R13. If $f(x) = x^2 sinx, find\ f'\left(\frac{\pi}{2}\right)$. product rule

$$f'(x) = 2x(\sin x) + (\cos x)(x^2)$$

$$f'\left(\frac{\pi}{2}\right) = \frac{2\pi}{2}\sin\frac{\pi}{2} + (\cos\frac{\pi}{2})\left(\frac{\pi}{2}\right)^2$$

$$= \pi(1) + 0 = \pi$$

R14. Find the slope of the tangent of $y = x - cosx\ at\ x = \frac{\pi}{6}$.

$$y' = 1 + sinx$$

$$y'\left(\frac{\pi}{6}\right) = 1 + \sin\frac{\pi}{6} = 1 + \frac{1}{2} = \frac{3}{2}$$

S. Derivatives of Inverse Trig

Example 1. $f'(x) = (1)(arc\tan x) + \left(\frac{1}{1+x^2}\right)(x)$

$= arc\tan x + \frac{x}{1+x^2}$

Example 2. $f'(x) = \frac{1}{1+x^2}(e^x) + (arc\tan x)(e^x)$

$= e^x\left[\frac{1}{1+x^2} + arc\tan x\right]$ (product rule)

vs. $f'(x) = \frac{e^x}{1+(e^x)^2} = \frac{e^x}{1+e^{2x}}$ (chain rule)

Example 3. $f'(x) = \frac{1}{3x\sqrt{(3x)^2-1}} = \frac{1}{3x\sqrt{9x^2-1}}$

Example 4. $f(x) = e^x\cos^{-1}x$

$f'(x) = e^x\cos^{-1}x + \left(\frac{-1}{\sqrt{1-x^2}}\right)e^x$

$= e^x\left[\cos^{-1}x - \frac{1}{\sqrt{1-x^2}}\right]$

Example 5. $f'(x) = \frac{1}{\sqrt{1-x^2}}(e^x) + e^x\sin^{-1}x$

$= e^x(\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x)$

Example 6. $f'(x) = \frac{e^x(arctanx)-\frac{1(e^x)}{1+x^2}}{(arctanx)^2}$

S1. $f'(x) = e^x \sin^{-1} x + \left(\frac{1}{\sqrt{1-x^2}}\right)(e^x)$

$= e^x \left[\sin^{-1} x + \frac{1}{\sqrt{1-x^2}}\right]$

S2. $f'(x) = 3x^2 \sin^{-1} x + \frac{1}{\sqrt{1-x^2}}(x^3)$

$= x^2 \left[3\sin^{-1} x + \frac{x}{\sqrt{1-x^2}}\right]$

S3. $f'(x) = 0 \quad \textit{since } e^{10} \textit{ is a constant}$

S4. $f'(x) = \frac{-1}{\sqrt{1-x^2}} + \cos x$

S5. $f'(x) = \frac{1}{1+x^2} + \sec^2 x$

S6. $f'(x) = \sec^2 x(arc \cos x) + \frac{-1}{\sqrt{1-x^2}}(\tan x)$

S7. $f'(x) = e^x(arc \sec x) + \frac{1}{x\sqrt{x^2-1}}(e^x)$

$= e^x \left[arc \sec x + \frac{1}{x\sqrt{x^2-1}}\right]$

S8. $f'(x) = 3x^2 \tan^{-1} x + \frac{1}{1+x^2}(x^3)$

$= x^2(3\tan^{-1} x + \frac{x}{1+x^2})$

S9. $f'(x) = \frac{1}{\sqrt{1-(e^x)^2}}(e^x)$

$= \frac{e^x}{\sqrt{1-e^{2x}}}$

S10. $f'(x) = \frac{3e^{3x}}{1+(e^{3x})^2} = \frac{3e^{3x}}{1+e^{6x}}$

$S11.\ f'(x) = sec^2(\text{arcsin }(e^x)\frac{1}{\sqrt{1-(e^x)^2}}(e^x)$

$$f'(x) = \frac{e^x sec^2(\text{arcsin }(e^x))}{\sqrt{1-e^{2x}}}$$

S12.
$h'(1)$

$h(x) = \tan^{-1}\big(f(x)\big)^3$

$h'(x) = \frac{3[f(x)]^2 \cdot f'(x)}{1+(f(x)^3)^2}$

$h'(1) = \frac{3[f(1)]^2 \cdot f'(1)}{1+\big(f(1)\big)^6}$

$h'(1) = \frac{3(-1)^2(2)}{1+(-1)^6} = \frac{6}{2} = 3$

T. Equation of a Tangent Line

Example 1. Find the slope of the tangent line to the curve $f(x) = 2x^3 + 5x + 4$ at x=1.

$$slope = f'(x) = 6x^2 + 5$$

$$at\ x = 1,\ \ f'(1) = 6(1)^2 + 5 = 11$$

Example 2. Find the slope of the tangent line to the curve $y = 4^{2x+1}$ at x=2.

$$y' = 4^{2x+1}(\ln 4)(2)$$

$$at\ x = 2,\ \ y' = 4^5(\ln 4)(2)$$

Example 3. Find the points on the curve $f(x) = 3x^3 + 9x$ where the slope of the tangent is parallel to the line$-18x + y - 6 = 0$.

$$parallel\ \ \therefore same\ slope\ or\ derivative$$

$$-18x + y - 6 = 0.$$

$$y = 18x + 6 \quad \therefore y' = 18$$

$$f(x) = 3x^3 + 9x$$

$$f'(x) = 9x^2 + 9 \quad \therefore 9x^2 + 9 = 18$$

$$9x^2 = 9$$

$$x^2 = 1 \quad x = 1, -1$$

$$x = 1 \quad f(1) = 12 \quad \therefore (1, 12)\ is\ one\ point$$

$$x = -1 \quad f(-1) = -12 \quad \therefore (-1, -12)\ is\ the\ other\ point$$

Example 4. Find the equation of a tangent line to $f(x) = sin2x\ at$

$x = \frac{\pi}{6}$. $f'(x) = 2cos2x$

$$f'\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$$

Find the value of y, when x=$\frac{\pi}{6}$

$f(x) = sin2x$

f($\frac{\pi}{6}$) = $sin2\left(\frac{\pi}{6}\right) = sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$y - y_1 = m(x - x_1) becomes\ y - \frac{\sqrt{3}}{2} = 1(x - \frac{\pi}{6})$$

$$y = x - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

Example 5. $g(3) = 2(3)^2 - 5(3) + 1 = 18 - 15 + 1 = 4, \quad x_1 = 3$

$$\therefore y_1 = 4$$

$g'(x) = 4x - 5 \qquad g'(3) = 4(3) - 5 = 7$

$\therefore M_{normal} = \frac{-1}{7}$

$y - y_1 = m(x - x_1)$

$$y - 4 = -\frac{1}{7}(x - 3)$$

$$y - 4 = -\frac{1}{7}x + \frac{3}{7}$$

$$y = -\frac{1}{7}x + \frac{3}{7} + \frac{28}{7}$$

$$y = -\frac{1}{7}x + \frac{31}{7}$$

T1. $y = x^2e^{3x}$ at x=1

$y' = 2xe^{3x} + 3e^{3x}(x^2)$

$y'(1) = 2e^3 + 3e^3 = 5e^3$

$Find\ y\ when\ x = 1$

$y = (1)^2e^{3(1)} = e^3$

$y - e^3 = 5e^3(x - 1)$

$y = 5e^3x - 5e^3 + e^3 = 5e^3x - 4e^3$

T2.
$y = 12(e^{3x} + 1) parallel\ to\ y = 4x - 1$

$y' = 12(3e^{3x}) = 36e^{3x}$ parallel to $y' = 4$...same slopes, so we get:

$36e^{3x} = \frac{4}{1}$

$e^{3x} = \frac{1}{9}$

$lne^{3x} = \ln\left(\frac{1}{9}\right)$

$3x = ln1 - ln9$

$3x = -ln9$

$x = -\frac{ln9}{3}$

T3. Horizontal tangent $\therefore slope = 0$

$$f(x) = 4x^2 - 8x + 3 - 4x + 17$$

$$f(x) = 4x^2 - 12x + 20$$

$$f'(x) = 8x - 12$$

$$\therefore 0 = 8x - 12$$

$$x = \frac{12}{8} = \frac{3}{2}$$

$$f\left(\frac{3}{2}\right) = 4\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) + 20 = 4\left(\frac{9}{4}\right) - 18 + 20 = 11$$

So, the point is (3/2, 11).

T4. $f(x) = 3x^3 - 6x \quad at\ x = 2$

$f(2) = 3(2)^3 - 6(2) = 12$ So y= 12

$$f'(x) = 9x^2 - 6$$

$f'(2) = 9(2)^2 - 6 = 30$...so the slope is m=30

$$y - y_1 = m(x - x_1)$$

$$y - 12 = 30(x - 2) \qquad y = 30x - 48$$

$T5.$ $y = (2x - 1)^4 \ \ at\ x = 0$

$$y' = 4(2x - 1)^3(2)$$

$y'_{x=0} = 4(-1)^3(2) = -8$...so slope=m= - 8

$$at\ x = 0 \quad y = (3(0) - 1)^4 = (-1)^4 = 1$$

$y - y_1 = m(x - x_1)$

$y - 1 = -8(x - 0)$ $\quad y = -8x + 1$

T6. $f'(x)$=$9x^2e^x + e^x(3x^3)$

$f'(2)$=$9(2)^2e^2 + e^2(3(2)^3) = 36e^2 + 24e^2 = 60e^2$

U. Second Derivatives

Example 1.

a) $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

b) $f(x) = xe^{2x}$

$f'(x) = (1)(e^{2x}) + (2e^{2x})(x) = e^{2x}(1 + 2x)$

$f''(x) = 2e^{2x}(1 + 2x) + (2)(e^{2x}) = 4xe^{2x} + 4e^{2x}$

$=4e^{2x}(x + 1)$

Example 2.

$f(x) = (2x + 1)^3$

$f'(x) = 3(2x + 1)^2(2) = 6(2x + 1)^2$

$f''(x) = 12(2x + 1)^1(2) = 24(2x + 1)^1$=48x+24

$f'''(x) = 48$

U1. $y'=3(2x-7)^2(2)=6(2x-7)^2$

$y''=12(2x-7)^1(2)=24(2x-7)$

$y'''=24(2)=48$

U2. $f'(x)=\frac{1}{2}(4x-1)^{-1/2}(4)=2(4x-1)^{-1/2}$

$f''(x)=-1(4x-1)^{-3/2}(4)=-4(4x-1)^{-3/2}$

$f'''(x)=6(4x-1)^{-5/2}(4)=24(4x-1)^{-5/2}$

U3. $If\ f(x)=cos2x, find\ f^6(x)$.

$f'(x)=-2\sin 2x$

$f''(x)=-2(\cos 2x\cdot 2)=-4\cos 2x$

$f^3(x)=-4(-\sin 2x\cdot 2)=8\sin 2x$

$f^4(x)=16\cos 2x$

$f^5(x)=-32sin2x$

$f^6(x)=-64\ cos2x$

U4. If $f(x) = (2x+1)^4$, find $f^6(x)$.

$f'(x) = 4(2x+1)^3(2) = 8(2x+1)^3$

$f''(x) = 24(2x+1)^2(2) = 48(2x+1)^2$

$f'''(x) = 96(2x+1)(2) = 192(2x+1)$

$f^4(x) = 192(2) = 384$

$f^5(x) = f^6(x) = 0$

U5. f(x) = sinx

$f'(x) = cosx$

$f"(x) = -sinx$

$f^3(x) = -cosx$

$f^4(x) = sinx$...keep going in this pattern... $f^{10}(x) = -sinx$

V. Tricky Domain Questions

Example 1. $x \neq \pm 2$

$x^2 - 4 \neq 0$ $D = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

$x^2 \neq 4$ So, $x \neq \pm 2$

R=$(-\infty, \infty)$

Example 2. $x - 5 \geq 0$

$x \geq 5$ $\therefore D = [5, \infty)$

R=$[0, \infty)$ (includes 0)

Example 3. $x^2 - 4 \neq 0$ $x^2 - 4 \geq 0$

$x \neq \pm 2$ $x^2 \geq 4$

$x \geq 2 \; or \; x \leq -2$

$\therefore D = (-\infty, -2) \cup (2, \infty)$

R=$(-\infty, 0) \cup (0, \infty)$since the bottom has to be above 0, x divided by a positive number can never give us 0.

Example 4. $x - 2 > 0$

$x > 2$ $\therefore D = (2, \infty)$

Example 5. $f(g(x)) = f(\sqrt{x-5}) = ln\sqrt{x-5}$

From the square root, we know x-5$\geq$ 0 but because of the "ln" we know x-5>0, so the domain is $D = (5, \infty)$ as x=5 is not included. The range is: R=$[0, \infty)$ (includes 0)

Example 6. $D\ of\ arc\cos x\ \ is\ \ [-1, 1]$

$D\ of\ e^x\ is\ R$

So, $D\ of\ f(x) is\ \ [-1, 1]$

Example 7. Domain of $\cos^{-1}(x)\ is\ \ [-1, 1]$

Solve $-1 \leq e^{3x} \leq 1 \quad but \quad e^{3x} > 0\ \ for\ all\ x, so$

$0 < e^{3x} \leq 1$

$\ln 0 < \ln e^{3x} \leq \ln 1$

$-\infty < 3x \leq 0$

$-\infty < x \leq 0$

$\therefore D = (-\infty, 0]$

V1. $x + 2 \neq 0$

$x \neq -2 \qquad \therefore D = (-\infty, -2) \cup (-2, \infty)$

V2. $D = [-1, 1]$

V3. Domain of $\cos^{-1}(x)\ is\ \ [-1,1]$

Solve $-1 \leq e^x \leq 1 \quad but \quad e^x > 0\ \ for\ all\ x,\ so$

$0 < e^x \leq 1$

$\ln 0 < \ln e^x \leq \ln 1$

$-\infty < x \leq 0$

$\therefore D = (-\infty, 0]$

V4. x+6≥ 0

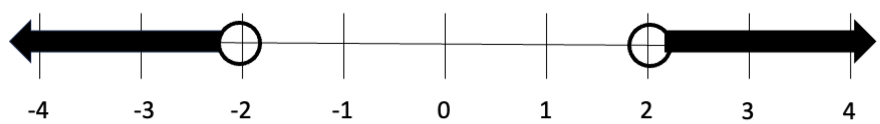
$x \geq -6 \qquad \therefore D = [-6, \infty)$

V5.

$y = \ln(x^2 - 4)$

$x^2 - 4 > 0$

$x^2 > 4$



$\therefore D = (-\infty, -2) \cup (2, \infty)$

V6.

I $f(x) = \sqrt{\dfrac{x}{x+4}}$ II $g(x) = \dfrac{\sqrt{x}}{\sqrt{x+4}}$

<u>No, they do NOT have the same domain</u>

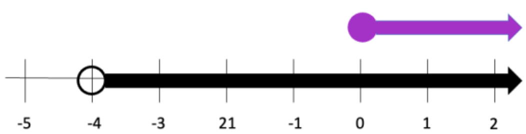
For I:

$\dfrac{x}{x+4} \geq 0 \quad and \;\; x + 4 \neq 0$

$$ie. x \neq -4$$

<u>Case 1</u>

$x \geq 0$ and $x + 4 > 0$, ie) $x > -4$

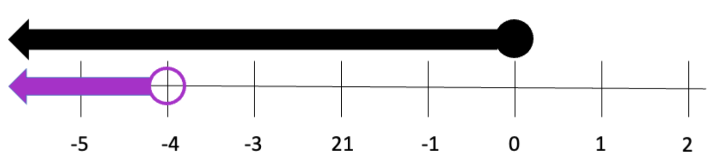


$\therefore x \geq 0$

OR

<u>Case 2</u>

$x \leq 0$ and $x + 4 < 0$, ie) $x < -4$



$\therefore x < -4$

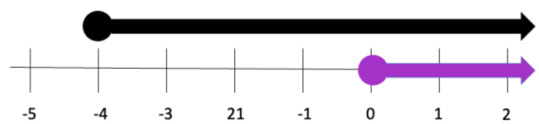
$\therefore$ solution $x \geq 0$ or $x < -4$ $(-\infty, -4) \cup [0, \infty)$ which is not the answer for II below.

For II:

D of $\sqrt{x}$ D of $\sqrt{x+4}$

$x \geq 0$ $x + 4 \geq 0$

$x \geq -4$



∴ combined we get:

Ie.) $x \geq 0$ or $[0, \infty)$

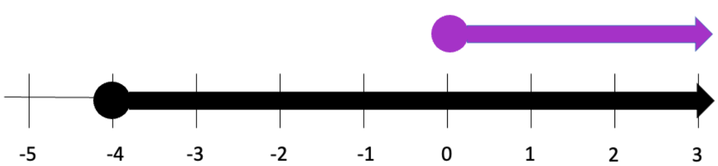
V7.

$h(x) = \sqrt{x(x+4)}$

$x(x+x) \geq 0$

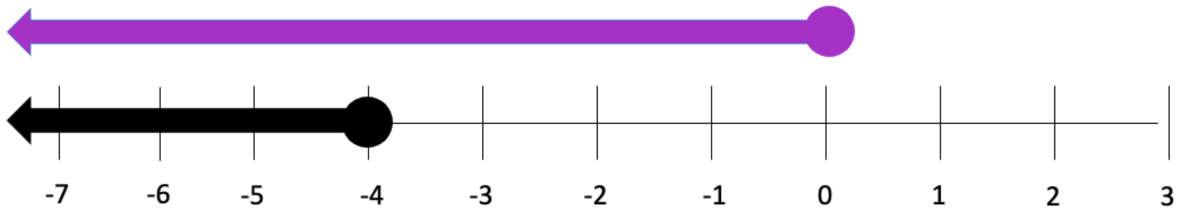
<u>Case 1</u> $x \geq 0$ *and* $x + 4 \geq 0$

$ie).\, x \geq -4$



∴ both are coloured in for $x \geq 0$ ie. $[0, \infty)$

<u>Case 2</u> $x \leq 0 \quad and\ x + 4 \leq 0, ie.)\ x \leq -4$



∴ both are coloured in for $x \leq -4$ ie. $(-\infty, -4]$

∴ final answer is $(-\infty, -4] \cup [0, \infty)$

W. Inverse Function Theorem

Example 1.

Use the inverse $function$ theorem to find derivative of $g(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

First, find the inverse of y=$x^{\frac{1}{3}}$

Switch x and y: x=$y^{\frac{1}{3}}$ cube both sides and get $x^3 = y$ and $f(x) = x^3$

The $function$ $g(x) = x^{\frac{1}{3}}$ is the inverse of $f(x) = x^3$

$$f'(x) = 3x^2$$

$$f'\left(g(x)\right) = f'\left(x^{\frac{1}{3}}\right) = 3(x^{\frac{1}{3}})^2 = 3x^{\frac{2}{3}}$$

$$\therefore g'(x) = \frac{1}{f'(g(x))} = \frac{1}{3x^{\frac{2}{3}}}$$

Example 2. $f(x) = x + \sqrt{x}$ $\quad Find \ (f^{-1})'(2)$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Step 1. Find $f'(x)$

$$f'(x) = 1 + \frac{1}{2}x^{-\frac{1}{2}} = 1 + \frac{1}{2\sqrt{x}}$$

Step 2. Find $f^{-1}(2)$ $\quad \leftarrow f(x) is \ 2$

$$2 = x + \sqrt{x}$$

$$f(x) = x + \sqrt{x} \quad \leftarrow true \ if \ x = 1$$

$$\therefore f^{-1}(2) = 1$$

Step 3. $(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}$

$$= \frac{1}{f'(1)} \leftarrow f'(x) = 1 + \frac{1}{2\sqrt{x}}$$

$$f'(1) = 1 + \frac{1}{2\sqrt{1}} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$= \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}}$$

W1.

Use the inverse $function$ theorem to find derivative of $g(x) = \frac{x+2}{x}$

$$y = \frac{x+2}{x}$$

$$x \leftrightarrow y \quad x = \frac{y+2}{y}$$

$$xy = y + 2$$

$$xy - y = 2$$

$$y(x-1) = 2$$

$$y = \frac{2}{x-1} \qquad \therefore f(x) = \frac{2}{x-1} = 2(x-1)^{-1}$$

Need $f'(g(x))$

$$f'(x) = -2(x-1)^{-2}$$

$$f'(g(x)) = f'\left(\frac{x+2}{x}\right) = -2\left[\frac{x+2}{x} - 1\right]^{-2}$$

$$= -2\left[1 + \frac{2}{x} - 1\right]^{-2}$$

$$= -2\left[\frac{2}{x}\right]^{-2} = -2\left[\frac{x}{2}\right]^{2}$$

$$= \frac{-2x^2}{4} = \frac{-x^2}{2}$$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\frac{-x^2}{2}} = \boxed{\frac{-2}{x^2}}$$

W2.

$f(x) = 2x^3 + 3x^2 + 7x + 4$

Find $(f^{-1})'(4)$

$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

Step 1. find $f'(x)$

$f'(x) = 6x^2 + 6x + 7$

Step 2. find $f^{-1}(4)$ $\quad 4\ is\ f(x) not\ x$

$\therefore f(x) = 2x^3 + 3x^2 + 7x + 4$

$4 = 2x^3 + 3x^2 + 7x + 4$ ← true if $x = 0$

$\therefore f^{-1}(4) = 0$

Step 3. $(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))}$

$= \frac{1}{f'(0)}$

$= \frac{1}{6(0)^2+6(0)+7}$ $\quad \leftarrow f'(x) = 6x^2 + 6x + 7$

$\therefore f'(0) = 7$

$= \boxed{\frac{1}{7}}$

X. Implicit Differentiation and Log Differentiation

Logarithmic Differentiation

Example 2.

$$y=x^x$$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y}\frac{dy}{dx}\left[(1)\ln x + \frac{1}{x}(x)\right]$$

$$\frac{dy}{dx} = y[\ln x + 1] = x^x(\ln x + 1)$$

Example 3. If $y = (4x+3)^x$, find $\frac{dy}{dx}$ at x=1.

$$\ln y = \ln(4x+3)^x$$

$$\ln y = x \cdot \ln(4x+3)$$

$$\frac{1}{y}\frac{dy}{dx} = (1)\ln(4x+3) + \frac{4x}{4x+3}$$

$$\frac{dy}{dx} = (4x+3)^x\left[\ln(4x+3) + \frac{4x}{4x+3}\right]$$

$$\frac{dy}{dx}|x = 1 \quad = 7^1\left[\ln 7 + \frac{4}{7}\right]$$

Example 4. If $f(x) = \ln(x^x)$, find $f'(x)$.

$$y = \ln x^x \quad use \log rules$$

$$y = x \ln x \quad not \log diff$$

$$y' = (1)\ln x + \frac{1}{x}(x)$$

$$y' = \ln x + 1$$

Example 5. If $y = (lnx)^x$, find $f'(e)$.

$$\ln y = \ln(\ln x)^x$$

$$\ln y = x \cdot \ln(\ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = (1)\ln(\ln x) + \frac{1\left(\frac{1}{x}\right)(x)}{\ln x}$$

$$\frac{dy}{dx} = (\ln x)^x \left[\ln(\ln x) + \frac{x}{x\ln x}\right]$$

$$\frac{dy}{dx} = (\ln x)^x \left[\ln(\ln x) + \frac{1}{\ln x}\right]$$

$$at\ x = e \quad \frac{dy}{dx} = (\ln e)^e \left[\ln(\ln e) + \frac{1}{\ln e}\right]$$

$$(1)^e \left[\ln 1 + \frac{1}{1}\right]$$

$$= \ln 1 + 1 \quad = 1$$

Example 6. $If\ f(x) = (x+5)^x x^{x+6}, find\ \frac{dy}{dx}.$

$$y = (x+5)^x x^{x+6}$$

$$\ln y = \ln[(x+5)^x \cdot x^{x+6}]$$

$$\ln y = x\ \ln(x+5) + (x+6)\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = (1)\ln(x+5) + \frac{x}{x+5} + (1)\ln x + \frac{1}{x}(x+6)$$

$$\frac{dy}{dx} = y\left[\ln(x+5) + \frac{x}{x+5} + \ln x + \frac{x+6}{x}\right]$$

$$\frac{dy}{dx} = (x+5)^x x^{x+6}\left[\ln(x+5) + \frac{x}{x+5} + \ln x + \frac{x+6}{x}\right]$$

Example 7. Find the derivative of $y(x) = \frac{x^2(x-6)^3}{\sqrt[3]{3-x}}$

$$y = x^2(x-6)^3(3-x)^{\frac{-1}{3}}$$

$$\ln y = \ln\left[x^2(x-6)^3(3-x)^{\frac{-1}{3}}\right]$$

$$\ln y = 2\ln x + 3\ln(x-6) - \frac{1}{3}\ln(3-x)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\left(\frac{1}{x}\right) + 3\left(\frac{1}{x-6}\right) - \frac{1}{3}\left(\frac{-1}{3-x}\right)$$

$$\frac{dy}{dx} = y\left[\frac{2}{x} + \frac{3}{x-6} + \frac{1}{9-3x}\right]$$

$$\frac{dy}{dx} = \frac{x^2(x-6)^3}{\sqrt[3]{3-x}}\left[\frac{2}{x} + \frac{3}{x-6} + \frac{1}{9-3x}\right]$$

Implicit Differentiation

Example 1.

$$x^3 + y^2 = 2$$

$$3x^2 + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{3x^2}{2y}$$

Example 3. Find the derivative $x^4 + 6y^3 = 12x + 6$.

$$4x^3 + 18y^2\frac{dy}{dx} = 12$$

$$18y^2\frac{dy}{dx} = 12 - 4x^3$$

$$\frac{dy}{dx} = \frac{12-4x^3}{18y^2}$$

dy/dx at $(0,1) = \frac{12-4(0)^3}{18(1)^2} = \frac{12}{18} = \frac{2}{3}$

Example 4. Differentiate: $x^3y^2 - 4x = 3 - 2y$

product rule

$$3x^2(y^2) + 2y \cdot \frac{dy}{dx}(x^3) - 4 = 0 - 2\frac{dy}{dx}$$

$$2x^3y\frac{dy}{dx} + 2\frac{dy}{dx} = 4 - 3x^2y^2$$

Factor $\frac{dy}{dx}(2x^3y + 2) = 4 - 3x^2y^2$

$$\frac{dy}{dx} = \frac{4-3x^2y^2}{2x^3y+2}$$

Example 5. Differentiate: $e^{xy} + 4x = 5 - lnx + y$

$$e^{xy}\left[(1)(y) + \frac{dy}{dx}(x)\right] + 4 = 0 - \frac{1}{x} + (1)\frac{dy}{dx}$$

$$ye^{xy} + xe^{xy}\frac{dy}{dx} + 4 = -\frac{1}{x} + \frac{dy}{dx}$$

$$xe^{xy}\frac{dy}{dx} - \frac{dy}{dx} = -\frac{1}{x} - 4 - ye^{xy}$$

factor

$$\frac{dy}{dx}(xe^{xy} - 1) = -\frac{1}{x} - 4 - ye^{xy}$$

$\frac{dy}{dx} = \frac{\frac{-1}{x} - 4 - ye^{xy}}{xe^{xy} - 1}$ OR $\frac{dy}{dx} = \frac{\frac{1}{x} + 4 + ye^{xy}}{-xe^{xy} + 1}$ (if you brought all the terms to the opposite sides)

Example 6. $x^3 - 4y^3 + 2y^2 = -1$ Find $y'(1,1) and\ y''(1,1)$.

$$3x^2 - 12y^2y' + 4yy' = 0$$

$$y'(-12y^2 + 4y) = -3x^2$$

$$y' = \frac{-3x^2}{-12y^2 + 4y}$$

$$y'(1,1) = \frac{-3}{-12 + 4} = \frac{-3}{-8} = \frac{3}{8}$$

$$\frac{d^2y}{dx^2} = \frac{(-6x)(-12y^2 + 4y) - (-24yy' + 4y')(-3x^2)}{(-12y^2 + 4y)^2}$$

$$\frac{d^2y}{dx^2}(1,1) = \frac{(-6)(-12+4) - \left(-24(1)\left(\frac{3}{8}\right) + 4\left(\frac{3}{8}\right)\right)(-3)}{(-12+4)^2}$$

$$\frac{d^2y}{dx^2}(1,1) = \frac{48 - (-9 + \frac{3}{2})(-3)}{64} = \frac{48 - \left(-\frac{18}{2} + \frac{3}{2}\right)(-3)}{64}$$
$$= \frac{48 - \left(-\frac{15}{2}\right)(-3)}{64}$$

$$\frac{d^2y}{dx^2}(1,1) = \frac{\frac{96}{2} - \frac{45}{2}}{64} = \frac{51}{2}\left(\frac{1}{64}\right) = \frac{51}{128}$$

Example 7.

$2x^2 + xy + 2y^2 = 4$ $\boxed{1}$

Horizontal tangent $y' = 0$

Implicit

$4x + 1(y) + xy' + 4yy' = 0$

$y'(x + 4y) = -4x - y$

$y' = -\dfrac{4x - y}{x + 4y}$

$y' = 0, so - 4x - y = 0 \ and \ make \ sure \ (x + 4y \neq 0)$

$y = -4x$

Substitute into $\boxed{1}$

$$2x^2 + xy + 2y^2 = 4$$

$$2x^2 + x(-4x) + 2(-4x)^2 = 4$$

$$2x^2 - 4x^2 + 2(16x^2) = 4$$

$$-2x^2 + 32x^2 = 4$$

$$30x^2 = 4$$

$$x^2 = \frac{4}{30}$$

$$x = \pm\sqrt{\frac{4}{30}} = \pm\frac{2}{\sqrt{30}}$$

$$x = \frac{2}{\sqrt{30}}$$

$$y = -4x$$

$$y = -4\left(\frac{2}{\sqrt{30}}\right)$$

$$y = \frac{-8}{\sqrt{30}}$$

$$x = \frac{-2}{\sqrt{30}}$$

$$y = -4x$$

$y = -4\left(\frac{-2}{\sqrt{30}}\right)$

$y = \frac{8}{\sqrt{30}}$

$\therefore$ points are $\left(\frac{2}{\sqrt{30}}, \frac{-8}{\sqrt{30}}\right)$ and $\left(\frac{-2}{\sqrt{30}}, \frac{8}{\sqrt{30}}\right)$

Practice Exam Questions on Log Differentiation and Implicit Differentiation

X1. y=$(x+2)^x$

$\ln y = \ln(x+2)^x$

$\ln y = x\ln(x+2)$

$\frac{1}{y}\frac{dy}{dx} = \left[(1)\ln(x+2) + \frac{1}{x+2}(x)\right]$

$\frac{dy}{dx} = (x+2)^x\left[\ln(x+2) + \frac{x}{x+2}\right]$

$\frac{dy}{dx}(x=1) = 3^1\left[\ln 3 + \frac{1}{3}\right]$

Therefore, answer is B).

X2. $\ln y = \ln(x+2)^{\ln x}$

$\ln y = \ln x \ln(x+2)$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(x+2) + \frac{1}{x+2}(\ln x)$

$$\frac{dy}{dx} = y\left[\frac{\ln(x+2)}{x} + \frac{\ln x}{x+2}\right]$$

$$\frac{dy}{dx} = (x+2)^{\ln x}\left[\frac{\ln(x+2)}{x} + \frac{\ln x}{x+2}\right]$$

$$\frac{dy}{dx}(x=1) = 3^{\ln 1}\left[\frac{ln3}{1} + \frac{\ln 1}{3}\right] = 3^0(\ln 3) = ln3$$

X3. y=$(5x+1)^x$

$$\ln y = \ln(5x+1)^x$$

$$\ln y = x\ln(5x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = (1)\ln(5x+1) + \frac{5}{5x+1}(x)$$

$$\frac{dy}{dx} = y\left[\ln(5x+1) + \frac{5x}{5x+1}\right]$$

$$\frac{dy}{dx} = (5x+1)^x\left[\ln(5x+1) + \frac{5x}{5x+1}\right]$$

$$\frac{dy}{dx}(x=1) = 6^1\left[\ln 6 + \frac{5}{6}\right] = 6ln6 + 5$$

X4. $lny = ln\left[\frac{x^{1/3}(3x-1)^3}{(3-x)^{\frac{1}{3}}}\right]$

$$lny = lnx^{1/3} + ln(3x-1)^3 - ln(3-x)^{\frac{1}{3}}$$

$$lny = \frac{1}{3}lnx + 3\ln(3x-1) - \frac{1}{3}\ln(3-x)$$

$$\frac{1}{y}y' = \frac{1}{3}\left(\frac{1}{x}\right) + 3\left(\frac{3}{3x-1}\right) - \frac{1}{3}\left(\frac{-1}{3-x}\right)$$

$$y' = \frac{x^{1/3}(3x-1)^3}{(3-x)^{\frac{1}{3}}}\left[\frac{1}{3x} + \frac{9}{3x-1} + \frac{1}{9-3x}\right]$$

X5.

$y = (x + e^x)^{\ln x}$

$\ln y = \ln(x + e^x)^{\ln x}$

$\ln y = \ln x \ln(x + e^x)$

$\frac{1}{y}y' = \frac{1}{x}\ln(x + e^x) + \frac{(1+e^x)\ln x}{x+e^x}$

$\frac{1}{y}y' = \frac{1}{x}\ln(x + e^x) + \frac{(1+e^x)\ln x}{x+e^x}$ multiply by y

$y' = (y)\left[\frac{\ln(x+e^x)}{x} + \frac{(1+e^x)\ln x}{x+e^x}\right]$

$$y' = (x + e^x)^{\ln x}\left[\frac{\ln(x + e^x)}{x} + \frac{(1 + e^x)\ln x}{x + e^x}\right]$$

X6.

$y = (\ln x)^{\ln x}$

$\ln y = \ln (\ln x)^{\ln x}$

$\ln y = \ln x \ln(\ln x)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}(\ln(\ln x)) + \frac{1}{lnx}\left(\frac{1}{x}\right)(\ln x)$$

$$\frac{dy}{dx} = y\left[\frac{\ln(\ln x)}{x} + \frac{\frac{\ln x}{x}}{\ln x}\right]$$

$\frac{dy}{dx} = (\ln x)^{\ln x}\left[\frac{\ln(\ln x)}{x} + \frac{\ln x}{x}\left(\frac{1}{\ln x}\right)\right] = (\ln x)^{\ln x}\left[\frac{\ln(\ln x)}{x} + \frac{1}{x}\right]$

At x=e, dy/dx=$(\ln e)^{\ln e}\left[\frac{\ln(\ln e)}{e} + \frac{1}{e}\right] = 1\left(\frac{ln1}{e} + \frac{1}{e}\right) = \frac{1}{e}$

X7. $y = x^{(2^x)}$

$$\ln y = \ln x^{(2^x)}$$

$$\ln y = 2^x \ln x$$

$$\frac{1}{y} \cdot y' = 2^x \ln 2(\ln x) + \frac{1}{x}(2^x)$$

$$y' = y\left[2^x \ln 2(\ln x) + \frac{2^x}{x}\right]$$

$$y' = x^{(2^x)}\left[2^x \ln 2(\ln x) + \frac{2^x}{x}\right]$$

X8. $x^2 + y^2 = 2$

$2x + 2yy' = 0$

$2yy' = -2x$

$y' = \frac{-2x}{2y} = \frac{-x}{y} \qquad subst\ x = y = 1$

$y' = -\frac{1}{1} = -1$

X9. $x^2 + y^2 + 3x - y = 5x^2$

$2x + 2y\,y' + 3 - 1y' = 10x$

$y'(2y - 1) = 10x - 2x - 3$

$y'(2y - 1) = 8x - 3$

$y' = \frac{8x-3}{2y-1}$

X10. $\sin(x + y) = xcosy$

$\cos(x + y)\,(1 + y') = (1)cosy + (x)(-siny)y'$

$\cos(x + y) + y'\cos(x + y) = cosy - xsinyy'$

$y'(\cos(x + y) + xsiny) = cosy - \cos(x + y)$

$$y' = \frac{cosy - \cos(x + y)}{(\cos(x + y) + xsiny)}$$

X11.

$x^3 + y^3 = 27$

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = -\frac{x^2}{y^2}$$

X12. $x^2 + y^2 = 5$

$2x + 2yy' = 0$

$2yy' = -2x$

$y' = \frac{-2x}{2y} = -\frac{x}{y} = -xy^{-1}$

$y'' = (-1)(y^{-1}) + (-y^{-2})y'(-x) = substitute\ y' = -xy^{-1}$

$y'' = (-1)(y^{-1}) + (-y^{-2})(-xy^{-1})(-x)$..you could leave your answer like this!!

$y'' = -y^{-1} - x^2y^{-3} = -y^{-3}(y^2 + x^2) \ldots factor$

Now, the tricky step is to substitute the original equation

in $x^2 + y^2 = 5$

Crazy, eh?

$y'' = -y^{-3}(5) = \frac{-5}{y^3}$

X13. $x^3 + y^2 = 6$

$3x^2 + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -3x^2$

$\frac{dy}{dx} = \frac{-3x^2}{2y}$

X14. $x^2y + x^3 = 12$

$$2x(y) + (1)\frac{dy}{dx}x^2 + 3x^2 = 0$$

$$2xy + x^2\frac{dy}{dx} = -3x^2$$

$$x^2\frac{dy}{dx} = -3x^2 - 2xy$$

$$\frac{dy}{dx} = \frac{-3x^2 - 2xy}{x^2} = \frac{-(3x + 2y)}{x}$$

$$\frac{dy}{dx}(2,1) = \frac{-(3x+2y)}{x} = -\frac{6+2}{2} = -4$$

$X15.$ $e^{xy} = 4x + 7y$

$$e^{xy}[(1)(y) + (1)y'(x)] = 4 + 7y'$$

$$ye^{xy} + xe^{xy}y' = 4 + 7y'$$

$$y'(xe^{xy} - 7) = 4 - ye^{xy}$$

$$y' = \frac{4 - ye^{xy}}{xe^{xy} - 7}$$

$X16.$ $x^2y + x^3 = 8 + y^3$

$$(2x)(y) + (1)\frac{dy}{dx}(x^2) + 3x^2 = 0 + 3y^2\frac{dy}{dx}$$

$$\frac{dy}{dx}(x^2 - 3y^2) = -3x^2 - 2xy$$

$$\frac{dy}{dx} = \frac{-3x^2 - 2xy}{x^2 - 3y^2}$$

X17. Find y' if $\sin(xy) = x^2 + \cos y + 3$.

$$\cos(xy)\left[1(y) + \frac{dy}{dx}(x)\right] = 2x - \sin y \cdot \frac{dy}{dx}$$

$$y\cos(xy) + x\cos(xy)\frac{dy}{dx} = 2x - \sin y \frac{dy}{dx}$$

$$x\cos(xy)\frac{dy}{dx} + \sin y \frac{dy}{dx} = 2x - y\cos(xy)$$

$$\frac{dy}{dx}[x\cos(xy) + \sin y] = 2x - y\cos(xy)$$

$$\frac{dy}{dx} = \frac{2x - y\cos(xy)}{x\cos(xy) + \sin y}$$

X18. Find y'

$$3x^2 - 4x(y) + y'(-2x^2) + 6yy' = 0$$

$$y'(6y - 2x^2) = -3x^2 + 4xy$$

$$y' = \frac{-3x^2 + 4xy}{6y - 2x^2} \qquad y'(1,0) = \frac{-3(1)^2 + 0}{0 - 2(1)^2} = \frac{3}{2}$$

$$\frac{d^2y}{dx^2} = y'' = \frac{[-6x + 4y + y'(4x)][6y - 2x^2] - [(6y' - 4x)(-3x^2 + 4xy)]}{(6y - 2x^2)^2}$$

At (1,0)

$$y'' = \frac{\left[-6 + 0 + \frac{3}{2}(4)(1)\right][0 - 2(1)^2] - \left[6\left(\frac{3}{2}\right) - 4(1)\right][-3(1)^2 + 0]}{(0-2)^2}$$

$$= \frac{0(-2) - (9-4)(-3)}{4}$$

$$= \frac{-(5)(-3)}{4}$$

$$= \frac{15}{4}$$

Y. Related Rates

Example 1. $given\ \frac{dx}{dt} = 2ft/s$

$find\ \frac{dy}{dt}\ when\ x = 5\ ft$

$8^2 = y^2 + 5^2$

$64 = y^2 + 25$

$y = \sqrt{39}$

$x^2 + y^2 = 8^2$

$x^2 + y^2 = 64$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

$5(2) + \sqrt{39}\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-10}{\sqrt{39}}\left(\frac{\sqrt{39}}{\sqrt{39}}\right) = \frac{-10\sqrt{39}}{39}ft/s$

Example 2. $given\ \frac{dv}{dt} = 120cm^3/s$

$find\ \frac{dr}{dt}\ \ when\ V = 36\pi$

$V = \frac{4}{3}\pi r^3\ \ \ sphere$

$find\ r$

$V = \frac{4}{3}\pi r^3$

$36\pi = \frac{4}{3}\pi r^3$

$\frac{36(3)}{4} = r^3$

$r^3 = 27$

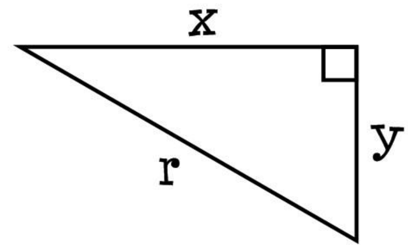
$r = 3$

$\frac{dV}{dt} = \frac{4\pi}{3}(3r^2)\frac{dr}{dt}$

$120 = 4\pi(3)^2\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{120}{4\pi(9)} = \frac{120}{36\pi} = \frac{10}{3\pi} cm/s$

Example 3. $given\ \frac{dx}{dt} = -50,\quad \frac{dy}{dt} = -60\ km/h$



$find\ \frac{dr}{dt}\ \ when\ x = 9\ and\ y = 12$

$r^2 = 9^2 + 12^2 = 81 + 144 = 225$

$r = 15$

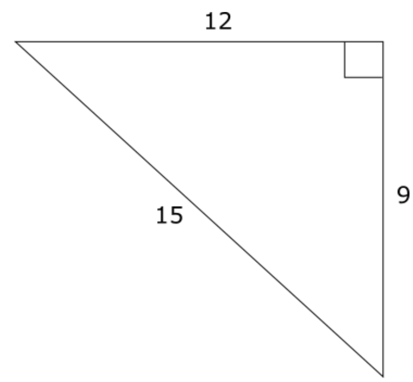
$x^2 + y^2 = r^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt} \quad (divide\ by\ 2)$

$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$

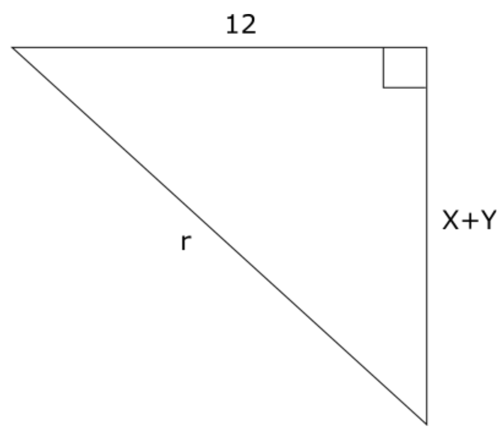
$9(-50) + 12(-60) = 15\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-450-720}{15} = \frac{-1170}{15}\ km/h$ or $-78\ km/h$



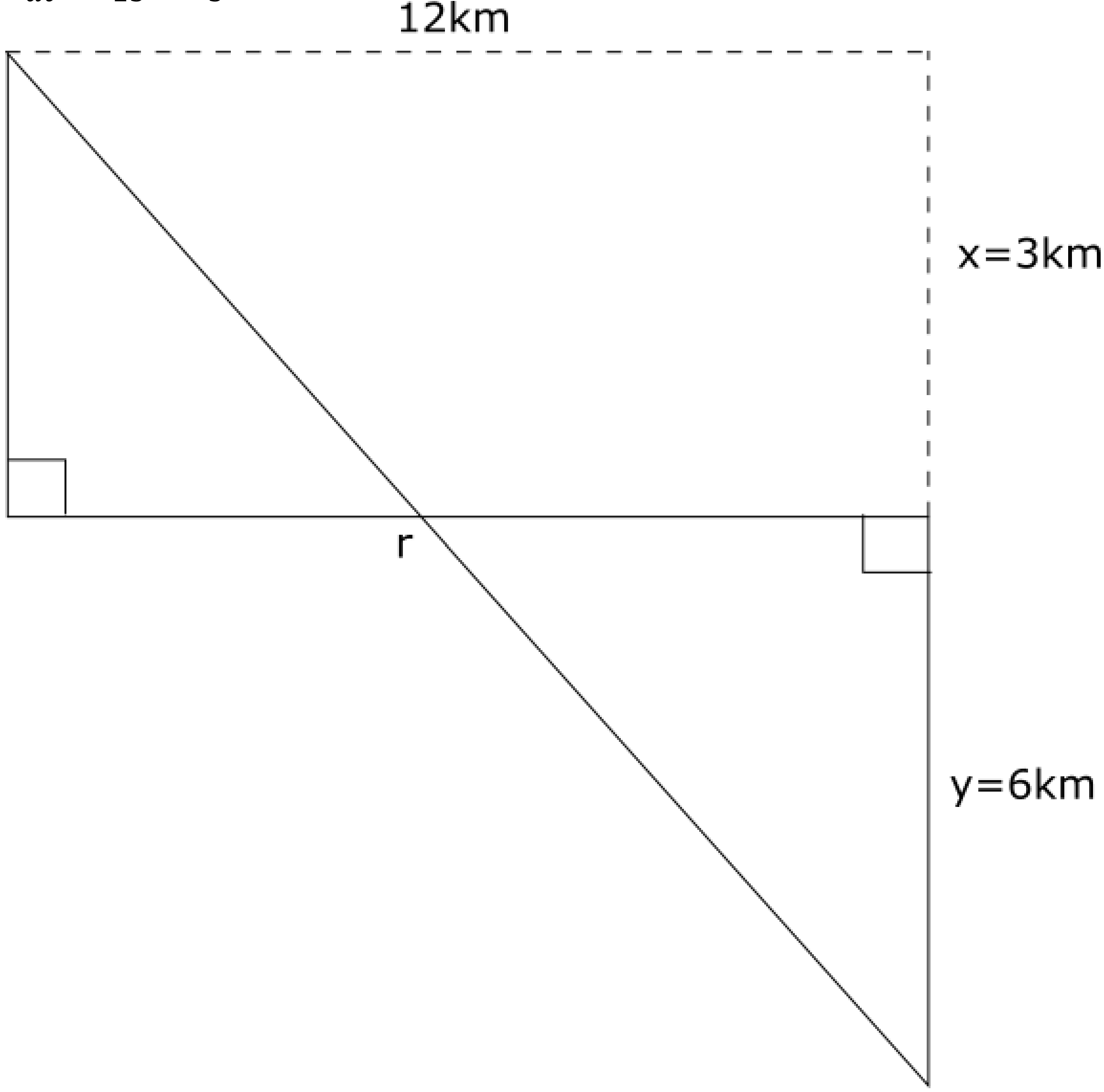
Example 4

$6\ km/hr \cdot 1hr = 6$
$3\ km/hr \cdot 1\ hr = 3$



$r^2 = 9^2 + 12^2$
$r^2 = 81 + 144$
$r^2 = 225$
$r = \sqrt{225} = 15$
$r^2 = 12^2 + (x + y)^2$
$2r\ \frac{dr}{dt} = 0 + 2(x + y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$
$15\ \frac{dr}{dt} = (3 + 6)(3 + 6)$
$15\ \frac{dr}{dt} = 81$

$$\frac{dr}{dt} = \frac{81}{15} = \frac{27}{5}\ km/hr$$



Solve each of the following:

Y1. $given\ \frac{dA}{dt} = -4\ cm^2/mm$ (melting ∴ decreasing)

$find\ \frac{dr}{dt}\quad when\ d = 1m = 100cm \qquad r = 50\ cm$

$A = 4\pi r^2\quad sphere$

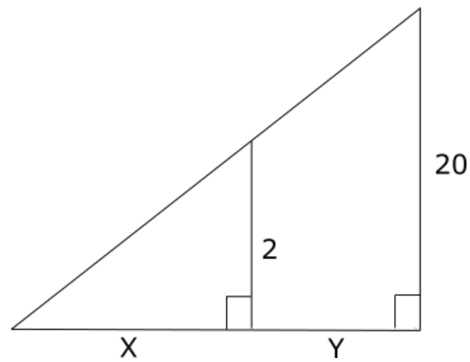
$\frac{dA}{dt} = 4\pi(2r)\frac{dr}{dt}$

$-4 = 4\pi(2)(50)\frac{dr}{dt}$

$-1 = \pi(100)\frac{dr}{dt}$

$\frac{-1}{100\pi} = \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-1}{100\pi}cm/min$



Y2. $given\ \frac{dy}{dt} = 3m/s$

$find\ \frac{dx}{dt}\quad when\ y = 10$

$using\ similar\ \Delta's \quad \frac{2}{x} = \frac{20}{x+y}$

$2x + 2y = 20x$

$2y = 18x \qquad y = 9x$

$\frac{dy}{dt} = 9\frac{dx}{dt}$

$3 = 9\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{1}{3}m/s$

$the\ shadow\ moves\ at \frac{1}{3}m/s.$ The tip of the shadow moves at 3+1/3 m/s.

Y3. $given\ \frac{dV}{dt} = 5\ cm^3/s$

$find\ \frac{dr}{dt}\ when\ V = 36\pi$

$V = \frac{4}{3}\pi r^3 \quad sphere$

$36\pi = \frac{4}{3}\pi r^3$

$\frac{36}{4} \times 3 = r^3 \quad r = 3$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} \quad substitute\ 5 = 4\pi(3)^2\frac{dr}{dt}$

$dr/dt = 5/36\pi\ cm/s$

Y4. $given\ \frac{dx}{dt} = 10\ ft/s$

$find\ \frac{dr}{dt}\ when\ r = 100 \quad 100^2 = 80^2 + x^2$

$x^2 = 3600 \quad x = 60ft$

$x^2 + 80^2 = r^2$

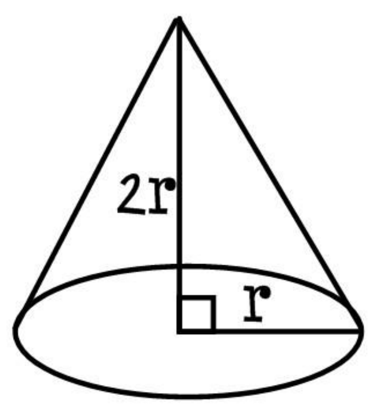
$2x\frac{dx}{dt} + 0 = 2r\frac{dr}{dt}$

$60(10) = 100\frac{dr}{dt}$

$\frac{dr}{dt} = 6\ ft/s$

Y5. $given\ \frac{dV}{dt} = 3ft^3/min$

$find\ \frac{dh}{dt}\ when\ h = 6ft$...note: h=2r and r=1/2 h



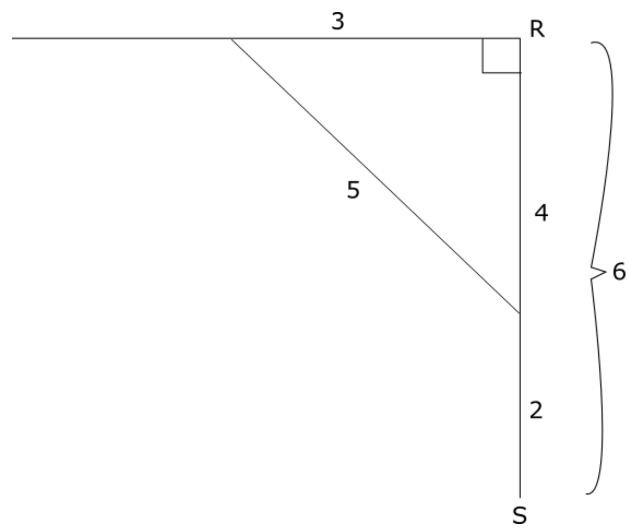
$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{1}{2}h)^2 h = \frac{1}{3}\pi(\frac{1}{4}h^3)$

$\frac{dV}{dt} = \frac{1}{12}\pi(3h^2)\frac{dh}{dt}$

$3 = \frac{1}{12}\pi(3(6)^2)\frac{dh}{dt}$

$3 = 9\pi\frac{dh}{dt} \quad \therefore \frac{dh}{dt} = \frac{1}{3\pi}ft/min$

Y6.



$given\ \frac{dx}{dt} = 1.5km/h \quad \frac{dy}{dt} = -1.0km/h$

$find\ \frac{dr}{dt}\ when\ x = 1.5km/h\ (2hr) = 3$

$y = (1.0km/h)(2) = 2$

6-2=4 inside the triangle

$x^2 + y^2 = r^2$

$(3)^2 + (4)^2 = r^2$

$r^2 = 25 \quad r = 5$

$x^2 + y^2 = r^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$

$3(1.5) + (4)(-1.0) = 5\frac{dr}{dt}$

4.5 – 4 = 5 dr/dt

$\therefore \frac{dr}{dt} = \frac{0.5}{5} = 0.1km/h$

Y7. $let\ \frac{dx}{dt}\ be\ Cara's\ speed \quad let\ \frac{dy}{dt}\ be\ Jocelyn's\ speed$

$given\ \frac{dx}{dt} = 4m/s \quad \frac{dy}{dt} = 3m/s$

$find\ \frac{dr}{dt}\ after\ 5s$

$y = 4m/s(5s) = 20m \quad x = 3m/s(5s) = 15m$

$x^2 + y^2 = r^2$

$20^2 + 15^2 = r^2$

$r^2 = 625$

$r = 25$

$x^2 + y^2 = r^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$

$15(3) + 20(4) = 25\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{45+80}{25} = \frac{\frac{125}{5}m}{s} = 5\ m/s$

Y8. $A = 4\pi r^2 \quad sphere$

$given \quad \frac{dr}{dt} = -3\text{ m/s} \quad find \quad \frac{dA}{dt} \quad when \ r = 4m$

$Formula \quad A = 4\pi r^2$

$derivative \quad \frac{dA}{dt} = 4\pi(2r)\frac{dr}{dt}$

$substitute \quad \frac{dA}{dt} = 4\pi(2)(4)(-3) \quad \therefore \frac{dA}{dt} = -96\pi m^2/\text{s}$

Y9. Sphere $V = \frac{4}{3}\pi r^3$

$given \quad \frac{dv}{dt} = -4cm^3/\text{min}$

$find \quad \frac{dr}{dt} \quad when \ V = 36\pi cm^3 \quad find\ r \quad V = \frac{4}{3}\pi r^3$

$$36\pi = \frac{4}{3}\pi(r)^3$$

$cancel\ \pi\ by\ dividing\ both\ sides\ by\ \pi$

$$36 = \frac{4}{3}r^3 \quad divide\ by\ 4$$

$$9 = \frac{1}{3}r^3 \quad multiply\ 9\ by\ 3$$

$$27 = r^3 \qquad r = 3$$

$formula \quad V = \frac{4}{3}\pi r^3$

$derivative \quad \frac{dv}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$

$substitute \quad -4 = 4\pi(3)^2\frac{dr}{dt} \qquad \therefore -4 = 36\pi\frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{-1}{9\pi}cm/\text{min}$$

Y10. $V = \pi r^2 h \quad (cylinder)$

$given \quad \frac{dv}{dt} = -36cm^3$/s (being drained)

$find \quad \frac{dh}{dt} \quad when \quad d = 60cm \quad (r = 30cm)$

$formula \quad V = \pi r^2 h$ (radius is constant)

$$V = \pi(30)^2 h \quad = 900\pi h$$

$$derivative \quad \frac{dv}{dt} = 900\pi\left[\frac{dh}{dt}\right]$$

$$-36 = 900\pi\frac{dh}{dt}$$

$$\frac{-36}{900\pi} = \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{-36}{900\pi} = \frac{-1}{25\pi}cm/\text{s}$$

Y11. $given \quad h = 2r, \ \frac{dv}{dt} = 12ft^3$/min

$find \ \frac{dr}{dt} \ when \ h = 6\ ft \quad (r = 3ft)$

$$formula \quad cone \ V = \frac{1}{3}\pi r^2 h \qquad substitute \quad h = 2r$$

$$we \ get \quad V = \frac{1}{3}\pi r^2(2r)$$

$$\therefore V = \frac{2}{3}\pi r^3$$

$$derivative \quad \frac{dv}{dt} = \frac{2}{3}\pi(3r^2)\frac{dr}{dt}$$

$$\frac{dv}{dt} = 2\pi r^2\frac{dr}{dt}$$

$$substitute \quad 12 = 2\pi(3)^2\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{12}{18\pi} = \frac{2}{3\pi}ft/\text{min}$$

Y12. $given: \quad x = 6$

$\frac{dx}{dt} = 2\ ft/s,\ \ x = 6ft,\ \ r = 9ft \qquad Find\ \frac{dy}{dt}$

$$x^2 + y^2 = r^2$$

$$y^2 = r^2 - x^2$$

$$y^2 = 81 - 36 = 45$$

$$y = \sqrt{45} = 3\sqrt{5}$$

$derivative \qquad x^2 + y^2 = 9^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

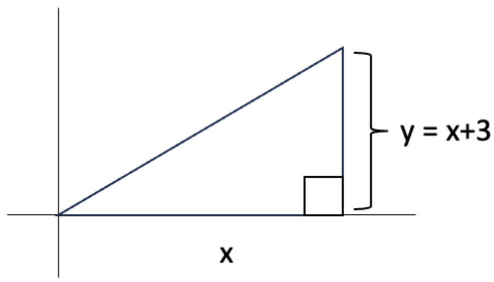
$$6(2) + 3\sqrt{5}\frac{dy}{dt} = 0$$

$$3\sqrt{5}\frac{dy}{dt} = -12$$

$$\frac{dy}{dt} = \frac{-12}{3\sqrt{5}}$$

$$\frac{dy}{dt} = -\frac{4}{\sqrt{5}}\ or - \frac{4\sqrt{5}}{5}\ ft/s$$

Y13.



Given: $\frac{dx}{dt} = 4 \text{ m/s}$

Find: $\frac{dA}{dt}\,at\ (3{,}6)$at x=3

Area= $\frac{\text{base x height}}{2}$

$$A = \frac{x\,(x+3)}{2} = \frac{1}{2}(x^2 + 3x)$$

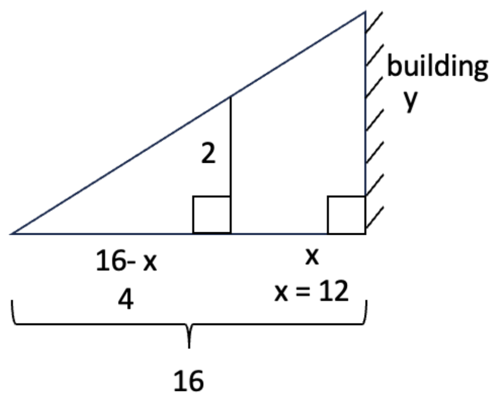
$$A(x) = \frac{1}{2}x^2 + \frac{3}{2}x$$

$\frac{dA}{dt} = \frac{1}{2}(2x)\left(\frac{dx}{dt}\right) + \frac{3}{2}(\frac{dx}{dt})$ Substitute $\frac{dx}{dt} = 4\ m/s$ and x=3:

$$\frac{dA}{dt} = \frac{1}{2}(2)(3)(4) + \frac{3}{2}(4)$$

$$= 12 + 6 = 18 \text{ m}^2\text{/ second}$$

Y14.



Given: $\frac{dx}{dt} = 1.4 \text{ m/s}$

Let y be the height of the building

Find $\frac{dy}{dt}$ when $x = 12$

<u>Similar Triangle's</u>

$$\frac{2}{4} = \frac{y}{16}$$

$$4y = 32$$

$$y = 8$$

In general → $\frac{2}{16-x} = \frac{y}{16}$

Flip the fractions: $\frac{16-x}{2} = \frac{16}{y}$

$$8 - \frac{1}{2}x = 16y^{-1}$$

<u>Do derivative</u>

$$-\frac{1}{2}(1.4) = -\frac{16}{(8)^2}\frac{dy}{dt}$$

$$-0.7 = -\frac{16}{64}\frac{dy}{dt}$$

$$-0.7 = -\frac{1}{4}\frac{dy}{dt}$$

$$-0.7(4) = -\frac{dy}{dt}$$

$$\frac{dy}{dt} = 2.8 \text{ m/s}$$

Y15. Given: $\frac{dr}{dt} = 6 \text{ cm/s}$

Find: $\frac{dV}{dt}$ when $r = 3$

Formula: $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{4}{3}\pi\,(3r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substitute: $\frac{dr}{dt} = 6, r = 3$

$$\frac{dV}{dt} = 4\pi(3)^2(6)$$

$$\frac{dV}{dt} = 4\pi(9)(6)$$

$$\frac{dV}{dt} = 216 \text{ cm}^3/\text{s}$$

Best of luck on your exam!!

NEW MATERIAL SINCE THE MIDTERM

A. L'Hospital's Rule

Example 3. $\lim_{x\to\infty} \frac{e^x}{x^2} \quad \frac{\infty}{\infty}$

$\text{H} = \lim_{x\to\infty} \frac{e^x}{2x} \quad \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$

Example 4. $\lim_{x\to 0} \frac{\sin(x^3)}{2x}$

$H = \lim_{x\to 0} \frac{\cos(x^3)(3x^2)}{2} = \frac{(cos0)(3(0))}{2} = 0$

Example 5. $\lim_{x\to\frac{\pi}{2}^-} (\sec x - \frac{1}{cotx})$

$= \lim_{x\to\frac{\pi}{2}^-} (\frac{1}{\cos x} - \frac{\sin x}{\cos x}) = \lim_{x\to\frac{\pi}{2}^-} \left(\frac{1-\sin x}{\cos x}\right) \qquad \frac{0}{0}$

$H = \lim_{x\to\frac{\pi}{2}^-} (\frac{-\cos x}{-\sin x}) = \lim_{x\to\frac{\pi}{2}^-} \left(\frac{\cos x}{\sin x}\right) = \frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}} = \frac{0}{1} = 0$

Example 6. $\lim_{x\to\infty} \frac{x^3+2x}{x^4-2x} \qquad \frac{\infty}{\infty}$

$H = \lim_{x\to\infty} \frac{3x^2+2}{4x^3-2} \qquad \frac{\infty}{\infty}$

$H = \lim_{x\to\infty} \frac{6x}{12x^2} \frac{\infty}{\infty}$

$H = \lim_{x\to\infty} \frac{6}{24x} = \frac{6}{24(\infty)} = 0$

Example 7. $\lim_{x\to 0^+} x^x \qquad$ indeterminate 0^0

Let $y=x^x$ take the ln of both sides NOTE: $x^x = e^{xlnx}$

$lny = lnx^x$

$lny = xlnx$ log rules

$\lim_{x\to 0^+} lny = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$

$H = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{1}{x}\left(\frac{-x^2}{1}\right) = \lim_{x\to 0^+} (-x) = -(0) = 0$

So, $\lim_{x\to 0^+} y = \lim_{x\to 0^+} x^x \quad = e^0 = 1$

$\therefore \lim_{x\to 0^+} x^x = 1$

Example 8.

$Let\ y = (x^4 + e^{3x})^{\frac{1}{x}}$...take the ln of both sides

$\ln y = ln(x^4 + e^{3x})^{\frac{1}{x}}$

$lny = \frac{1}{x}\ln(x^4 + e^{3x})$

$\lim_{x\to\infty} \ln y$

$= \lim_{x\to\infty} \ln\,(x^4 + e^{3x})^{\frac{1}{x}}$

$\lim_{x\to\infty} lny = \lim_{x\to\infty} \frac{ln(x^4+e^{3x})}{x} \quad \frac{\infty}{\infty}$

$(\mathrm{H}) = \lim_{x\to\infty} \frac{(4x^3+3e^{3x}}{(x^4+e^{3x})}) \quad \frac{\infty}{\infty}$

$(\mathrm{H}) = \lim_{x\to\infty} \frac{12x^2+9e^{3x}}{(4x^3+3e^{3x}} \quad \frac{\infty}{\infty}$

$(\mathrm{H}) = \lim_{x\to\infty} \frac{24x+27e^{3x}}{12x^2+9e^{3x}} \quad \frac{\infty}{\infty}$

$(\mathrm{H}) = \lim_{x\to\infty} \frac{24+81e^{3x}}{24x+27e^{3x}} \quad \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{243e^{3x}}{24+81e^{3x}} \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{729e^{3x}}{243e^{3x}} = \frac{729}{243} = 3$

$\therefore \lim_{x\to\infty} y = e^3$

Practice Exam Questions on L'Hospital's Rule

A1. $\lim_{x\to\infty} \frac{x^3+x}{x^4-x} \quad \frac{\infty}{\infty}$

$H = \lim_{x\to\infty} \frac{3x^2+1}{4x^3-1} \quad \frac{\infty}{\infty}$

$H = \lim_{x\to\infty} \frac{6x}{12x^2} \frac{\infty}{\infty}$

$H = \lim_{x\to\infty} \frac{6}{24x} = 0$

A2. $\lim_{x\to 0} \frac{\tan x - x}{3x^2}$

$H = \lim_{x\to 0} \frac{\sec^2 x - 1}{6x} \qquad \frac{0}{0} \qquad \sec^2 0 = (\sec 0)^2 = (\frac{1}{\cos 0})^2 = (\frac{1}{1})^2 = 1$

$H = \lim_{x\to 0} \frac{2\sec x(\sec x \tan x)}{6} = \frac{2\sec^2 0 \tan 0}{6} = \frac{0}{6} = 0$

A3. $\lim_{x\to 0}\frac{2\cos x\tan x}{4\sin x}=\lim_{x\to 0}\frac{2\cos x(\frac{\sin x}{\cos x})}{4\sin x}=\lim_{x\to 0}\frac{2\sin x}{4\sin x}=\lim_{x\to 0}\left(\frac{2}{4}\right)=\frac{1}{2}$

A4. $\lim_{x\to \pi}\frac{\sin x}{4+\cos x}=\frac{\sin \pi}{4+\cos \pi}=\frac{0}{4+(-1)}=\frac{0}{3}=0$

A5. $\lim_{x\to 0^+} 9^{\ln x}=9^{\ln 0^+}=9^{-\infty}=\frac{1}{9^{\infty}}=0$

A6. $\lim_{x\to 0}\frac{e^x-1-2x-\frac{x^3}{6}}{x^3}$ $\qquad \frac{0}{0}$

$H=\lim_{x\to 0}\frac{e^x+0-2-\frac{3x^2}{6}}{3x^2}=-\frac{1}{0}=-\infty$

A7. $\lim_{x\to \infty}\frac{5x^2-x}{5x+15}$

$H=\lim_{x\to \infty}\frac{10x-1}{5}=\frac{10(\infty)-1}{5}=\infty$

A8. $\lim_{x\to 2\pi}\frac{\sin x}{1-\cos x}=\frac{0}{0}$

$H=\lim_{x\to 2\pi}\frac{cosx}{sinx}=\frac{1}{0}=\infty$

A9. $\lim_{x\to 2\pi}\frac{\sin x}{1-\cos x}=\frac{0}{0}$

$H=\lim_{x\to 2\pi}\frac{cosx}{sinx}=\frac{1}{0}=\infty$

A10. $\lim_{x\to 0}\frac{5^x-e^x}{2x}$ $\qquad \frac{1-1}{0}=\frac{0}{0}$

$H=\lim_{x\to 0}\frac{5^x\ln 5-e^x}{2}=\frac{5^0\ln 5-e^0}{2}=\frac{\ln 5-1}{2}$

A11.

$\lim_{x\to0^+} x^{\sqrt{x}}$

indeterminate 0^0

Let y=$x^{\sqrt{x}}$ take the ln of both sides NOTE: $x^{\sqrt{x}} = e^{\ln x^{\sqrt{x}}} = e^{\sqrt{x}\ln x}$

$lny = lnx^{\sqrt{x}}$ log rules

$lny = \sqrt{x}lnx$

$\lim_{x\to0^+} lny = \lim_{x\to0^+} \sqrt{x}lnx$

$= \lim_{x\to0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}}$

$= \lim_{x\to0^+} \frac{\ln x}{x^{\frac{-1}{2}}}$

$H = \lim_{x\to0^+} \frac{\frac{1}{x}}{\frac{-1}{2}x^{\frac{-3}{2}}}$

$= \lim_{x\to0^+} \frac{x^{-1}}{\frac{-1}{2}x^{\frac{-3}{2}}} = \lim_{x\to0^+} -2x^{\frac{1}{2}} = -2(0)^{\frac{1}{2}} = 0$

$\therefore e^0 = 1 \quad \therefore \lim_{x\to0^+} y = \lim_{x\to0^+} x^{\sqrt{x}} = 1$

A12. $\lim_{x\to4^-} \frac{1}{e^{x-4}} = \frac{1}{e^{4-4}} = \frac{1}{e^0} = \frac{1}{1} = 1$

A13. $\lim_{x\to0} \frac{e^{2x}-1}{\cos x} = \frac{e^0-1}{\cos 0} = \frac{1-1}{1} = \frac{0}{1} = 0$

A14. $\lim_{x\to0^+} \ln\left(\frac{1}{cotx}\right) = \lim_{x\to0+} \ln(tanx) = \ln(tan0^+) = \ln(0^+) = -\infty$ from the y=lnx graph

A15. $\lim_{x\to\infty} x^2 sin(\frac{1}{x^2})$

$=\lim_{x\to\infty} \frac{\sin(\frac{1}{x^2})}{\frac{1}{x^2}}$ $\frac{0}{0}$

(H)$= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x^2}\right)(-2x^{-3})}{(-2x^{-3})}$

$= \lim_{x\to\infty} \cos\left(\frac{1}{x^2}\right)$

$=\cos 0$

$=1$

A16. $\lim_{x\to 0^+} (cosx)^{\frac{1}{x}}$ let y=$(cosx)^{\frac{1}{x}}$

$\lim_{x\to 0^+} lny = \lim_{x\to 0^+} ln(cosx)^{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\ln(\cos x)}{x}$ $\frac{0}{0}$

(H)$= \lim_{x\to 0^+} \frac{-\frac{1}{cosx}sinx}{1}$

$= \lim_{x\to 0^+} \frac{-tanx}{1}$

$= 0$

$\lim_{u\to 0^+} y = e^0 = 1$

B. Critical Numbers

Example 1. $D_f = (-\infty, \infty)$

$f'(x) = 3x^2 - 3$

$f'(x) = 0 \qquad 3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = 1, -1$

Example 2. $D_f = (-\infty, \infty)$

$f'(x) = \frac{2}{3}(x-2)^{\frac{-1}{3}} = 0$

$\frac{2}{3(x-2)^{\frac{1}{3}}} = 0 \qquad no\ solution$

$f'(x)\ is\ undefined\ at\ x = 2\ and\ x = 2\ is\ in\ DF$

$\therefore x = 2\ is\ a\ critical\ number$

Example 3. $f'(x) = \cos x - \sin x = 0$

$\frac{\cos x}{\cos x} = \frac{\sin x}{\cos x}$

$1 = \tan x$

$x = \frac{\pi}{4}$

By CAST, $\tan x$ is positive in T.

$\therefore x = \pi + \frac{\pi}{4} = \frac{4\pi}{4} + \frac{\pi}{4}$

$x = \frac{5\pi}{4}$ $\quad \therefore$ *critical number's are* $\frac{\pi}{4}, \frac{5\pi}{4}$

Practice Exam Questions on Critical Numbers

B1. $D_f = (-\infty, 3) \cup (3, \infty)$

$$f'(x) = \frac{2x(x-3)-(1)(x^2-5)}{(x-3)^2}$$

$$= \frac{2x^2-6x-x^2+5}{(x-3)^2}$$

$$= \frac{x^2-6x+5}{(x-3)^2}$$

$f'(x) = 0$

$x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0$

$x = 5{,}1$ ∴ critical numbers are 5, 1

NOTE: the derivative is undefined at x=3, but so is f(x) so since x=3 is NOT in the domain of f(x), it is not a critical number

B2. $D_f = (-\infty, \infty)$

$$f'(x) = \frac{4}{3}x^{\frac{-1}{3}}(2-x) + (-1)\left(2x^{\frac{2}{3}}\right) = 0$$

$$x^{\frac{-1}{3}}\left[\frac{4}{3}(2-x) - 2x\right] = 0$$

$x^{-\frac{1}{3}} = 0$ *no solution*

$$\frac{8}{3} - \frac{4}{3}x - 2x = 0$$

$$\frac{8}{3} = \frac{6}{3}x + \frac{4}{3}x$$

$$8 = 10x$$

$x = \frac{8}{10} = \frac{4}{5}$ Also, $f'(x)$ is undefined at x=0 and x=0 is in D_f

$\therefore 0\ is\ also\ a\ critical\ number$. The critical numbers are 4/5 and 0.

B3. $f'(x) = -4\sin x + 4\sin x(-\cos x) = 0$

$-4\sin x(1+\cos x) = 0$

$\sin x = 0$

$x = 0, \pi, 2\pi$ $\quad\quad$ $1 + \cos x = 0$

$\cos x = -1$

$x = \pi$

$\therefore$ critical numbers are $0, \pi\ and\ 2\pi$

C. Maximum and Minimum

Example 1. $D_f = (-\infty, 2) \cup (2, \infty)$

$f(x) = (x-2)^{-1}$

$f'(x) = -1(x-2)^{-2}$

$f'(x) = 0 \quad \frac{-1}{(x-2)^2} = 0 \quad \therefore no\ solution$

$f'(x) is\ undefined\ at\ x = 2 \quad but\ x = 2\ is\ undefined\ in\ f$

$\therefore it\ is\ not\ a\ critical\ number$

$\therefore x = 2\ is\ a\ vertical\ asymptole$

2

Test point 0	Test point 3
⊟	⊟

Put $x = 2$ in chart (critical and VA)

$\therefore decreasing\ (-\infty, 2) \cup (2, \infty)$

Example 2. $f(x) = 4x^3 - 6x^2 - 72x$ D_f=(-∞, ∞)

$f'(x) = 12x^2 - 12x - 72 = 0$

$12(x^2 - x - 6) = 0$

$12(x-3)(x+2) = 0$

$x = 3, -2$

-2 3

Test point -3	Test point 0	Test point 4
+(-)(-)	+ (-)(+)	+ (+) (+)
+	−	+

Increasing $(-\infty, -2) \cup (3, \infty)$

Decreasing $(-2, 3)$

Example 3.

$f(x) = x^3 - 3x^2 + 2$

$f'(x) = 3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0,2$ $(both\ in\ [-1,4])$

$f(x) = x^3 - 3x^2 + 2$

$f(0) = 2$

$f(2) = 2^3 - 3(2)^2 + 2 = 8 - 12 + 2 = -2$ min

$f(-1) = (-1)^3 - 3(-1)^2 + 2 = -1 - 3 + 2 = -2$ min

$f(4) = 4^3 - 3(4)^2 + 2 = 64 - 48 + 2 = 18$ max

$\therefore absolute\ minimum\ is -2\ and\ absolute\ maximum\ is\ 18$

Practice Exam Questions on Maximum and Minimum

C1. $f'(x) = 8x^3 - 64x = 0$

$8x(x^2 - 8) = 0$

$8x(x - \sqrt{8})(x - \sqrt{8}) = 0$

$x = 0, \quad x = \sqrt{8}, -\sqrt{8}$

$-\sqrt{8} \qquad 0 \qquad \sqrt{8}$

-3	-1	1 test point	3
(-)(-)(-)	(-)(-)(+)	+(-)(+)	+(+)(+)
$\boxed{-}$	$\boxed{+}$	$\boxed{-}$	$\boxed{+}$
min	max	min	

$f(0) = 0$

$f(\sqrt{8}) = 2(\sqrt{8})^4 - 32(\sqrt{8})^2 = 2(8)(8) - 32(8) = 128 - 256 = -128$

$f(-\sqrt{8}) = 2(-\sqrt{8})^4 - 32(-\sqrt{8})^2 = 2(8)(8) - 32(8) = -128$

$-\sqrt{8} = -2\sqrt{2}$ and $\sqrt{8} = 2\sqrt{2}$

$\therefore \textit{local} \min(-2\sqrt{2}, -128) \textit{and } (2\sqrt{2}, -128), \quad \textit{local} \max(0,0)$

Increasing $(-2\sqrt{2}, 0) \cup (2\sqrt{2}, \infty)$, Decreasing $(-\infty, -2\sqrt{2}) \cup (0, 2\sqrt{2})$

C2. $f'(x) = 3x^2 - 12x = 0$

$3x(x - 4) = 0$

$x = 0, 4 \qquad \therefore 4 \textit{ is not in } [-3,2] \textit{ so don't check it}$

$f(x) = x^3 - 6x^2 + 5$

$f(0) = 5$

$f(2) = 2^3 - 6(2)^2 + 5 = 8 - 24 + 5 = -11$

$f(-3) = (-3)^3 - 6(-3)^2 + 5 = -27 - 54 + 5 = -76$

absolute maximum = 5 absolute minimum = -76

C3. a) $a = -1 \quad b = 1$

$f'(x) = 3x^2 - 6x$
$f'(x) = 0 \qquad 3x^2 - 6x = 0$
$factor \qquad 3x(x-2) = 0$
$x = 0,2 \quad [2\ is\ not\ in\ the\ interval\ \therefore\ not\ tested\ in\ f(x)]$ check endpoints and critical values in f
$f(x) = x^3 - 3x^2 + 4$
$f(-1) = (-1)^3 - 3(-1)^2 + 4 = 0$
$f(0) = 4$
$f(1) = 1^3 - 3(1)^2 + 4 = 2 \qquad \therefore abs\ max = 4$
$abs\ min = 0$

b) $a = 0 \quad b = 4$
$f'(x) = 3(x^2-1)^2(2x)$
$f'(x) = 0 \qquad 3(x^2-1)^2(2x) = 0$
$3(x-1)(x+1)(2x) = 0$
$x = 0, \pm 1 \qquad -1\ is\ not\ in\ [0,4]$
$f(x) = (x^2-1)^3$
$f(0) = -1 \quad \leftarrow abs\ \text{min} = -1$
$f(1) = 0$
$f(4) = 15^3 \quad \leftarrow abs\ \text{max} = 15^3$

c) $f'(x) = -8\sin x + 8\cos 2x = 0$
$-8\sin x + 8(1 - 2\sin^2 x) = 0$
$-8\sin x + 8 - 16\sin^2 x = 0$
$0 = 16\sin^2 x + 8\sin x - 8$
$0 = 8(2\sin^2 x + \sin x - 1)$
$0 = 2(2\sin x - 1)(\sin x + 1)$

$\sin x = -1 \qquad \sin x = \frac{1}{2}$

No solution in $\left[0, \frac{\pi}{2}\right] \qquad x = \frac{\pi}{6}$

$f(0) = 8$
$f\left(\frac{\pi}{2}\right) = 8cos\frac{\pi}{2} + 4\sin\left(\frac{2\pi}{2}\right) = 0 \quad abs\ min$

$f\left(\frac{\pi}{6}\right) = 8cos\frac{\pi}{6} + 4\sin\left(\frac{\pi}{6}\right) = \frac{8\sqrt{3}}{2} + \frac{4\sqrt{3}}{2} = 6\sqrt{3}$ *abs max*

d) $f(x) = x - lnx$

$f'(x) = 1 - \frac{1}{x} = 0$

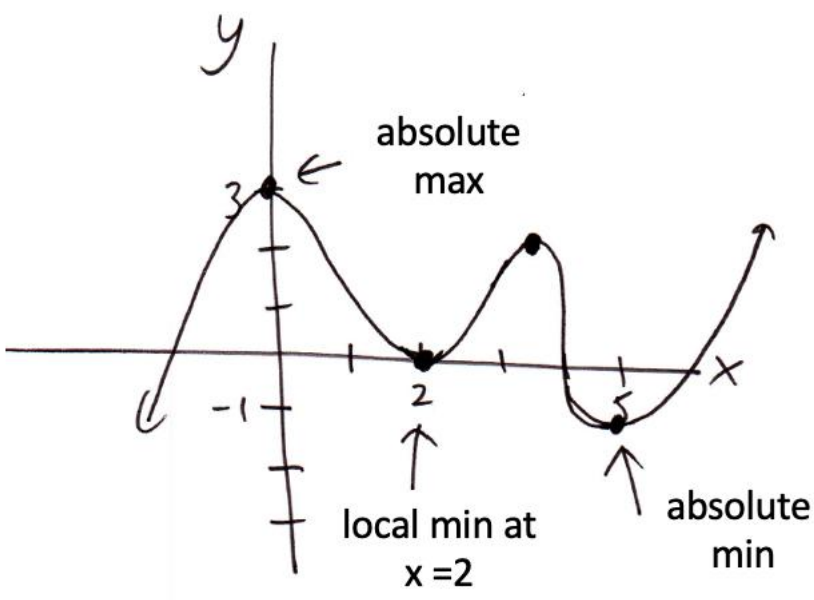
$1 = \frac{1}{x}$ $x = 1$ *critical value*

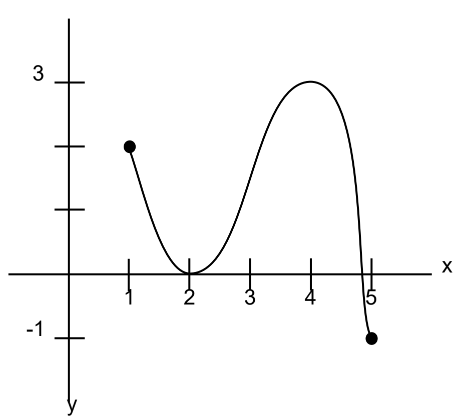
$f(1) = 1 - \ln 1 = 1$

$f(6) = 6 - \ln 6 = 4.2$

abs max $(6 - \ln 6)$ *abs min* $= 1$

C4. Answers vary. Two examples are given below:





D. Concavity and Points of Inflection and The Second Derivative Test

Example 1. $f'(x) = 6x^2 - 24x + 18$

$6(x^2 - 4x + 3) = 0$

$6(x-3)(x-1) = 0$

$x = 3, 1$

1	3	
Test point 0	Test point 2	Test point 4
+(-)(-)	+(-)(+)	+(+)(+)
+	−	+
max	min	

$local$ max(1,12) $f(1) = 2 - 12 + 18 + 4 = 12$

$local$ min(3,4) $f(3) = 2(3)^3 - 12(3)^2 + 18(3) + 4 = 4$

$f''(x) = 12x - 24 = 0$

$12x = 24$

$x = 2$

2	
Test point 0	Test point 3
-	+

$\therefore POI\ at\ (2, 8)$

$f(2) = 2(2)^3 - 12(2)^2 + 18(2) + 4 = 8$

$concave\ up\ \ (2, \infty)\quad concave\ down\ \ (-\infty, 2)$

Example 2. $f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

$f''(x) = 0$

$\frac{-\sin x}{\cos x} = \frac{\cos x}{\cos x}$

$-\tan x = 1$

$\tan x = -1$

From CAST, $\tan x$ is negative in C & S

From special $\Delta's$, $\tan x = 1$ *gives* $x = \frac{\pi}{4}$

$\therefore x = 2\pi - \frac{\pi}{4} = \frac{8\pi}{4} - \frac{\pi}{4} = \frac{7\pi}{4}$ $(C\ quadrant)$

$\therefore x = \pi - \frac{\pi}{4} = \frac{4\pi}{4} - \frac{\pi}{4} = \frac{3\pi}{4}$ $(S\ quadrant)$

$\frac{3\pi}{4}$ $\frac{7\pi}{4}$

Test point 0	π	2π
-sinx – cosx =-sin0-cos0 -1 ⊟	-sinπ -cosπ =1 ⊞	-sin$2\pi - cos2\pi$ =-1 ⊟

There is a point of inflection at x=$\frac{3\pi}{4}$ *and* $\frac{7\pi}{4}$

concave up $(\frac{3\pi}{4}, \frac{7\pi}{4})$ concave down $[0, \frac{3\pi}{4})$ *or* $(\frac{7\pi}{4}, 2\pi]$

Example 3. $f'(x) = 15x^4 - 15x^2 = 0$

$$15x^2(x^2 - 1) = 0$$

$$x = 0,\ 1, -1$$

$$f''(x) = 60x^3 - 30x$$

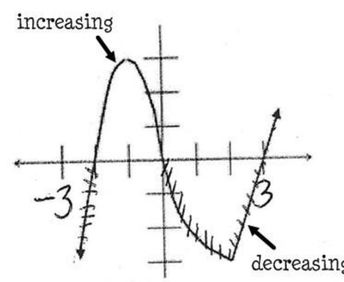
$f''(0) = 0$ *inconclusive*

$f''(1) = 60 - 30 = 30 > 0$ $\therefore$ *local* min *at* $x = 1$ sub x=1 into f(x) and get (1, -2)

$f''(-1) = 60(-1) + 30 = -30 < 0$ $\therefore$ *local* max *at* $x = -1$ sub x= -1 into f(x) and get (-1,2)

Making Conclusions about Graphs Based on First or Second Derivatives

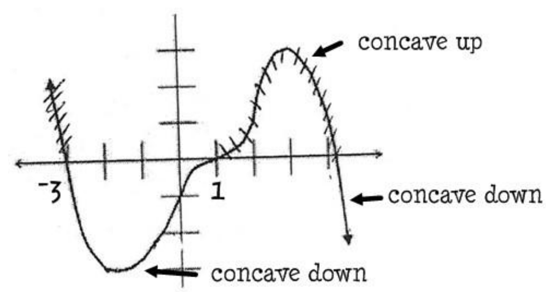
Example 4. Part (1)
local max at x=0, local min at x=3,-2



Increasing $(-2,0) \cup (3, \infty)$
Decreasing $(-\infty, -2) \cup (0,3)$

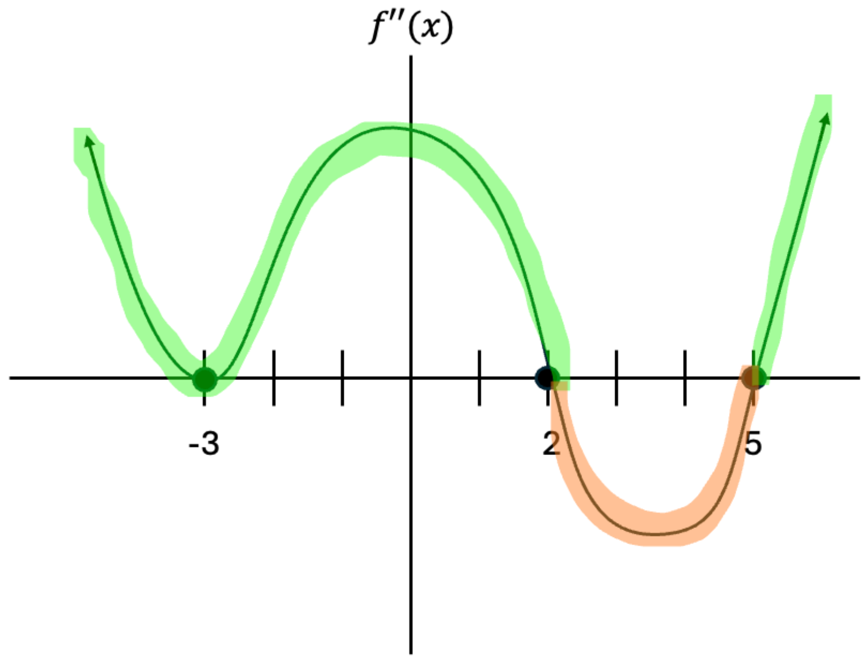
We can also tell by the slope of the tangent where it is concave up and down. It is concave up if the slope is positive, ie. up and to the right. It is concave down if the slope is down and to the right (negative). So, here it is concave up from $(-\infty, -1) \cup (2, \infty)$ and down for (-1,2).

Example 5. $f''(x) < 0$ *concave down*
$f''(x) > 0$ *concave up*



Points of Inflection →
occur when graph switches from concave
a)
up to down or concave down to concave up
(3) *concave up* $(-\infty, -3) \cup (1,4)$
concave down $(-3,1) \cup (4, \infty)$
POI at $x = -3, x = 1, x = 4$

b) concave up $(-\infty, -3) \cup (-3,2) \cup (5, \infty)$
concave down (2,5)
Points of inflection at x=-3, 2 and



<u>Practice Exam Questions on Concavity and Points of Inflection and The Second Derivative Test</u>

D1. $f(x) = 4x^3 + 6x^2 - 72x$

$$f'(x) = 12x^2 + 12x - 72$$

$$12(x^2 + x - 6) = 0$$

$$12(x+3)(x-2) = 0$$

$$x = -3, 2$$

-3	2	
Test point -4	0	3
+(-)(-)	+(+)(-)	+(+)(+)
+	−	+

Increasing $(-\infty, -3) \cup (2, \infty)$

Decreasing $(-3, 2)$

$f''(x) = 24x + 12 = 0$

$24x = -12$

$x = \frac{-1}{2}$

$\frac{-1}{2}$	
-1	1
$\boxed{-}$	$\boxed{+}$

POI at $(\frac{-1}{2}, 37)$

$f\left(\frac{-1}{2}\right) = 4(-\frac{1}{2})^3 + 6(\frac{-1}{2})^2 - 72(-\frac{1}{2})$

$= 4\left(-\frac{1}{8}\right) + 6\left(\frac{1}{4}\right) + \frac{72}{2}$

$= \frac{-1}{2} + \frac{3}{2} + \frac{72}{2}$

$= \frac{74}{2} = 37$

D2. $f(x) = 3x^4 - 6x^2 + 9$

$f'(x) = 12x^3 - 12x$

$12x(x^2 - 1) = 0$

$12x(x-1)(x+1) = 0$

$x = 0, 1, -1$

-1	0	1	
Test point -2	$\frac{-1}{2}$	$\frac{1}{2}$	2
$\boxed{-}$	$\boxed{+}$	$\boxed{-}$	$\boxed{+}$

Increasing $(-1,0) \cup (1,\infty)$

Decreasing $(-\infty,-1) \cup (0,1)$

$f''(x) = 36x^2 - 12 = 0$

$36x^2 = 12$

$x^2 = \frac{1}{3}$

$x = \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

$\frac{-1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	
-2	0	2
$\boxed{+}$	$\boxed{-}$	$\boxed{+}$

$concave\ up\ \left(-\infty, \frac{-1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$

$concave\ down\ \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

D3. $f(x) = 3x + 6sinx$ on $[0,2\pi]$

$f'(x) = 3 + 6\cos x = 0$

$6\cos x = -3$

$\cos x = \frac{-1}{2}$

$\cos x\ \ is\ negative\ in\ S\ \&\ T\ quadrants$

From Special △

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$

$\therefore x = \pi - \frac{\pi}{3} = \frac{3\pi}{3} - \frac{\pi}{3} = \frac{2\pi}{3} \quad (S\ quadrant)$

$x = \pi + \frac{\pi}{3} = \frac{3\pi}{3} + \frac{\pi}{3} = \frac{4\pi}{3}$ $(T\ quadrant)$

	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$
Test point 0	π	2π
$3 + 6\cos 0$	$3 + 6\pi$	$3 + 6\cos x = 3 + 6\cos 2\pi$
$\boxed{+}$	$\boxed{-}$	$\boxed{+}$
local max	local min	

Increasing $\left(0, \frac{2\pi}{3}\right) \cup (\frac{4\pi}{3}, 2\pi)$

Decreasing $(\frac{2\pi}{3}, \frac{4\pi}{3})$

Local maximum $(\frac{2\pi}{3}, 2\pi + 3\sqrt{3})$

Local minimum $(\frac{4\pi}{3}, 4\pi - 3\sqrt{3})$

$f(x) = 3x + 6\sin x$

$$f\left(\frac{2\pi}{3}\right) = 3\left(\frac{2\pi}{3}\right) + 6\sin\frac{2\pi}{3}$$

$$= 2\pi + 6(\frac{\sqrt{3}}{2})$$

$$= 2\pi + 3\sqrt{3}$$

$$f\left(\frac{4\pi}{3}\right) = 3\left(\frac{4\pi}{3}\right) + 6\sin\frac{4\pi}{3}$$

$$= 4\pi + 6(-\frac{\sqrt{3}}{2})$$

$$= 4\pi - 3\sqrt{3}$$

D4. $f(x) = 6x^5 - 40x^3$

$f'(x) = 30x^4 - 120x^2$

$f''(x) = 120x^3 - 240x = 0$

$120x(x^2 - 2) = 0$

$120x(x - \sqrt{2})(x + \sqrt{2}) = 0$

$x = 0, \sqrt{2}, -\sqrt{2}$

$-\sqrt{2}$	0	$\sqrt{2}$	
-2	-1	1	2
(-)(-)(-)	(-)(-)(+)	(+)(-)(+)	+(+)(+)
$\boxed{-}$	$\boxed{+}$	$\boxed{-}$	$\boxed{+}$

$POI\ at - \sqrt{2}, 0\ and\ \sqrt{2}$

D5. $f''(x) = 12x^2 - 48 = 0$

$12x^2 = 48$

$x^2 = 4$

$x = 2, -2$

-2	2	
-3	0	3
$\boxed{+}$	$\boxed{-}$	$\boxed{+}$

$POI\ at\ \ x = 2, -2$

Concave up $(-\infty, -2) \cup (2, \infty)$

Concave down $(-2,2)$

D6. a) $f(x) = 9x^5 - 15x^3 + 9$

$f'(x) = 45x^4 - 45x^2 = 0$
$45x^2(x^2 - 1) = 0$

$x = 0, 1 - 1$

$f''(x) = 180x^3 - 90x$

$f''(0) = 0 \quad inconclusive$

$f''(1) = 90 > 0 \quad \therefore local \min at\ x = 1$ f(1)=3, so the local minimum is (1,3)

$f''(-1) = 180(-1) - 90(-1) = -90 < 0 \quad \therefore local \max at\ x = -1$

f(-1)=-15, so the local maximum is (-1, 15)

b) $f(x) = 12x^3 - 32x$
$f'(x) = 36x^2 - 32 = 0$

$36x^2 = 32$

$x^2 = \frac{32}{26}$ so, $x = \pm\sqrt{\frac{32}{36}} = \pm\frac{\sqrt{16}\sqrt{2}}{6} = \pm\frac{2\sqrt{2}}{3}$

$f''(x) = 16x$

$f''\left(\frac{2\sqrt{2}}{3}\right) = 16\left(\frac{2\sqrt{2}}{3}\right) > 0 \quad \therefore local \min at\ x = \frac{2\sqrt{2}}{3}$ sub. into f(x) to find y

$f''(-4) = 16(\frac{-2\sqrt{2}}{3}) < 0 \quad \therefore local \max at\ x = -\frac{2\sqrt{2}}{3}$

D7. NOTE $D_f =$ doesn't include 0 $\quad D_f(-\infty, 0) \cup (0, \infty)$

a) $f'(x) = (1)e^{1/x} + e^{1/x}(-x^{-2})(x) = e^{1/x} - e^{1/x}x^{-1}$

$e^{1/x}\left(1 - \frac{1}{x}\right) = 0$

$1 - \frac{1}{x} = 0$

$1 = \frac{1}{x}$

$x = 1$

Include where it is undefined

b) 0 1

-1	0.5	2
$e^{1/x}(1 - \frac{1}{x})$ $e^{1/-1}(1 - \frac{1}{-1})$ $e^{-1}(1 + 1)$ $= \frac{2}{e}$ $\boxed{+}$	$e^{1/x}(1 - \frac{1}{x})$ $e^{1/0.5}(1 - \frac{1}{0.5})$ $e^{2}(1 - 2)$ $= \boxed{-}$	$e^{1/x}(1 - \frac{1}{x})$ $e^{1/2}(1 - \frac{1}{2})$ $\boxed{+}$

Decreasing (0, 1) and local minimum (1,e)

c) $f'(x) = e^{\frac{1}{x}}(1 - x^{-1})$

$f''(x) = e^{\frac{1}{x}}(-x^{-2})(1 - x^{-1}) + (x^{-2})e^{\frac{1}{x}}$

$= e^{\frac{1}{x}}(-x^{-2} + x^{-3}) + x^{-2}e^{\frac{1}{x}}$

$= e^{\frac{1}{x}}(-\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^2})$

$= e^{\frac{1}{x}}(\frac{1}{x^3})$

$f''(x) = 0$ test points around $x = 0$ where the graph is undefined

$e^{\frac{1}{x}}\left(\frac{1}{x^3}\right) = 0$

↓ ↘

No solution $\frac{1}{x^3} = 0$

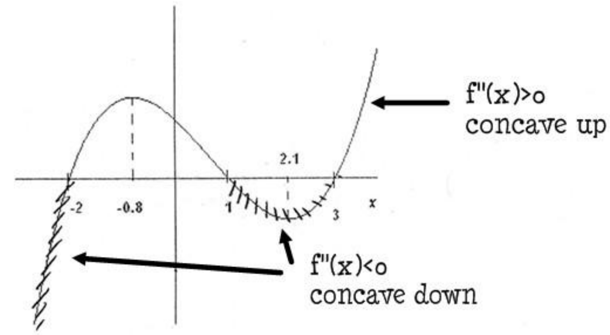
no solution $\therefore$ *no POI*

0

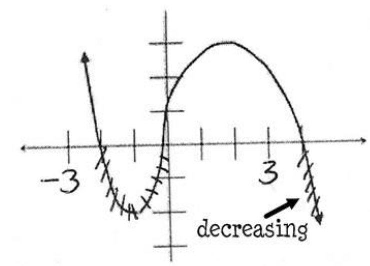
−1	1
$e^{\frac{1}{x}}(\frac{1}{x^3})$ $e^{\frac{1}{-1}}(\frac{1}{(-1)^3}$ $= \frac{-1}{e}$ $\boxed{-}$ Concave down $(-\infty, 0)$	$e^{\frac{1}{x}}(\frac{1}{x^3})$ $e^{\frac{1}{1}}(\frac{1}{1})$ $= \boxed{+}$ Concave up $(0, \infty)$

D8. $f''(x)>0$ concave up $f''(x) < 0$ $concave\ down$

$concave\ up\ (-2,1) \cup$
$(3,\infty)$ $concave\ down\ (-\infty,-2) \cup (1,3)$
$POI\ at\ x = -2,1,3$

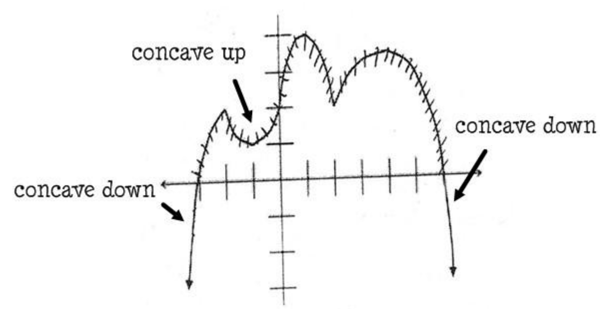


D9. Part (2)
$local$ max $at\ x = -2,4$ $local$ min $at\ x = -0.25$
$If\ f'(x) < 0 \rightarrow decreasing$ $If\ f'(x) > 0 \rightarrow increasing$
$local$ max $\Longrightarrow increasing\ to\ decreasing$
$local$ min $\Longrightarrow decreasing\ to\ increasing$

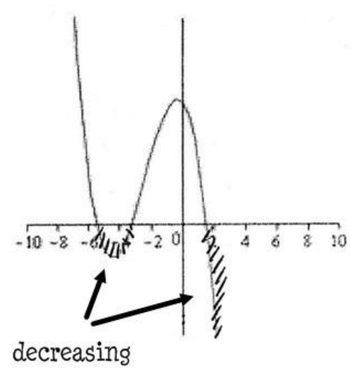


Increasing $(-\infty,-2) \cup (-0.25,4)$
Decreasing (-2,-0.25) $\cup$ $(4,\infty)$

D10. $(4) concave\ up\ (-3,6)$
$concave\ down\ (-\infty,-3) \cup (6,\infty)$
$POI\ at\ x = -3\ and\ x = 6$



D11.

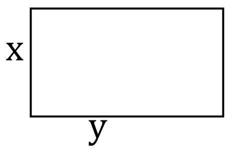


Plot 2 $f'(x) > 0$ $f(x)\ is\ increasing$
$f'(x) < 0$ $f(x)\ is\ decreasing$
$from\ increasing\ to\ decreasing \rightarrow$
$local\ max$
$from\ decreasing\ to\ increasing \rightarrow$
$local\ min$

$local$ max $at\ x = -5$ and 1.8 (approx..)
$local$ min $at\ x = -3$

E. Optimization

Example 1.



$$2x + 2y = 128$$

$$x + y = 64$$

$$y = 64 - x$$

FIND max $A = xy$

$$A = x(64 - x)$$

$$A = 64x - x^2$$

$$A'(x) = 64 - 2x = 0$$

$$64 = 2x$$

$$x = 32$$

$$\therefore y = 64 - x$$

$$= 64 - 32$$

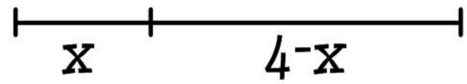
$$= 32$$

$\therefore$ length and width are both 32 and Area is $32 \times 32 = 1024\ m^2$.

Prove it is a maximum

$A''(x) = -2 < 0, \mathit{so\ it\ is\ a\ maximum}$

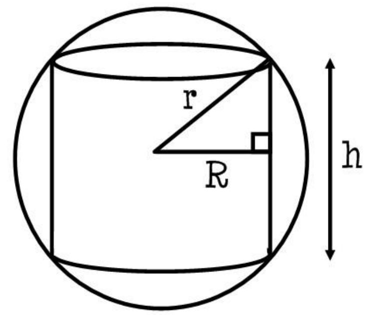
Example 2. $Let\ x\ be\ the\ number\ of\ feet\ that\ goes\ toward\ the\ circle$
$Let\ 4-x\ be\ the\ number\ of\ feet\ that\ goes\ toward\ the\ square$
$C = 2\pi r$
$x = 2\pi r$
$r = \frac{x}{2\pi}$



Square $\frac{4-x}{4}$ $goes\ to\ each\ side\ (4-x\ total\ for\ 4\ sides)$
$Area = A_{circle} + A_{square}$
$A = \pi r^2 + x^2 = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{4-x}{4}\right)\left(\frac{4-x}{4}\right)$
$A = \frac{x^2}{4\pi} + \frac{16-8x+x^2}{16}$
$A' = \frac{1}{4\pi}(2x) + \frac{1}{16}(-8+2x) = 0$
$\frac{-8+2x}{16} = \frac{-2x}{4\pi}$
$\frac{-8+2x}{4} = \frac{-2x}{\pi}$
$-8\pi + 2\pi x = -8x$
$-8\pi = -8x - 2\pi x$
$8\pi = 8x + 2\pi x$
$8\pi = x(8+2\pi)$ $x = \frac{8\pi}{8+2\pi}\ to\ circle\ or\ \frac{4\pi}{4+\pi}$
for a minimum, $\frac{4\pi}{4+\pi}\ ft\ to\ the\ circle, 4 - x = 4 - \frac{4\pi}{4+\pi} = \frac{4(4+\pi)}{4+\pi} - \frac{4\pi}{4+\pi} = \frac{16}{4+\pi}$ to the square

<u>Prove it is a minimum</u> $A'' = \frac{1}{2\pi} + \frac{1}{16}(2) > 0, so\ it\ is\ a\ minimum$

Example 3.



$Let\ R = radius\ of\ cylinder$
$h = height\ of\ cylinder$

From Pythagorean Theorem:

$r^2 = R^2 + \left(\frac{1}{2}h\right)^2 = R^2 + \frac{h^2}{4} \quad \therefore R^2 = r^2 - \frac{h^2}{4}$

$$Volume\ of\ cylinder\ \ V = \pi R^2 h = \pi\left(r^2 - \frac{h^2}{4}\right)h$$

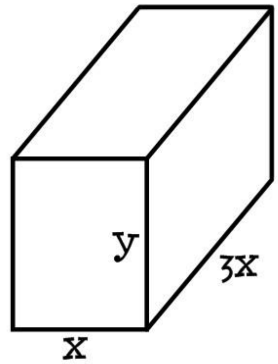
$= \pi\left(r^2 h - \frac{h^3}{4}\right) a\ function\ of\ h\ alone$

$$\frac{dV}{dh} = \pi\left(r^2 - \frac{3h^2}{4}\right) = 0 \quad when\ r^2 = \frac{3h^2}{4} \quad h^2 = \frac{4r^2}{3}$$

$$h = \sqrt{\frac{4}{3}r^2} = \frac{2r}{\sqrt{3}}$$

$R^2 = r^2 - \frac{h^2}{4} = r^2 - \frac{(\frac{2r}{\sqrt{3}})^2}{4} = r^2 - \frac{1}{3}r^2 = \frac{2}{3}r^2$

$\therefore \max Volume\ \ V = \pi R^2 h = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}$



Example 4. V=90

$let\ x\ be\ width \quad 3x\ be\ length \quad y\ be\ height$

$V = 3x(x)(y) = 3x^2y$

$90 = 3x^2y \quad y = \frac{90}{3x^2} = \frac{30}{x^2}$ $\boxed{1}$

$A = 2(3x^2) + 2(xy) + 2(3xy) = 6x^2 + 8xy$

$Cost\ C = 10(6x^2) + 5(8xy) = 60x^2 + 40xy \quad sub\ \boxed{1}$

$$C = 60x^2 + 40x\left(\frac{30}{x^2}\right) = 60x^2 + \frac{1200}{x}$$

$$= 60x^2 + 1200x^{-1}$$

$C' = 120x - 1200x^{-2}$

$C' = 0 \quad 120x - 1200x^{-2} = 0$

$$120x = \frac{1200}{x^2}$$

$$120x^3 - 1200 = 0$$

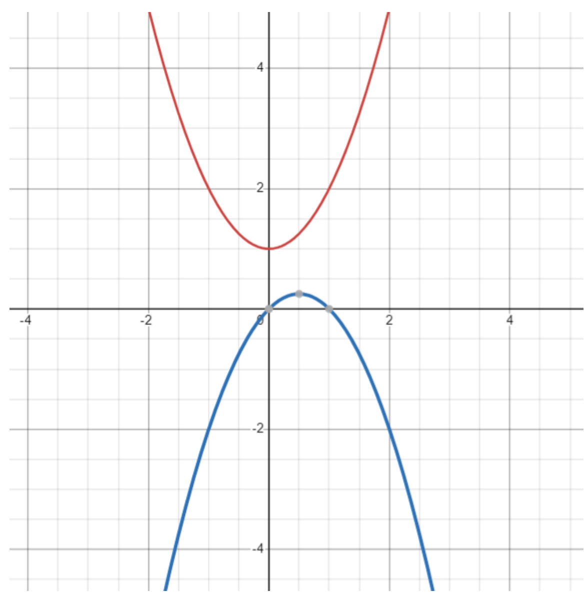
$$120(x^3 - 10) = 0 \quad x = \sqrt[3]{10}$$

$y = \frac{30}{(\sqrt[3]{10})^2}$

$dimensions\ are\ 3\sqrt[3]{10} \times \sqrt[3]{10} \times \frac{30}{(\sqrt[3]{10})^2}\ feet$

$$C''(x) = 120 + \frac{2400}{x^2} > 0, since\ x > 0, so\ it\ is\ a\ minimum$$

Example 5.



$\frac{df}{dx} = 4x - 1$

$\frac{df}{dx} = 0\ when\ x = \frac{1}{4} for\ minimum\ f(x)$

$f\left(\frac{1}{4}\right) = 2(\frac{1}{4})^2 - \frac{1}{4} + 1 = \frac{2}{16} - \frac{4}{16} + \frac{16}{16} = \frac{14}{16}$

$= \frac{7}{8}$

Prove Minimum $f''(x) = 4>0$, so it is a minimum.

Practice Exam Questions on Optimization

E1. $let\ x, y\ be\ the\ numbers$

$x + 2y = 24 \quad x = 24 - 2y$

$max = xy = (24 - 2y)y = 24y - 2y^2$

max $f(y) = 24y - 2y^2$

$f'(y) = 24 - 4y = 0$

$24 = 4y \quad y = \frac{24}{4} = 6$

$\therefore x = 24 - 2(6) \qquad x = 12$

The numbers are 12 and 6.

E2. $Max\ A = xy$

$A(x) = x(150 - 2x) = 150x - 2x^2 \qquad y + 2x = 150$

$A'(x) = 150 - 4x = 0 \qquad y = 150 - 2x$

$150 = 4x \quad x = 37.5$

$\therefore y = 150 - 2(37.5) = 75$

$\therefore Max\ A = (75)(37.5) = 2812.50\ m^2$

E3.

$V = \pi r^2 h$

$1000 = \pi r^2 h$

$h = \frac{1000}{\pi r^2}$

$MinA = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$

$A = 2\pi r^2 + 2000r^{-1}$

$A'(r) = 4\pi r - 2000r^{-2}$

$let\ A'(r) = 0 \quad 4\pi r = \frac{2000}{r^2}$

$r^3 = \frac{2000}{4\pi} \quad r = \sqrt[3]{\frac{2000}{4\pi}} = \sqrt[3]{\frac{500}{\pi}}$

E4. Perimeter is 100...

$2x + 2y = 100$

$x + y = 50$ $\quad substitute\ y = 50 - x$ into equation for area

$A = xy = x(50 - x)$

$A(x) = 50x - x^2$

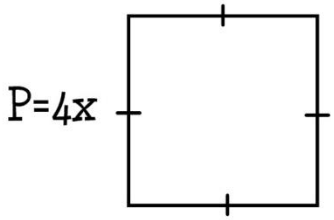
$A'(x) = 50 - 2x = 0$

$\therefore 50 = 2x \quad x = 25$

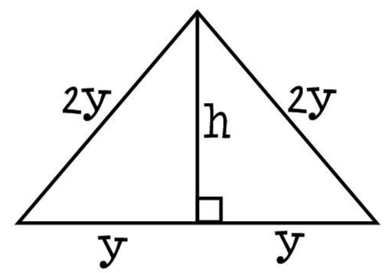
$y = 50 - 25 \quad y = 25$

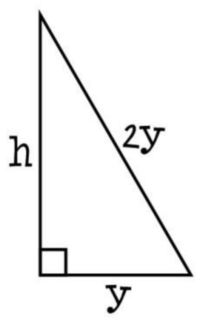
$maxA = 25(25) = 625cm^2$

E5. First, take the 2m and put x of it on each side of the square



Now, there is $2 - 4x = left\ for\ \Delta$ and it has three sides, so let each side be 2y, and that way you don't end up with any fractions





Solve for h using Pythagorean Theorem:

$$h^2 + y^2 = (2y)^2$$
$$h^2 = 3y^2$$
$$h = \sqrt{3}y$$

Since we wrap the "2-4x" around three sides, each equal to "2y" we get:

$2 - 4x = 6y$ and solving for y:

$$y = \frac{2-4x}{6} = \frac{1-2x}{3}$$

The minimum Area= Area square + Area triangle

$$A = x^2 + \frac{1}{2}(base)(height)$$
$$A = x^2 + \frac{1}{2}(2y)(\sqrt{3}y)$$
$$A = x^2 + \sqrt{3}y^2$$
$$\therefore A = x^2 + \sqrt{3}\left(\frac{1-2x}{3}\right)^2 = x^2 + \frac{\sqrt{3}}{9}(1 - 4x + 4x^2)$$
$$A' = 2x + \frac{\sqrt{3}}{9}(-4 + 8x) = 0$$
$$2x = \frac{-\sqrt{3}}{9}(-4 + 8x)$$
$$18x = 4\sqrt{3} - 8\sqrt{3}x$$
$$18x + 8\sqrt{3}x = 4\sqrt{3}$$
$$x\left(18 + 8\sqrt{3}\right) = 4\sqrt{3} \qquad x = \frac{4\sqrt{3}}{18+8\sqrt{3}} = 0.217\ sides\ of\ square$$
$$y = \frac{1-2(0.217)}{3} = 0.19$$
$$sides\ of\ \Delta = 2y = 2(0.19) = 0.38m$$

E6. $4x + 3y = 120 \qquad y = \frac{120-4x}{3}$

$A = 2xy = 2x\left(\frac{120-4x}{3}\right) = \frac{1}{3}(240x - 8x^2)$

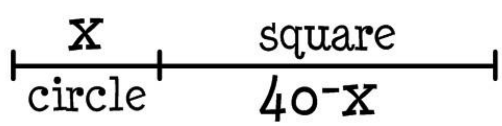
$A' = \frac{1}{3}(240 - 16x) = 0$

$\therefore 240 = 16x \qquad x = 15$

$\therefore y = \frac{120-4(15)}{3} = 20$

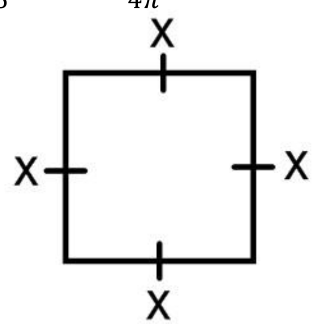
$\max area = 2xy = 2(15)(20) = 600m^2$

E7. $square\ (each\ side\ \frac{40-x}{4})$



$2\pi r = x \qquad r = \frac{x}{2\pi}$

$A = \left(\frac{40-x}{4}\right)\left(\frac{40-x}{4}\right) + \pi\left(\frac{x}{2\pi}\right)^2 = \frac{1600-80x+x^2}{16} + \frac{\pi x^2}{4\pi^2}$



$A' = 0 \quad \frac{1}{16}(-80 + 2x) + \frac{1}{4\pi}(2x) = 0$

$\frac{-40+x}{8} = \frac{-x}{2\pi}$

$-80\pi + 2\pi x = -8x$

$80\pi = x(8 + 2\pi) \qquad x = \frac{80\pi}{8+2\pi} = 17.59\ to\ circle$

$(40 - 17.59) to\ square$

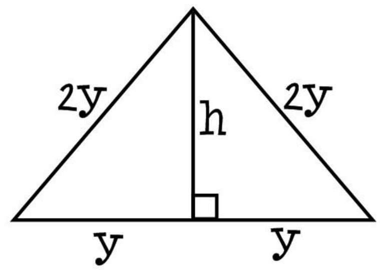
E8. Solve for h $\quad h^2 + y^2 = (2y)^2$

$h^2 + y^2 = 4y^2 \quad h^2 = 3y^2 \quad h = \sqrt{3}y$

$Min\ area \quad 20 - 4x = left\ for\ equilateral\ \Delta$

$20 - 4x = 6y \qquad y = \frac{20-4x}{6} = \frac{10-2x}{3}$

$A = x^2 + \frac{1}{2}(base)(height) = x^2 + \frac{1}{2}(2y)(\sqrt{3}y)$
$= x^2 + \sqrt{3}y^2$
$A = x^2 + \sqrt{3}\left(\frac{10-2x}{3}\right)\left(\frac{10-2x}{3}\right) = x^2 + \frac{\sqrt{3}}{9}(100 - 40x + 4x^2)$
$A'(x) = 2x + \frac{\sqrt{3}}{9}(-40 + 8x) = 0$
$\frac{\sqrt{3}}{9}(-40 + 8x) = -2x$
$-18x = -40\sqrt{3} + 8\sqrt{3}x$
$40\sqrt{3} = 18x + 8\sqrt{3}x$
$40\sqrt{3} = x(18 + 8\sqrt{3})$
$\therefore x = \frac{40\sqrt{3}}{18+8\sqrt{3}} = 2.17 \quad sides\ of\ square$



E9. $4x + 2\pi r = 12 \quad perimeter$
$\frac{2\pi r}{2\pi} = \frac{12-4x}{2\pi} \qquad \therefore r = \frac{12-4x}{2\pi}$
$minA = x^2 + \pi r^2 = x^2 + \pi\left(\frac{12-4x}{2\pi}\right)\left(\frac{12-4x}{2\pi}\right)$
$A = x^2 + \frac{1}{4\pi}(144 - 96x + 16x^2)$
$A'(x) = 2x + \frac{1}{4\pi}(-96 + 32x) = 0$
$\frac{-96+32x}{4\pi} = -2x$
$-96 + 32x = -8\pi x$
$-96 = -8\pi x - 32x$
$x = \frac{-96}{-8\pi-32} = \frac{-4(24)}{-4(2\pi+8)} = \frac{12}{\pi+4}$

$\therefore sides\ of\ square = \frac{12}{\pi+4}$
$r = \frac{12-4\left(\frac{12}{\pi+4}\right)}{2\pi} \quad \rightarrow circle$

E10. $A = 120 = x^2 + 4xy$

$\therefore y = \frac{120-x^2}{4x}$

Max $V = x^2 y = x^2\left(\frac{120-x^2}{4x}\right) = \frac{1}{4}(120x - x^3)$

$V'(x) = \frac{1}{4}(120 - 3x^2) = 0$

$120 = 3x^2 \quad x^2 = 40 \quad x = \sqrt{40} = 2\sqrt{10}$

$\therefore y = \frac{120-\sqrt{40}^2}{4\sqrt{40}} = \frac{120-40}{4\sqrt{40}} = \frac{20}{\sqrt{40}} = \frac{20}{2\sqrt{10}} = \frac{10}{\sqrt{10}} = \frac{10\sqrt{10}}{10} = \sqrt{10}$

$\therefore$ *dimensions are* $2\sqrt{10} \times 2\sqrt{10} \times \sqrt{10}$ *inches*

E11.

let x, y be the two numbers

$xy = 300$ $\boxed{1}$ $\quad \therefore y = \frac{300}{x}$

$\min = x + 3y$

$M = x + 3\left(\frac{300}{x}\right) = x + 900x^{-1}$

$M' = 1 - 900x^{-2} = 0$

$1 = 900x^{-2}$

$1 = \frac{900}{x^2} \quad \therefore x^2 = 900 \quad x = 30$

$\therefore y = \frac{300}{x} = \frac{300}{30} = 10 \quad \therefore y = 10$

The two numbers are 30 and 10.

E12. V=120

let x be width *4x be length* *y be height*

$V = 4x(x)(y) = 4x^2 y$

$120 = 4x^2 y \quad y = \frac{120}{4x^2} = \frac{30}{x^2}$ $\boxed{1}$

$A = 2(4x^2) + 2(xy) + 2(4xy) = 8x^2 + 10xy$

$Cost\ C = 5(8x^2) + 3(10xy) = 40x^2 + 30xy$ *sub* $\boxed{1}$

$$C = 40x^2 + 30x\left(\frac{30}{x^2}\right) = 40x^2 + \frac{900}{x}$$
$$= 40x^2 + 900x^{-1}$$

$C' = 80x - 900x^{-2}$

$C' = 0 \quad 80x - 900x^{-2} = 0$

$$80x = \frac{900}{x^2}$$
$$80x^3 - 900 = 0$$
$$20(4x^3 - 45) = 0 \qquad x = \sqrt[3]{11.25}$$

$$y = \frac{30}{(\sqrt[3]{11.25})^2}$$

$$dimensions\ are\ 4\sqrt[3]{11.25} \times \sqrt[3]{11.25} \times \frac{30}{(\sqrt[3]{11.25})^2}\ feet$$

F. Sums and Sigma Notation

Example 1. $= \frac{4}{n}\sum_{i=1}^{n}\left(\frac{1}{n}\right)^i = \frac{4}{n^2}(\sum_{i=1}^{n} i) = \frac{4}{n^2}(\frac{n(n+1)}{2})$

$= \frac{2n(n+1)}{n^2} = \frac{2(n+1)}{n} = 2 + \frac{2}{n}$

Example 3.

$$S_n = \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(i^3 + 4i^2 + 6i)$$
$$= \frac{1}{n^4}\left[\boxed{\sum_{i=1}^{n} i^3} + 4\boxed{\sum_{i=1}^{n} i^2} + 6\boxed{\sum_{i=1}^{n} i}\right]$$
$$= \left[\frac{n^2(n+1)^2}{4n^4} + \frac{4n(n+1)(2n+1)}{6n^4} + \frac{6n(n+1)}{2n^4}\right]$$
$$= \frac{(n+1)(n+1)}{4n^2} + \frac{4(2n^2+3n+1)}{6n^3} + \frac{6n+6}{2n^3}$$
$$= \boxed{\frac{3n(n^2+2n+1)}{12n^3} + \frac{2(8n^2+12n+4)}{12n^3} + \frac{6(6n+6)}{12n^3}}$$

Get common denominator$= \frac{3n^3+6n^2+3n}{12n^3} + \frac{16n^2+24n+8}{12n^3} + \frac{36n+36}{12n^3}$

$$S_n = \frac{3n^3+22n^2+63n+44}{12n^3}$$
$$\lim_{n\to\infty}\left(\frac{3n^3}{12n^3} + \frac{22n^2}{12n^3} + \frac{63n}{12n^3} + \frac{44}{12n^3}\right)$$
$$= \lim_{n\to\infty}\left(\frac{1}{4} + \frac{22}{12n} + \frac{63}{12n^2} + \frac{44}{12n^3}\right)$$
$$= \boxed{\frac{1}{4}}$$

Example 4. $a = 1 \quad r = 3 \qquad n = 9$

$S_n = \frac{a(r^n - 1)}{r-1}$

$S_9 = \frac{1(3^9-1)}{3-1} \quad = \frac{(3^9-1)}{2}$

Example 5. $= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \quad = 1 - \frac{1}{n+1} \quad = \frac{n+1}{n+1} - \frac{1}{n+1} \quad = \frac{n}{n+1}$

Example 6.

a) $\sum_{i=3}^{25} \sqrt{3i+1}$

$\downarrow \qquad \downarrow$

$m = 3 \qquad\qquad j = i - m + 1$

$\downarrow \qquad\qquad j = i - 3 + 1$

$j = i - 2 \quad solve\ for\ i$

$i = j + 2$

$for\ i = 25 \qquad j = i - 2$

$j = 25 - 2$

$j = 23$

$\therefore \sum_{j=1}^{23} \sqrt{3(j+2)+1}$

$= \sum_{j=1}^{23} \sqrt{3j+7}$

Example 7. Suppose $\sum_{k=1}^{125} a_k = -8 \quad and \quad \sum_{k=1}^{125} b_k = 5$

Find $\sum_{k=1}^{125} (2a_k + 3b_k + 10)$

$$= 2\sum_{k=1}^{125} a_k + 3\sum_{k=1}^{125} b_k + \sum_{k=1}^{125} 10$$

$$= 2(-8) + 3(5) + 125(10) = -16 + 15 + 1250 = 1249$$

Example 8. a) $A_n = \sum_{k=1}^{n}\left(2+\frac{2k}{n}+\frac{k^2}{n^2}\right) = \sum_{k=1}^{n} 2 + \frac{2}{n}\sum_{k=1}^{n} k + \frac{1}{n^2}\sum_{k=1}^{n} k^2$

$= 2n + \frac{2}{n}\frac{(n)(n+1)}{2} + \frac{1}{n^2}\frac{n(n+1)(2n+1)}{6} = \frac{2n}{1} + \frac{n+1}{1} + \frac{2n^2+3n+1}{6n}$

$= \frac{12n^2+6n^2+6n+2n^2+3n+1}{6n} = \frac{20n^2+9n+1}{6n}$

b) $\lim_{n\to\infty} \frac{20n^2+9n+1}{6n} = \lim_{n\to\infty}\left(\frac{20}{6}n + \frac{9}{6} + \frac{1}{6n}\right) = \infty$

Practice Exam Questions on Sums and Sigma Notation

F1. $= \frac{1}{n^2}\sum_{i=1}^{n}(4i^2+3i)$

$= \frac{1}{n^2}\left[\left(4\sum_{i=1}^{n} i^2\right) + \left(3\sum_{i=1}^{n} i\right)\right] = \frac{1}{n^2}\left[\frac{4n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2}\right]$

$= \frac{4n(n+1)(2n+1)}{6n^2} + \frac{3n(n+1)}{2n^2} = \frac{4(2n^2+3n+1)}{6n} + \frac{3n+3}{2n}$

$= \frac{8n^2+12n+4+9n+9}{6n} = \frac{8n^2+21n+13}{6n}$

F2. $= \sum_{k=1}^{10} a_k + \sum_{k=1}^{10} b_k + \sum_{k=1}^{10} 10$

$= -20 + 30 + 100 = 110$

F3. $= 4\sum_{i=1}^{20} a_i - 2\sum_{i=1}^{20} bi$

$= 4(10) - 2(15) = 40 - 30 = 10$

F4. $1^2 + 2^2 + 3^2 + \cdots + 120^2$

$= \sum_{i=1}^{120} i^2$

F5. $= \sum_{i=1}^{n} \frac{2}{n}(i^2+4i+4) = \frac{1}{n}\sum_{i=1}^{n}(i^2+4i+4)$

$= \frac{2}{n}\left(\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 4\right) = \frac{2}{n}\left[\frac{n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} + 4n\right]$

$= \frac{n(n+1)(2n+1)}{3n} + \frac{4n(n+1)}{n} + 8 = \frac{(n+1)(2n+1)}{3} + \frac{4(n+1)}{1} + 8$

$= \frac{2n^2+3n+1}{3} + \frac{12n+12}{3} + \frac{24}{3} = \frac{2n^2+15n+37}{3}$

F6. $= \frac{2}{n}\sum_{i=1}^{n}\left(i^2 + \frac{6i}{n} + \frac{9}{n^2}\right) = \frac{2}{n}\left[\sum_{i=1}^{n} i^2 + \frac{6}{n}\sum_{i=1}^{n} i + \frac{9}{n^2}\sum_{i=1}^{n} 1\right]$
$= \frac{2}{n}\left[\frac{n(n+1)(2n+1)}{6} + \frac{6}{n}\frac{(n)(n+1)}{2} + \frac{9}{n^2}(n)\right] = \frac{(n+1)(2n+1)}{3} + 6\left(\frac{n+1}{n}\right) + \frac{18}{n^2}$
$= \frac{n^2(2n^2+3n+1)}{3n^2} + \frac{18n^2+18n}{3n^2} + \frac{54}{3n^2} = \frac{2n^4+3n^3+n^2+18n^2+18n+54}{3n^2}$
$= \frac{2n^4+3n^3+19n^2+18n+54}{3n^2} = \frac{2n^2}{3} + n + \frac{19}{3} + \frac{6}{n} + \frac{18}{n^2}$

F7. $= 100(0.6) = 60$

F8. a) $\frac{1}{n}\left(\sum_{i=1}^{n} 4 + \frac{1}{n}\sum_{i=1}^{n} i + \frac{1}{n^2}\sum_{i=1}^{n} i^2\right)$
$= \frac{1}{n}\left[4n + \frac{1}{n}\frac{(n)(n+1)}{2} + \frac{1}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$
$= \frac{1}{n}\left[4n + \frac{n+1}{2} + \frac{2n^2+3n+1}{6n}\right] = \left[4 + \frac{n+1}{2n} + \frac{2n^2+3n+1}{6n^2}\right]$
$= \frac{24n^2+3n^2+3n+2n^2+3n+1}{6n^2} = \frac{29n^2+6n+1}{6n^2}$

c) $\lim_{n\to\infty} \frac{29n^2+6n+1}{6n^2} = \lim_{n\to\infty}\left(\frac{29}{6} + \frac{1}{n} + \frac{1}{6n^2}\right) = \frac{29}{6}$

F9. $= \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n}\left(\frac{i^2}{n^2} - 2\right)\right] = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \frac{i^2}{n^2} - \sum_{i=1}^{n} 2\right]$
$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n^2}\frac{n(n+1)(2n+1}{6} - 2n\right]$
$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{(n+1)(2n+1)}{6n} - \frac{12n^2}{6n}\right] = \lim_{n\to\infty} \frac{1}{n}\left[\frac{2n^2+3n+1-12n^2}{6n}\right]$
$= \lim_{n\to\infty}\left(\frac{-10n^2+3n+1}{6n^2}\right) = \lim_{n\to\infty}\left(-\frac{10}{6} + \frac{1}{2n} + \frac{1}{6n^2}\right) = -\frac{5}{3}$

F10. $= \lim_{n\to\infty} \frac{1}{n^7}\left[\sum_{i=1}^{n} i^2 i^3\right] = \lim_{n\to\infty} \frac{1}{n^7}\left[\sum_{i=1}^{n} i^2 \sum_{i=1}^{n} i^3\right]$
$= \lim_{n\to\infty} \frac{1}{n^7}\left[\left(\frac{n(n+1)(2n+1)}{6}\right)\left(\frac{n^2(n+1)^2}{4}\right)\right] = \lim_{n\to\infty}\left[\frac{n^3(n+1)(2n+1)(n+1)^2}{24n^7}\right]$
$= \lim_{n\to\infty}\left[\left(\frac{2n^2+3n+1}{24n^4}\right)\left(\frac{n^2+2n+1}{1}\right)\right]$

$$= \frac{1}{24}\lim_{n\to\infty}\frac{(2n^4+4n^3+2n^2+3n^3+6n^2+3n+n^2+2n+1)}{n^4}$$

$$= \frac{1}{24}\lim_{n\to\infty}\frac{(2n^4+7n^3+9n^2+5n+1)}{n^4}$$

$$=\frac{1}{24}\lim_{n\to\infty}\left(2+\frac{7}{n}+\frac{9}{n^2}+\frac{5}{n^3}+\frac{1}{n^4}\right)=\frac{2}{24}=\frac{1}{12}$$

F11. *Write the sum starting at* 1: $\sum_{i=7}^{30}\sqrt{4i+5}$

$m = 7 \qquad j = i - m + 1$

$j = i - 7 + 1$

$j = i - 6 \quad solve\ for\ i$

$i = j + 6$

$for\ i = 30 \qquad j = i - 6$

$j = 30 - 6 = 24$

$j = 24$

$\therefore \sum_{j=1}^{24}\sqrt{4(j+6)+5}$

$= \sum_{j=1}^{24}\sqrt{4j+29}$

F12. $\sum_{i=4}^{27} i^2$

m=4 $\qquad j = i - m + 1$

$j = i - 4 + 1$

$j = i - 3 \quad solve\ for\ i$

$i = j + 3$

$for\ i = 27 \qquad j = i - 3$

$j = 27 - 3$

$j = 24$

$$\sum_{i=4}^{27} i^2 = \sum_{j=1}^{24}(j+3)^2$$

G. Riemann Sums

Example 2.

$$\Delta x = \frac{b-a}{n} = \frac{12-0}{3} = 4$$

$$Sum = \sum \Delta x f(x_i) = 4f(2) + 4f(6) + 4f(10)$$
$$= 4(7) + 4(39) + 4(103) = 596$$

Example 4.

$\int_0^4 x^2 dx$

f(x)=x^2

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$$

Area= $\lim\limits_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

= $\lim\limits_{n\to\infty} \sum_{i=1}^{n} f(a + \Delta x i)\Delta x$

= $\lim\limits_{n\to\infty} \sum_{i=1}^{n} f\left(0 + \frac{4i}{n}\right)(\frac{4}{n})$

= $\lim\limits_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\left(\frac{4}{n}\right)$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n}\right)^2 \left(\frac{4}{n}\right)$$

$$= \frac{4}{n} \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n}\right)^2$$

=$\frac{4}{n} \lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{16i^2}{n^2}$

= $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{64i^2}{n^3}$

= $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{64n(n+1)(2n+1)}{6n^3}$

= $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{64(n+1)(2n+1)}{6n^2}$

= $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{(64n+64)(2n+1)}{6n^2}$

= $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{128n^2+128n+64n+64}{6n^2}$

= $\lim\limits_{n\to\infty} \sum_{i=1}^{n}(\frac{128}{6} + \frac{192}{6n} + \frac{64}{6n^2}) = \frac{128}{6} = \frac{64}{3}$

Example 5.

a) $\frac{b-a}{n} = \frac{5}{n} \quad \therefore b - a = 5$

$xi = a + \Delta xi = a + \frac{5}{n}i \quad \therefore a = 3 \quad b = 8$

$\therefore \int_3^8 (x^3)dx$

b) $\frac{b-a}{n} = \frac{7}{n} \quad \therefore b - a = 7$

$xi = a + \Delta xi = a + \frac{7}{n}i \quad \therefore a = 0, b = 7$

$let\ f(x) = \cos(x^2)$

$\int_0^7 \cos(x^2)dx$

Example 6. Estimate the area under the graph of the function

$f(x) = \ln(x^2 + 1)$from x=0 to x=8 using four approximating rectangles and right endpoints. Repeat using left end points and midpoints.

$\Delta x = \frac{8-0}{4} = 2$

$R4 = \Delta x[f(2) + f(4) + f(6) + f(8)]$

$= 2[\ln 5 + \ln 17 + \ln 37 + \ln 65]$

$L4 = \Delta x[f(0) + f(2) + f(4) + f(6)]$

$= 2[\ln 1 + \ln 5 + \ln 17 + \ln 37]$

$= 2[\ln 5 + \ln 17 + \ln 37]$

M4=$\Delta x[f(1) + f(3) + f(5) + f(7)]$

$= 2[\ln 2 + \ln 10 + \ln 26 + \ln 50]$

Practice Exam Questions on Reimann Sums

G1. $\Delta x = \frac{b-a}{n} = \frac{5}{n} \quad b - a = 5$

$f(x) = e^x$

$xi = a + \Delta xi$

$xi = a + \frac{5}{n}i \quad a = 0,\ b = 5$

$\therefore \int_0^5 e^x dx$

G2. $f(xi) = f(a + i\Delta x) = f(a + \frac{\pi i}{n})$

$\frac{b-a}{n} = \frac{2\pi}{n} \quad \therefore b - a = 2\pi$

$a = 0 \quad b = 2\pi \quad f(x) = \sin(x^2)$

$\therefore \int_0^{2\pi} \sin(x^2) dx$

G3. $\Delta x = \frac{b-a}{n} = \frac{1}{n} \quad b - a = 1$

$f(xi) = f(a + \frac{1}{n}i)$

$= \sqrt{\sec(\frac{i}{n})} \quad \therefore a = 0 \quad b = 1$

$let\ f(x) = \sqrt{secx}$

$\int_0^1 \sqrt{\sec x}\, dx$

G4. $\frac{b-a}{n} = \frac{6}{n} \quad b - a = 6$

$f(a + i\Delta x)$

$= f\left(a + \frac{6}{n}i\right) \quad \therefore a = 1,\ b = 7$

$\int_1^7 \ln(x) dx \quad or \quad use \ \int_0^6 \ln(1 + x) dx$

G5. $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(xi)\Delta x$

$\Delta x = \frac{b-a}{n} = \frac{4}{n} \quad a = 1,\ b = 5$

$f(x) = \frac{1}{x}$

$f(xi) = f(1 + i\left(\frac{4}{n}\right))$

$= \lim_{n\to\infty} \left[\frac{4}{n} \sum_{i=1}^{n} \frac{1}{1 + i(\frac{4}{n})}\right]$

G6. $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$

$xi = a + i\Delta x = 0 + i\left(\frac{3}{n}\right) = \frac{3}{n}i$

$\therefore \lim_{n\to\infty} \sum_{i=1}^{n} f(xi)\Delta x$

$\lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 3\left(\frac{3i}{n}\right)\right] = \lim_{n\to\infty} \frac{3}{n}\left[\sum_{i=1}^{n} \frac{27i^3}{n^3} - \frac{9i}{n}\right]$

$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{27n^2(n+1)^2}{4n^3} - \frac{9n(n+1)}{2n}\right]$

$= \lim_{n\to\infty} \left[\frac{3(27)(n^2+2n+1)}{4n^2} - \frac{27n+27}{2n}\right]$

$= \lim_{n\to\infty} \left[\frac{81n^2+162n+81}{4n^2} - (\frac{27n+27}{2n})\right]$

$= \lim_{n\to\infty} \left[\frac{81}{4} + \frac{162}{4n} + \frac{81}{4n^2} - (\frac{27}{2} + \frac{27}{2n})\right] = \frac{81}{4} - \frac{27}{2} = \frac{81}{4} - \frac{54}{4} = \frac{27}{4}$

G7.

$\Delta x = \frac{b-a}{n} = \frac{8-0}{4} = 2$

$Sum = \sum \Delta x f(x_i) = 2f(1) + 2f(3) + 2f(5) + 2f(7)$

$= 2((1)^3 + 1) + 2((3)^3 + 1) + 2((5)^3 + 1) + 2((7)^3 + 1)$

= 2(2) + 2(28)+2(126)+2(344)=1000

G8.

$\int_0^2 x^3 dx$

f(x)=x³

$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

Area= $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

= $\lim_{n\to\infty} \sum_{i=1}^{n} f(a+\Delta x i)\Delta x$

= $\lim_{n\to\infty} \sum_{i=1}^{n} f\left(0+\frac{2i}{n}\right)(\frac{2}{n})$

= $\lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^3 \left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{8i^3}{n^3}\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{16i^3}{n^4}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{16}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$

= $\lim_{n\to\infty} \frac{16}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$

= $\lim_{n\to\infty} \frac{16}{n^2}\left[\frac{(n^2+2n+1)}{4}\right]$

= $\lim_{n\to\infty} \frac{16}{n^2}(\frac{16n^2+32n+16}{4n^2})$

= $\lim_{n\to\infty} (4+\frac{8}{n}+ \frac{4}{n^2})$

G9. $\Delta x = \frac{6-2}{4} = \frac{4}{4} = 1$

$R4 = \Delta x[f(3)+f(4)+f(5)+f(6)]$

$= 1[\ln 3 + \ln 4 + \ln 5 + \ln 6]$

$= \ln 3 + \ln 4 + \ln 5 + \ln 6$

H. Indefinite Integrals or Antiderivatives

Example 1. Find the indefinite integral.

a) $\int x^{12}dx$

$= \frac{x^{13}}{13} + c$

b) $\int \frac{1}{2}\sqrt{x^3}dx$

$= \frac{1}{2}\int x^{\frac{3}{2}}dx$

$= \frac{1}{2}\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c$

$= \frac{1}{5}x^{\frac{5}{2}} + c$

c) $\int \frac{1}{4\sqrt{x}}dx$

$= \frac{1}{4}\int x^{-\frac{1}{2}}dx$

$= \frac{1}{4}\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{1}{2}\sqrt{x} + c$

d) $\int 6e^{2x}dx$

$= \frac{6e^{2x}}{2} + c$

$=3e^{2x} + c$

Example 2. Integrate each of the following:

a) $\int 2sin3xdx$

$= -\frac{2\cos 3x}{3} + c$

$= -\frac{2}{3}\cos 3x + c$

b) $\int \frac{sec^2 4x}{2}dx$

$= \frac{1}{2}\int \sec^2 4x$

$= \frac{1}{2}\frac{\tan 4x}{4} + c$

$= \frac{\tan 4x}{8} + c$

c) $\int \frac{1}{5x} dx$

$= \frac{1}{5} \int \frac{1}{x} dx$

$= \frac{1}{5} \ln|x| + c$

d) $\int e^{7x} dx$

$= \frac{e^{7x}}{7} + c$

e) $\int \frac{x^5}{3} dx \quad = \frac{1}{3} \int x^5 dx = \frac{1}{3}\frac{x^6}{6} + c \; = \frac{x^6}{18} + c$

f) $\int (2x-1)(3x+2)dx$

$= \int (6x^2 + x - 2)dx = \frac{6x^3}{3} + \frac{x^2}{2} - 2x + c = 2x^3 + \frac{x^2}{2} - 2x + c$

g) $\int \frac{x^7+4x^6}{2x^2} dx = \int \left(\frac{1}{2}x^5 + 2x^4\right) dx = \frac{\frac{1}{2}x^6}{6} + \frac{2x^5}{5} + c = \frac{1}{12}x^6 + \frac{2}{5}x^5 + c$

<u>Practice Exam Questions on Indefinite Integrals or Antiderivatives</u>

H1. $\int \frac{1}{2x^4} dx = \frac{1}{2} \int x^{-4} dx$

$= \frac{1}{2}\frac{x^{-3}}{-3} + c$

$= -\frac{1}{6x^3} + c$

H2. $\int 2e^{7x} dx$

$= 2\frac{e^{7x}}{7} + c$

Integrate each of the following:

H3. $\int sin3x dx$

$= \frac{-cos3x}{3} + c$

H4. $\int \frac{sec^2 2x}{2} dx$

$= \frac{1}{2}\frac{tan2x}{2} + c$

$= \frac{1}{4}tan2x + c$

H5. $\int \frac{1}{8x} dx$

$= \frac{1}{8}ln|x| + c$

H6. $\int e^{6x} dx$

$= \frac{e^{6x}}{6} + c$

H7. $\int \frac{x^5}{3} dx$

$= \frac{1}{3}\frac{x^6}{6} + c \ or \frac{x^6}{18} + c$

H8. Integrate: $\int secxtanxdx$=secx+c

H9. Integrate: $\int -5sinxdx$=5cosx+c

H10. Integrate: $\int (2sec^2 x + \frac{3x^6+2\sqrt{x}}{x}) dx$

$$= 2tanx + \int \frac{3x^6}{x}dx + \int 2\frac{\sqrt{x}}{x}dx$$

$$= 2tanx + 3\int x^5dx + 2\int x^{-\frac{1}{2}}dx$$

$$= 2tanx + \frac{3x^6}{6} + \frac{2x^{\frac{1}{2}}}{1/2} + c$$

$$= 2tanx + \frac{x^6}{2} + 4\sqrt{x} + c$$

H11. Integrate: $\int \frac{1}{5} sec^2 x dx$=1/5 tanx+c

H12. Integrate: $\int \frac{1}{2x}dx = \frac{1}{2}ln|x| + c$

H13. Integrate: $\int \frac{1}{2} sec4xtan4xdx$=$\frac{1}{2}\frac{sec4x}{4} + c = \frac{1}{8}sec4x + c$

H14. Integrate: $\int cos8xdx = \frac{sin8x}{8} + c$

H15. $= 6\tan^{-1} x + c$

H16. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

$2\int \frac{1}{\sqrt{4^2-x^2}}dx = 2\sin^{-1}\left(\frac{x}{4}\right) + c$

H17. "a"=5 $\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x + c$

$\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

$= \sin^{-1}\left(\frac{x}{5}\right) + c$

I. Definite Integrals and The Fundamental Theorem of Calculus

Example 4. a) a) $\int_1^2 x^3 dx$

$=\left[\frac{x^4}{4}\right]_1^2 = \frac{2^4}{4} - \frac{1^4}{4} = \frac{16-1}{4} = \frac{15}{4}$

b) $\int_0^1 3e^x dx = 3[e^x]_0^1 = 3[e^1 - e^0] = 3(e-1)$

c) $\int_0^1 (x^3 - 4x)dx$

$\left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_0^1 = \left[\frac{x^4}{4} - 2x^2\right]_0^1$

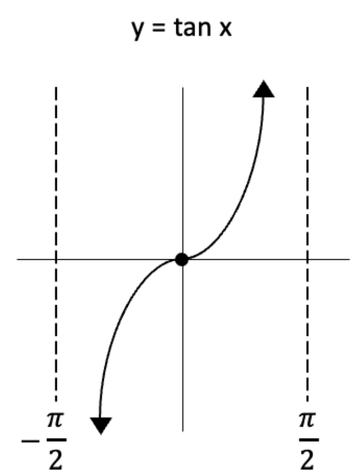
$= \left(\frac{1}{4} - 2\right) - (0-0) = \frac{1}{4} - \frac{8}{4} = -\frac{7}{4}$

d)

$$\int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$



To solve $\tan^{-1} 1$:

Ask yourself tan ? = 1 ? = $\frac{\pi}{4}$ (special triangle's)

To solve $\tan^{-1} 0$:

Ask yourself tan? = 0 ? = 0 (special triangle's)

Recall, arctanx is defined in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Example 5. Evaluate each of the following:

a) $\frac{d}{dx}\left(\int_0^{2/\pi} 3cos^2xdx\right)$=0 since the derivative of a constant is 0

b) $\frac{d}{dx}\int_2^x \sqrt{t^2+4}dt$=$\sqrt{x^2+4}$

c) $\frac{d}{dx}\int_x^5 sintdt = -\frac{d}{dx}\int_5^x sintdt = -sinx$

Example 6.

a) $= (3x^2)\tan(x^3) - \tan(x)(1)$
$= 3x^2\tan(x^3) - tanx$

b) $= e^{x^2}(2x) - e^{3x}(3)$
$= 2xe^{x^2} - 3e^{3x}$

Example 7.

$f'(x) = \int(1+cosx)dx = x + sinx + c$ substitute $f'(0) = 5$ $and\ get$
$5 = 0 + sin0 + c$
c=5
$f'(x)$= $x + sinx + 5$
$$f(x) = \int(x + sinx + 5)dx = \frac{x^2}{2} - cosx + 5x + k$$
Substitute f(0)=0
0=0 – cos 0 +0+ k
k=1
f(x)= $\frac{x^2}{2} - cosx + 5x + 1$

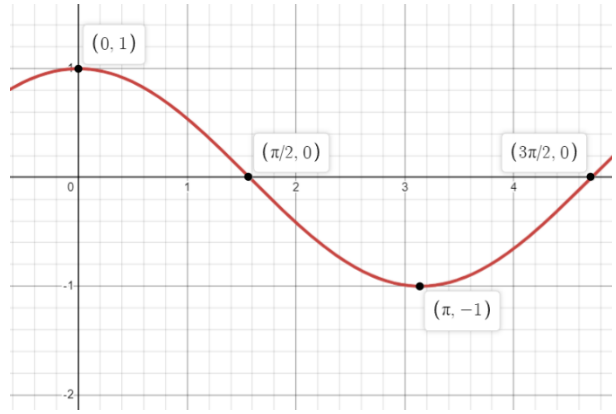
Example 8.

a) 0

b) $[e^x]_2^4 = e^4 - e^2$

c) $[sinx]_{\pi/2}^{\pi} = sin\pi - \sin\left(\frac{\pi}{2}\right) = 0 - 1 = -1$

Example 9.



$\cos x > 0 \; for \; 0 < x < \pi$

$\cos x < 0 \quad for \; \pi < x < \frac{3\pi}{2}$

$|\cos x| = \cos x \quad 0 < x < \pi$

$|\cos x| = -\cos x \;\; \pi < x < \frac{3\pi}{2}$

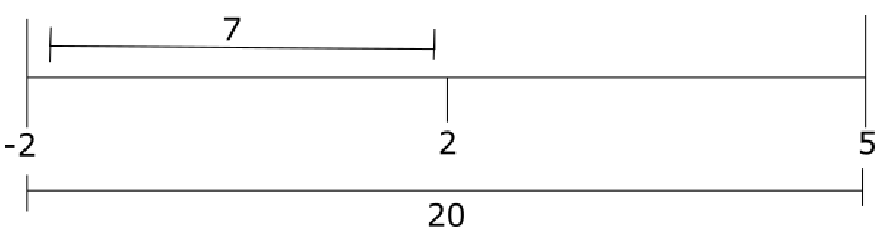
$$= \int_0^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\cos x dx = [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \left(\sin\frac{\pi}{2} - \sin 0\right) - \left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right) = (1-0) - (-1-1) = 3$$

Example 10.

$$\int_{-2}^{2} 3f(x)dx = 21, \quad \int_{-2}^{5} f(x)dx = 20, \quad find \int_{2}^{5} f(x)dx$$

$\int_{-2}^{2} f(x)dx = \frac{21}{3} = 7$



$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

$$\therefore \int_{-2}^{5} f(x)dx = \int_{-2}^{2} f(x)dx + \int_{2}^{5} f(x)dx$$

$$20 = 7 + \int_2^5 f(x)dx$$

$$\int_2^5 f(x)dx = 20 - 7 = 13$$

Example 11.

$= \lim_{x\to 0} \frac{\int_0^x e^{t^2} dt}{3x} \qquad (\frac{0}{0})$

$H = \lim_{x\to 0} \frac{e^{x^2}}{3} = \frac{1}{3}$

Example 12.

a) $F'(x) = \frac{d}{dx}\int_0^x (5t^5 + t^4 - 5t^3 - t^2 - 3t - 5)dt$

$F'(x) = 5x^5 + x^4 - 5x^3 - x^2 - 3x - 5$

$F''(x) = 25x^4 + 4x^3 - 15x^2 - 2x - 3$

b) $F'(x)\frac{d}{dx}\int_{x^4}^{x^5} (-4t-6)^2 dt$

$= (-4x^5 - 6)^2(5x^4) - (-4x^4 - 6)^2(4x^3)$

Example 13.

Net Area$= \int_{-4}^{4} |2x| dx = A1 - A2$

$= \int_{-4}^{4} 2x dx$

$= [x^2]_{-4}^{4}$

$= 16 - (-4)^2$

$= 16 - 16 = 0 \qquad OR$

$A1 = \frac{bh}{2} = \frac{4(8)}{2} = 16 \qquad A2 = \frac{bh}{2} = \frac{4(8)}{2} = 16$

$\therefore Net\ area = A1 - A2 = 16 - 16 = 0$

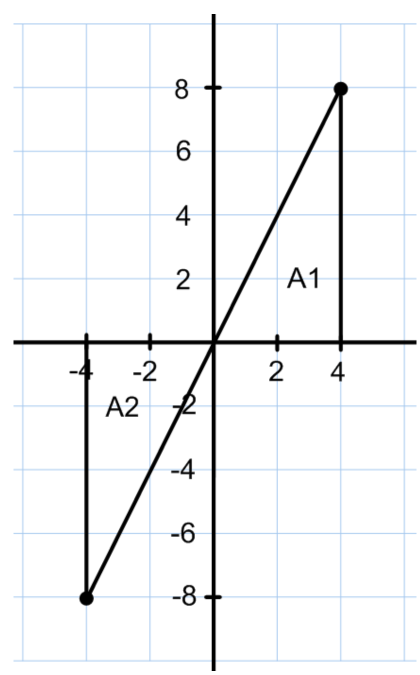
Total Area$= \int_{-4}^{0} -2x dx + \int_0^4 2x dx$

$A1 + A2 = 16 + 16 = 32 \quad or\ DO$

$2\int_0^4 2x dx = 2[x^2]_0^4$

$= 2[4^2 - 0]$

$= 32$



Example 14.

$f(x) = 2x - x^2$

$g(x) = x^2$

$f(x) \geq g(x) \quad for \quad 0 \leq x \leq 1$

$\therefore \int_0^1 (2x - x^2)dx \geq \int_0^1 x^2 dx$

Practice Exam Questions on Definite Integrals and The Fundamental Theorem of Calculus

I1. $= \sqrt{x^8 + 1}(4x^3) - \sqrt{\cos^2 x + 1}\,(-sinx)$

I2. $= [(x^2)^2 + x^2](2x) - (x^2 + x)(1)$

$= 2x(x^4 + x^2) - (x^2 + x) or\ 2x^5 + 2x^3 - x^2 - x$

I3. $= \frac{d}{dx}(constant) = 0$

I4. FTC $= arccosx$

I5. FTC $= 7\tan^{-1}\sqrt{7x} - 2\tan^{-1}\sqrt{2x}$

I6. $\cos x > 0\ for\ 0 < x < \pi/2$

$\cos x < 0 \quad for\ \pi/2 < x < \frac{3\pi}{2}$

$|\cos x| = \cos x \quad 0 < x < \pi/2$

$|\cos x| = -\cos x \quad \pi/2 < x < \frac{3\pi}{2}$

$= \int_0^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\cos x dx = [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$

$= \left(\sin\frac{\pi}{2} - \sin 0\right) - \left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right) = (1 - 0) - (-1 - 1) = 3$

I7. $= \left[\frac{3^x}{\ln 3 \ln 3}\right]_0^1 = \left[\frac{3^x}{(ln3)^2}\right]_0^1 = \frac{3^1}{(\ln 3)^2} - \frac{3^0}{(\ln 3)^2} = \frac{2}{(\ln 3)^2}$

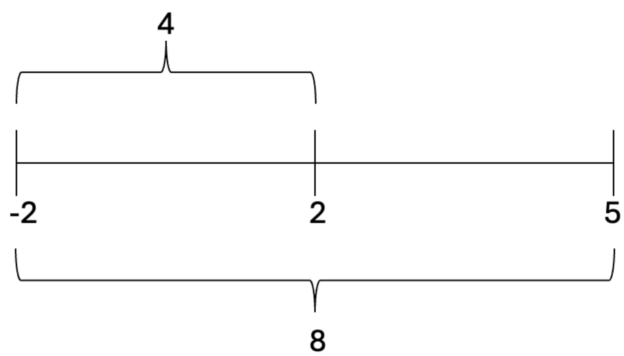
I8. $\int_{-2}^{2} f(x)dx = \frac{12}{3} = 4$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

$$\int_{-2}^5 f(x)dx = \int_{-2}^2 f(x)dx + \int_2^5 f(x)dx$$

$$\therefore \int_2^5 f(x)dx = \int_{-2}^5 f(x)dx - \int_{-2}^2 f(x)dx$$

$= 8 - 4 = 4$



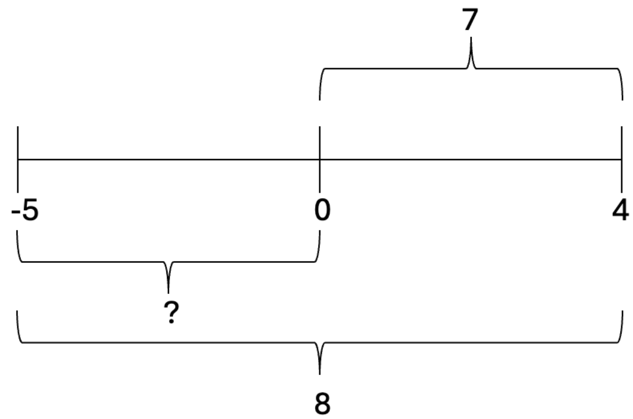
I9. $[4tan^{-1}(x)]_0^1 = 4tan^{-1}(1) - 4tan^{-1}(0) = 4\left(\frac{\pi}{4}\right) - 0 = \pi$

I10. Evaluate $\int_0^1 10^x dx = \left[\frac{10^x}{ln10}\right]_0^1 = \frac{10^1}{ln10} - \frac{10^0}{ln10} = \frac{9}{ln10}$

I11. Integrate $\int_0^\pi 3cosxdx = [3sinx]_0^\pi = 3sin\pi - 3sin0 = 0 - 0 = 0$

I12. Suppose f is continuous on the interval [-5,4], $\int_0^4 f(x)dx = 7$ and $\int_{-5}^4 2f(x) = 16$.Find $\int_{-5}^0 3f(x)dx$.

$\int_{-5}^4 f(x) = \frac{16}{2} = 8$



$\therefore 3\int_{-5}^{0} f(x)dx$

$= 3\left[\int_{-5}^{4} f(x)dx - \int_{0}^{4} f(x)dx\right]$

$= 3\,[8-7] = 3$

I13. Find $f'\left(\frac{\pi}{2}\right)$ $where\ f(t) = \int_{-\pi}^{\pi}\sqrt{sint}dt$

$$f'(t) = \frac{d}{dt}\left(\int_{-\pi}^{\pi}\sqrt{sint}dt\right) = 0\ by\ FTC$$

$\downarrow a\ constant$

The answer is C).

I14. Evaluate $\int_{1}^{3}\frac{1}{1+x}dx$

$[\ln(1+x)]_1^3 = \ln(1+3) - \ln(1+1)$

$$= \ln 4 - \ln 2 = \ln(\frac{4}{2}) = ln2$$

The answer is C).

I15. Evaluate $\int_{-2}^{1}|x|dx$

$|x| = x \ \ if \ x > 0$

$\quad = -x \ \ if \ x < 0$

$$\int_{-2}^{1}|x|dx \ = \int_{-2}^{0} -xdx + \int_{0}^{1} xdx$$

$$= \left[\frac{-x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1}$$

$$= \left(0 + \frac{(-2)^2}{2}\right) + \left(\frac{1}{2} - 0\right) = 2 + \frac{1}{2} = \frac{5}{2}$$

The answer is B)

I16. Integrate $\int_{0}^{\pi/6} secxtanxdx = [secx]_{0}^{\pi/6} = \sec\left(\frac{\pi}{6}\right) - sec0$

$= \dfrac{2}{\sqrt{3}} - 1$

Using Special Triangles

Recall, $\sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

I17. $|x-5| = x - 5 \ \ if \ x - 5 \geq 0$

$\quad = -(x-5) \ \ if \ x < 5$

$= \int_{0}^{5} -(x-5)dx + \int_{5}^{6}(x-5)dx$

$= \left[\frac{-x^2}{2} + 5x\right]_{0}^{5} + \left[\frac{x^2}{2} - 5x\right]_{5}^{6}$

$= \left(-\frac{5^2}{2} + 5(5)\right) - (0) + \left(\frac{6^2}{2} - 5(6)\right) - (\frac{5^2}{2} - 5(5))$

$= -\frac{25}{2} + 25 + 18 - 30 - \frac{25}{2} + 25$

$= 13$

I18. Integrate $\int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

I19. Integrate $\int_0^{\pi/4} sin8xdx = \left[\frac{-cos8x}{8}\right]_0^{\pi/4} = -\frac{cos2\pi}{8} + \frac{cos0}{8} = -\frac{1}{8} + \frac{1}{8} = 0$

I20. Integrate $\int x^2 \left(x + \frac{1}{x^3}\right) dx = \int \left(x^3 + \frac{1}{x}\right) dx = \frac{x^4}{4} + ln|x| + c$

I21. If $F(x) = \int_x^5 sintdt = -\int_5^x sintdt, \quad find\ F'(-\frac{\pi}{4})$

$F'(x) = -sinx$

$F'\left(-\frac{\pi}{4}\right) = -\sin\left(-\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

I22. $\frac{d}{dx}\int_7^x tan\sqrt{2t}dt = tan\sqrt{2x}$

I23. Integrate $\int_0^{\frac{3\pi}{2}} |sinx|dx$

$|\sin x| = \sin x \quad if\ 0 < x < \pi$

$\qquad = -\sin x \quad if\ \pi < x < \frac{3\pi}{2}$

$= \int_0^{\pi} \sin x dx + \int_{\pi}^{\frac{3\pi}{2}} -\sin x dx = [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{\frac{3\pi}{2}}$

$= (-\cos\pi + \cos 0) + (\cos\frac{3\pi}{2} - \cos\pi)$

$= (-(-1) + 1) + \big(0 - (-1)\big) = 3$

I24. If $f'(x) = 3e^x + 2, and\ f(0) = 6, find\ f(2)$.

$f(x) = \int (3e^x + 2)dx$

$f(x)$=$3e^x + 2x + c$ substitute x=0 and f(x)=6

6=$3e^0 + 2(0) + c$

c=6-3=3

$f(x)$=$3e^x + 2x + 3$

$f(2)$=$3e^2 + 2(2) + 3 = 3e^2 + 7$

I25. F(x)=$\int (3x^2 - 6x + 1)dx = \frac{3x^3}{3} - \frac{6x^2}{2} + x + c$

F(x) = $x^3 - 3x^2 + x + c$

-4 = c

F(x)= $x^3 - 3x^2 + x - 4$

F(1) = 1-3+1-4= -5

I26. Total Area= $\int_0^3 -(x-3)dx + \int_3^5 (x-3)dx$

OR DO A= $\frac{bh}{2}$

Total Area= $A1 + A2 = \frac{3(3)}{2} + \frac{2(2)}{2}$

$= \frac{9}{2} + \frac{4}{2}$

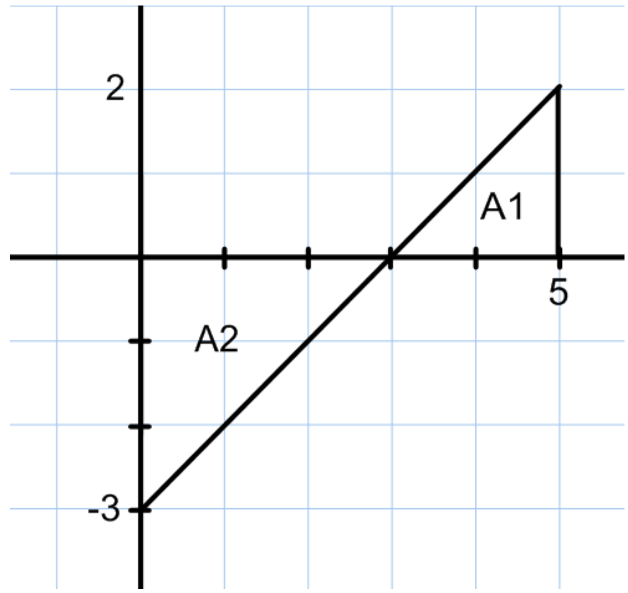
$= \frac{13}{2}$

Net Signed Area= $A1 - A2$

$= \frac{2(2)}{2} - \frac{3(3)}{2}$

$= \frac{4}{2} - \frac{9}{2}$

$= -\frac{5}{2}$



I27. Net Area= $\int_{-2}^2 3x dx$

$= \left[\frac{3x^2}{2}\right]_{-2}^2$

$= \frac{3}{2}(2^2) - \frac{3}{2}(-2)^2$

$= 6 - 6 = 0$

Total Area= $2\int_0^2 3x dx \quad or \quad \int_{-2}^0 (-3x)dx + \int_0^2 3x dx$

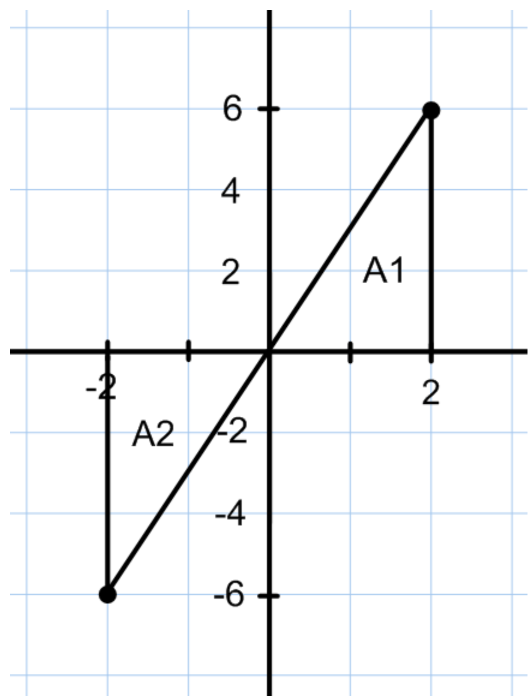
$= 2\left[\frac{3x^2}{2}\right]_0^2$

$= 2\left[\frac{3}{2}(2)^2 - 0\right] = 12$

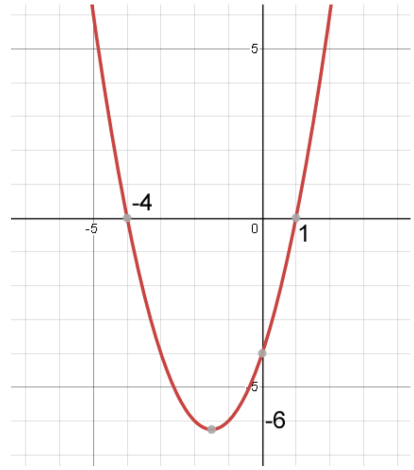
$t^2 + 3t - 4 = 0$

$(t+4)(t-1) = 0$

$t = 1, -4$



I28.



a) $s(2) - s(0) = \int_0^2 (t^2 + 3t - 4)dt$

$$= \left[\frac{t^3}{3} + \frac{3t^2}{2} - 4t\right]_0^2$$

$$= \frac{2^3}{3} + \frac{3(2)^2}{2} - 4(2) - 0$$

$$= \frac{8}{3} + 6 - 8$$

$$= \frac{8}{3} - \frac{6}{3}$$

$$= \frac{2}{3}$$

b) $\int_0^2 |v(t)|dt = \int_0^1 -v(t)dt + \int_1^2 v(t)dt$

$$= \int_0^1 (-t^2 - 3t + 4)dt + \int_1^2 (t^2 + 3t - 4)dt$$

$$= \left[\frac{-t^3}{3} - \frac{3t^2}{2} + 4t\right]_0^1 + \left[\frac{t^3}{3} + \frac{3t^2}{2} - 4t\right]_1^2$$

$$= \left(\frac{-1}{3} - \frac{3}{2} + 4\right) - 0 + \left[\left(\frac{2^3}{3} + \frac{3(2)^2}{2} - 4(2) - \left(\frac{1^3}{3} + \frac{3(1)^2}{2} - 4(1)\right)\right]$$

$$= \frac{-1}{3} - \frac{3}{2} + 4 + \frac{8}{3} + 6 - 8 - \frac{1}{3} - \frac{3}{2} + 4$$

$$= \frac{-1}{3} + \frac{8}{3} - \frac{1}{3} - \frac{3}{2} - \frac{3}{2} + 4 + 6 - 8 + 4$$

$$= \frac{6}{3} - \frac{6}{2} + 6$$

$$= 2 - 3 + 6 = 5$$

I29. $If\ f''(x) = \frac{2}{x} + 3\ where\ f'(1) = 2 + f, find\ f'(3)$

$f'(x) = \int f''(x)dx = \int \left(\frac{2}{x} + 3\right) dx$

$\therefore f'(x) = 2\ln|x| + 3x + c$

Substitution (1,2) $x, f'(x)$

$2 = 2\ln 1 + 3(1) + c$

$2 = 3 + c$

$c = -1$

$\therefore f'(x) = 2\ln|x| + 3x - 1$

$f'(3) = 2\ln|3| + 3(3) - 1$

$= 2\ln 3 + 8$

I30. $If\ f''(x) = 2e^{2x} + 5\sin x + 1, f'(0) = 1, f(0) = 2, find\ f(x)$

$f'(x) = \int f''(x)dx = \int (2e^{2x} + 5\sin x + 1)dx$

$\therefore f'(x) = \frac{2e^{2x}}{2} - 5\cos x + x + c = e^{2x} - 5\cos x + x + c$

Substitution $f'(0) = 1$

$\therefore 1 = e^0 - 5\cos 0 + 0 + c$

$1 = 1 - 5 + c$

$c = 5$

$\therefore f'(x) = e^{2x} - 5\cos x + x + 5$

$f(x) = \int f'(x)\,dx = \int (e^{2x} - 5\cos x + x + 5)\,dx$

$f(x) = \frac{e^{2x}}{2} - 5\sin x + \frac{x^2}{2} + 5x + k$

Substitution $f(0) = 2$

$$2 = \frac{e^0}{2} - 5\sin 0 + \frac{0^2}{2} + 5(0) + k$$

$$2 = \frac{1}{2} - 0 + 0 + 0 + k$$

$$k = 2 - \frac{1}{2} = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

$$\therefore f(x) = \frac{e^{2x}}{2} - 5\sin x + \frac{x^2}{2} + 5x + \frac{3}{2}$$

I31. a)

$$\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx = [\sin^{-1} x]_{-1}^{0}$$

$$= \sin^{-1} 0 - \sin^{-1}(-1)$$

$$= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

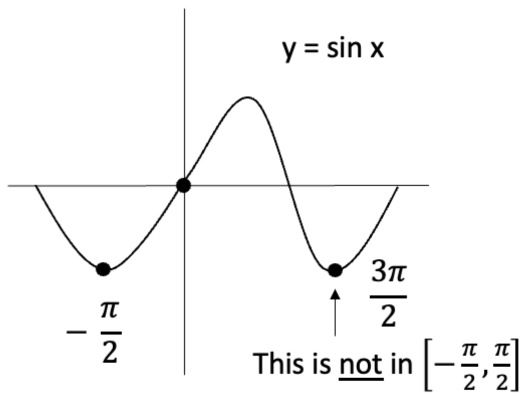
Recall, arcsin x is defined in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

To solve $\sin^{-1}(-1)$

Ask yourself sin ? = -1

?= $-\frac{\pi}{2}$

$\sin^{-1}(0) = 0$ since $\sin 0 = 0$



b) $\int_0^1 \frac{-1}{\sqrt{1-x^2}}\,dx = [\cos^{-1} x]_0^1$

$$= \cos^{-1} 1 - \cos^{-1} 0 = 0 - \frac{\pi}{2} = -\frac{\pi}{2}$$

To solve $\cos^{-1} 1$

Ask yourself cos ? = 1

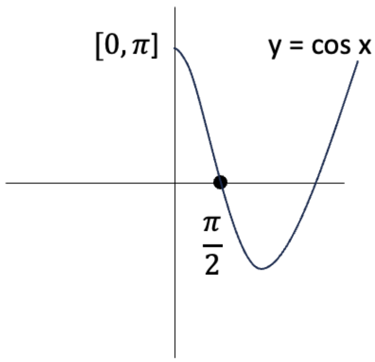
? = 1

To solve $\cos^{-1} 0$

Ask yourself cos ? = 0

? = $\frac{\pi}{2}$

Recall, we draw arccos x in A,S



I32. $If\ f'(x) = 3x^2 + 2\ and\ f(0) = 3, then\ f(1) =$_______________.

$f(x) = \int(3x^2 + 2)dx$

$f(x) = x^3 + 2x + c \qquad sub\ \ x = 0 \quad f(0) = 3$

$3 = 0 + 0 + c$

$\therefore c = 3$

$f(x) = x^3 + 2x + 3 \qquad \therefore f(1) = 1^3 + 2(1) + 3 = 6$

I33. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ in metres per second.

a) Find the net displacement of the particle from 0 to 4 seconds

$s(4) - s(0) = \int_0^4 (t^2 - t - 6)dt$

$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_0^4$

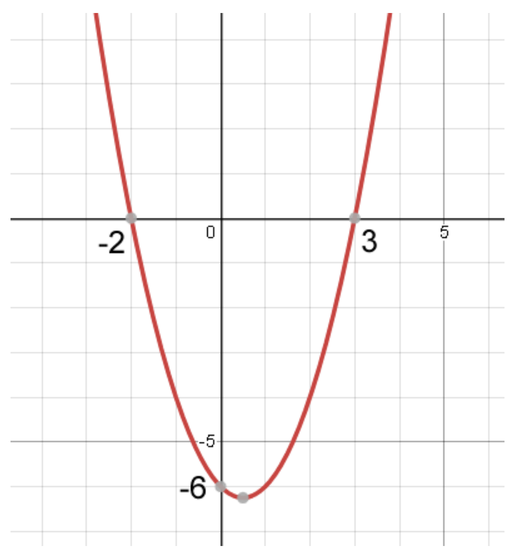
$= \frac{4^3}{3} - \frac{4^2}{2} - 6(4) - 0$

$= \frac{64}{3} - 8 - 24$

$= \frac{64}{3} - \frac{32}{1}$

$= \frac{64}{3} - \frac{96}{3}$

$= -\frac{32}{3}$



b) Find the distance traveled during 0 to 4 seconds

$t^2 - t - 6 = 0$

$(t-3)(t+2) = 0$

$t = 3, -2$

$$\int_0^4 |v(t)|dt = \int_0^3 -v(t) + \int_3^4 v(t)dt$$
$$= \int_0^3 (-t^2 + t + 6)dt + \int_3^4 (t^2 - t - 6)dt$$
$$= \left[\frac{-t^3}{3} + \frac{t^2}{2} + 6t\right]_0^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$
$$= \left[\frac{-3^3}{3} + \frac{3^2}{2} + 6(3) - 0\right] + \left[\left(\frac{4^3}{3} - \frac{4^2}{2} - 6(4)\right) - \frac{3^3}{3} - \frac{3^2}{2} - 6(3)\right]$$
$$= \frac{-27}{3} + \frac{9}{2} + 18 + \frac{64}{3} - \frac{16}{2} - 24 - \frac{27}{3} + \frac{9}{2} + 18$$
$$= 16\frac{1}{3}$$

J. Substitution

Example 1. $\int \frac{(3x^2+1)}{(x^3+x)^4}dx \quad subst \quad let\ u = x^3+x \qquad du = (3x^2+1)dx$

$$= \int \frac{1}{u^4 du} = \int u^{-4}du = \frac{u^{-3}}{-3} + c = \frac{(x^3+x)^{-3}}{-3} + c = \frac{1}{-3(x^3+x)^3} + c$$

b) $\int (40x-16)(5x^2-4x)^6\, dx$

$= 4\int u^6 du \qquad subst \quad let\ u = 5x^2-4x$

$= \frac{4u^7}{7} + c \qquad du = (10x-4)dx$

$= \frac{4(5x^2-4x)^7}{7} + c \qquad 4du = (40x-16)dx$

c) $\int \frac{dx}{(2x+3)^8} \qquad subst \quad u = 2x+3 \quad du = 2dx \quad \frac{du}{2} = dx$

$= \int (2x+3)^{-8}dx$

$= \frac{1}{2}\int u^{-8}du$

$= \frac{1}{2}\left(\frac{u^{-7}}{-7}\right) + c$

$= \frac{-1}{14u^7} + c = \frac{-1}{14(2x+3)^7} + c$

d) $\int \frac{-2}{x(1+lnx)^2} dx \qquad subst \quad u = 1 + \ln x \qquad du = \frac{1}{x} dx$

$$= -2 \int \frac{1}{u^2} du$$

$$= -2 \int u^{-2} du$$

$$= -2 \left[\frac{u^{-1}}{-1}\right] + c$$

$$= \left[\frac{2}{u}\right] + c$$

$$= \frac{2}{1+lnx} + c$$

e) $\int 4xe^{x^2} dx \qquad subst \quad u = x^2 \qquad du = 2x\, dx$ 2du=4xdx

$$= 2 \int e^u du$$

$$= 2e^u + c$$

$$= 2e^{x^2} + c$$

Example 2. Integrate $\int \frac{e^x}{\sqrt{e^x+4}} dx$

$$u = e^x + 4 \qquad du = e^x dx$$

$$= \int \frac{1}{\sqrt{u}} du$$

$$= \int u^{-\frac{1}{2}} du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{u} + c = 2\sqrt{e^x + 4} + c$$

Example 3. Integrate $\int 2sinx(1+cosx)^5dx$

$$u = 1+\cos x$$

$$du = -\sin x dx$$

$$-du = \sin x dx$$

$= -2\int u^5 du$

$= -2\left[\frac{u^6}{6}\right] + c$

$= \left[\frac{-1}{3}u^6\right] + c \; = \frac{-1}{3}(1+cosx)^6 + c$

Example 4. Integrate $\int 2tan^6xsec^2xdx$

$$u = \tan x \qquad du = \sec^2 x dx$$

$= 2\int u^6 du$

$= \frac{2u^7}{7} + c$

$= \frac{2}{7}\tan^7 x + c$

Example 5. Integrate $\int \frac{8}{(arc\tan 2x)(1+4x^2)}dx$

$$u = arc\tan 2x \qquad du = \frac{1}{1+(2x)^2}(2)dx = \frac{2}{1+4x^2}dx$$

$$4du = \frac{1}{1+(2x)^2}(2)dx = \frac{8}{1+4x^2}dx$$

$= 4\int \frac{1}{u}du = [4ln|u|] + c = 4ln\ |arctan2x| + c$

Example 6. $\int sin^3xcos^3xdx$ Let u=sinx and du=cosx dx

$= \int sin^3xcos^3xdx = \int sin^3xcos^2xcosxdx$ use the identity $sin^2x+cos^2x = 1$

$= \int sin^3x(1 - sin^2x)cosxdx = \int u^3(1 - u^2)du = \int (u^3 - u^5)du$

$= \frac{u^4}{4} - \frac{u^6}{6}$+c = $\frac{(sinx)^4}{4} - \frac{(sinx)^6}{6}$+c=$\frac{1}{4}sin^4x - 16\ sin^6x + c$

Example 7. Evaluate: $\int_1^2 \frac{x}{1+x^2}dx$

$$u = 1 + x^2 \quad du = 2x\ dx$$

$$\frac{du}{2} = xdx$$

$$x = 1 \quad u = 1 + 1^2 = 2$$

$$x = 2 \quad u = 1 + 2^2 = 5$$

$= \frac{1}{2}\int_2^5 \frac{1}{u}du$

$= \frac{1}{2}[\ln|u|]_2^5$

$= \frac{1}{2}[\ln 5 - \ln 2]$

$= \frac{1}{2}\ln(\frac{5}{2})$

Example 8. Evaluate: $\int_e^{e^3} \frac{lnx}{x}dx$

$$subst\ \ let\ u = \ln x \qquad du = \frac{1}{x}dx$$

$$x = e \qquad u = \ln e = 1$$

$$x = e^3 \quad u = 3$$

$= \int_1^3 u\ du$

$= \left[\frac{u^2}{2}\right]_1^3 = \frac{3^2}{2} - \frac{1^2}{2} \ = \frac{9}{2} - \frac{1}{2} \ \ = 4$

Example 9. Evaluate: $\int_0^{ln2} \frac{e^x}{e^x+1} dx$

$$subst \quad u = e^x + 1$$

$$du = e^x dx$$

$$x = 0 \quad u = e^0 + 1 = 1 + 1 = 2$$

$$x = \ln 2 \quad u = e^{\ln 2} + 1 = 2 + 1 = 3$$

$= \int_2^3 \frac{1}{u} du$

$= [\ln u]_2^3 \ = \ln 3 - \ln 2 \ = \ln\left(\frac{3}{2}\right)$

Example 10.

$$\int_0^1 2x(x-1)^4 dx$$

$$u = x - 1 \qquad x = 0$$

$$x = u + 1 \qquad u = 0 - 1 = -1$$

$$x = 1$$

$$u = 1 - 1 = 0$$

$$2\int_{-1}^{0} (u+1)u^4 du$$

$$= 2\int_{-1}^{0} (u^5 + u^4) du$$

$$= 2\left[\frac{u^6}{6} + \frac{u^5}{5}\right]_{-1}^{0}$$

$$= 2\left(0 - \left[\frac{(-1)^6}{6} + \frac{(-1)^5}{5}\right]\right)$$

$$= 2\left(-\left[\frac{1}{6} - \frac{5}{5}\right]\right)$$

$$= -2\left(\frac{5}{30} - \frac{6}{30}\right)$$

$$= -2\left(-\frac{1}{30}\right)$$

$$= \frac{2}{30} \; or \; \frac{1}{15}$$

Practice Exam Questions on Substitution

J1. $u = arc\tan x \qquad du = \frac{1}{1+x^2}dx$

$$= \int u^4 du = \frac{u^5}{5} + c \quad or \quad \frac{(arc\tan x)^5}{5} + c$$

J2. Integrate: $\int \tan x dx \quad \tan x = \frac{\sin x}{\cos x} \quad subst \; let \; u = \cos x$

$du = -\sin x dx$

$-du = \sin x dx$

$$= \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du = -\ln|u| + c = -\ln|\cos x| + c$$

J3.

$= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx \quad trick \quad multiply \; by \; \frac{\sec x + \tan x}{\sec x + \tan x}$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \qquad let \; u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)dx$

$= \int \frac{1}{u} du = \ln|u| + c \quad or \quad \ln|\sec x + \tan x| + c$

J4. Evaluate $\int_1^e \frac{(lnx)^2}{x} dx$

$$u = \ln x \quad du = \frac{1}{x} dx$$

$$x = 1 \quad u = \ln 1 = 0$$

$$x = e \quad u = \ln e = 1$$

$$= \int_0^1 u^2 du = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

J5. Integrate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$$u = \sqrt{x} \qquad du = \frac{1}{2} x^{-\frac{1}{2}} dx \qquad 2du = \frac{1}{\sqrt{x}} dx$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^u du = 2e^u + c = 2e^{\sqrt{x}} + c$$

J6. Integrate $\int 6x^2(x^3 + 1)^4 dx$

$$u = x^3 + 1 \qquad du = 3x^2 dx \qquad 2du = 6x^2 dx$$

$$= 2 \int u^4 du = 2\frac{u^5}{5} + c = 2\frac{u^5}{5} + c = 2\frac{(x^3+1)^5}{5} + c$$

J7. Evaluate $\int_0^{\frac{\pi}{4}} tan^3 x sec^2 x dx$

$$u = \tan x \quad du = \sec^2 x dx$$

$$x = 0 \quad u = \tan 0 = 0$$

$$x = \frac{\pi}{4} \quad u = \tan\frac{\pi}{4} = 1$$

$$= \int_0^1 u^3 du = \left[\frac{u^4}{4}\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$$

J8. Integrate $\int_0^1 \frac{x}{(x^2+1)^4}dx$

$u = x^2 + 1 \quad du = 2xdx \qquad \frac{du}{2} = xdx$

$x = 0, u = 1$

$x = 1, u = 2$

$$\int_0^1 \frac{x}{(x^2+1)^4}dx = \frac{1}{2}\int_1^2 u^{-4}du = \frac{1}{2}\left[\frac{u^{-3}}{-3}\right]_1^2 = \frac{1}{2}\left[-\frac{1}{3u^3}\right]_1^2$$

$$= -\frac{1}{6}\left[\frac{1}{u^3}\right]_1^2 = -\frac{1}{6}\left(\frac{1}{8}-\frac{1}{1}\right) = -\frac{1}{6}\left(-\frac{7}{8}\right) = \frac{7}{48}$$

J9. Integrate $\int_0^1 \frac{2x^3}{x^4+1}dx$

$u = x^4 + 1 \qquad du = 4x^3dx \qquad \frac{du}{2} = 2x^3dx$

$x = 0, u = 0 + 1 = 1$

$x = 1, u = 1 + 1 = 2$

$$\int_0^1 \frac{x^3}{x^4+1}dx = \frac{1}{2}\int_1^2 \frac{1}{u}du = \frac{1}{2}[ln|u|]_1^2 = \frac{1}{2}(ln2 - ln1) = \frac{1}{2}(ln2 - 0)$$

$$= \frac{1}{2}ln2 = ln2^{1/2} = ln\sqrt{2}$$

J10. Integrate $\int_0^2 \frac{x}{x^2+1}dx$

$u = x^2 + 1 \quad du = 2xdx \quad \therefore \frac{du}{2} = xdx$

$x = 0 \quad u = o^2 + 1 = 1$

$x = 2 \quad u = 2^2 + 1 = 5$

$= \frac{1}{2}\int_1^5 \frac{du}{u} = \frac{1}{2}\int_1^5 \frac{1}{u}du = \frac{1}{2}[\ln u]_1^5 = \frac{1}{2}(\ln 5 - \ln 1) = \frac{1}{2}(ln5 - 0)$

$= \frac{1}{2}ln5 \text{ or } \frac{ln5}{2}$

J11. $u = 1 + cosx \;\; du = -sinxdx$

$x = 0 \quad u = 1 + \cos 0 = 2$

$x = \frac{\pi}{3} \quad u = 1 + \cos\frac{\pi}{3} \qquad u = 1 + \frac{1}{2} = \frac{3}{2}$

$= \int_2^{\frac{3}{2}} \frac{du}{u} = [\ln u]_2^{\frac{3}{2}} = \left[\ln\frac{3}{2} - \ln 2\right]$

$= \ln\left(\frac{3/2}{2}\right) = \ln\left(\frac{3}{4}\right)$

J12. Integrate $\int_1^2 \frac{3x^2}{(x^3+1)[\ln(x^3+1)]^4}dx$

$$u = \ln(x^3 + 1) \qquad du = \frac{1}{x^3 + 1}(3x^2)dx$$

$x = 1 \;\rightarrow\; u = ln2 \qquad\qquad x = 2 \;\rightarrow\; u = \ln 9$

$= \int_{ln2}^{ln9} \frac{1}{u^4}du = \int_{ln2}^{ln9} u^{-4}du = \left[\frac{u^{-3}}{-3}\right]_{ln2}^{ln9} = \left[\frac{-1}{3u^3}\right]_{ln2}^{ln9}$

$= \left[\frac{-1}{3(ln9)^3} + \frac{1}{3(\ln 2)^3}\right]$

J13. Integrate $\int_0^1 \frac{6x}{x^2+1}dx$

u=x^2+1 x=0 u=$0^2+1=1$

du=2xdx x=1 u=$1^2+1=2$

3du=6xdx

=3 $\int_1^2 \frac{du}{u} = 3[\ln u]_1^2 = 3(\ln 2 - \ln 1) = 3\ln 2 - 0 = 3ln2 = ln2^3 = ln8$

J14. Integrate $\int_2^3 \frac{1}{x\sqrt{lnx}}dx$

$u = \ln x \qquad du = \frac{1}{x}dx \qquad du = \frac{1}{x}dx$

$x = 2 \quad u = ln2$

$x = 3 \quad u = ln3$

$= \int_{ln2}^{ln3} \frac{1}{\sqrt{u}}du = \int_{ln2}^{ln3} u^{\frac{-1}{2}} = \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_{ln2}^{ln3}$

$= \left[2\sqrt{u}\right]_{ln2}^{ln3} = \left[2\sqrt{\ln 3} - 2\sqrt{\ln 2}\right]$

J15. Find $\int_1^9 \frac{(1+\sqrt{x})^3}{\sqrt{x}}dx$

$u = 1 + \sqrt{x} \qquad du = \frac{1}{2}x^{-\frac{1}{2}}dx \qquad 2du = \frac{1}{\sqrt{x}}dx$

$x = 1, u = 2$

$x = 9, u = 1 + \sqrt{9} = 1 + 3 = 4$

$$\int_1^9 \frac{(1+\sqrt{x})^3}{\sqrt{x}}dx = 2\int_2^4 u^3 du = 2\left[\frac{u^4}{4}\right]_2^4 = 2\left[\frac{4^4}{4} - \frac{2^4}{4}\right]$$

J16. $\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$u = \sqrt{x} \qquad du = \frac{1}{2} x^{-\frac{1}{2}} dx \qquad 2du = \frac{1}{\sqrt{x}} dx$

$x = 1, u = \sqrt{1} = 1$

$x = 9, u = \sqrt{9} = 3$

$$\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2\int_1^3 e^u du = 2[e^u]_1^3 = 2(e^3 - e)$$

J17. Evaluate: $\int_0^{\pi/4} \frac{\sec^2 x}{\sqrt{1+\tan x}} dx$

$subst \quad u = 1 + \tan x$

$du = \sec^2 x dx$

$x = 0 \quad u = 1 + \tan 0 = 1$

$x = \frac{\pi}{4} \quad u = 1 + \tan\frac{\pi}{4} = 1 + 1 = 2$

$= \int_1^2 \frac{1}{\sqrt{u}} du \quad = \int_1^2 u^{-\frac{1}{2}} du = \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^2 = \left[2\sqrt{u}\right]_1^2$

$= 2\sqrt{2} - 2\sqrt{1} = 2\sqrt{2} - 2$

J18. Evaluate $\int tan^3 x sec^2 x dx$

$u = \tan x \qquad du = \sec^2 x dx$

$\int u^3 du = \frac{u^4}{4} + c = \frac{tan^4 x}{4}$+c

J19. Integrate $\int \frac{x}{(x^2+1)^4} dx$

$u = x^2 + 1 \quad du = 2xdx \qquad \frac{du}{2} = xdx$

$$\int \frac{x}{(x^2+1)^4} dx = \frac{1}{2}\int u^{-4} du = = \frac{1}{2}\left(\frac{u^{-3}}{-3}\right) + c = -\frac{1}{6u^3} + c$$
$$= -\frac{1}{6(x^2+1)^3} + c$$

J20. Integrate $\int \frac{2x^3}{x^4+1} dx$

$u = x^4 + 1 \qquad du = 4x^3dx \qquad \frac{du}{2} = 2x^3dx$

$$\int \frac{x^3}{x^4+1} dx = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ ln|u| + c = \frac{1}{2} ln|x^4+1| + c$$

J21. Integrate $\int \frac{8x}{x^2+1} dx$

$u = x^2 + 1 \qquad du = 2xdx \quad \therefore\ 4du = 8xdx$

$$= 4\int \frac{1}{u} du = 4ln|u| + c = 4ln|x^2+1| + c$$

J22. Integrate $\int 8x^3\sqrt{x^4+1}dx$

$$subst\ \ u = x^4 + 1 \qquad du = 4x^3dx$$

$$2du = 8x^3dx$$

$= 2\int \sqrt{u}du$

$= 2\int u^{\frac{1}{2}}du \ = 2\ \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = 2\left(\frac{2}{3}\right)u^{\frac{3}{2}} + c = \frac{4}{3}(x^4+1)^{\frac{3}{2}} + c$

J23. Integrate $\int \frac{sec^2(lnx)}{2x}dx$

$$subst\ \ u = \ln x \qquad du = \frac{1}{x}dx \qquad \frac{1}{2}du = \frac{1}{2x}dx$$

$= \frac{1}{2}\int \sec^2 u du$

$= \frac{1}{2}\tan u + c$

$= \frac{1}{2}\tan(\ln x) + c$

J24. Integrate $\int_0^{\pi/2} \cos^3 x \, dx$

Substitution

$u = \sin x$

$du = \cos x \, dx$

$x = 0 \quad u = \sin 0 = 0$

$x = \frac{\pi}{2} \quad u = \sin\frac{\pi}{2} = 1$

Identity

$\sin^2 x + \cos^2 x = 1$

$\therefore \cos^2 x = 1 - \sin^2 x$

$$\int_0^{\pi/2} \cos^3 x \, dx$$

$$= \int_0^{\pi/2} \cos^2 x \cos x \, dx$$

$$= \int_0^{\pi/2} (1 - sin^2 x) \cos x \, dx$$

$$= \int_0^{1} (1 - u^2) du$$

$$= \left[u - \frac{u^3}{3}\right]_0^1$$

$$= \left(1 - \frac{1}{3}\right) - (0 - 0)$$

$$= \frac{2}{3}$$

J25. Evaluate: $\int_0^2 6x^2 e^{x^3} dx$

$$subst \quad u = x^3 \quad du = 3x^2 dx \qquad 2du = 6x^2 dx$$

$$x = 0 \quad u = 0^3 = 0$$

$$x = 2 \quad u = 2^3 = 8$$

$= 2\int_0^8 e^u du$

$= 2[e^u]_0^8 \ = 2(e^8 - e^0) = 2(e^8 - 1)$

K. Other Types of Integrals

Example 1. Integrate $\int_{-5}^{5} \sqrt{25 - x^2} dx$

If we write $y = \sqrt{25 - x^2}$ and square both sides, the square root will disappear and we get:
$y^2 = 25 - x^2$ simplifying, we get...
$x^2 + y^2 = 25$...which we recognize as half of a circle, since originally, we only had the positive square root of the function

This circle has centre at (0,0) and radius=5

$$A = \frac{\pi r^2}{2} = \frac{\pi(5^2)}{2} = 12.5\pi$$

Example 2. Determine whether each of the following functions are odd, even, or neither.

a) $f(x) = x^3 - x$

$$\begin{aligned} f(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -(x^3 - x) \\ &= -f(x) \end{aligned}$$

$since\ in\ this\ case\ f(-x) = -f(x), the\ function\ is\ odd$

b) $f(x) = x^2 + 1$

$$\begin{aligned} f(-x) &= (-x)^2 + 1 \\ &= x^2 + 1 \\ &= f(x) \end{aligned}$$

$since\ f(-x) = f(x), the\ function\ is\ even$

c) $f(x) = 3x - x^2$

$$\begin{aligned} (-x) &= 3(-x) - (-x)^2 \\ &= -3x - x^2 \end{aligned}$$

$since\ f(-x) \neq f(x) and\ f(-x) \neq -f(x), this\ function\ is\ neither\ odd\ or\ even.$

Example 3. Determine whether the following functions are odd or even:

a) $f(x) = x^3 \sin x$

x^3 *is an odd function, since* $(-x)^3 = -x^3$

$\sin x$ *is an odd function, since* $\sin(-x) = -\sin x$

Thus, $f(x) = x^3 \sin x$

$= odd \times odd = even$

b) $f(x) = x^3 \cos x$

x^3 *is once again an odd function, however, this time* $\cos x$ *is an even function, since* $\cos(-x) = \cos x$.

thus, $f(x) = x^3 \cos x$

$odd \times even = odd$

L. Practice With All Types of Integration

L1. $\Delta x = \frac{b-a}{n} = \frac{2\pi}{n} \quad \therefore b - a = 2\pi$

$f(xi) = f(a + i\Delta x) = f\left(a + \frac{2\pi}{n} i\right) \quad \therefore a = 0,\ b = 2\pi$

$\therefore \int_0^{2\pi} \cos x dx$

L2. $\frac{b-a}{n} = \frac{1}{n} \quad \therefore b - a = 1 \quad a = 0 \quad b = 1$

$f(xi) = f\left(0 + \frac{1}{n} i\right) = f\left(\frac{1}{n} i\right) \quad let\ f(x) = \sqrt{\csc x}$

The answer is B.

L3. $[\ln(1 + x)]_1^4 = \ln(1 + 4) - \ln(1 + 1)$

$= \ln 5 - \ln 2 = \ln(\frac{5}{2})$

The answer is E.

L4. $|x| = x \ \ if\ x > 0$

$= -x \ \ if\ x < 0$

$= \int_{-2}^{0} -x dx + \int_0^2 x dx$

$= \left[\frac{-x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_0^2$

$= \left(0 + \frac{(-2)^2}{2}\right) + (2 - 0) = 2 + 2 = 4$

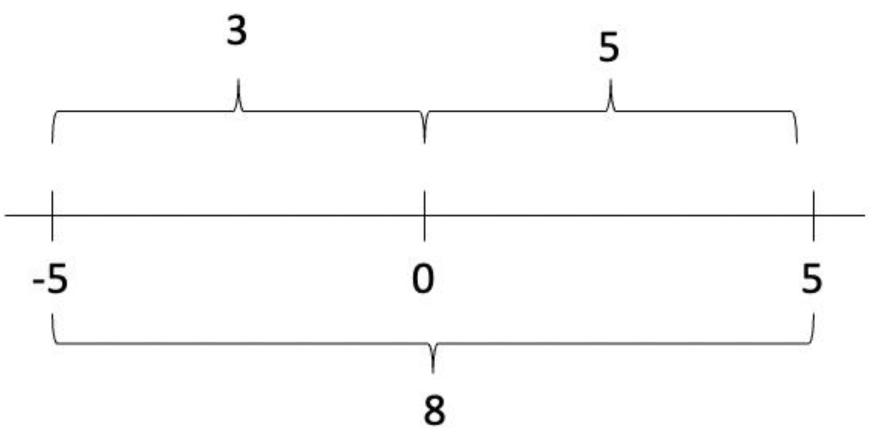
L5. $f'(t) = \frac{d}{dt}(\int_{\frac{\pi}{2}}^{\pi} \sqrt{1 + \cos t}\, dt) = 0 \quad FTC$

$\downarrow a\ constant$

The answer is C.

L6. $\int_{-5}^{5} f(x) = \frac{24}{3} = 8$

$= 5\int_{-5}^{0} f(x)dx = 5(8-5) = 5(3) = 15$



L7. $= \sin^{-1}\sqrt{(x^2+1)^2+1}(2x) - \sin^{-1}\sqrt{x^2+1}(1)$

$= 2x\sin^{-1}\sqrt{(x^2+1)^2+1} - \sin^{-1}\sqrt{x^2+1}$

L8. $|x-5| = x-5 \quad if\ x-5 \geq 0$

$= -(x-5) \quad if\ x < 5$

$= \int_{-1}^{5} -(x-5)dx + \int_{5}^{6}(x-5)dx$

$= \left[\frac{-x^2}{2} + 5x\right]_{-1}^{5} + \left[\frac{x^2}{2} - 5x\right]_{5}^{6}$

$= \left(-\frac{5^2}{2} + 5(5)\right) - \left(-\frac{(-1)^2}{2} + 5(-1)\right) + \left(\frac{6^2}{2} - 5(6)\right) - \left(\frac{5^2}{2} - 5(5)\right)$

$= -\frac{25}{2} + 25 + \frac{1}{2} + 5 + 18 - 30 - \frac{25}{2} + 25$

$= 43 - \frac{25}{2} - \frac{25}{2} + \frac{1}{2} = 43 - \frac{49}{2} = \frac{86}{2} - \frac{49}{2} = \frac{37}{2}$

L9. $u = \tan^{-1} x \qquad du = \frac{1}{1+x^2}dx$

$x = 0 \quad u = \tan^{-1} 0 = 0$

$x = 1 \quad u = \tan^{-1} 1 = \frac{\pi}{4}$

$= 2\int_{0}^{\frac{\pi}{4}} udu = 2\left[\frac{u^2}{2}\right]_{0}^{\frac{\pi}{4}} = 2\left[\frac{\left(\frac{\pi}{4}\right)^2}{2} - 0\right] = \frac{2\pi^2}{32} = \frac{\pi^2}{16}$

L10. $= \int_0^5 (\cos^2\sqrt{2x} + \sin^2\sqrt{2x})dx = \int_0^5 1dx = [x]_0^5$
$= 5 - (0) = 5$

L11. $\sqrt{49 - x^2}$ $\frac{1}{2}$ *circle* with radius 7

$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(7)^2 = \frac{49}{2}\pi$

L12. $u = \tan^{-1}(x^4)$ $du = \frac{(4x^3)}{1+(x^4)^2} = \frac{(4x^3)}{1+x^8}$
$x = 0$ $u = \tan^{-1} 0 = 0$
$x = 1$ $u = \tan^{-1} 1 = \frac{\pi}{4}$
$= \int_0^{\frac{\pi}{4}} udu = \left[\frac{u^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{\left(\frac{\pi}{4}\right)^2}{2} - 0 = \frac{\pi^2}{32}$

L13. $u = 1 + x^2$ $du = 2xdx$ $\frac{du}{2} = xdx$
$x = 1$ $u = 1 + 1^2 = 2$
$x = 2$ $u = 1 + 2^2 = 5$
$= \frac{1}{2}\int_2^5 \frac{1}{u} du = \frac{1}{2}[\ln u]_2^5 = \frac{1}{2}[\ln 5 - \ln 2] = \frac{1}{2}\ln(\frac{5}{2})$

L14. $= [\tan^{-1} x]_0^1$ $= \tan^{-1} 1 - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

L15. $F'(x) = \sin(0)(0) - \sin x(1)$
$F'(x) = -\sin x$
$F'\left(-\frac{\pi}{4}\right) = -\sin\left(-\frac{\pi}{4}\right) = -(-\sin\frac{\pi}{4}) = sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

L16. $u = \tan x$ $du = \sec^2 xdx$
$x = 0$ $u = \tan 0 = 0$
$x = \frac{\pi}{4}$ $u = \tan\frac{\pi}{4} = 1$
$= 7\int_0^1 u^2 du = 7\left[\frac{u^3}{3}\right]_0^1 = 7(\frac{1}{3} - 0) = \frac{7}{3}$

L17. $u = \ln x \qquad du = \frac{1}{x}dx$

$x = e \quad u = \ln e = 1$

$x = e^3 \quad u = \ln e^3 = 3\ln e = 3$

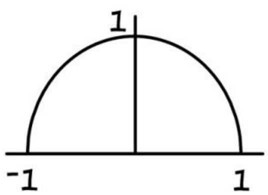
$= \int_1^3 u du = \left[\frac{u^2}{2}\right]_1^3 = \frac{3^2}{2} - \frac{1^2}{2} = \frac{8}{2} = 4$

L18. $= \cos^{-1}\sqrt{4x}(4) - \cos^{-1}\sqrt{x}(1)$

L19. $\sqrt{1-x^2}\ \frac{1}{2}\ circle \ \ (from - 1\ to\ 1) \qquad y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1 \qquad r = 1$



Between -1 and 0 is $\frac{1}{4}$ of the circle $\quad A = \frac{\pi r^2}{4} = \frac{\pi(1)^2}{4} = \frac{\pi}{4}$

L20. $u = \ln x \qquad du = \frac{1}{x}dx$

$x = 1 \quad u = \ln 1 = 0$

$x = 4 \quad u = \ln 4$

$= \int_0^{\ln 4} u du = \left[\frac{u^2}{2}\right]_0^{\ln 4} = \frac{(\ln 4)^2}{2} - 0 = \frac{(\ln 4)^2}{2}$

L21. $u = \cos^{-1} x \qquad du = \frac{-1}{\sqrt{1-x^2}}dx \qquad -du = \frac{1}{\sqrt{1-x^2}}dx$

$x = -1 \quad u = \cos^{-1}(-1) = \pi$

$x = 0 \quad u = \cos^{-1} 0 = \frac{\pi}{2}$

$= -\int_{\pi}^{\frac{\pi}{2}} u du = \int_{\frac{\pi}{2}}^{\pi} u du = \left[\frac{u^2}{2}\right]_{\frac{\pi}{2}}^{\pi} = \frac{(\pi)^2}{2} - \frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{\pi^2}{2} - \frac{\pi^2}{8} = \frac{4\pi^2 - \pi^2}{8} = \frac{3\pi^2}{8}$

L22. $u = x^2 \qquad du = 2xdx \qquad 3du = 6xdx$

$x = 0 \quad u = 0^3 = 0$

$x = 2 \quad u = 2^2 = 4$

$= 3\int_0^4 e^u du = [3e^u]_0^4 = 3e^4 - 3e^0 = 3e^4 - 3$

L23. $u = sinx \quad du = cosxdx$

$= \int \frac{1}{1+u^2} du = \tan^{-1} u + c = \tan^{-1}(sinx) + c$

L24. $u = \sin x \quad du = \cos x dx$

$x = \frac{\pi}{6} \quad u = \sin\frac{\pi}{6} = \frac{1}{2}$

$x = \frac{\pi}{2} \quad u = \sin\frac{\pi}{2} = 1$

$= \int_{\frac{1}{2}}^{1} u^4 du = \left[\frac{u^5}{5}\right]_{\frac{1}{2}}^{1} = \frac{1}{5} - \frac{(0.5)^5}{5} = \frac{1}{5} - \frac{1}{160} = \frac{32}{160} - \frac{1}{160} = \frac{31}{160}$

L25. $u = \ln x \quad du = \frac{1}{x} dx$

$x = 1 \quad u = \ln 1 = 0$

$x = e \quad u = \ln e = 1$

$= \int_0^1 u^5 du = \left[\frac{u^6}{6}\right]_0^1 = \frac{1}{6} - 0 = \frac{1}{6}$

L26. $u = 4 + e^x \quad du = e^x dx$

$= \int \sqrt{u} du = \int u^{\frac{1}{2}} du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \quad or \quad \frac{2u^{\frac{3}{2}}}{3} + c$

$= \frac{2(4+e^x)^{\frac{3}{2}}}{3} + c$

L27. $= \left[\frac{\tan 5x}{5}\right]_0^{\frac{\pi}{4}} = \frac{\tan\frac{5\pi}{4}}{5} - \frac{\tan 0}{5} = \frac{\tan\frac{5\pi}{4}}{5} = \frac{\tan\frac{\pi}{4}}{5} = \frac{1}{5}$

L28. $u = \sec x \quad du = \sec x \tan x \, dx$

$= \int \sec^4 x \sec x \tan x dx = \int u^4 du = \frac{u^5}{5} + c \;\; or \quad \frac{\sec^5 x}{5} + c$

L29. $= \frac{1}{n} \lim_{n\to\infty} \left[4n + \frac{n(n+1)}{2n} + \frac{n(n+1)(2n+1)}{6n}\right]$

$= \frac{1}{n} \lim_{n\to\infty} \left[\frac{4n}{1} + \frac{n+1}{2} + \frac{2n^2+3n+1}{6}\right]$

$= \frac{1}{n} \lim_{n\to\infty} \left[\frac{24n+3n+3+2n^2+3n+1}{6}\right] = \lim_{n\to\infty} \left[\frac{2n^2+30n+4}{6n}\right]$

$= \lim_{n\to\infty} \left[2n + 5 + \frac{4}{6n}\right] = 2(\infty) + 5 + 0 = \infty$

L30. $\Delta x = \frac{b-a}{n} = \frac{6}{n} \quad \therefore b - a = 6$

$f(xi) = f(a + i\Delta x) = f(a + \frac{6i}{n})$

$let\ a = 1 \quad b - 1 = 6 \quad b = 7$

$\therefore \int_1^7 \sqrt{x}dx$

L31. $\int \frac{6x^2}{1+x^3}\ dx$

$u = 1 + x^3$

$du = 3x^2 dx$

$2du = 6x^2 dx$

$2\int \frac{1}{u} du = 2\ln|u| + c$

or $2\ln|1 + x^3| + c$

L32. $\int \frac{\sin\sqrt{x}}{3\sqrt{x}}\ dx$

$u = \sqrt{x} = x^{\frac{1}{2}}$

$du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$

$2du = \frac{1}{\sqrt{x}}\ dx$

$= \frac{1}{3}\int \frac{\sin\sqrt{x}}{\sqrt{x}}\ dx$

$= \frac{2}{3}\int \sin u\ \ du$

$= \frac{2}{3}[-\cos u] + c$

$= -\frac{2}{3}\cos\sqrt{x} + c$

L33.

$$\int x^2 e^{x^3} dx$$

$$u = x^3$$

$$du = 3x^2 dx$$

$$\frac{1}{3} du = x^2 dx$$

$$= \frac{1}{3} \int e^u du$$

$$= \frac{1}{3} e^u + c$$

$$= \frac{1}{3} e^{x^3} + c$$

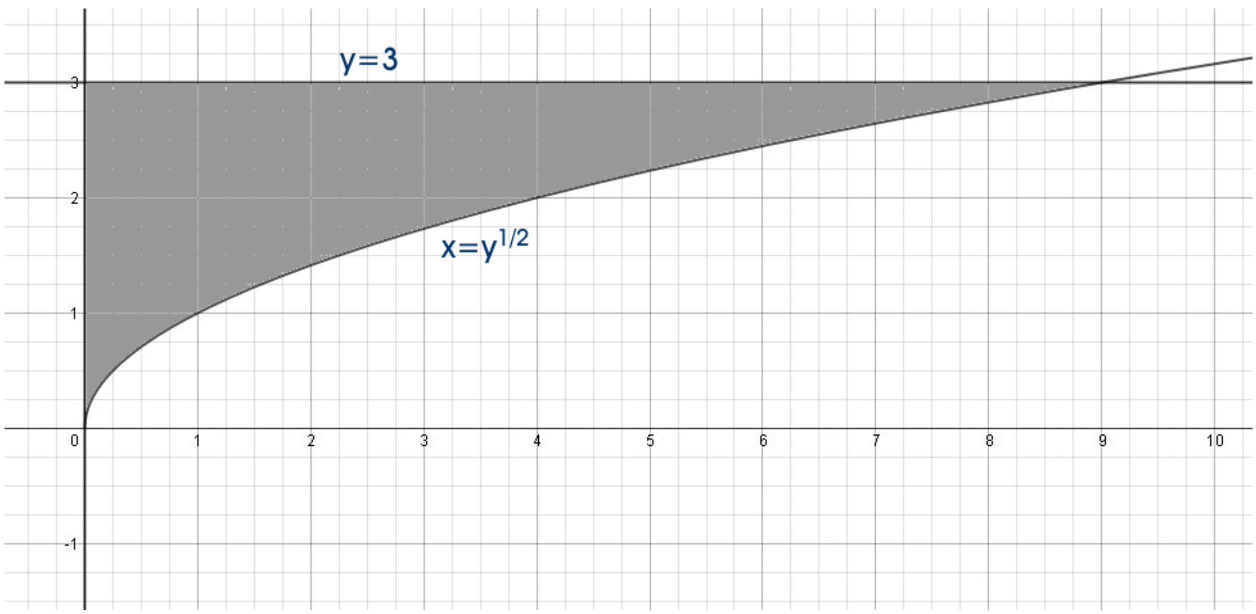
M. Area Between Curves

Example 1.

$$A = \int_a^b (top - bottom)dx$$

$$A = \int_0^9 (3 - \sqrt{x})dx = \left[3x - \frac{2}{3}x^{3/2}\right]_0^9 = \left(27 - \frac{2}{3}(9^{\frac{3}{2}}\right) - (0 - 0) =$$

$$27 - \frac{2}{3}(27) = \frac{81}{3} - \frac{54}{3} = 9$$



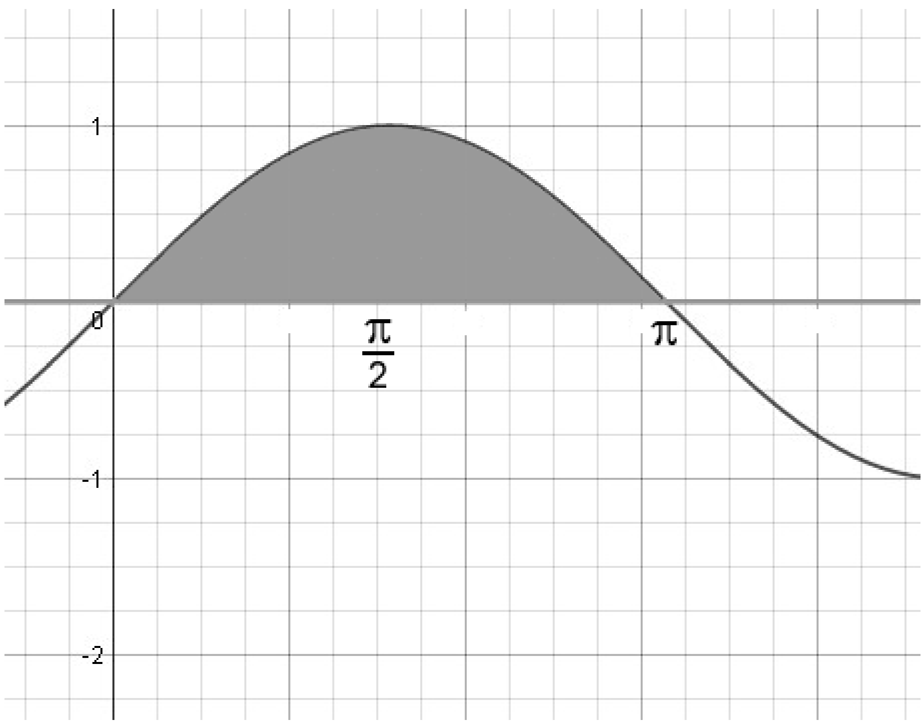
or use top and bottom with 0 to 9 for x-values

Example 2.

$$A = \int_a^b (top - bottom)dx$$

$$A = \int_0^\pi (\sin x - 0)dx = [-\cos x]_0^\pi$$

$$= -\cos\pi + \cos 0 = 1 + 1 = 2$$



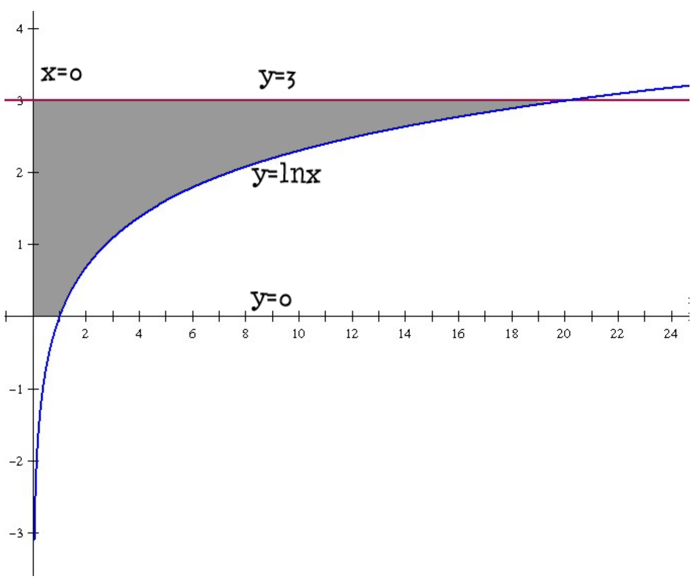
Example 3. Find the area of the region bounded by $y = lnx,\ y = 3, the\ x - axis\ and\ the\ y = axis.$

Point of Intersection $3 = \ln x \qquad x = e^3$

$A = \int_0^3 (e^y - 0)dy$

$= \int_0^3 e^y dy$

$A = [e^y]_0^3 = e^3 - e^0 = e^3 - 1$



Example 4.

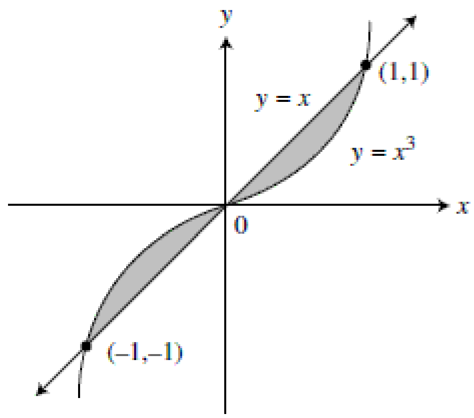


Figure 12.6-2

$$A = 2\int_0^1 (x - x^3)dx \quad using\ symmetry$$

$$= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$$

Example 5.

$x = 2y^2 - 2$

x	y
-2	0
0	1
0	-1

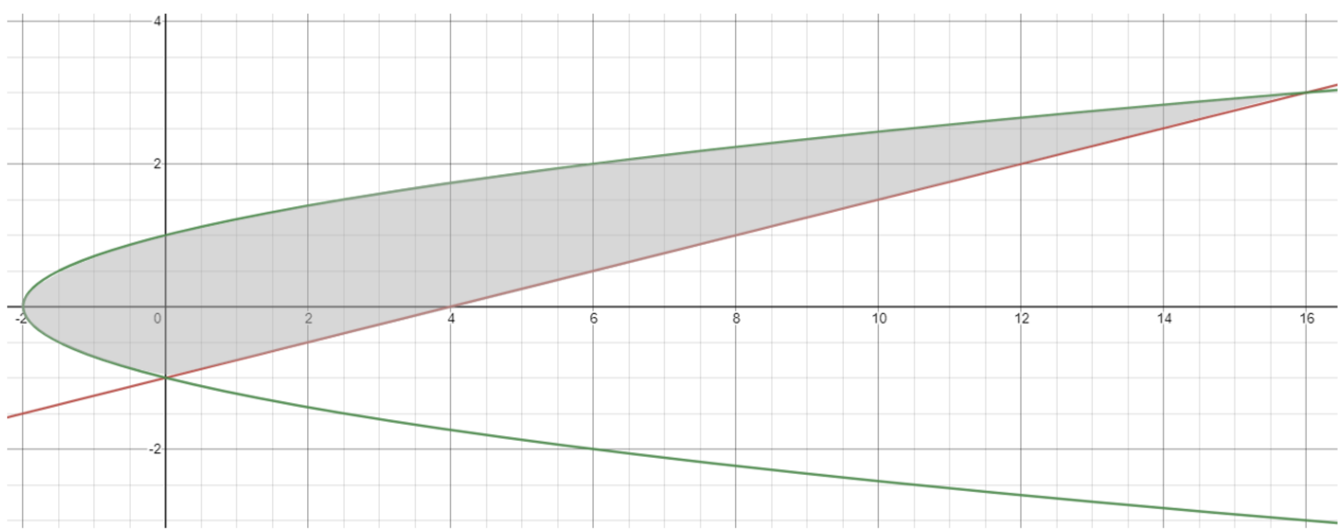
Point of intersection (Set x=x)

$4y + 4 = 2y^2 - 2$

$2(y^2 - 2y - 3) = 0$ factor

$(y - 3)(y + 1) = 0$

$y = 3, -1$



$$A = \int_{-1}^{3} (4y + 4) - (2y^2 - 2)dy = \int_{-1}^{3} (-2y^2 + 4y + 6)dy$$

Practice Exam Questions on Area between Curves

M1. $0.5y^2$ $\therefore$ use right and left

$0.5y^2 - 1 = y + 3$

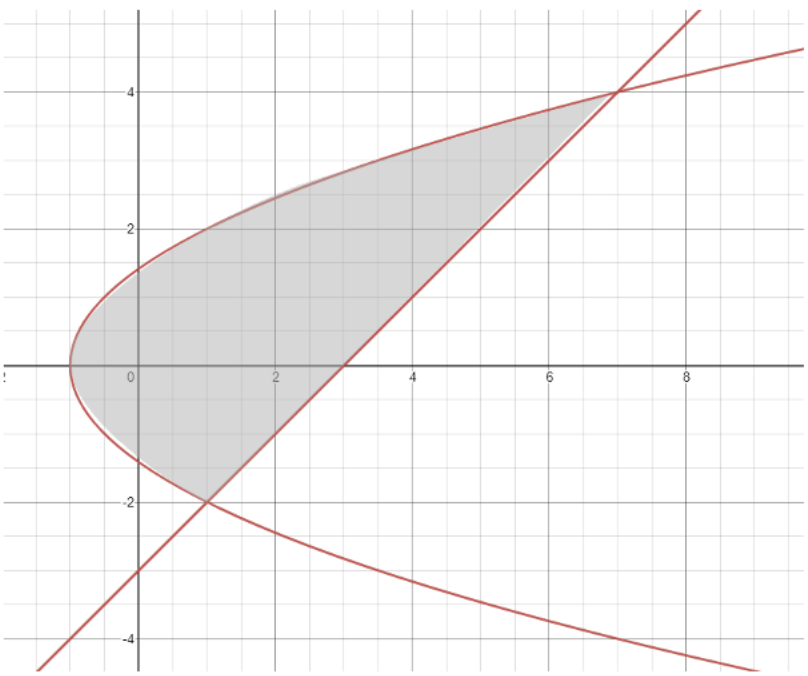
y^2-2y-8=0

(y-4)(y+2)=0

y=4, -2

$$A = \int_{-2}^{4} \left[\left(\frac{1}{2}y^2 - 1\right) - (y + 3)\right] dy$$

$$= \int_{-2}^{4} \left[\left(\frac{1}{2}y^2 - y - 3\right)\right] dy$$



M2.

Point of intersection

$$(y-1)^2 = -y^2 + 1$$

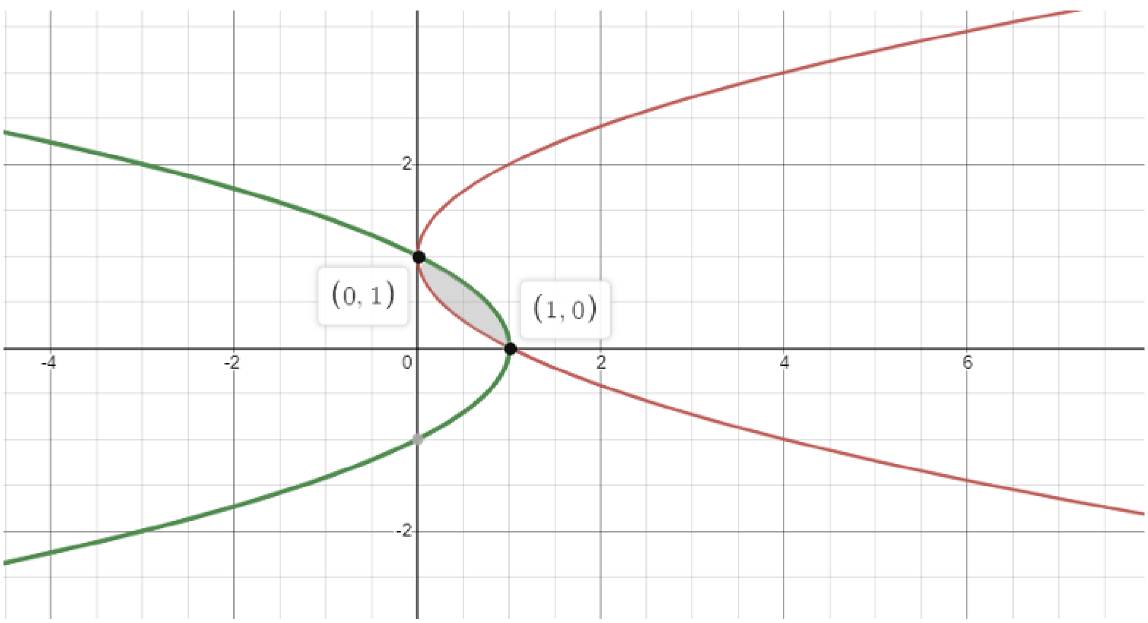
$$y^2 - 2y + 1 = -y^2 + 1$$

$$2y^2 - 2y = 0$$

$$2y(y-1) = 0$$

$$y = 0,1$$

$A = \int_0^1 [(y-1)^2 - (-y^2+1)]dy = \int_0^1 (y^2 - 2y + 1 + y^2 - 1)dy =$
$\int_0^1 (2y^2 - 2y)dy$



A=$\left(-2/3\left(3^3 + 2(3^2) + 6(3)\right) - \left(-2/3(-1)^3 + 2(-1)^2 + 6(-1)\right)\right)$

$= -\frac{2}{3}(27) + 18 + 18 - \left(\frac{2}{3} + 2 - 6\right) = -\frac{54}{3} + 36 - \frac{2}{3} + 4 = -\frac{56}{3} + 40$

$= -\frac{56}{3} + \frac{120}{3} = \frac{64}{3}$

M3. Intersection $x^2 = 2x - x^2$

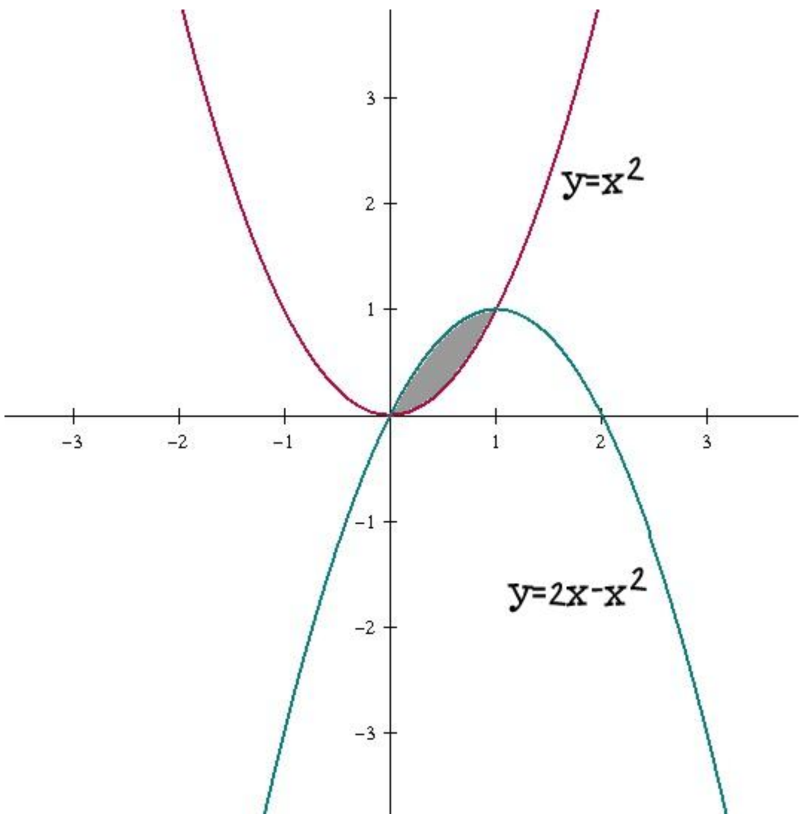
$$2x^2 - 2x = 0$$

$$2x(x-1) = 0 \quad x = 0,\ 1$$

$$A = \int_0^1 [(2x - x^2) - (x^2)]dx$$

$$= \left[x^2 - \frac{x^3}{3} - \frac{x^3}{3}\right]_0^1 = \left[x^2 - \frac{2x^3}{3}\right]_0^1$$

$$= \left(1 - \frac{2}{3}\right) - (0 - 0) = \frac{1}{3}$$



M4. $y = \sec^2 x \qquad y = 2$

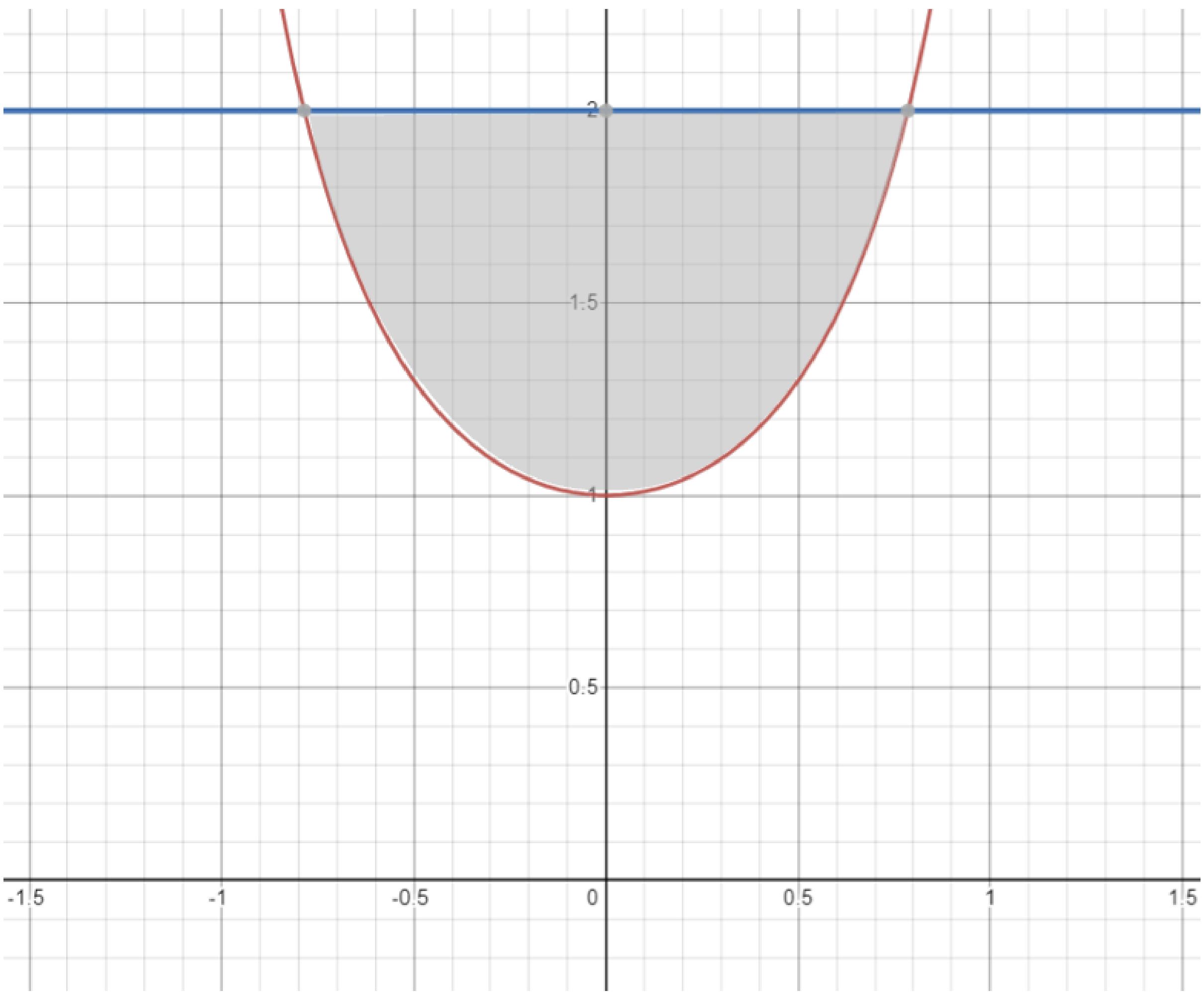
$$x = -\frac{\pi}{4} \quad to \quad \frac{\pi}{4}$$

$$A = \int_{-\pi/4}^{\pi/4} (2 - \sec^2 x)dx \quad or \quad 2\int_0^{\pi/4} (2 - \sec^2 x)dx$$

$$= 2[2x - \tan x]_0^{\frac{\pi}{4}}$$

$$= 2\left[\left(\frac{2\pi}{4} - \tan\frac{\pi}{4}\right) - (2(0) - \tan 0)\right]$$

$$= \pi - 2$$



M5. point of intersection $\sqrt{x} = \frac{x}{2}$

$$(2\sqrt{x})^2 = (x)^2$$

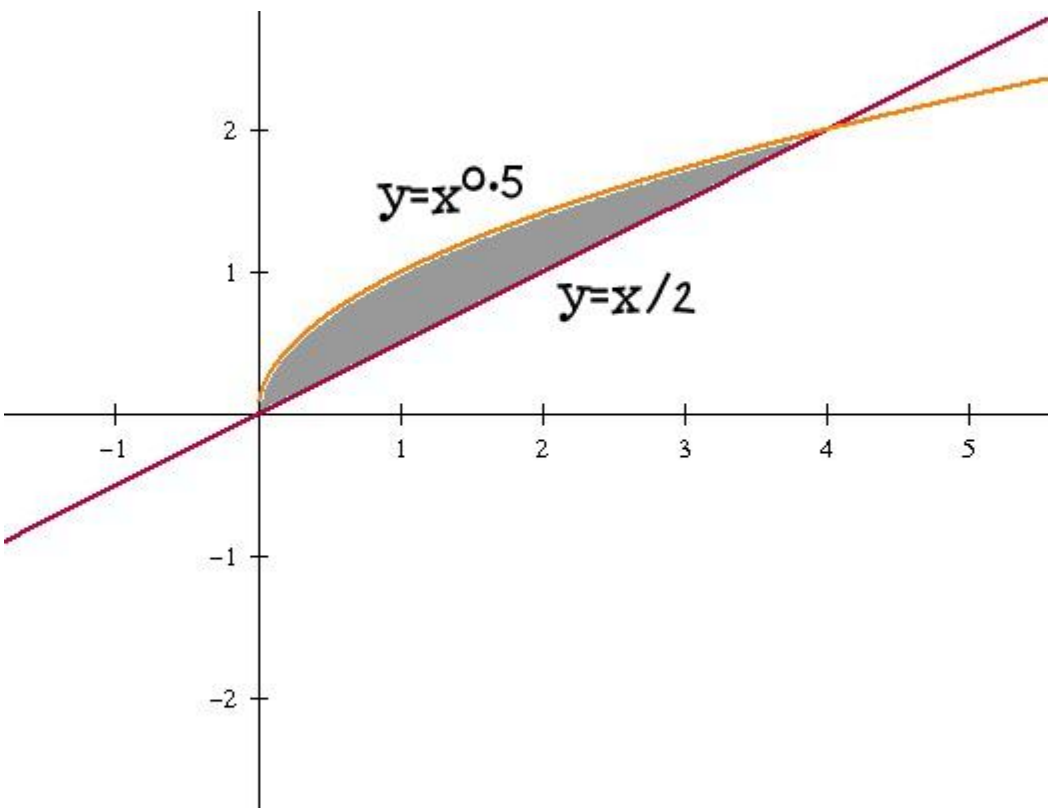
$$4x = x^2$$

$$0 = x^2 - 4x$$

$$0 = x(x-4) \quad x = 0,4$$

$$A = \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{4}\right]_0^4 = \left(\frac{2}{3}(4)^{\frac{3}{2}} - \frac{4^2}{4} - (0) = \frac{16}{3} - \frac{4}{1} = \frac{16-12}{3} = \frac{4}{3}$$



M6. point of intersection

$e^x = e^{3x}$

$0 = e^{3x} - e^x$

$0 = e^x(e^{2x} - 1)$

$e^x = 0 \ \ no\ solution$

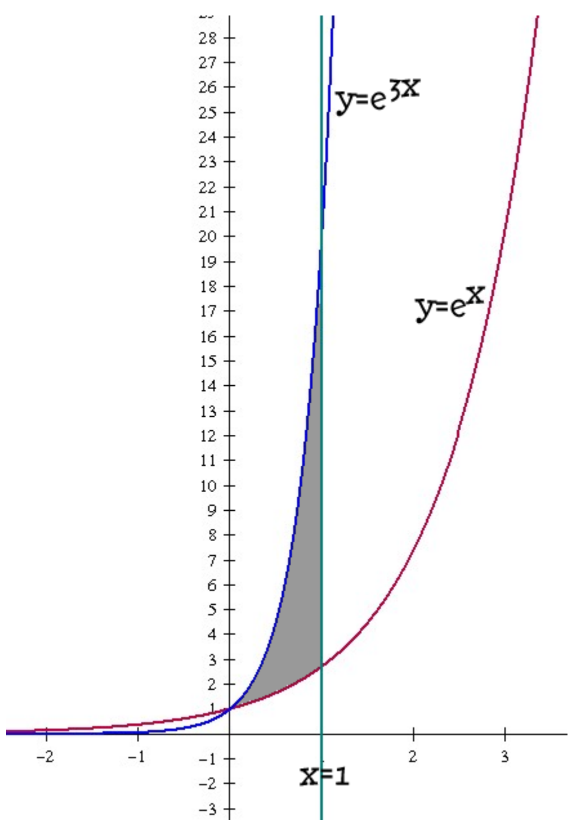
$e^{2x} = 1$

$lne^{2x} = ln1$

$x = 0$

$$A = \int_0^1 (e^{3x} - e^x)dx = \left[\frac{e^{3x}}{3} - e^x\right]_0^1 = \left(\frac{e^3}{3} - e^1\right) - \left(\frac{e^0}{3} - e^0\right)$$

$$= \frac{e^3}{3} - e - \frac{1}{3} + 1 = \frac{e^3}{3} - e + \frac{2}{3}$$



M7. point of intersection $8 = e^x$

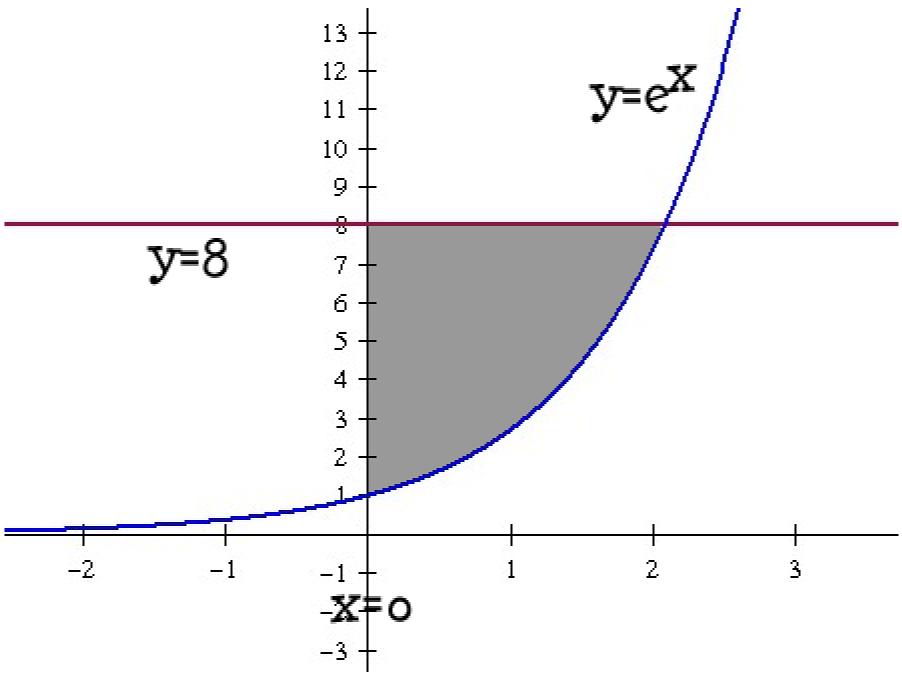
$$\ln 8 = \ln e^x$$

$$x = \ln 8$$

$$A = \int_0^{\ln 8} (8 - e^x)dx$$

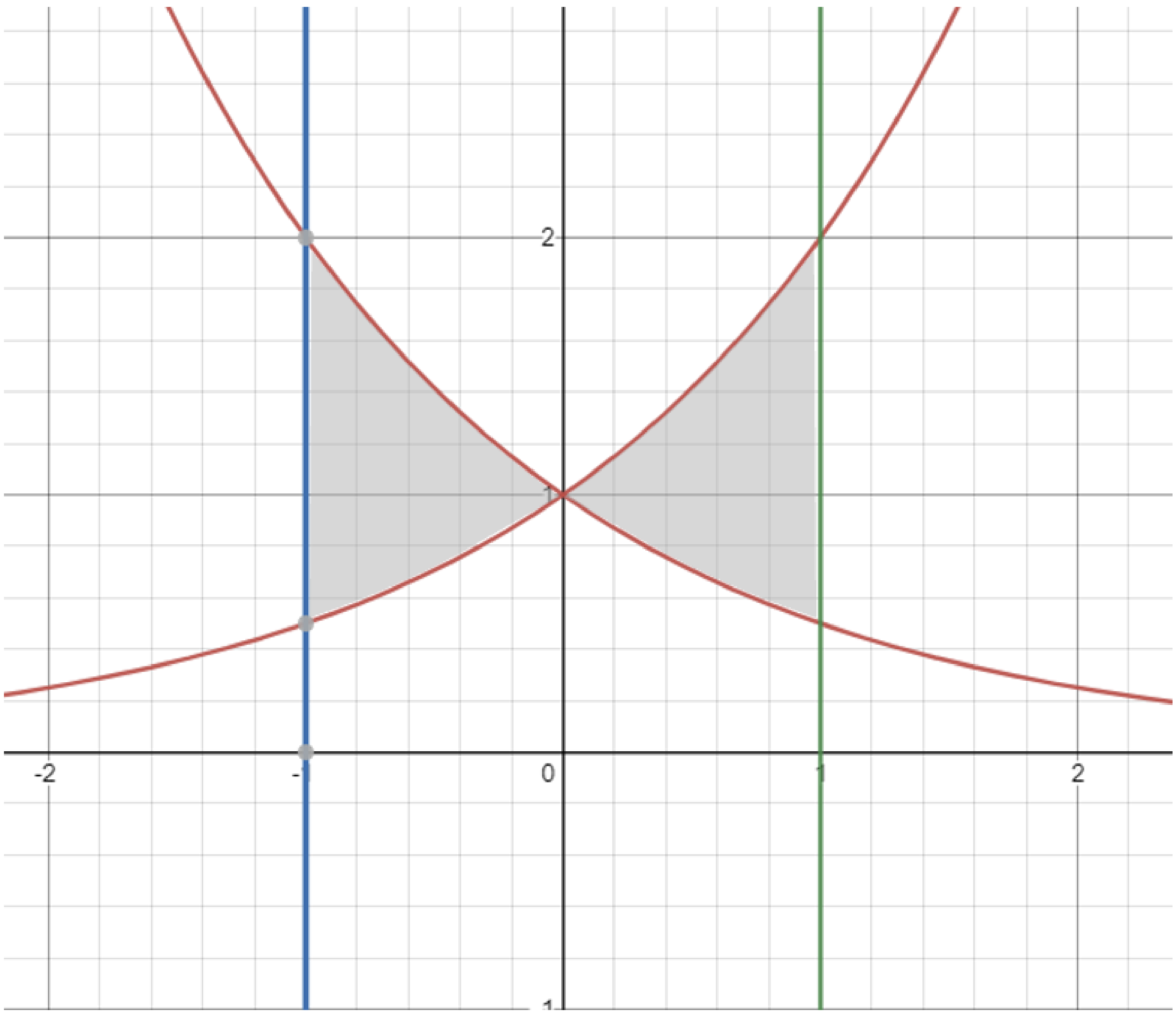
$$= [8x - e^x]_0^{\ln 8}$$

$$= (8\ln 8 - e^{\ln 8}) - (0 - e^0) = 8\ln 8 - 8 + 1 = 8\ln 8 - 7$$

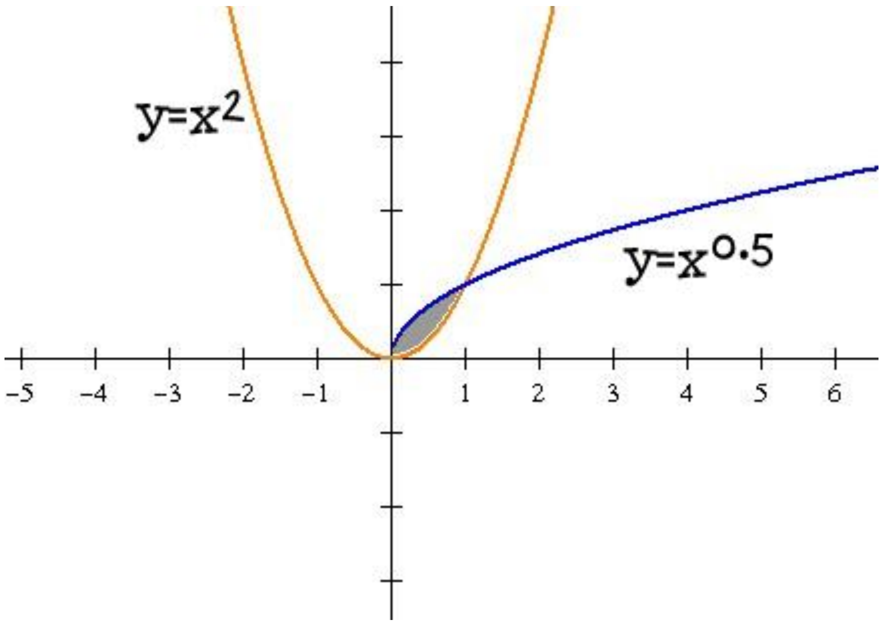


M8. $y = 2^x, \quad y = 2^{-x} \quad from\ -1\ to\ 1$

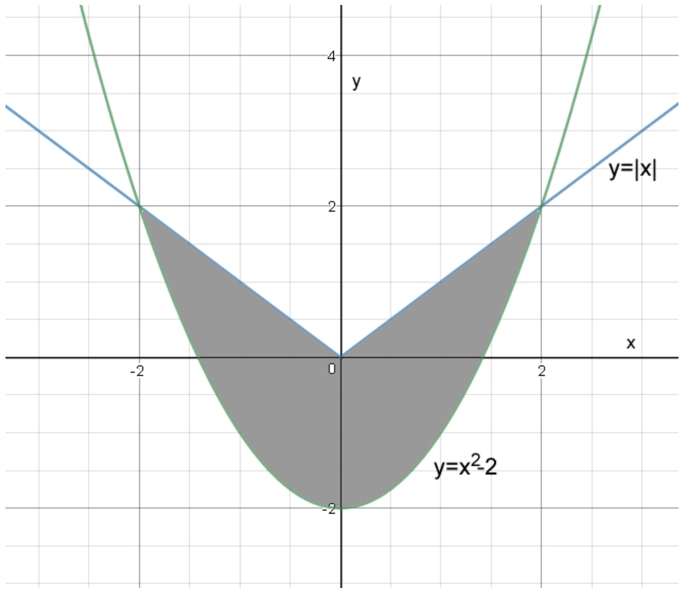
Set up only $A = 2\int_0^1 (2^x - 2^{-x})dx$ since it is symmetrical



M9. A=$\int_0^1(\sqrt{x}-x^2)dx=\left[\frac{2x^{3/2}}{3}-\frac{x^3}{3}\right]_0^1=\frac{2}{3}(1)^{3/2}-\frac{1}{3}-(0)=\frac{1}{3}$



M10.



$y = |x|$

x	y
0	0
1	1
-1	1
2	2

$y = x^2 - 2$

x	y
0	-2
1	-1
-1	-1
2	2

Absolute value of x is just x here since x is positive

$$A = 2\int_0^2 [x - (x^2 - 2)]dx$$

$$= 2\int_0^2 (x - x^2 + 2)dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_0^2$$

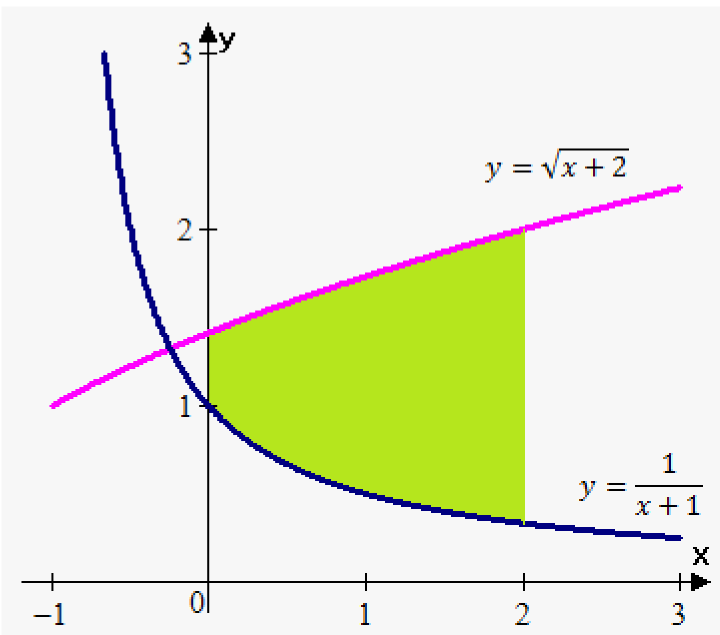
$$= 2\left[\left(\frac{2^2}{2} - \frac{2^3}{3} + 2(2)\right) - 0\right]$$

$$= 2(2 - \frac{8}{3} + 4)$$

$$= 2\left(6 - \frac{8}{3}\right) = 2\left(\frac{18}{3} - \frac{8}{3}\right) = 2\left(\frac{10}{3}\right) = \frac{20}{3}$$

M11.

$$A = \int_0^2 \left(\sqrt{x+2} - \frac{1}{x+1}\right) dx$$
$$= \left[\frac{2}{3}(x+2)^{3/2} - ln|x+1|\right]_0^2$$
$$= \left[\frac{2}{3}\left(4^{3/2}\right) - ln3\right] - \left[\frac{2}{3}(2)^{3/2} - ln1\right]$$
$$= \frac{2}{3}(8) - ln3 - \frac{2}{3}\sqrt{8}$$
$$= \frac{16}{3} - ln3 - \frac{2}{3}\sqrt{8}$$



M12.

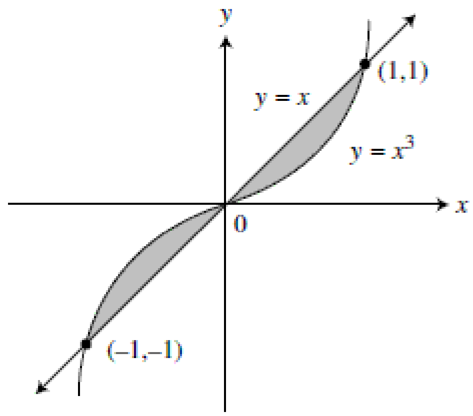


Figure 12.6-2

$$A = 2\int_0^1 (x - x^3)dx \quad using\ symmetry$$

$$= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$$

M13. $y = \cos 2x, \quad y = \sin x \quad from\ x = 0,\ \frac{\pi}{4}$

$$A = \int_0^{\pi/6}(\cos 2x - \sin x)dx + \int_{\pi/6}^{\pi/4}(\sin x - \cos 2x)dx$$

$$= \left[\frac{\sin 2x}{2} + \cos x\right]_0^{\frac{\pi}{6}} + \left[-\cos x - \frac{\sin 2x}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \left(\frac{\sin\frac{\pi}{3}}{2} + \cos\frac{\pi}{6}\right) - (\sin 0 + \cos 0) + \left(-\cos\frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2}\right) - \left(-\cos\frac{\pi}{4} - \frac{\sin\frac{\pi}{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} - 1 - \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{4}$$

$$= \frac{2\sqrt{3}}{4} + \frac{\sqrt{3}}{2} - \frac{3}{2}$$

$y = \cos 2x$

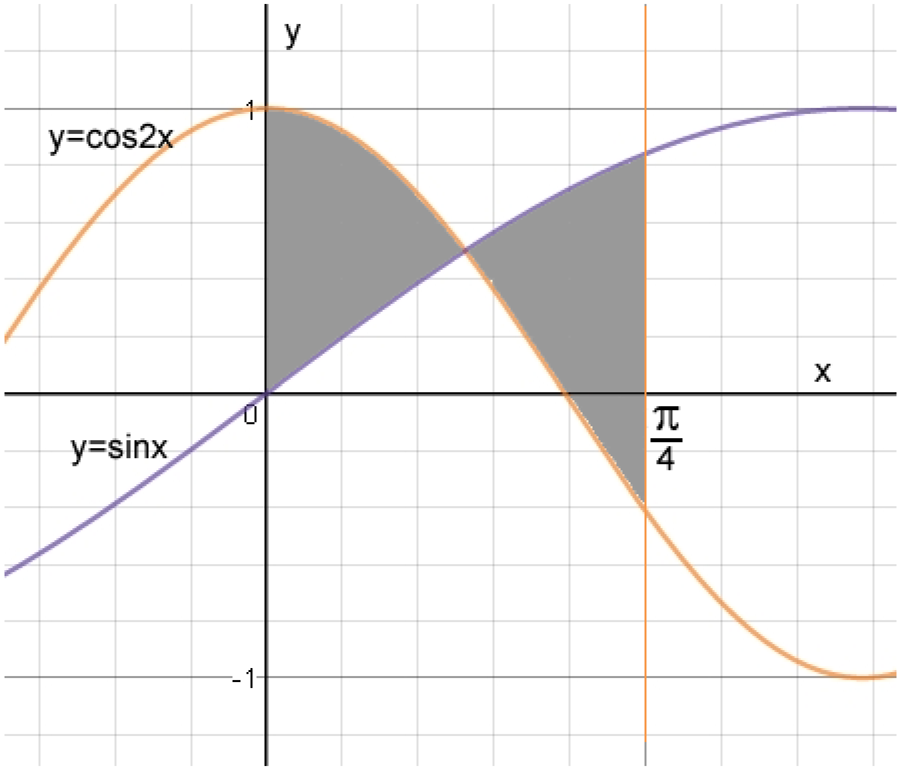
x	y
0	1
$\frac{\pi}{2}$	$\cos\pi = -1$
$\frac{\pi}{4}$	$\cos\frac{\pi}{2} = 0$
$\frac{\pi}{6}$	$\cos\frac{\pi}{3} = \frac{1}{2}$

↑

Point of intersection

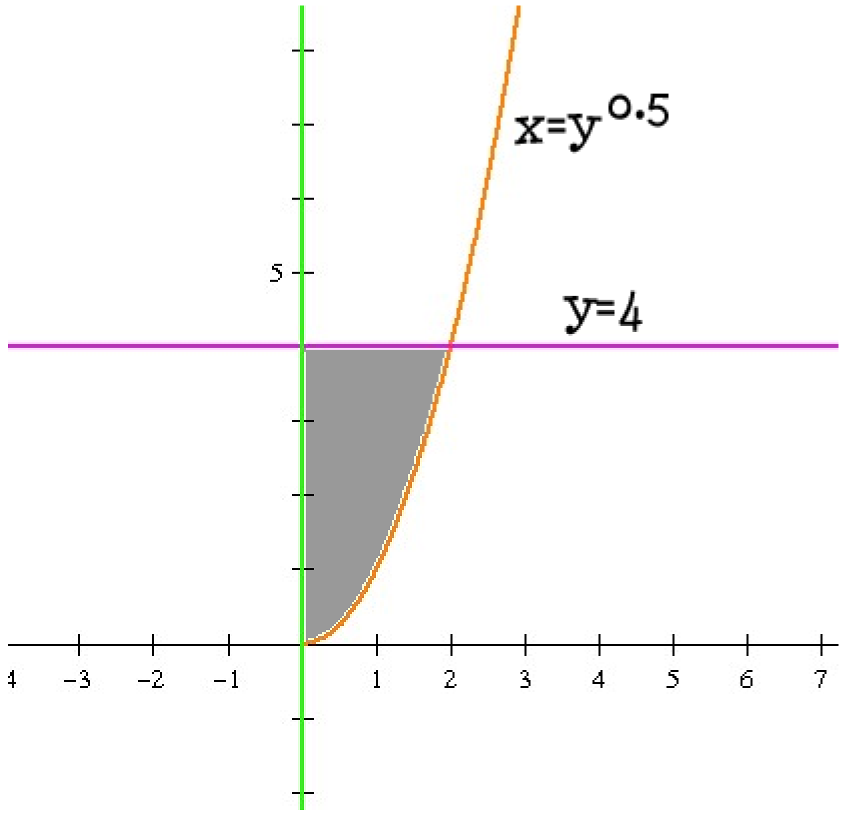
$y = \sin x$

x	y
0	0
$\frac{\pi}{2}$	1
$\frac{\pi}{6}$	$\frac{1}{2}$



y
x
1
-1
0
y=cos2x
y=sinx
π/4

M14.



$$A = \int top - bottom$$

$$A = \int_0^2 (4 - x^2)dx$$

OR

$$A = \int right - left$$

$$A = \int_0^4 (\sqrt{y} - 0)dy$$

Point of intersection

$$x = \sqrt{y} \; and \; y = 4$$

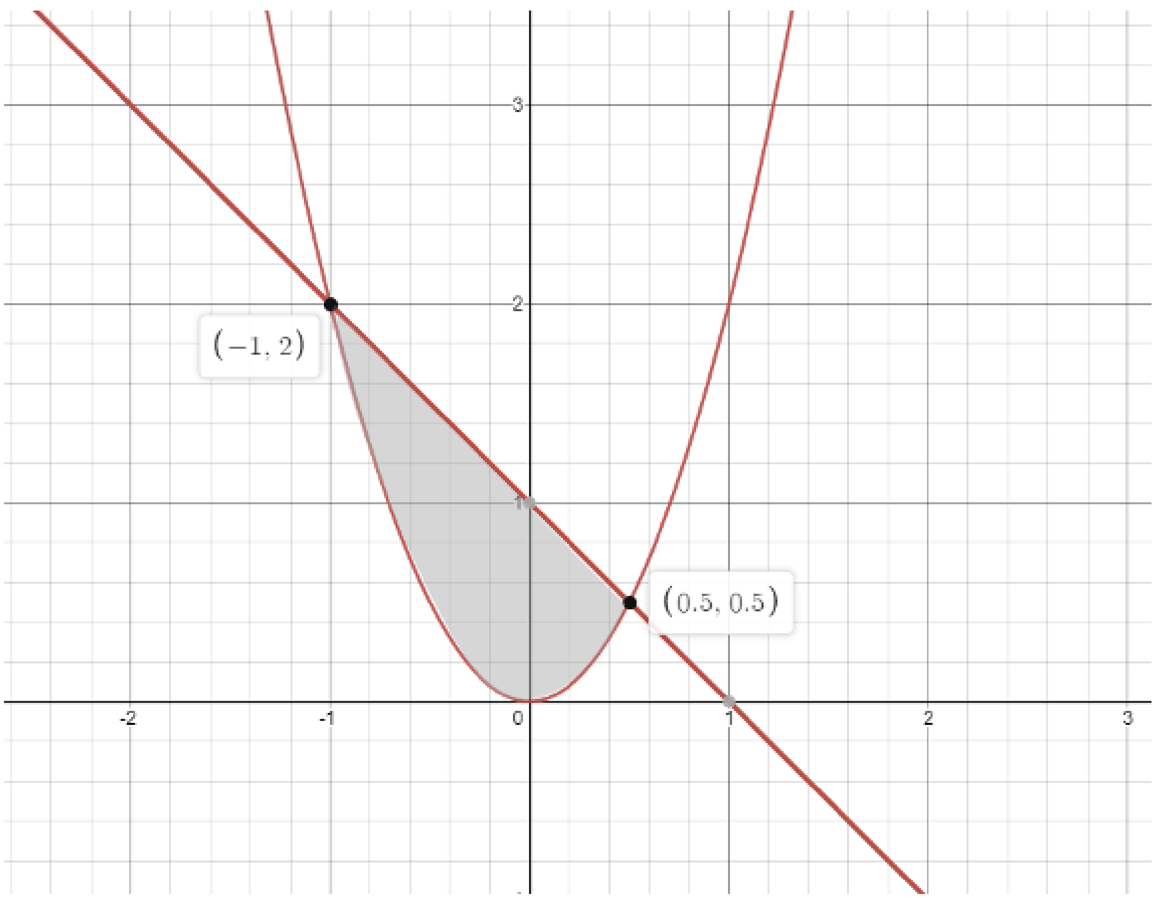
$$\therefore x = \sqrt{4}$$

$$A = \left[4x - \frac{x^3}{3}\right]_0^2 = 4(2) - \frac{2^3}{3} = \frac{8}{1} - \frac{8}{3} = \frac{24-8}{3} = \frac{16}{3}$$

M15. *Point of Intersection*

$$2x^2 = -x + 1$$
$$2x^2 + x - 1 = 0$$
$$(2x-1)(x+1) = 0$$
$$x = -1, \frac{1}{2}$$

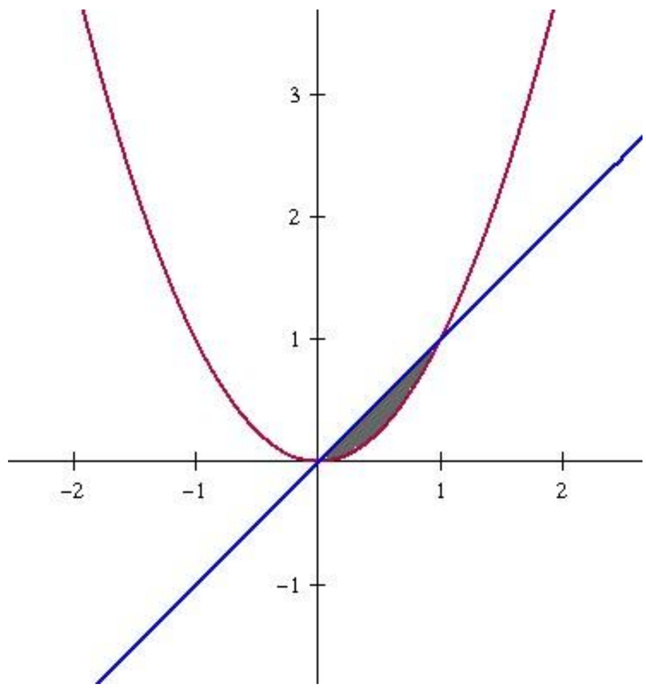
$$A = \int top - bottom = \int_{-1}^{1/2} [(-x+1) - 2x^2]dx = \int_{-1}^{1/2} (-2x^2 - x + 1)dx$$



M16. Find the area of the region bounded by $y = x^2$ and $y = x$.

$A = \int_0^1 (x - x^2) dx \quad x - values$

$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$



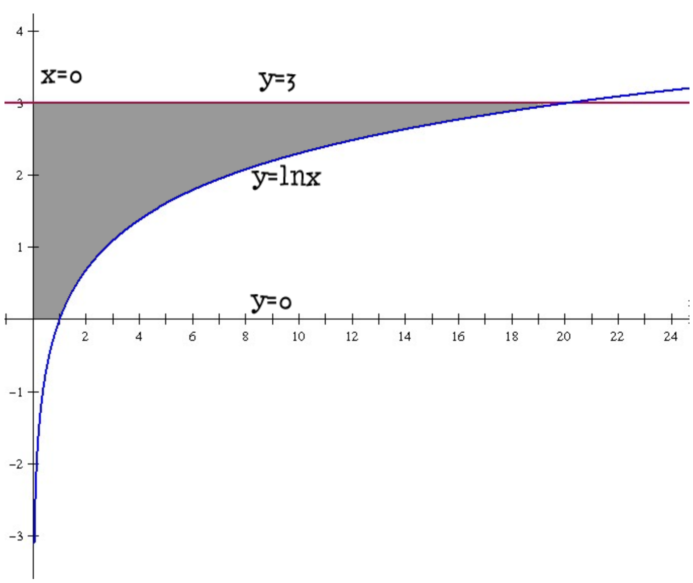
M17. Find the area of the region bounded by $y = lnx,\ y = 3, the\ x - axis\ and$ $the\ y - axis$.

Point of Intersection $3 = \ln x \qquad x = e^3$

$A = \int_0^3 (e^y - 0)dy$

$= \int_0^3 e^y dy$

$A = [e^y]_0^3 = e^3 - e^0 = e^3 - 1$



M18. Find the area of the region bounded by $y = e^x \; and \; y = e^{-x} \; from$

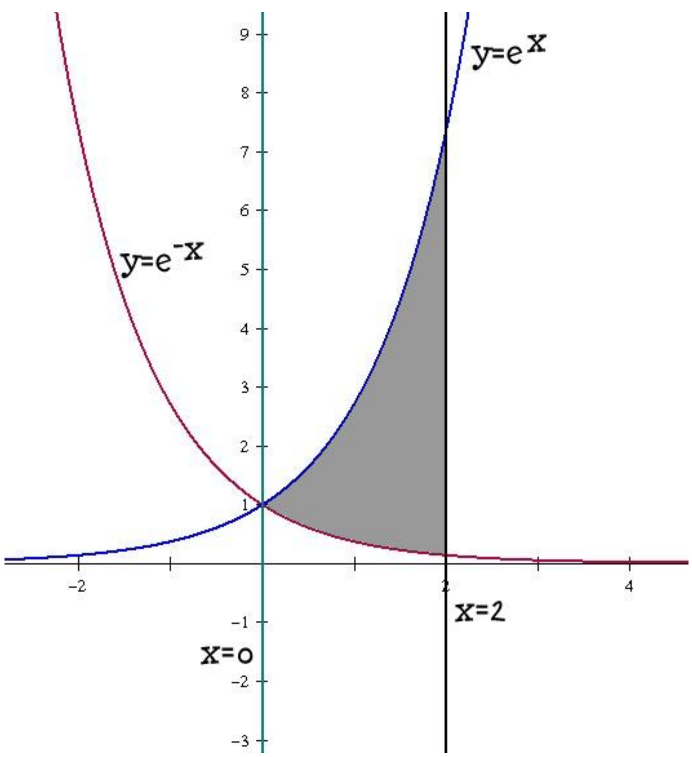
$x = 0 \; to \; x = 2.$

$A = \int_0^2 (e^x - e^{-x})dx$

$= [e^x + e^{-x}]_0^2$

$= (e^2 + e^{-2}) - (e^0 + e^0)$

$= e^2 + \frac{1}{e^2} - 2$



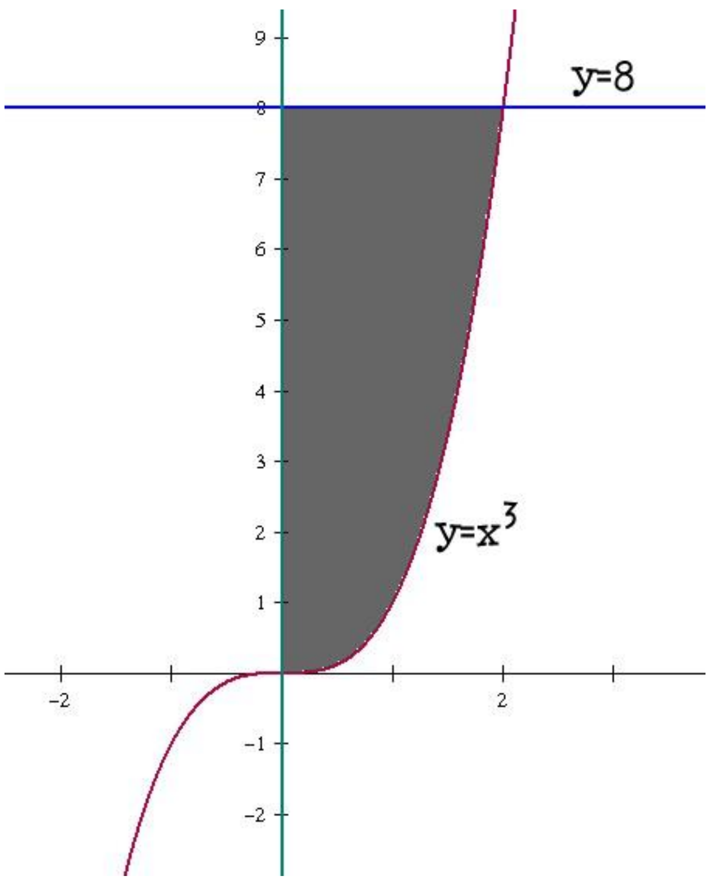
M19. Set up the integral representing the area of the region bounded by $y = x^3$, $y = 8 \; and \; x = 0$.

Point of Intersection $x^3 = 8 \quad x = 2$

$$A = \int_0^8 \left(y^{\frac{1}{3}} - 0\right) dy \quad right - left$$

$$= \left[\frac{3}{4} y^{\frac{4}{3}}\right]_0^8 = \frac{3}{4}(8)^{\frac{4}{3}} - 0 = \frac{3}{4}(16) = 12$$

OR $\int_0^2 (8 - x^3) dx = \left[8x - \frac{x^4}{4}\right]_0^2 = 16 - 4 = 12 \quad top - bottom$



M20. y=x and y^2=x + 6.

$$y^2 = x + 6 \quad \text{or} \quad x = y^2 - 6$$

x	y
-6	0
-2	2
-2	-2
3	3

$A = \int_{-2}^{3}(y) - (y^2 - 6)dy = \int_{-2}^{3}(-y^2 + y + 6)dy$

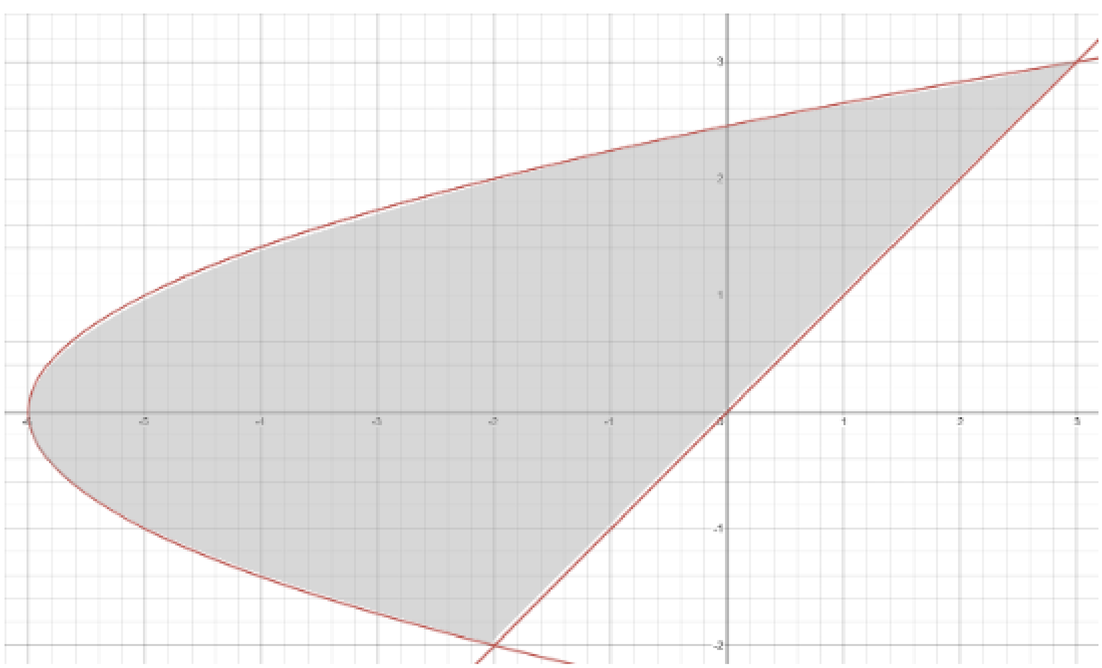
find point of intersection, set x=x

$y = y^2 - 6$

$y^2 - y - 6 = 0$

$0 = (y - 3)(y + 2)$

$y = 3, -2$



M21. Find the area bounded by the curves $x = y^2 + 3y$ *and*

$y = x - 3$

x	y
3	0
0	-3

$x = y^2 + 3y = 0$

$y(y + 3) = 0$

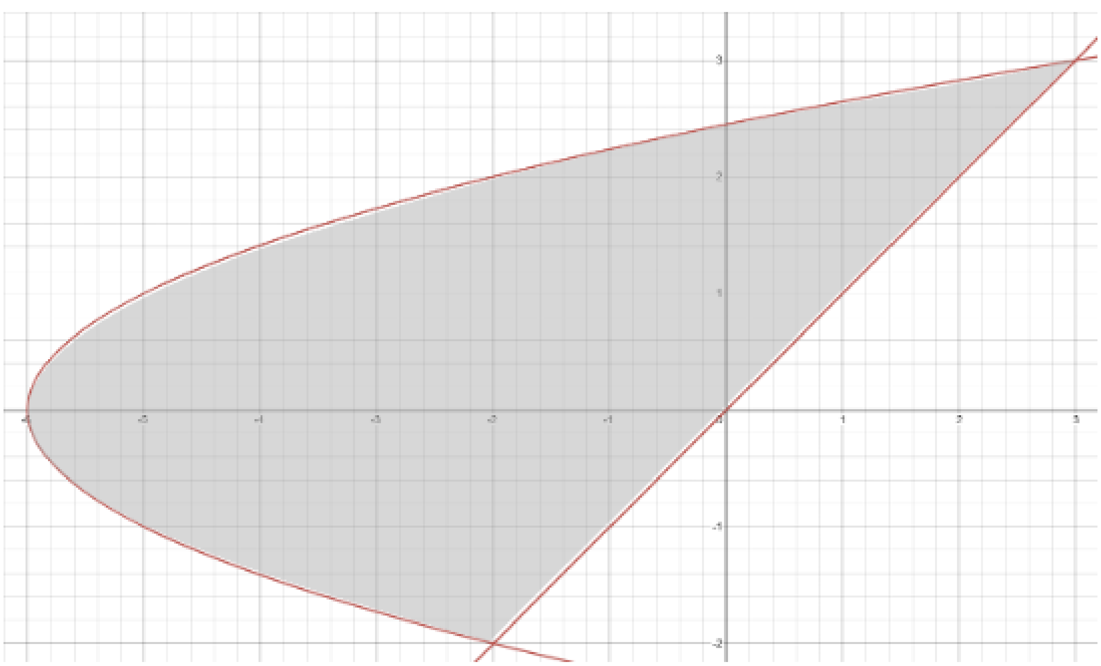
$y = 0, -3$ $\qquad$ $A = \int_{-3}^{1}[(y + 3) - (y^2 + 3y]dy = \int_{-3}^{1}(-y^2 - 2y + 3)dy$

Point of intersection $\quad y^2 + 3y = y + 3$

$y^2 + 2y - 3 = 0$

$(y + 3)(y - 1) = 0$

$y = -3, 1$



M22.

$$\sin x = \cos x$$

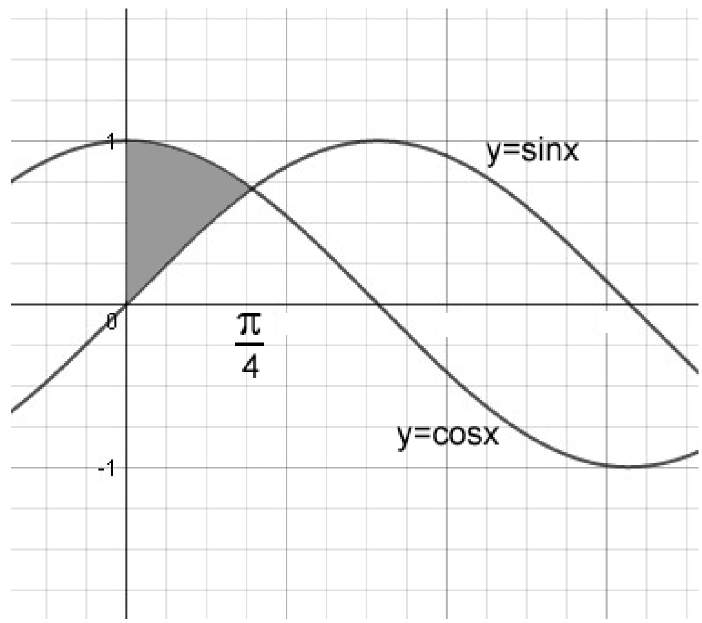
$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

$$x = \frac{\pi}{4} \quad from\ special\ \Delta's$$

$$A = \int_a^b (top - bottom)dx$$

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx$$



M23. It is bounded by the x-axis

$y = x - 6 \quad or \quad x = y + 6$

$y = \sqrt{x} \quad or \quad x = y^2$

Point of intersection $\quad y = y$

$$(\sqrt{x})^2 = (x-6)^2$$

$$x = x^2 - 12x + 36$$

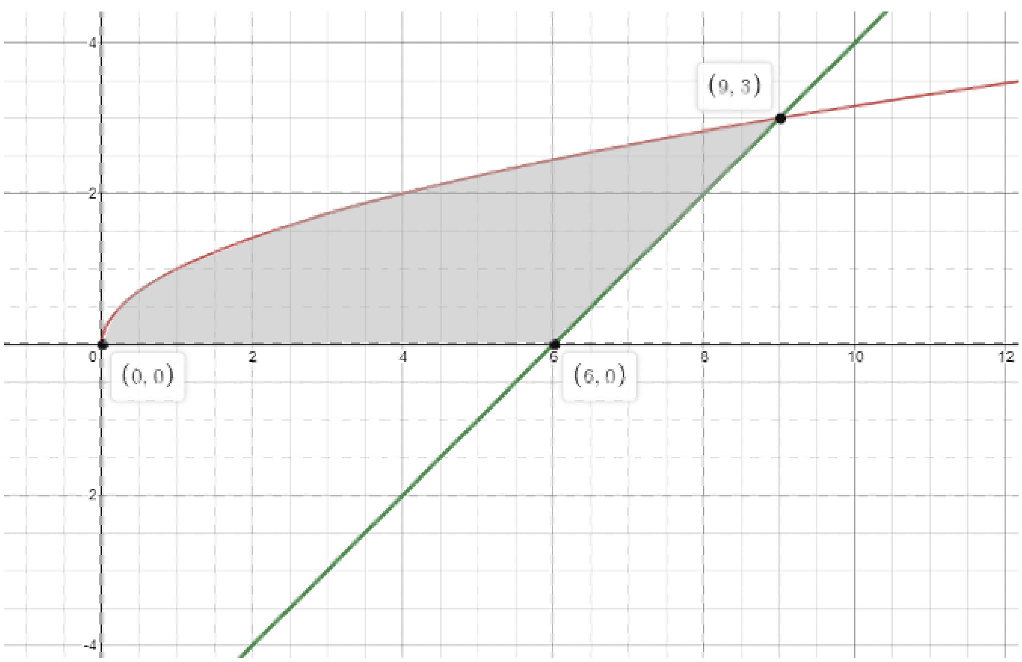
$$0 = x^2 - 13x + 36$$

$$0 = (x-4)(x-9)$$

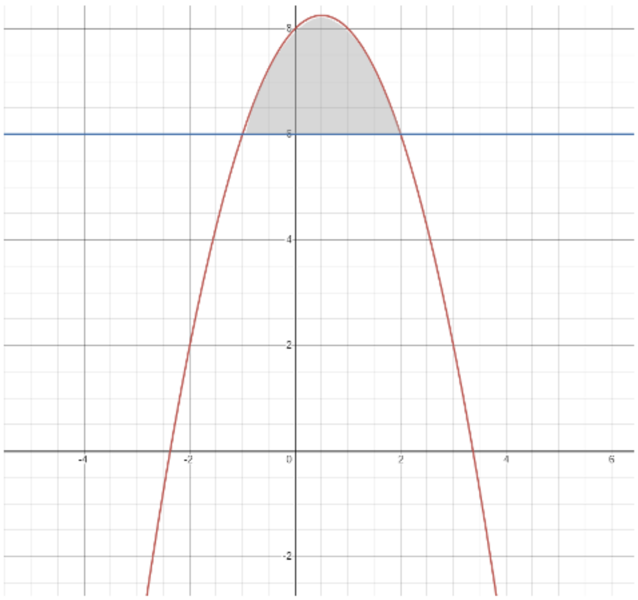
$$x = 4,9 \qquad y = \sqrt{4} = 2$$

$A = \int_0^3 \begin{bmatrix} right - left \\ (y+6) - y^2 \end{bmatrix} dy$ or do $\int_0^9 \begin{bmatrix} top - bottom \\ (\sqrt{x}) - (x-6)^2 \end{bmatrix} dy$

$= \int_0^3 (-y^2 + y + 6) dy$



M24.



Point of intersection

$-x^2 + x + 8 = 6$

$0 = x^2 - x - 2$

$0 = (x-2)(x+1)$

$x = 2, -1$

$A = \int_{-1}^{2}[(-x^2 + x + 8) - 6]dx = \int_{-1}^{2}(-x^2 + x + 2)dx$

$= \left[\frac{-x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2} = \left(\frac{-8}{3} + \frac{4}{2} + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$

$= 6 - \frac{8}{3} - \frac{1}{3} - \frac{1}{2} + 2 = 8 - 3 - \frac{1}{2}$

$= 5 - \frac{1}{2} = \frac{9}{2}$

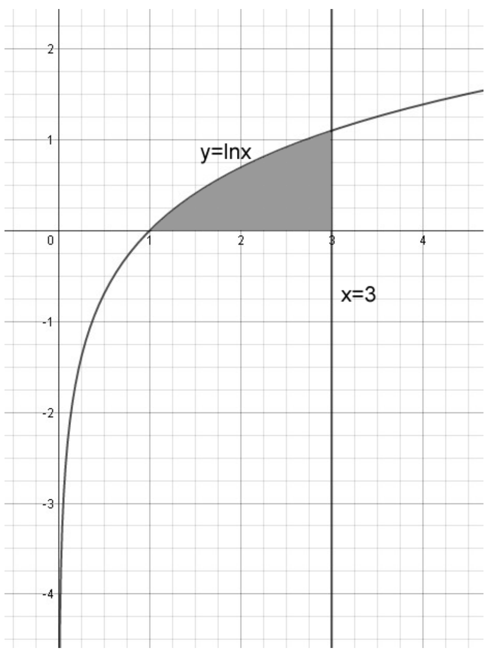
M25.

$$y = \ln x \quad intersects\ with\ x = 3$$

$$A = \int_a^b (top - bottom)dx \quad at\ \ln 3$$

$$A = \int_1^3 (\ln x - 0)dx = [xlnx - x]_1^3 =$$
$$(3ln3 - 3) - (ln1 - 1) = 3ln3 - 2$$

Integration by parts would be use u=lnx and dv=dx, so that du=1/xdx and v=x

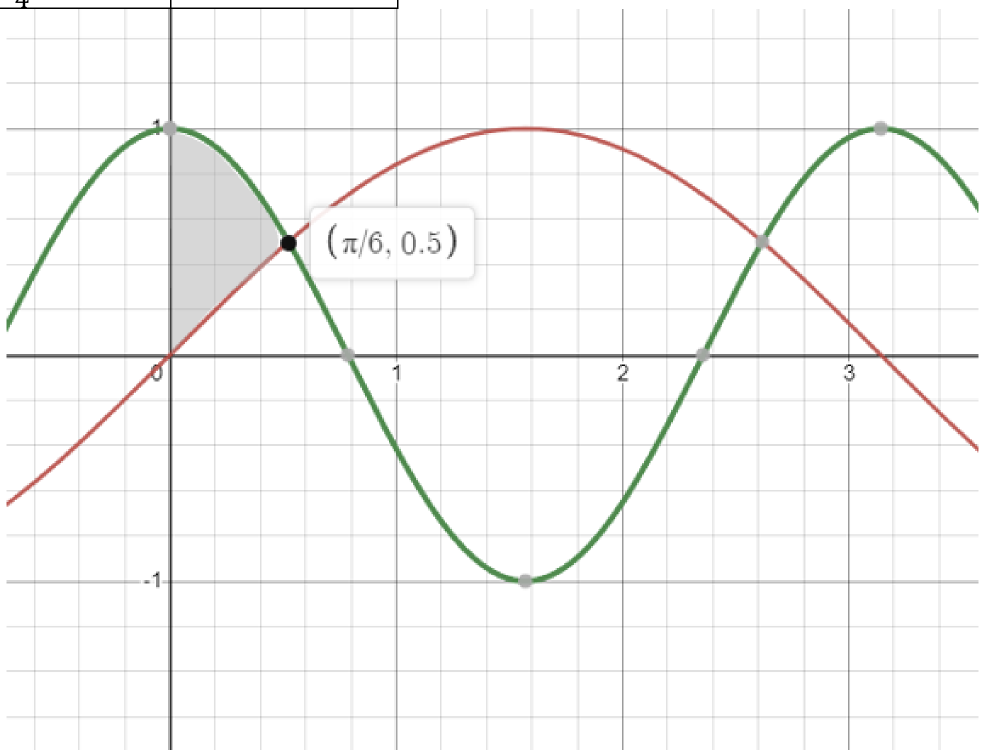


You can also use right/left and get $\int_0^{ln3} (3 - e^y)dy = [3y - e^y]_0^{ln3}$

M26.

$y = \cos 2x$

x	y
0	1
$\frac{\pi}{6}$	1/2
$\frac{\pi}{4}$	0



$$A = \int_0^{\pi/6} (cos2x - sinx)dx$$

$$= \left[\frac{sin2x}{2} + cosx\right]_0^{\pi/6}$$

$$= (\frac{\sin(\pi/3)}{2} + cos\,{}^{\pi}/_{6} - (0+1)$$

$$= \left(\frac{\sqrt{3}}{2}\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{2}) - 1\right) = \frac{\sqrt{3}+2\sqrt{3}-4}{4} = \frac{3\sqrt{3}-4}{4}$$

N. Volume of Solids of Revolution

Example 1.

Interval= $[1,4]$

Thickness=Δx

Top=$x^2 - 4x + 5$

Bottom=0

Volume=$\pi \int_1^4 ((x^2 - 4x + 5)^2 - (0)^2)dx$

$V\ of\ one\ disc = \pi(x^2 - 4x + 5)^2 \Delta x$

Example 2.

Interval=[0,1]

Thickness=Δx

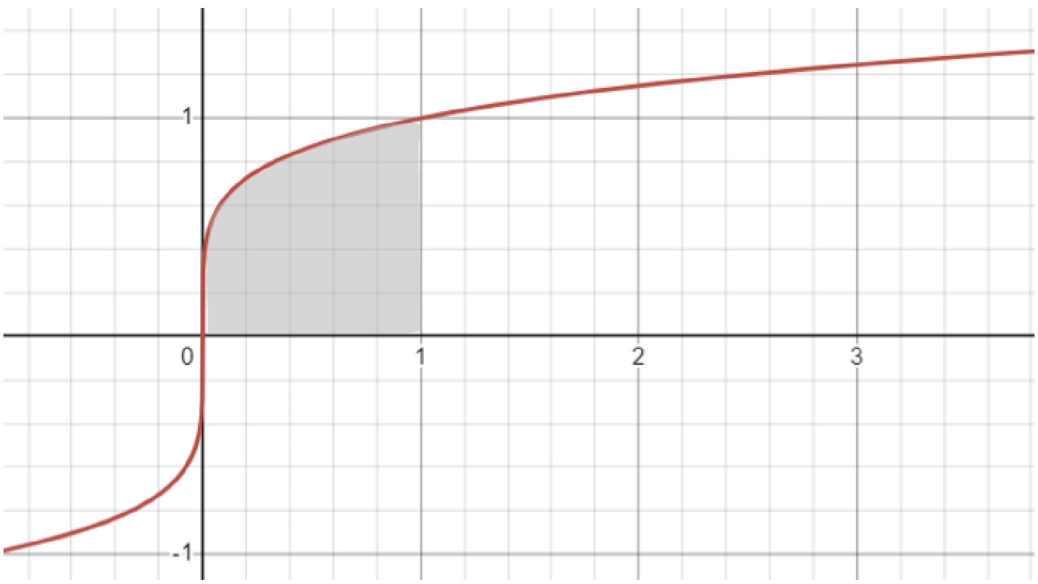
Top=$\sqrt[5]{x}$

Bottom= 0

Volume of one disc=$\pi\left(\sqrt[5]{x}\right)^2\Delta x$

$$V=\pi\int_0^1\left[\left(\sqrt[5]{x}\right)^2-0^2\right]dx=\pi\int_0^1 x^{\frac{2}{5}}dx=\pi\left[\frac{5}{7}x^{\frac{7}{5}}\right]_0^1$$

$$=\pi\left[\frac{5}{7}-0\right]=\frac{5\pi}{7}$$



Example 3.

Interval=$[1, 2]$

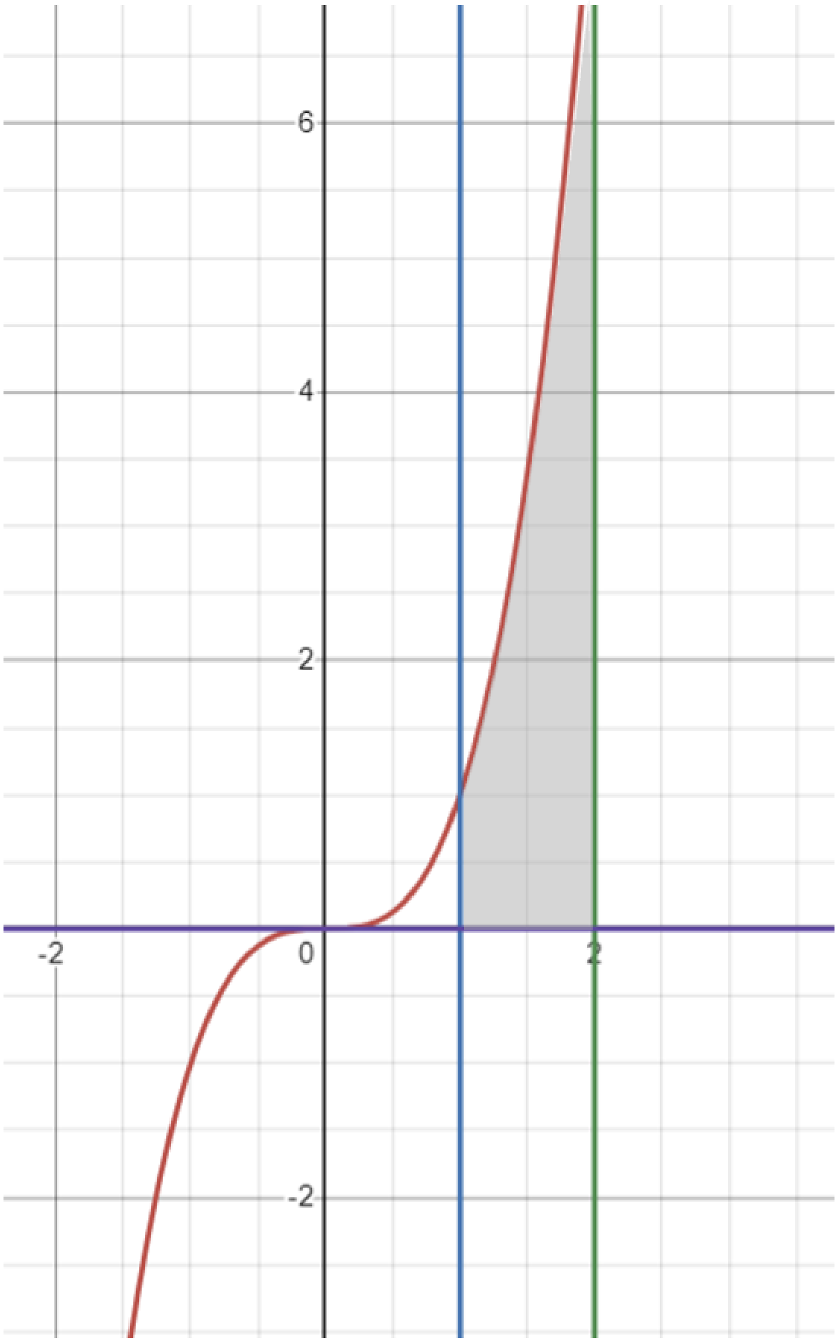
Thickness=Δx

Top=x^3

Bottom=0

Volume of one disc=$\pi(x^3)^2\Delta x$

$$V = \pi \int_1^2 (x^3)^2\, dx = \pi \int_1^2 x^6 dx = \pi \left[\frac{x^7}{7}\right]_1^2 = \pi \left[\frac{2^7}{7} - \frac{1}{7}\right]$$



Example 4.

About y-axis interval=$[0,4]$ $\leftarrow y - values$

Thickness=Δy

Top= $y^{\frac{1}{3}}$

Bottom= 0

Volume of one disc=$\pi(y^{\frac{2}{3}})\Delta y$

$$V = \pi \int_0^4 (y^{\frac{1}{3}})^2 dy = \pi \int_0^4 y^{\frac{2}{3}} dy = \pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_0^4 = \pi \left[\frac{3}{5}(4)^{\frac{5}{3}}\right]$$

Example 5. Set up, but do not evaluate $y = \cos x$

$x = \cos^{-1} y$

Interval=$[0,1]$

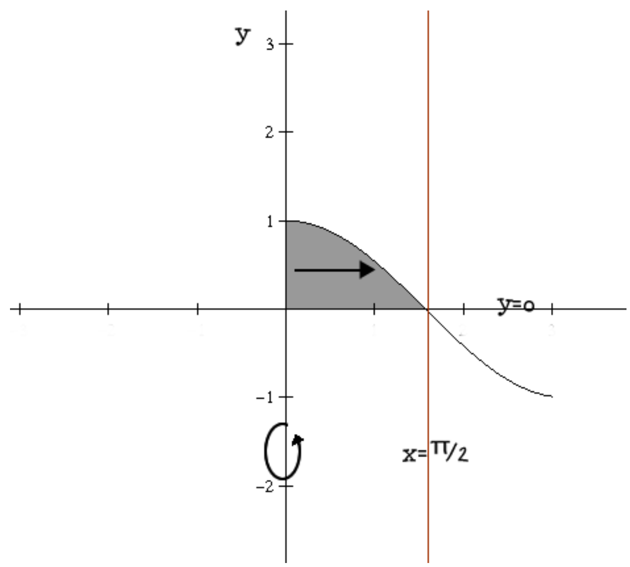
Thickness=Δy

Top=$\cos^{-1} y$

Bottom = 0

Volume of one disc=$\pi(\cos^{-1} y)^2 \Delta y$

$$V = \pi \int_0^1 (\cos^{-1} y)^2 dy$$



Washer's Method

Example 1. $y = \sqrt[3]{x}$ $y = \frac{x}{4}$ $\sqrt[3]{x} = \frac{x}{4}$ $\frac{x^3}{64} = x$

$x^3 - 64x = 0$

$x(x^2 - 64) = 0$

$x = 0, 8, -8$

$y = \frac{x}{4} = \frac{8}{4} = 2$

Interval=$[0,2]$

Thickness=Δy

Outer radius=$4y$

Inner radius= y^3

Volume =$\pi \int_a^b A(y)dy = \pi \int_0^2 [(4y)^2 - (y^3)^2]dy$= $\pi \int_0^2 [16y^2 - y^6]dy$

Example 2.

Point of intersection $x^2 + 1 = x + 3$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$ $x = -1, 2$

Interval=$[-1,2]$

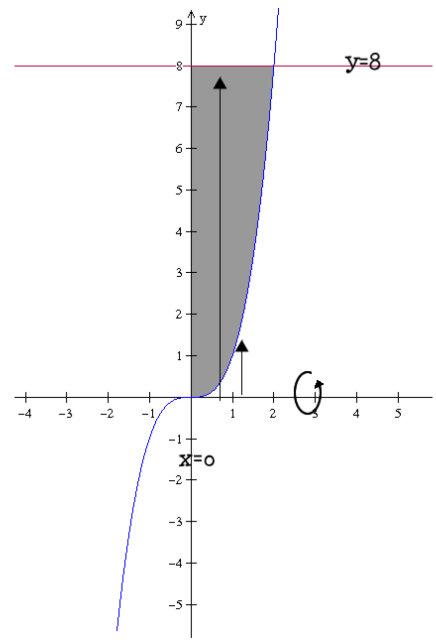
Thickness=Δx

Outer radius= $x + 3$

Inner radius=$x^2 + 1$

$V = \pi \int_a^b A(y)dy = \pi \int_{-1}^2 [(x + 3)^2 - (x^2 + 1)^2)]dx$

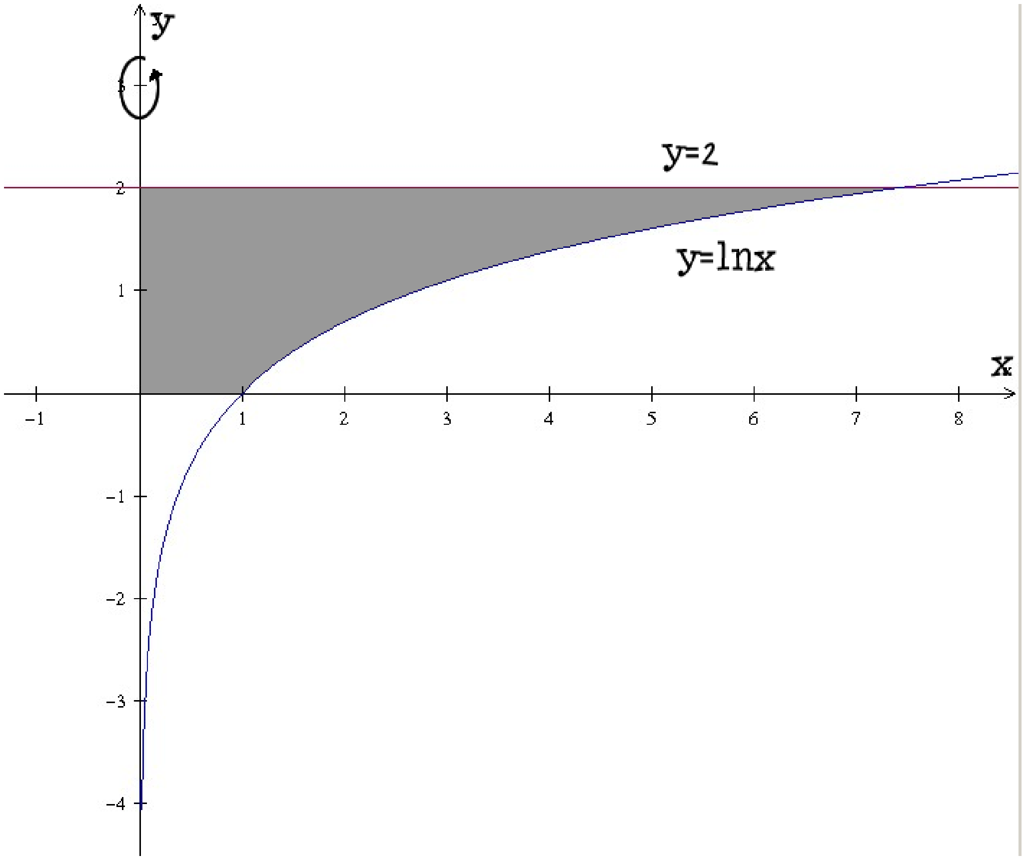
Example 3. Point of intersection $x^3 = 8 \quad x = 2$
Interval=[0,2]
Thickness=Δx
Outer radius= 8
Inner radius= x^3



$$V = \pi \int_0^2 [8^2 - (x^3)^2]dx = \pi \int_0^2 (64 - x^6)dx = \pi \left[64x - \frac{x^7}{7}\right]_0^2$$
$$= \pi \left[64(2) - \frac{2^7}{7} - (0 - 0)\right] = \pi \left(128 - \frac{128}{7}\right) = \frac{768\pi}{7}$$

Example 4.

y=lnx is the same graph as x=e^y



Example 4. a) $V = \pi \int_0^2 [(e^y)^2 - 0^2] dy$

$= \pi \int_0^2 e^{2y} dy = \pi \left[\frac{e^{2y}}{2}\right]_0^2 = \pi \left[\frac{e^4}{2} - \frac{e^0}{2}\right] = \pi \left[\frac{e^4}{2} - \frac{1}{2}\right] = \pi (\frac{e^4 - 1}{2})$

b) $V = \pi \int_{-\infty}^2 [(e^y)^2 - 0^2] dy$

Revolving About other Lines

Example 5.

Interval=$[0,3]$

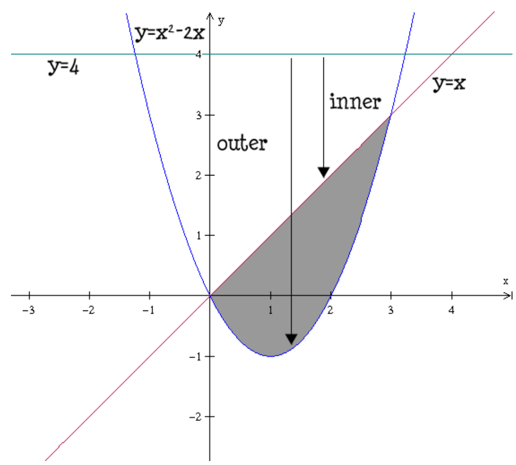
Thickness=Δx

Outer radius= $4-(x^2-2x)=-x^2+2x+4$

Inner radius=$4-x$

$$x^2-2x=x$$
$$x^2-3x=0$$
$$x(x-3)=0$$
$$x=0,3$$

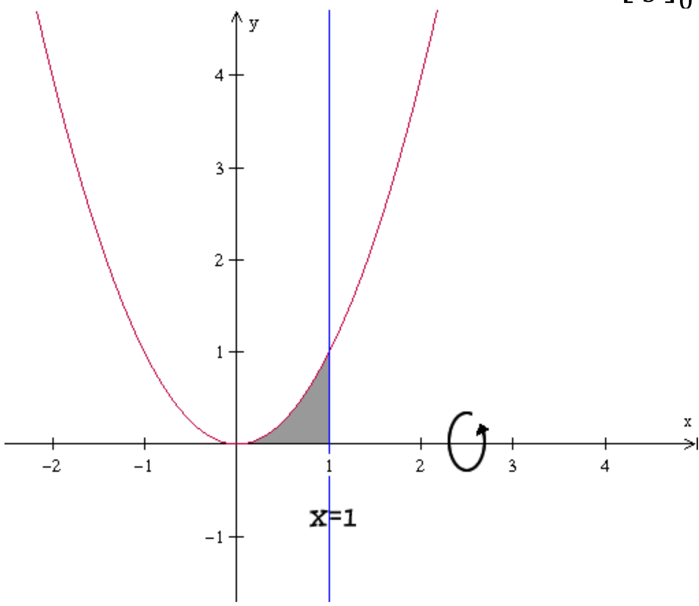
$$V=\pi\int_0^3[(-x^2+2x+4)^2-(4-x)^2]dx$$



Practice Exam Questions on Volumes of Solids of Revolution

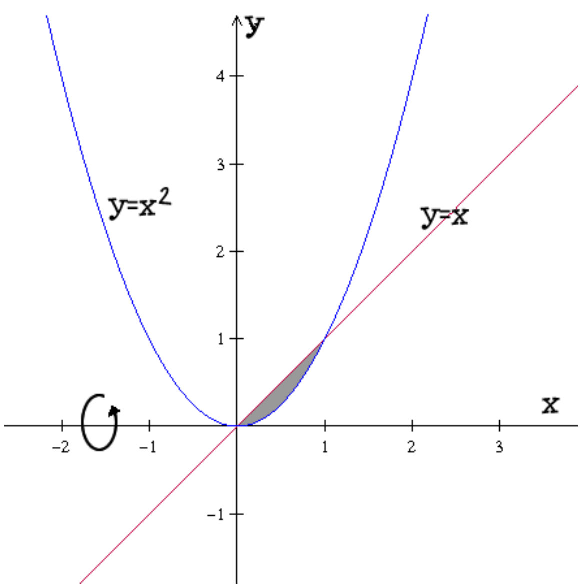
Find the Volume of each of the following using either the Disc method or the Washer method.

N1. $V = \pi \int_0^1 (x^2)^2 dx = \pi \int_0^1 x^4 dx = \pi \left[\frac{x^5}{5}\right]_0^1 = \pi \left[\frac{1}{5} - 0\right] = \frac{\pi}{5}$

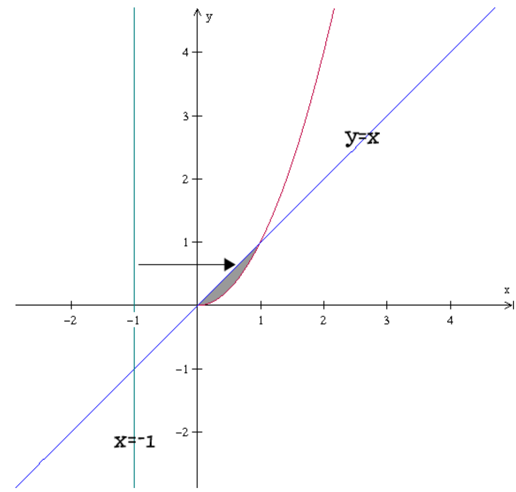


N2. $V = \pi \int_0^1 [(x)^2 - (x^2)^2] dx = \pi \int_0^1 (x^2 - x^4) dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$

$= \pi \left[\frac{1}{3} - \frac{1}{5}\right] = \frac{2\pi}{15}$



N3. $x = -1$ $parallel\ to\ y - axis$ $\therefore y, dy$ $x = \sqrt{y}$



Interval=[0,1]
Thickness=Δy
Outer radius (right-left)=$\sqrt{y} + 1$
Inner radius= $y + 1$

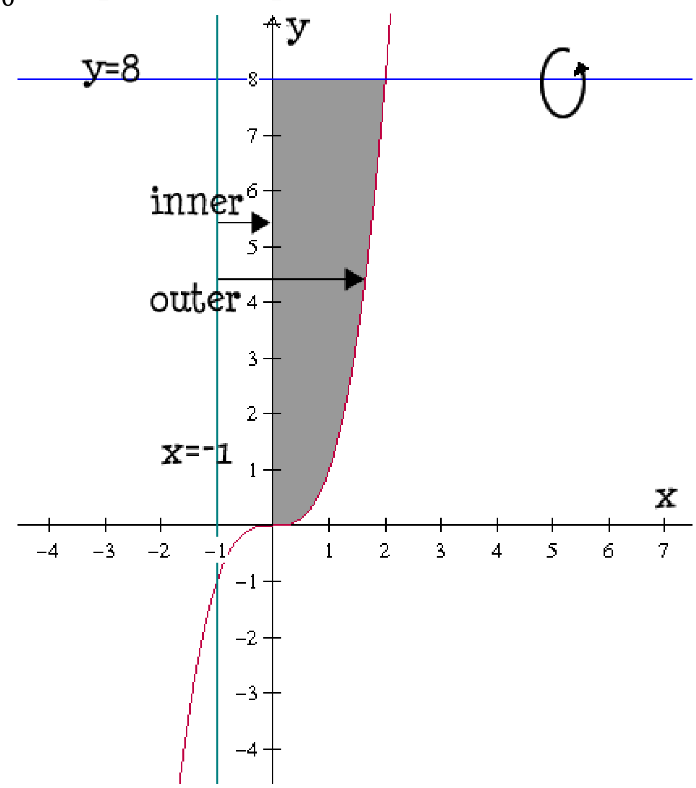
$$x^2 = x$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0,1 \quad \therefore y = 0,1 \ \ since\ y = x$$

$$V = \pi \int \left[\left(\sqrt{y} + 1\right)^2 - (y+1)^2 \right] dy$$

N4. "right – left"

$$V = \pi \int_0^8 \left[\left(y^{\frac{1}{3}} + 1\right)^2 - (0+1)^2 \right] dy = \pi \int_0^8 \left[y^{\frac{2}{3}} + 2y^{\frac{1}{3}} \right] dy$$

$$= \pi \left[\frac{3}{5} y^{\frac{5}{3}} + \frac{2y^{\frac{4}{3}}}{\frac{4}{3}} \right]_0^8 = \pi \left[\frac{3}{5}(8)^{\frac{5}{3}} + \frac{3^{\frac{4}{3}}}{2} \right] = \pi \left[\frac{96}{5} + \frac{48}{2} \right]$$



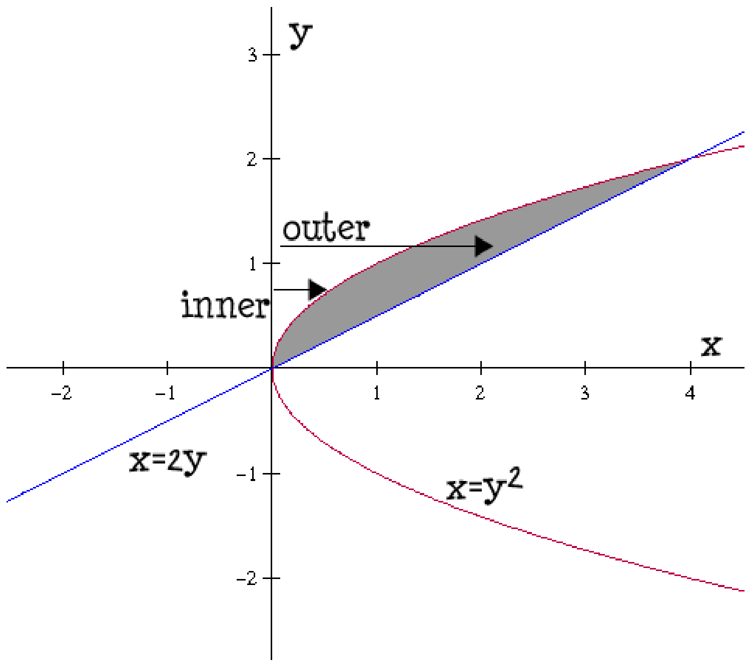
N5. $2y = y^2$

$0 = y^2 - 2y$

$0 = y(y-2) \quad \therefore y = 0,2$

$V = \pi \int_0^2 [(2y)^2 - (y^2)^2]dy = \pi \int_0^2 (4y^2 - y^4)dy = \pi \left[\frac{4y^3}{3} - \frac{y^5}{5}\right]_0^2$

$= \pi \left[\frac{4(2)^3}{3} - \frac{2^5}{5}\right] = \frac{64\pi}{15}$



N6. Parallel to y-axis $(to\ x = -1) \quad \therefore y, dy$

$x = 1 \quad y = 1^3 = 1$

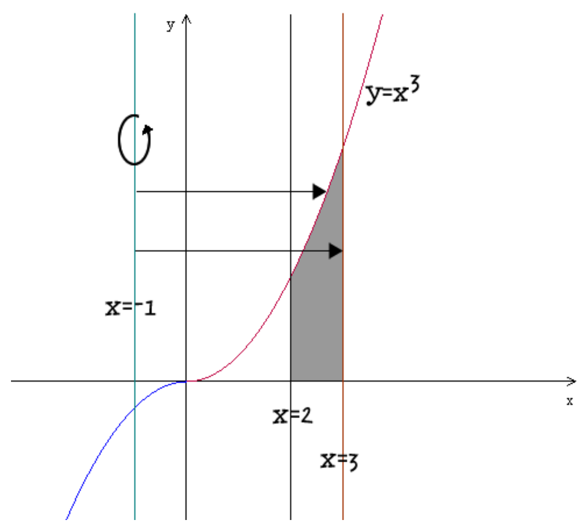
$x = 2 \quad y = 2^3 = 8$

Pt of intersection

y=x^3 $and\ x = 3\ is\ y = 3^3 = 27$

$V = \pi \int_0^{27} \left[(3+1)^2 - \left(y^{\frac{1}{3}} + 1\right)^2\right] dy$

$= \pi \int_0^{27} \left(4^2 - \left(y^{\frac{1}{3}} + 1\right)^2\right) dy$



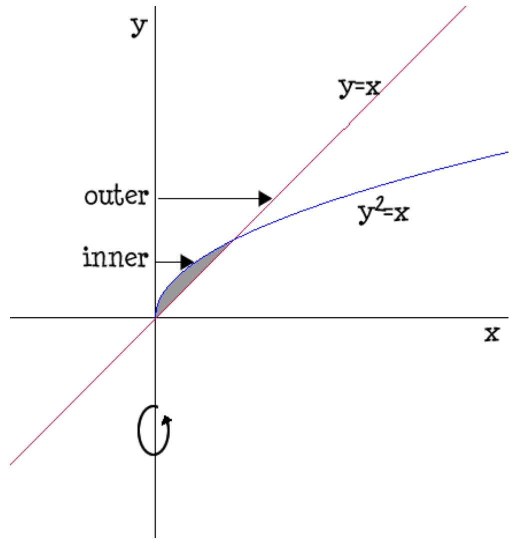
N7. Point of intersection $y^2 = y$

$$y^2 - y = 0$$

$$y(y-1) = 0 \quad y = 0,1$$

$$V = \pi\int_0^1 [(y)^2 - (y^2)^2]dy = \pi\int_0^1 (y^2 - y^4)dy = \pi\left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1$$

$$= \pi\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{2\pi}{15}$$



N8. Find the area of the region bounded by the curves $y = x + 2$ and $y = x^2$. Let R be the region. Find the volume formed by revolving the region R about the x-axis.

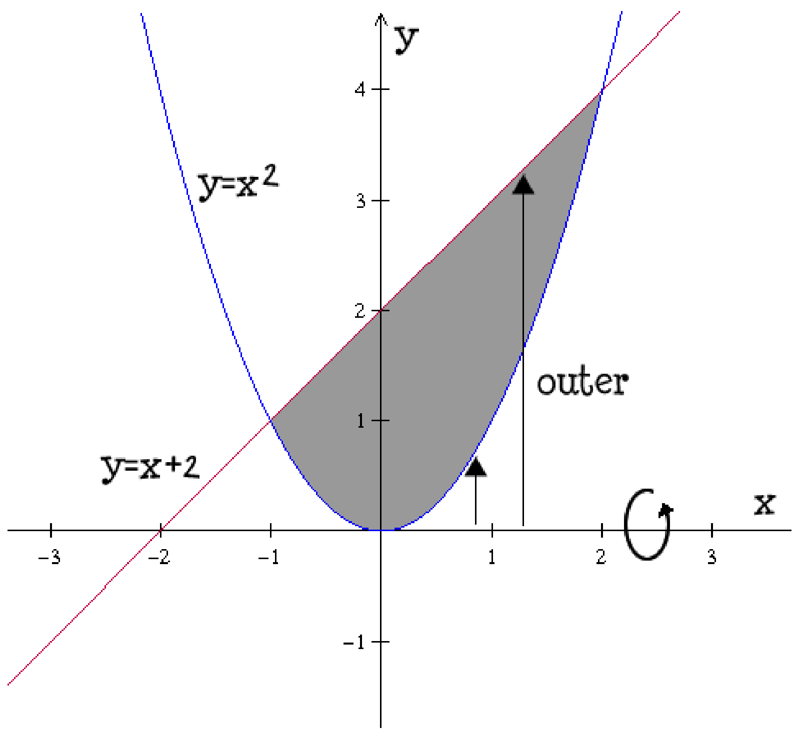
$x + 2 = x^2$

$0 = x^2 - x - 2$

$0 = (x - 2)(x + 1) \quad x = 2, -1$

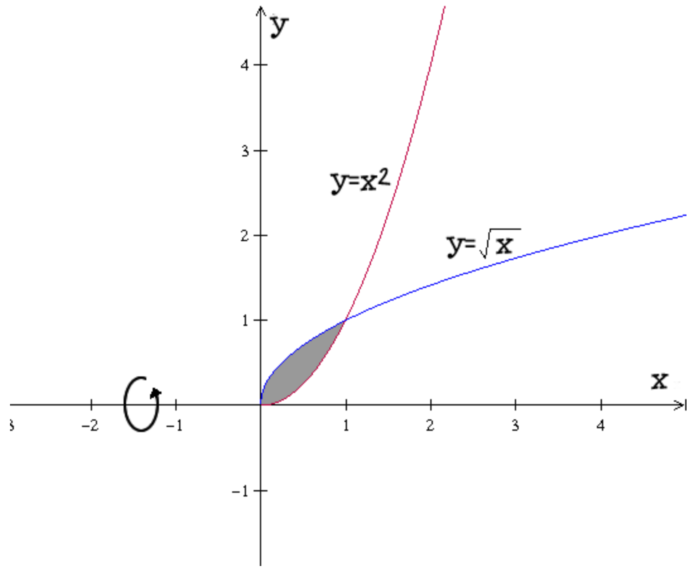
$V = \pi \int [(x + 2)^2 - (x^2)^2]\, dx = \pi \int_{-1}^{2} (x^2 + 4x + 4 - x^4) dx$

$= \pi \left[\frac{x^3}{3} + 2x^2 + 4x - \frac{x^5}{5}\right]_{-1}^{2} \quad etc$

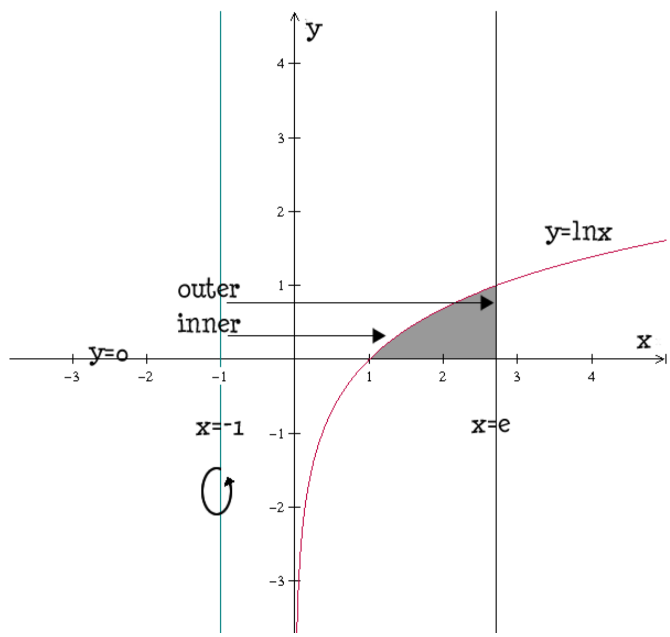


N9. $V = \pi \int_0^1 \left[\left(\sqrt{x} \right)^2 - (x^2)^2 \right] dx = \pi \int_0^1 (x - x^4) dx$

$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} \right) = \frac{3\pi}{10}$

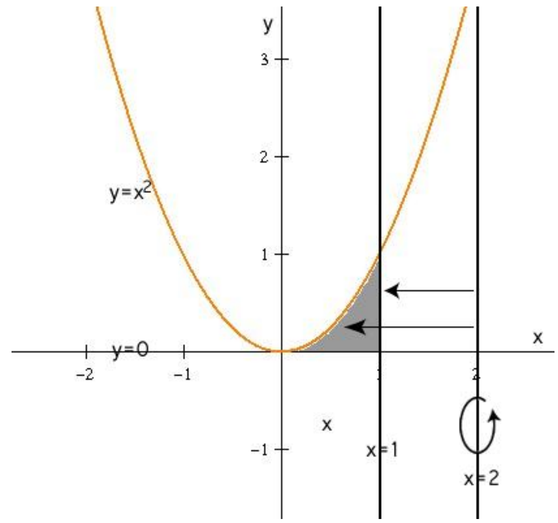


N10. $V = \pi \int_0^1 [(e+1)^2 - (e^y + 1)^2] dy \ldots etc$



N11. $V = \pi \int_0^4 \left[\left(2-\sqrt{y}\right)^2 - (2-1)^2\right] dy = \pi \int_0^4 (4 - 2\sqrt{y} + y - 1)dy$

$= \pi \int_0^4 (3 - 2\sqrt{y} + y)dy = \pi \left[3y - \frac{2y^{\frac{3}{2}}}{\frac{3}{2}} + \frac{y^2}{2}\right]_0^4 \; etc$

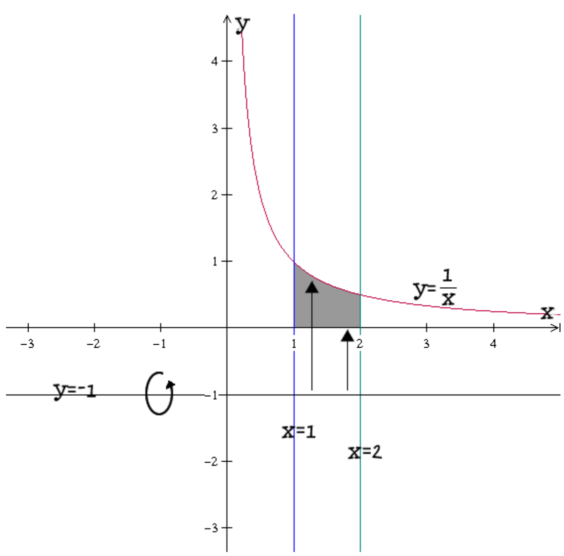


N12. $V = \pi \int_1^2 \left[\left(\frac{1}{x}+1\right)^2 - (0+1)^2\right] dx$

$= \pi \int_1^2 \left[\left(\frac{1}{x}+1\right)\left(\frac{1}{x}+1\right) - 1\right] dx = \pi \int_1^2 [x^{-2} + 2x^{-1} + 1 - 1]dx$

$= \pi \left[\frac{x^{-1}}{-1} + 2\ln x\right]_1^2 = \pi \left[\frac{-1}{x} + 2\ln x\right]_1^2$

$= \pi \left[\left(\frac{-1}{2} + 2\ln 2\right) - (1 + 2\ln 1)\right) = \pi(\frac{1}{2} + \ln 4)$



N13. Point of intersection $x^4 = x$

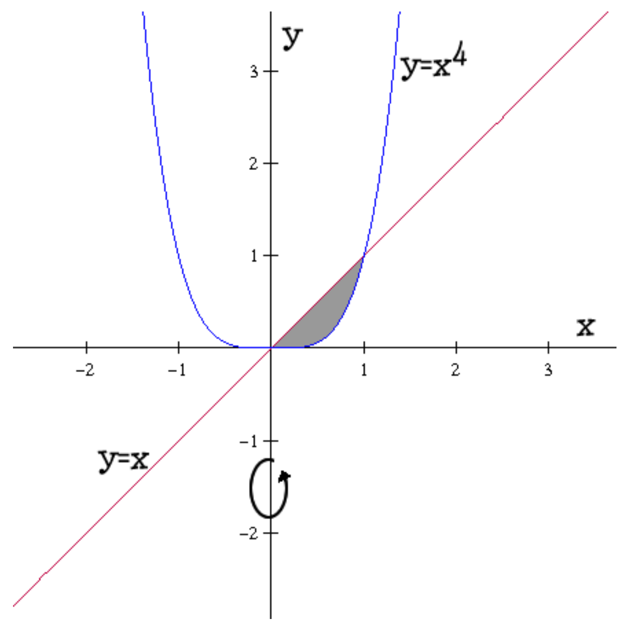
$$x^4 - x = 0$$

$$x(x^3 - 1) = 0 \quad x = 0,1$$

$$y = x^4 \quad \therefore y = 0,1$$

$$V = \pi \int_0^1 \left[\left(y^{\frac{1}{4}}\right)^2 - (y)^2\right] dy = \pi \int_0^1 \left[y^{\frac{1}{2}} - y^2\right] dy$$

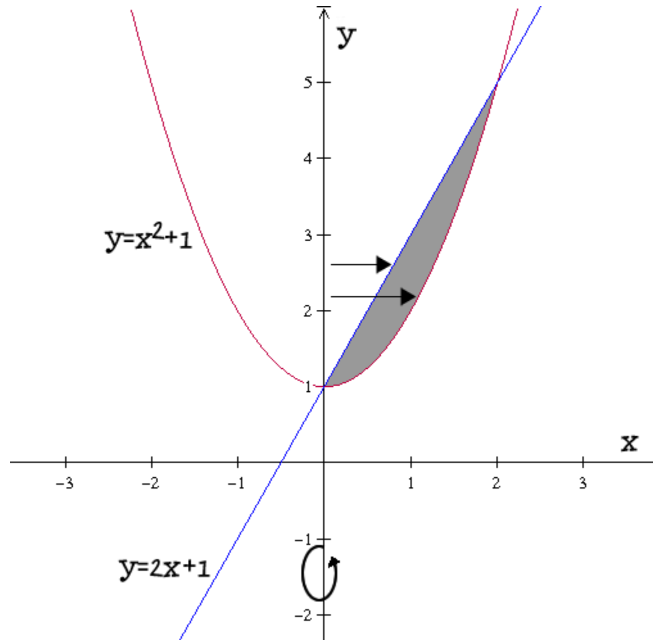
$$= \pi \left[\frac{2}{3} y^{\frac{3}{2}} - \frac{y^3}{3}\right]_0^1 = \pi \left[\frac{2}{3}(1)^{\frac{3}{2}} - \frac{1}{3}\right] = \frac{\pi}{3}$$



N14. Point of intersection $x^2+1=2x+1$

$$x^2-2x=0$$
$$x(x-2)=0 \quad x=0,2$$
$$y=x^2+1 \quad \therefore y=0^2+1=1$$
$$y=2^2+1=5$$

$$V=\pi\int_1^5\left[\left(\sqrt{y-1}\right)^2-\left(\frac{1}{2}y-\frac{1}{2}\right)^2\right]dy$$
$$=\pi\int_1^5\left[y-1-\left(\frac{1}{4}y-\frac{1}{2}y+\frac{1}{4}\right)\right]dy=$$
$$\pi\int_1^5\left(\frac{5}{4}y-\frac{5}{4}\right)dy \quad etc..$$



Pt of intersection $x^2+1 \quad =2x+1$

$x^2-2x=0$

x=0,2 substitute into either equation for y and get y=1 (x=0) and y=5 (x=2).

In the following two questions, the region R is the region shown below.

$V = \pi \int_0^2 [(x^2)^2 - (0)^2]dx = \pi \int_0^2 x^4 dx$

N15. The answer is A.

N16. $V = \pi \int_0^4 \left[(2)^2 - \left(\sqrt{y}\right)^2\right] dy = \pi \int_0^4 (4 - y)dy = \pi \left[4y - \frac{y^2}{2}\right]_0^4$

$= \pi \left[4(4) - \frac{4^2}{2}\right] = \pi(16 - 8) = 8\pi$

N17.

Interval=$[0,1]$

Thickness=Δy

Outer radius (right –left)=$2 - \sqrt{y}$

Inner radius (right – left)= 2-1 =1

$A(y) = \pi(outer\ radius)^2 - \pi(inner\ radius)^2$

$= \pi(2 - \sqrt{y})^2 - \pi(1)^2$

$V = \pi \int_0^1 \left[\left(2 - \sqrt{y}\right)^2 - (1)^2\right] dy = \pi \int_0^1 \left(4 - 2\sqrt{y} + y - 1\right) dy$

$= \pi \int_0^1 \left(-2\sqrt{y} + y + 3\right) dy$

Best of luck on the exam!!