AMATH 1201 Final Exam Booklet Solutions (2026)

Material Covered on the Midterm

A. Introduction to Single Equation Models

Example 1. a) 3
b) 2

Example 2. Match each of the following with the most appropriate differential equation listed.
I Second order, non-linear b)
II Second order, linear c)
III First order, linear, d)
IV Bernoulli differential equation a)

a) $u'(t) = 3u(t) + (u(t))^5$
b) $u''(t) - 3t^2u(t) = e^{u(t)}$
c) $u''(t) = e^t u(t)$
d) $tu'(t) - \frac{1}{t}u(t) = 2\sin(t)$

Example 3.
a) Non-linear (Bernoulli)
b) Linear
c) Non-linear
d) Linear

Example 4.
Non-linear
Linear
Non-linear

Example 5.
LS= $\frac{dx}{dt}$ $RS = 1 + 2x$
$x(t) = \frac{-1}{2} + \frac{3}{2}e^{2t}$ $= 1 + 2(\frac{-1}{2} + \frac{3}{2}e^{2t})$
$\therefore \frac{dx}{dt} = 0 + \frac{3}{2}e^{2t}(2)$ $= 1 - 1 + \frac{6}{2}e^{2t}$
$= 3e^{2t}$ $= 3e^{2t}$
$\therefore LS = RS \quad \therefore it\ is\ a\ solution$
To find the initial size, substitute t=0 into the x(t)

$$x(t) = -\frac{1}{2} + \frac{3}{2}e^{2t}$$

$$x(0) = -\frac{1}{2} + \frac{3}{2}e^{0} = -\frac{1}{2} + \frac{3}{2} = 1$$

Example 6. The answer is D. Per capita means it is a rate that is proportion to the # of individuals in a population

Example 7.

Which of the following functions are solutions to the differential equation:

$$u'(t) = t^2u(t)?$$

a) $u'(t) - t^2u(t) = 0$

$f(t) = e^{t^3}$ $\quad g(t) = e^{\frac{t^3}{3}}$ $\quad h(t) = 2e^{\frac{t^3}{3}}$

$u'(t) - t^2u(t)$
$= e^{t^3}(3t^2) - t^2\left(e^{t^3}\right)$
$= 3t^2e^{t^3} - t^2e^{t^3}$
$= 2t^2e^{t^3}$
$\neq 0$
$\therefore f(t)$ is **<u>not</u>** a solution

b) $u'(t) - t^2u(t)$
$= e^{\frac{t^3}{3}}\left(\frac{3}{3}t^2\right) - t^2\left(e^{\frac{t^3}{3}}\right)$
$= t^2e^{\frac{t^3}{3}} - t^2e^{\frac{t^3}{3}}$
$= 0$
$\therefore g(t)$ **<u>is</u>** a solution

c) $u'(t) - t^2u(t)$
$= 2e^{\frac{t^3}{3}}\left(\frac{3t^2}{3}\right) - t^2\left(2e^{\frac{t^3}{3}}\right)$
$= 2t^2e^{\frac{t^3}{3}} - 2t^2e^{\frac{t^3}{3}}$
$= 0$
$\therefore h(t)$ **<u>is</u>** a solution

Example 8

$f(t) = 3e^{\frac{t^3}{3}}$ $g(t) = e^{\frac{t^3}{3}}$ $h(t) = 2e^{\frac{t^3}{3}}$

$u(0) = 1$
$\therefore t = 0$ means $u(t) = 1$

$f(0) = 3e^0 = 3 \neq 1$ $\therefore$ **no**

$g(0) = e^0 = 1$ $\therefore$ **yes, s**ince it satisfies the initial condition and we already know it was a solution from example 7.

$h(0) = 2e^0 = 2 \neq 1$ $\therefore$ **no**

Practice Exam Questions on Introduction to Single Equation Models

A1.

$LS = \frac{db}{dt}$ $\qquad RS = 3b = 3(10e^{3t})$

$b(t) = 10e^{3t}$ $\qquad = 30e^{3t}$

$\therefore \frac{db}{dt} = 10e^{3t}(3) = 30e^{3t}$

$LS = RS \quad \therefore it\ is\ a\ solution$

A2.

$\frac{dG}{dt} = G - 1$

$$LS = \frac{dG}{dt} = 0 + e^t = e^t$$

$RS = G - 1 = (1 + e^t) - 1 = e^t$

$\therefore LS = RS \quad \therefore it\ is\ a\ solution$

A3. Match each of the following with the most appropriate differential equation listed.
I Second order, non-linear b)
II Third order, linear, homogeneous c)
III First order, linear, inhomogeneous d)
IV Bernoulli differential equation a)

a)$u'(t) = 4u(t) + (u(t))^6$
b)$u''(t) - 6t^3u(t) = \sin u(t)$
$c)u'''(t) = e^t u(t)$
$d)\ t^2u'(t) - \frac{1}{t}u(t) = \cos(t)$

A4. a) linear, since everything in front of u(t) and derivatives are only functions of t
It is inhomogeneous.
b) linear and homogeneous
c) non-linear since there is cos u(t)

A5. $e^x, x+1, x^4-2, 2x^4+x^3$

Check if $y(0)=0$

$y=e^x$
$y(0)=e^0=1\neq 0$
∴ **no**

$y=x+1$
$y(0)=0+1=1\neq 0$
∴ **no**

$y=x^4-2$
$y(0)=0-2=-2\neq 0$
∴ **no**

$y=2x^4+x^3$
$y(0)=0$
∴ **yes**

So, only y=$2x^4+x^3$ is a possible solution to this initial-value problem since it is the only one that satisfies the initial condition y(0)=0.

We will learn later how to find the solution to this linear differential equation

B. Isometry, Allometry, Log-log Plots, Dimensional Homogeneity

Example 1.

$$A = 6x^2 \quad x^2 = \frac{A}{6} \quad \therefore x = (\frac{A}{6})^{\frac{1}{2}}$$

$$v = x^3$$

$$V = x^3 = \left[(\frac{A}{6})^{\frac{1}{2}}\right]^3 = (\frac{1}{6})^{\frac{3}{2}} A^{\frac{3}{2}}$$

Let the new area be 2A. Let V* be the new volume

$$V^* = (\frac{1}{6})^{\frac{3}{2}}(2A)^{\frac{3}{2}}$$

$$= (\frac{1}{6})^{\frac{3}{2}} \cdot 2^{\frac{3}{2}} A^{\frac{3}{2}}$$

$$= \underbrace{(\frac{1}{6})^{\frac{3}{2}} A^{\frac{3}{2}}}_{V} \; 2^{\frac{3}{2}}$$

$$\therefore \textit{the volume increases by a factor of } 2^{\frac{3}{2}} \textit{ or } \sqrt{8}$$

$$V = \left(\frac{1}{6}\right)^{\frac{3}{2}} A^{\frac{3}{2}}$$

Example 2. $\frac{dI}{dt} = \frac{individual}{year} \qquad aI^2 = a \;\, individual^2$

$$\therefore a = \frac{1}{year\ (individual)}$$

Example 3. $P = \rho g h$

LS=$\frac{F}{L^2}$ RS= $\rho g h$

$$=\frac{FT^2}{L^4} \times (\quad) \times L$$

Therefore, the units of g must be $(\quad) = \frac{L}{T^2}$

Example 4. From the graph $m = \frac{2}{3} \qquad y - int = -2$

$$m \neq 1 \quad \therefore allometric$$

$$\ln y = m \ln x + \ln k$$

$$\therefore \ln k = -2 \quad since\ y - int = -2$$

$$k = e^{-2}$$

Example 5.

a) $\frac{\text{Mass of 2nd dinosaur}}{\text{Mass of 1st dinosaur}} = \frac{(density)\times c\times(\text{2nd footprint length})^3}{(density)\times c\times(\text{1st footprint length})^3}$

$$= \frac{65^3 cm^3}{80^3 cm^3} = 0.54$$

∴ the second dinosaurs’ mass is approximately 0.54 times the first.

Practice Exam Questions on Isometry, Allometry, Log-log Plots, Dimensional Homogeneity

$B1. \quad aI^3 = a\,(individual)^3$

$$\frac{dT}{dt} = \frac{individual}{year}$$

$$\therefore a = \frac{1}{year(individual)^2}$$

B2.

Let V* be new volume

$$V^* = \left(\frac{1}{6}\right)^{\frac{3}{2}} (3A)^{\frac{3}{2}}$$

$$= \left(\frac{1}{6}\right)^{\frac{3}{2}} 3^{\frac{3}{2}} \cdot A^{\frac{3}{2}}$$

$$V^* = \underbrace{\left(\frac{1}{6}\right)^{\frac{3}{2}} A^{\frac{3}{2}}}_{V} \cdot 3^{\frac{3}{2}}$$

$$\therefore volume\ increases\ by\ a\ factor\ of\ 3^{\frac{3}{2}}\ or \sqrt{27}$$

B3. What is the equation of in log $-log$ space? Find the slope and y-intercept.

$$In\ V = \ln\left[\left(\frac{1}{6}\right)^{\frac{3}{2}} A^{\frac{3}{2}}\right]$$

$$= \ln\left(A^{\frac{3}{2}}\right) + \ln\left(\frac{1}{6}\right)^{\frac{3}{2}}$$

$$InV = \frac{3}{2}\ln A + \ln\left(\frac{1}{6}\right)^{\frac{3}{2}}$$

Slope is $\frac{3}{2}$ $\quad and\ y - intercept\ is\ \ln\left(\frac{1}{6}\right)^{\frac{3}{2}}$

B4. $lny = mlnx + \ln(k)$ where the slope is m and the y-intercept is ln(k)

$$lny = \frac{3}{4}lnx + ln4$$

$$lny - lnx^{\frac{3}{4}} + ln4$$

$$lny = \ln(4x^{\frac{3}{4}})$$

$y = 4x^{\frac{3}{4}}$ $\quad m \neq 1$ so it is allometric

B5. LS=v

=m/s

RS=u+at

$$= \frac{m}{s} + \frac{m}{s^2}(s)$$

$$= \frac{m}{s} + \frac{m}{s}$$

$$= \frac{m}{s}$$

Therefore, a must be in $\frac{m}{s^2}$.

C. Recursion Models

Example 1.

a) A recurrence is often written as one line of integers.

$$\underset{\text{Initial Condition}}{\underset{\uparrow}{P_0 = 2}} \qquad \underbrace{P_t = \tfrac{1}{3}P_{t-1} - \tfrac{1}{3} \text{ for } t \geq 1.}_{\text{Recurrence Relation}}$$

b) A recurrence can also be written as a piecewise function

$$\underbrace{\begin{cases} a_0 = 2 \xleftarrow{\text{Initial Condition}} \\ a_t = a_{t-1} + 3, \text{ for integers } t \geq 1. \end{cases}}_{\text{Recurrence Relation}}$$

c) A recurrence can involve any algebraic operations

$b_0 = 1$ is the initial condition *and* $b_t = t\sqrt[3]{b_{t-1}}$ is the recurrence relation

d) Here, there is no initial condition given:

$F_t = F_{t-1} + 1$ recurrence relation

e) A recurrence relation can involve **more** than one previous state.

Example. $F_0 = 2,\ F_1 = 3$ (these are both initial conditions needed to find further terms)

$F_t = F_{t-1} + F_{t-2}$ (this is the recurrence relation)

Example 2.

a) $P_t = P_{t-1} + 0.15P_{t-1} = 1.15P_{t-1}$ $\quad b = 1.15 \quad P_0 = 200$

b) General solution is $P_t = b^t \times P_o$

$$P_t = (1.15)^t(200) \quad or \quad 200(1.15)^t$$

c) $P_5 = 200(1.15)^5 = 402.2 \qquad \therefore 402\ deer$

Example 3.

Notation: Let m_t be the balance in the savings account t years after the initial deposit.
Initial Condition: $m_0 = 1000$

Balance at end of t years = interest rate X balance at beginning of t years − withdrawal

$\boxed{m_0 = 1000}$ $\boxed{m_t = 1.03m_{t-1} - 60}$ for integers t≥ 1

a) $f(t) = 1000\,(1.03)^t$

$\boxed{m_0 = 1000}$ $\boxed{m_t = 1.03m_{t-1} - 60}$ $\boxed{1}$ for integers t≥ 1

Is $m_t - (1.03m_{t-1} - 60) = 0$? ← Substitute $f(t)$ for m_t and $f(t-1)$ for m_{t-1} into $\boxed{1}$

$f(t) - (1.03\,f(t-1) - 60)$ ← now substitute $f(t) = 1000\,(1.03)^t$ and

$f(t-1) = 1000\,(1.03)^{t-1}$

$= 1000\,(1.03)^t - [1.03(1000)(1.03)^{t-1} - 60]$

Multiplying with the same base ∴ add exponents

$= 1000\,(1.03)^t - (1000(1.03)^t - 60)$

$= \cancel{1000\,(1.03)^t} - \cancel{1000(1.03)^t} + 60$

$= 60 \neq 0$

∴ $f(t)$ is **not** a solution to this recurrence.

b) $m_0 = 1000 \quad m_t = 1.03m_{t-1} - 60$

Verify that $b_t = -1000\,(1.03)^t + 2000$ $\boxed{1}$
Is a solution to the above recurrence

1. Initial Condition
   $b_0 = -1000(1.03)^0 + 2000 = 1000$
   $\therefore\; b_0 = m_0$ ✓

2. Check that for $t \geq 1$, b_t satisfies the recurrence equation $\boxed{1}$

   i.e.) is $m_t - (1.03m_{t-1} - 60) = 0$?
   Substitute b_t for m_t and b_{t-1} for m_{t-1}

   $b_t - (1.03b_{t-1} - 60)$

   $= -1000(1.03)^t + 2000 - [1.03^1(-1000(1.03)^{t-1} + 2000) - 60]$

   $= -1000(1.03)^t + 2000 - [-1000(1.03)^t + 1.03(2000) - 60]$

   $= -1000(1.03)^t + 2000 - [-1000(1.03)^t + 2060 - 60]$

   $= -\cancel{1000(1.03)^t} + \cancel{2000} + \cancel{1000(1.03)^t} - \cancel{2060} + \cancel{60}$

   $= 0$

   $\therefore$ yes, it is a solution.

c) <u>Find the solution</u>

Method 1: Using the equilibrium

First, find the equilibrium:

$m_t = 1.03\,m_{t-1} - 60$

$\hat{m} = 1.03\,\hat{m} - 60$

$1\hat{m} - 1.03\hat{m} = -60$

$-0.03\,\hat{m} = -60$

$\hat{m} = \frac{60}{0.03} = \frac{60}{3/100} = 60 \times \frac{100}{3} = 2000$

∴ the equilibrium is 2000.

$\boxed{m_0 = 1000}$ $\boxed{m_t = 1.03m_{t-1} - 60}$ $\boxed{1}$

Equilibrium is $\hat{m} = 2000$.

Let $u_t = m_t - \hat{m}$

$u_t = m_t - 2000$

∴ $\boxed{m_t = u_t + 2000}$ $\boxed{2}$

$m_{t-1} = u_{t-1} + 2000$ ← Substitute into $\boxed{1}$

$m_t = 1.03(u_{t-1} + 2000) - 60$ ← Substitute $m_t = u_t + 2000$

$u_t + 2000 = 1.03u_{t-1} + 2060 - 60$

$u_t = 1.03u_{t-1}$

$m_t = u_t + 2000$

$u_t = m_t - 2000$

$u_0 = m_0 - 2000$ Recall, $m_0 = 1000$

$u_0 = 1000 - 2000 = -1000$

$u_t = u_0(b)^t$

$u_t = -1000(1.03)^t$

∴ $m_t = u_t + 2000$ from $\boxed{2}$

$$\boxed{m_t = -1000\,(1.03)^t + 2000}$$

Method 2: Change of Variable: $m_t = 1.03m_{t-1} - 60$

Now, sadly they are always changing the variables they give you, so let's practice making the original question $\boldsymbol{P_t = 1.03P_{t-1} - 60}$

Use $\boxed{\boldsymbol{P_t = u_t - C}}$ and set C to be the number so that the constant term is 0

Change of Variable Let $P_t = u_t - C$ and $P_{t-1} = u_{t-1} - C$

Substitute into $P_t = 1.03P_{t-1} - 60$

$$u_t - C = 1.03P_{t-1} - 60$$
$$u_t - C = 1.03(u_{t-1} - C) - 60$$
$$u_t = 1.03u_{t-1} \underbrace{- 1.03C - 60 + C}$$

Solve for C by setting this equal to 0

$\therefore -1.03C - 60 + C = 0$

$-60 = 1.03C - C$

$-60 = 0.03C$

$C = -2000$

And we get $\boxed{P_t = u_t + 2000}$…

And then the rest is the same as the other method. (do the last two steps!)

Method 3: Short cut Method for Multiple Choice Exam Questions

$$m_t = 1.03m_{t-1} - 60$$

$$a = 1.03,\ \ b = -60$$

$$m_t = \left(m_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a} \quad a \neq 1$$

$$m_t = \left(1000 - \left(\frac{-60}{1-1.03}\right)\right)(1.03)^t + \left(\frac{-60}{1-1.03}\right)$$

$$m_t = (1000 - 2000)(1.03)^t + 2000$$

$$m_t = -1000(1.03)^t + 2000$$

This method allows you to be done in a minute, rather than doing a page of calculations!
But you can only use it for MULTIPLE CHOICE!

Example 4.

Solution:

Method 1: Using Equilibrium

$p_0 = 500$
$p_t = 1.04p_{t-1} - 80$
$\hat{p} = 1.04\hat{p} - 80$
$80 = 0.04\hat{p}$
$\hat{p} = 2000$

Use $p_t = u_t + 2000$ and substitute (and $p_{t-1} = u_{t-1} + 2000$) into
$p_t = 1.04p_{t-1} - 80$
$\therefore u_t + 2000 = 1.04p_{t-1} - 80$
$u_t + 2000 = 1.04(u_{t-1} + 2000) - 80$
$u_t + \cancel{2000} = 1.04u_{t-1} + \cancel{2080} - \cancel{80}$
$\boxed{u_t = 1.04u_{t-1}}$

$p_0 = 500$ and $p_0 = u_0 + 2000$
$\therefore u_0 = p_0 - 2000$
$u_0 = 500 - 2000 = \boxed{-1500}$ ← Initial condition

$\therefore$ Now, we have $u_t = 1.04u_{t-1}$ and $u_0 = -1500$

And the solution is
$u_t = (a)\lambda^t$

$\therefore u_t = -1500(1.04)^t$ for all $t \geq 0$.

$\therefore p_t = -1500(1.04)^t + 2000.$

Solution:

Method 2: Change of Variable

$$P_t = u_t - c \qquad \text{(and } P_{t-1} = u_{t-1} - c\text{)}$$

Substitute into $P_t = 1.04P_{t-1} - 80$

$u_t - c = 1.04P_{t-1} - 80$

$u_t - c = 1.04(u_{t-1} - c) - 80$

$u_t = 1.04u_{t-1} \underbrace{- 1.04c - 80 + c}_{}$

Solve for C by setting this equal to 0
$\therefore -1.04c - 80 + c = 0$
$-80 = 1.04c - c$
$-80 = 0.04c$

$c = -2000$

And we get $\boxed{p_t = u_t + 2000}$

And then the rest is the same as the other method

Example 5.
Find all equilibrium solutions for the recurrence equation $u_t = \sqrt{4u_{t-1} + 5}$

Solution:
Replace u_t *and* u_{t-1} *by* $\hat{u}$
$u_t = \sqrt{4u_{t-1} + 5}$
$\hat{u} = \sqrt{4\hat{u} + 5}$ square both sides
$(\hat{u})^2 = 4\hat{u} + 5$
$(\hat{u})^2 - 4\hat{u} - 5 = 0$
$(\hat{u} - 5)(\hat{u} + 1) = 0$
$\hat{u} = 5, \ \hat{u} = -1$
This means there are two possible initial conditions so if $\hat{u} = 5$ or $\hat{u} = -1$ then the solution of the equation will be constant.

Example 6.

i) a) $P_t = bP_{t-1}$ $\therefore b = 0.4$

let $P_o = 10$

$$P_1 = 0.4P_o = 0.4(10) = 4$$
$$P_2 = 0.4P_1 = 0.4(4) = 1.6$$
$$P_3 = 0.4P_2 = 0.4(1.6) = 0.64$$

b) $P_t = b^t P_o$

$P_t = 0.4^t(10)\ or\ \ 10(0.4)^t$

c) $10(0.4)^\infty = 0$ $\quad as\ t \to \infty, \quad P_t \to 0$

In other words, $\lim_{t\to\infty} 10(0.4)^t = \infty,$

Example 7. $u(t) = 20(\frac{1}{2})^{\frac{t}{1600}}$

$$u(2000) = 20(\frac{1}{2})^{\frac{2000}{1600}} = 20\sqrt[4]{(\frac{1}{2})^5} = 20\sqrt[4]{\frac{1}{32}}8\text{ g}$$

Example 8. **Short Cut Method for Multiple Choice**

$$P_t = (P_{t-1} + 0.04P_{t-1}) - 50$$
$$P_t = 1.04P_{t-1} - 50 \qquad a = 1.04 \qquad b = -50$$

$$Pt = \left(1000 - \frac{-50}{1-1.04}\right)(1.04)^t + (\frac{-50}{1-1.04})$$
$$= \left(1000 + \frac{50}{-0.04}\right)1.04^t + (\frac{-50}{-0.04})$$
$$= (1000 - 1250)1.04^t + 1250$$
$$= -250(1.04)^t + 1250$$

How long until it is worth \$250?

$$250 = -250(1.04^t) + 1250$$
$$-1000 = -250(1.04)^t$$
$$4 = 1.04^t$$
$$t = \frac{ln4}{ln1.04}\ years$$

Example 8.

Long method: equilibrium

$$p_t = 1.04p_{t-1} - 50 \quad \boxed{1}, \qquad p_0 = 1000$$

Let $p_t = p_{t-1} = \hat{p}$ find equilibrium
$\hat{p} = 1.04\hat{p} - 50$
$-0.04\hat{p} = -50$
$\hat{p} = 1250$

Let $p_t = u_t + 1250$ and $p_{t-1} = u_{t-1} + 1250$

Substitute into $\boxed{1}$

$p_t = 1.04p_{t-1} - 50$
$u_t + 1250 = 1.04(u_{t-1} + 1250) - 50$
$u_t = 1.04u_{t-1} + 1300 - 50 - 1250$
$\boxed{u_t = 1.04u_{t-1}}$ ← b

$p_0 = 1000$
$p_t = u_t + 1250$
$u_t = p_t - 1250$
$u_0 = p_0 - 1250$
$u_0 = 1000 - 1250$
$u_0 = -250$

∴ Solution is $u_t = b^t u_0$

$u_t = -250(1.04)^t$

$\boxed{p_t = -250(1.04)^t + 1250}$

Example 8. **Long Method: Change of variables**

Let $p_t = u_t - c$ and $p_{t-1} = u_{t-1} - c$

Substitute into $p_t = 1.04\, p_{t-1} - 50$ $\boxed{1}$

$u_t - c = 1.04(u_{t-1} - c\,) - 50$

$u_t = 1.04u_{t-1} - 1.04c\, - 50 + c$

$-1.04c\, - 50 + c = 0$

$-0.04c = 50$

$c = 1250$

$p_t = u_t + 1250$ $\boxed{2}$

Substitute $p_t = u_t + 1250$ and $p_{t-1} = u_{t-1} + 1250$ into $\boxed{1}$

$u_t + 1250 = 1.04(u_{t-1} + 1250) - 50$

$u_t = 1.04u_{t-1} + 1300 - 50 - 1250$

$\boxed{u_t = 1.04u_{t-1}}$ ← $b = 1.04$

$p_0 = 1000$ $\quad u_0 = p_0 - 1250$

$u_0 = 1000 - 1250 = \boxed{-250}$

Solution: $u_t = b^t u_0$

$u_t = -250(1.04)^t$

From $\boxed{2}$

$p_t = u_t + 1250$

$\boxed{p_t = -250(1.04)^t + 1250}$

Example 8. **Long Method 3:**

$P_t = 1.04P_{t-1} - 50 \quad \boxed{1} \qquad b = -50 \quad a = 1.04 \quad P_o = 1000$

$P_t = P_{t-1} + 0.04P_{t-1} - 50$

$\Delta P_t = P_t - P_{t-1}$

$= P_{t-1} + 0.04P_{t-1} - 50 - P_{t-1}$

$\Delta P_t = 0.04P_{t-1} - 50$

Consider when there is no change i.e. equilibrium

$\Delta P_t = 0 \qquad 0 = 0.04P_{t-1} - 50$

$50 = 0.04P_{t-1}$

$P_{t-1} = 1250$

Define $u_t = P_t - 1250$ which means $P_t = u_t + 1250 \; \boxed{2}$

$u_{t-1} = P_{t-1} - 1250$ which means $P_{t-1} = u_{t-1} + 1250 \; \boxed{2}$

From $\boxed{1}$ $P_t = 1.04P_{t-1} - 50$ *substitute* $\boxed{2}$

$u_t + 1250 = 1.04(u_{t-1} + 1250)$ - 50

$u_t = 1.04u_{t-1} + 1300 - 50 - 1250$

$u_t = 1.04u_{t-1}$, so b=1.04

$\boxed{2} \quad u_t = P_t - 1250$

Substitute $t = 0 \qquad u_o = P_o - 1250 \qquad P_o = 1000$ substitute

$u_o = 1000 - 1250$

$u_o = -250$

$u_t = u_o(b)^t$

$u_t = -250(1.04)^t \quad \boxed{3}$

From $\boxed{2}$ $\overbrace{u_t = P_t - 1250}$

Substitute $\boxed{3}$ *into here* $P_t = u_t + 1250$

$P_t = -250(1.04)^t + 1250$

Example 9. $6U_n = 3U_{n-1} + 2$

$U_n = \frac{3}{6}\ U_{n-1} + \frac{2}{6}$

$U_n = \frac{1}{2} U_{n-1} + \frac{1}{3}$

$U_t = \frac{1}{2} U_{t-1} + \frac{1}{3}$

Same as $U_{t+1} = a\, U_t + b$

$a = \frac{1}{2} \quad b = \frac{1}{3}$

$$U_t = \left(U_0 - \frac{b}{1-a}\right)(a)^t + \frac{b}{1-a}$$

$$U_t = \left(2 - \frac{\frac{1}{3}}{1-\frac{1}{2}}\right)(\frac{1}{2})^t + \frac{\frac{1}{3}}{1-\frac{1}{2}}$$

$$= \left(2 - \frac{\frac{1}{3}}{\frac{1}{2}}\right)\left(\frac{1}{2}\right)^t + \frac{\frac{1}{3}}{\frac{1}{2}}$$

$$= \left(2 - \frac{2}{3}\right)\left(\frac{1}{2}\right)^t + \frac{2}{3}$$

$$U_t = \frac{4}{3}(\frac{1}{2})^t + \frac{2}{3} \quad \textit{explicit solution}$$

$$\lim_{t\to\infty}(\frac{4}{3}\left(\frac{1}{2}\right)^t + \frac{2}{3}\right) = \frac{2}{3}$$

Example 10. **Note: Regardless of method to find the solution, the part b) of finding t when $P_t = 12$ is the same in each case!**

Long Method 1: Change of Variables

$P_t = 3P_{t-1} + 1$
Find t such that $P_t = 2000$.
Let $P_0 = 1$

Let $P_t = u_t - c$ and $P_{t-1} = u_{t-1} - c$

Substitute into $P_t = 3P_{t-1} + 1$ $\boxed{1}$

$\therefore u_t - c = 3(u_{t-1} - c) + 1$
$u_t = 3u_{t-1} - 3c + 1 + c$

$$\text{set} \quad -\underbrace{3c + 1 + c}\ = 0$$

$$1 = 2c$$
$$c = \frac{1}{2}$$

$\therefore P_t = u_t - 1/2$
Substitute $P_t = u_t - 1/2$ and $P_{t-1} = u_{t-1} - 1/2$ into $\boxed{1}$

$$u_t - \frac{1}{2} = 3\left(u_{t-1} - \frac{1}{2}\right) + 1$$

$$u_t - \frac{1}{2} = 3u_{t-1} - \frac{3}{2} + 1$$

$$\boxed{u_t = 3u_{t-1}}$$

$P_0 = 1 \qquad \therefore P_0 = u_0 - c$
$1 = u_0 - \frac{1}{2}$
$u_0 = \frac{3}{2}$

$\therefore$ we have $u_t = 3u_{t-1}, u_0 = \frac{3}{2}$
The solution is $u_t = (a)\lambda^t$

$u_t = \frac{3}{2}(3)^t, t \geq 0.$

$\therefore P_t = u_t - \frac{1}{2}$

$\boxed{P_t = \frac{3}{2}(3)^t - \frac{1}{2}}, t \geq 0.$

Now, Let $P_t = 12$

$$12 = \frac{3}{2}(3)^t - \frac{1}{2}$$

$\frac{25}{2} = \frac{3}{2}(3)^t$ *multiply by* 2 *and divide by* 3 *then* take the ln of both sides
25 = 3 $(3)^t$
25/3 =$(3)^t$
$\ln\left(\frac{25}{3}\right) = \ln(3)^t$

$$t = \frac{\ln\left(\frac{25}{3}\right)}{ln3}$$

Short Cut Method for Multiple Choice

$P_t = 3P_{t-1} + 1 \quad \leftarrow a = 3 \quad b = 1 \ \ and \ P_o = 1$

$$P_t = \left(P_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}$$
$$Pt = \left(1 - \frac{1}{1-3}\right)(3)^t + (\frac{1}{1-3})$$
$$Pt = \frac{3}{2}(3^t) - \frac{1}{2})$$

Let $P_t = 12$

$$12 = \frac{3}{2}(3)^t - \frac{1}{2}$$

$\frac{25}{2} = \frac{3}{2}(3)^t$ *multiply by* 2 *and divide by* 3 *then* take the ln of both sides
25 = 3 $(3)^t$
25/3 =$(3)^t$
$\ln\left(\frac{25}{3}\right) = \ln(3)^t$

$$t = \frac{\ln\left(\frac{25}{3}\right)}{ln3}$$

Long Method 2: Find the equilibrium

$P_t = 3P_{t-1} + 1$
$\hat{p} = 3\hat{p} + 1$
$\hat{p} = -1/2$

Use $P_t = u_t - 1/2$ and substitute (and $P_{t-1} = u_{t-1} - 1/2$) into

$p_t = 3P_{t-1} + 1$

$\therefore u_t - \frac{1}{2} = 3p_{t-1} + 1$

$$u_t - \frac{1}{2} = 3(u_{t-1} - 1/2) + 1$$

$u_t - \frac{1}{2} = 3u_{t-1} - \frac{3}{2} + 1$

$\boxed{u_t = 3u_{t-1}}$ Now, the rest is the same as Method 1 above.

Long Method 3:

$P_t = 3P_{t-1} + 1 \quad \leftarrow a = 3 \quad b = 1 \ and\ P_o = 1$
Find t $so\ that$ $P_t = 12$
$P_t = 1P_{t-1} + 2P_{t-1} + 1$ $\boxed{1}$
$\Delta\, P_t = P_t - P_{t-1}$
$= 1P_{t-1} + 2P_{t-1} + 1 - P_{t-1}$

$\Delta\, P_t = 2P_{t-1} + 1 \quad Consider\ \Delta P_t = 0$
$0 = 2P_{t-1} + 1$
$-1 = 2P_{t-1}$
$P_{t-1} = -\frac{1}{2}$
Define $u_t = P_t - \left(-\frac{1}{2}\right) = P_t + \frac{1}{2}$ $which\ means\ P_t = u_t - \frac{1}{2}$ $\boxed{2}$
And $u_{t-1} = P_{t-1} + \frac{1}{2}$ $which\ means\ P_{t-1} = u_{t-1} - \frac{1}{2}$ $\boxed{2}$

From $\boxed{1}$ $P_t = 3P_{t-1} + 1 \ \ substitute\ from$ $\boxed{2}$

$$u_t - \frac{1}{2} = 3(u_{t-1} - \frac{1}{2}) + 1$$
$$u_t = 3u_{t-1} - \frac{3}{2} + 1 + \frac{1}{2}$$
$$u_t = 3u_{t-1} \quad \leftarrow b = 3$$

From $\boxed{2}$ $u_t = P_t + \frac{1}{2}$

Substitute $t = 0$ $P_o = 1$

$u_o = P_o + \frac{1}{2}$

$u_o = 1 + \frac{1}{2} = \frac{3}{2}$

$u_t = u_o(b)^t$

$\therefore u_t = \frac{3}{2}(b)^t$

$u_t = \frac{3}{2}(3)^t$ $\boxed{3}$ substitute

From $\boxed{2}$ $u_t = P_t + \frac{1}{2}$

$\frac{3}{2}(3)^t = P_t + \frac{1}{2}\backslash$

$\therefore P_t = \frac{3}{2}(3)^t - \frac{1}{2}$

Let $P_t = 12$

$$12 = \frac{3}{2}(3)^t - \frac{1}{2}$$

$\frac{25}{2} = \frac{3}{2}(3)^t$ *multiply by* 2 *and divide by* 3 *then* take the ln of both sides

25 = 3 $(3)^t$

25/3 =$(3)^t$

$\ln\left(\frac{25}{3}\right) = \ln\,(3)^t$

$$t = \frac{\ln\left(\frac{25}{3}\right)}{ln3}$$

NOTE: For all homework, you can use any of the long methods and I have included several methods for lots of homework questions. If you want to do a particular question a different way than I did, that is fine as long as you get the SAME answer!!! 😊

Practice Exam Questions on Recursion Models

C1. i) $let\ P_0 = 10 \qquad P_0 = 10$

a) $P_1 - P_0 = 6$

$P_1 - 10 = 6$

$P_1 = 16 \qquad\qquad P_2 - P_1 = 6$

$P_2 - 16 = 6 \qquad P_2 = 22$

b) $same\ as\ P_t - P_{t-1} = 6$

$P_t = P_{t-1} + 6 \qquad a = 1 \quad \therefore$

$no\ general\ solution\ using\ the\ formula\ as\ we\ can't$ divide by 1-a if a=1

Since the first term is $\mathrm{P}_0 = 10$, and we are adding 6(1) to it to get 16 and then the next term is adding 6(2) to 10 to get 22, etc.

The solution could be written as $P_t = 6t + 10\ by\ inspection.$

c) $as\ \ t \to \infty, \quad P_t \to \infty$

ii) $let\ P_0 = 10$

a) $P_t = 2P_{t-1} + 5 \qquad P_0 = 10$

$P_1 = 2P_0 + 5 = 2(10) + 5 = 25$

$P_2 = 2P_1 + 5 = 2(25) + 5 = 55$

b) Short-Cut Method for Multiple Choice

$a = 2, \ \ b = 5$

$$P_t = \left(P_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a} = \left(10 - \frac{5}{1-2}\right)(2)^t + \frac{5}{1-2}$$
$$= 15(2)^t - 5$$

C1. ii) Long-method 2: equilibrium

$p_t = 2p_{t-1} + 5$ $\boxed{1}$, $p_0 = 10$

Let $p_t = p_{t-1} = \hat{p}$ and find equilibrium
$\hat{p} = 2\hat{p} + 5$
$-\hat{p} = 5$
$\therefore \hat{p} = -5$

Let $p_t = u_t - 5$ and $p_{t-1} = u_{t-1} - 5$

Substitute into $\boxed{1}$

$p_t = 2p_{t-1} + 5$
$u_t - 5 = 2(u_{t-1} - 5) + 5$
$u_t - 5 = 2u_{t-1} - 10 + 5$
$u_t = 2u_{t-1} - 10 + 5 + 5$
$\boxed{u_t = 2u_{t-1}}$ ← b

$p_0 = 10$
$p_t = u_t - 5$
$u_t = p_t + 5$
$u_0 = p_0 + 5 = 10 + 5 = 15$

Solution is $u_t = b^t u_0$
$u_t = 2^t(15)$
$\therefore p_t = u_t - 5$
$\boxed{p_t = 2^t(15) - 5}$

c) $as\ \ t \to \infty,\ P_t \to \infty$

C1. iii) $P_0 = 10$

a) $P_t = -1.2P_{t-1}$ $P_1 = -1.2(10) = -12$
$P_2 = -1.2P_1 = -1.2(-12) = 14.4$

b) $P_t = b^t P_0 \ = (-1.2)^t(10)$

C2.a) $P_t = 0.85P_{t-1}$ $P_0 = 250$

a) $P_t = b^t P_0 = (0.85)^t(250)$

b) $P_1 = (0.85)^1(250) = 212.5\,mg$

C3. Short-Cut Method:

Since 25 are removed each year, we have: $b = -25$ $P_0 = 1000$ So, the equation would be $P_t = aP_{t-1} + b$ and we get: $P_t = 1.10P_{t-1} - 25$ since the population is increasing by 10% per year. $P_t = \left(P_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}$

$$P_t = \left(1000 - \frac{-25}{1-1.1}\right)(1.1)^t + \frac{-25}{1-1.1}$$
$$P_t = (1000 - 250)(1.1)^t + 250$$
$$P_t = 750(1.1)^t + 250$$

$p_t = 1.10\,p_{t-1} - 25$ $p_0 = 1000$

Long Method 1: Change of variables

Let $p_t = u_t - c$ and $p_{t-1} = u_{t-1} - c$

Substitute into $p_t = 1.10\,p_{t-1} - 25$ $\boxed{1}$

$\therefore u_t - c = 1.10(u_{t-1} - c\,) - 25$

$u_t = 1.10u_{t-1} - 1.1c\ - 25 + c$

$-1.1c\ - 25 + c = 0$

$-0.1c = 25$

$c = -250$

$\therefore p_t = u_t + 250$ $\boxed{2}$

Substitute $p_t = u_t + 250$ and $p_{t-1} = u_{t-1} + 250$ into $\boxed{1}$

$u_t + 250 = 1.10(u_{t-1} + 250) - 25$

$u_t + 250 = 1.10u_{t-1} + 275 - 25$

$\boxed{u_t = 1.10u_{t-1}}$ ← b

$p_0 = 1000$ $\therefore u_0 = p_0 + c$

$p_t = u_t - c$ $u_0 = 1000 - 250$

$\boxed{u_t = p_t + c}$ $\boxed{u_0 = 750}$

$\therefore$ solution is $u_t = b^t u_0$

$u_t = 750(1.10)^t$

$\therefore$ from $\boxed{2}$, $p_t = u_t + 250$

$$\boxed{p_t = 750(1.10)^t + 250}$$

C4. $a)P_0 = 10$ $P_1 = 1.2(10) = 12$

$P_2 = 1.2P_1 = 1.2(12) = 14.4$

$P_3 = 1.2P_2 = 1.2(14.4) = 17.28$

$P_t = b^t P_0$

$\therefore P_t = (1.2)^t(10)\ \ or\ \ 10(1.2)^t$

$b)P_0 = 10 \qquad P_t = aP_{t-1} + b \qquad a = 0.2 \quad b = 4$

$$P_1 = aP_0 + b = 0.2P_0 + 4 = 0.2(10) + 4 = 6$$
$$P_2 = 0.2P_1 + 4 = 0.2(6) + 4 = 5.2$$
$$P_3 = 0.2P_2 + 4 = 0.2(5.2) + 4 = 5.04$$

Short-Cut Method: $P_t = \left(P_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}$

$$= \left(10 - \frac{4}{1-0.2}\right)^{(0.2)^t} + \frac{4}{1-0.2}$$
$$= (10 - 5)(0.2)^t + 5$$
$$P_t = 5(0.2)^t + 5$$

C4. b) $p_t = 0.2p_{t-1} + 4$

Long Method: Change of variables

Let $p_t = u_t - c$ and $p_{t-1} = u_{t-1} - c$

Substitute into $p_t = 0.2p_{t-1} + 4 \quad \boxed{1}$

$$u_t - c = 0.2(u_{t-1} - c) + 4$$
$$u_t - c = 0.2u_{t-1} - 0.2c + 4$$
$$u_t = 0.2u_{t-1} - 0.2c + 4 + c$$
$$-0.2c + 4 + c = 0$$
$$-0.8c = -4$$
$$\boxed{c = -5}$$

$\therefore p_t = u_t + 5 \quad \boxed{2}$

Substitute $p_t = u_t + 5$ and $p_{t-1} = u_{t-1} + 5$ into $\boxed{1}$

$$u_t + 5 = 0.2(u_{t-1} + 5) + 4$$
$$u_t + 5 = 0.2u_{t-1} + 1 + 4$$
$$\boxed{u_t = 0.2u_{t-1}} \leftarrow b = 0.2$$

$$p_t = u_t + 5$$
$$u_t = p_t - 5$$
$$p_0 = 10$$
$$u_0 = p_0 - 5 = 10 - 5 = \boxed{5}$$

$\therefore$ solution is $u_t = b^t u_0$

$$u_t = 5(0.2)^t$$

$\therefore$ from $\boxed{2}$, $p_t = u_t + 5$

$$\boxed{p_t = 5(0.2)^t + 5}$$

C4. b)

Long Method: equilibrium

$\boxed{p_t = 0.2p_{t-1} + 4}$ $\boxed{1}$, $p_0 = 10$

Let $p_t = p_{t-1} = \hat{p}$ and find equilibrium

$\hat{p} = 0.2\hat{p} + 50$

$-0.8\hat{p} = 4$

$\hat{p} = 5$

Let $p_t = u_t + 5$ $\boxed{2}$ and $p_{t-1} = u_{t-1} + 5$

Substitute into $\boxed{1}$

$p_t = 0.2p_{t-1} + 4$

$u_t + 5 = 0.2(u_{t-1} + 5) + 4$

$u_t + 5 = 0.2u_{t-1} + 1 + 4$

$\boxed{u_t = 0.2u_{t-1}} \leftarrow b = 0.2$

$p_0 = 10$

$u_0 = p_0 - 5$

$u_0 = 10 - 5 = 5$

∴ Solution is $u_t = b^t u_0$

$u_t = 5(0.2)^t$

∴ from $\boxed{2}$ $p_t = u_t + 5$

∴ $\boxed{p_t = 5(0.2)^t + 5}$

C5. i) $P_t = 3P_{t-1} + 4 \qquad a = 3 \quad b = 4$

$P_o = 10$

$P_1 = 3P_o + 4 = 3(10) + 4 = 34$

$P_2 = 3P_1 + 4 = 3(34) + 4 = 106$

Short-Cut Method:

b)$P_t = \left(P_0 - \frac{b}{1-a}\right) a^t + \frac{b}{1-a}, a \neq 1 \qquad = \left(10 - \frac{4}{1-3}\right)(3)^t + \frac{4}{1-3}$

$= \left(10 - \frac{4}{-2}\right)(3)^t - 2$

$= (12)(3)^t - 2$

c) $as\ t \to \infty,\ P_t \to \infty$

In other words, $\lim_{t\to\infty}[\,(12)(3)^t - 2\,] = \infty$

$p_t = 3p_{t-1} + 4$

C5. i)**Long Method 1: Change of variables**

Let $p_t = u_t - c$ and $p_{t-1} = u_{t-1} - c$

Substitute into $p_t = 3p_{t-1} + 4 \quad \boxed{1}$

$u_t - c = 3(u_{t-1} - c) + 4$

$u_t - c = 3u_{t-1} - 3c + 4$

$u_t = 3u_{t-1} - 3c + 4 + c$

$-3c + 4 + c = 0$

$-2c = -4$

$\boxed{c = 2}$

$\therefore p_t = u_t - 2 \quad \boxed{2}$

Substitute $p_t = u_t - 2$ and $p_{t-1} = u_{t-1} - 2$ in $\boxed{1}$

$u_t - 2 = 3(u_{t-1} - 2) + 4$

$u_t = 3u_{t-1} - 6 + 4 + 2$

$\boxed{u_t = 3u_{t-1}} \leftarrow$ b

$p_0 = 10$

$p_t = u_t - 2$

$u_t = p_t + 2$

$u_0 = p_0 + 2 = 10 + 2 = 12$

$\therefore$ solution is $u_t = b^t u_0$

$u_t = (3)^t(12)$

$\therefore$ from $\boxed{2}$, $p_t = u_t - 2$

$\boxed{p_t = 12(3^t) - 2}$

C5. ii) **Short-Cut Method**

let $P_o = 10$ $\quad P_o = 10$

$$P_1 = P_o + 5 = 10 + 5 = 15$$

$$P_2 = P_o + 5 = 15 + 5 = 20$$

$$P_t = aP_{t-1} + b \qquad a = 1 \quad b = 5$$

$$\text{b)}P_t = \left(P_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}, a \neq 1$$

$$= 10 - \frac{5}{1-1} \qquad - divide\ by\ 0$$

$no\ general\ solution\ using\ the\ formula\ as\ we\ can't$ divide by 1-a if a=1

Since the first term is $P_0 = 10$, and we are adding $5(1)$ to it to get 15 and then the next term is adding $5(2)$ to 10 to get 20, etc.

The solution could be written as $P_t = 5t + 10\ by\ inspection$.

c) $as\quad t \to \infty,\ P_t \to \infty$ (numbers keep increasing in part a)

C6. **Short-Cut Method**

$$P_t = (P_{t-1} + 0.04P_{t-1}) - 100$$

$$P_t = 1.04P_{t-1} - 100 \qquad a = 1.04 \qquad b = -100$$

$$Pt = \left(Po - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}$$

$$= \left(1000 - \frac{(-100)}{1-1.04}\right)(1.04)^t + \left(\frac{-100}{1-1.04}\right)$$

$$= \left(1000 + \frac{100}{-0.04}\right)(1.04)^t + \left(\frac{-100}{-0.04}\right)$$

$$= (1000 - 2500)1.04^t + 2500$$

$$= -1500(1.04)^t + 2500$$

$$worthless \qquad 0 = -1500(1.04)^t + 2500$$

$$1500(1.04)^t = 2500$$

$$1.04^t = 1.\overline{6}$$

$$\ln 1.04^t = \ln 1.\overline{6}$$

$$t = 13\ years$$

C6. Long Method: equilibrium

$\boxed{p_t = 1.04p_{t-1} + 100}$ $\boxed{1}$, $p_0 = 1000$

Let $p_t = p_{t-1} = \hat{p}$ and find equilibrium
$\hat{p} = 1.04\hat{p} - 100$
$-0.04\hat{p} = -100$
$\hat{p} = 2500$

Let $\boxed{p_t = u_t + 2500}$ $\boxed{2}$ and $p_{t-1} = u_{t-1} + 2500$

Substitute into $\boxed{1}$

$p_t = 1.04p_{t-1} - 100$
$u_t + 2500 = 1.04(u_{t-1} + 2500) - 100$
$u_t = 1.04u_{t-1} + 2600 - 100 - 2500$

$\boxed{u_t = 1.04u_{t-1}}$ $\leftarrow b = 1.04$

$p_0 = 1000$
$u_t = p_t - 2500$
$u_0 = p_0 - 2500$
$u_0 = 1000 - 2500$
$u_0 = -1500$

Solution: $u_t = b^t u_0$
$u_t = -1500(1.04)^t$
$\therefore$ from $\boxed{2}$ $p_t = u_t + 2500$
$\therefore$ $\boxed{p_t = -1500(1.04)^t + 2500}$

Long Method 3:

C6. $P_t = P_{t-1} + 0.04P_{t-1} - 100$ $\quad a = 1.04 \quad b = -100$

$P_o = 1000$

$P_t = 1.04P_{t-1} - 100 \ \boxed{1}$

$\Delta P_t = P_t - P_{t-1} \quad sub\,\boxed{1}$

$= 1.04P_{t-1} - 100 - P_{t-1}$

$\Delta P_t = \underbrace{P_{t-1} + 0.04P_{t-1} - 100}_{\boxed{3}} - P_{t-1}$

$\Delta P_t = 0.04P_{t-1} - 100$

Consider $\Delta P_t = 0$

$0.04P_{t-1} - 100 = 0$

$0.04P_{t-1} = 100$

$P_{t-1} = 2500$

Define $U_t = P_t - 2500$ $\boxed{2}$ which means $P_t = U_t + 2500$

$U_{t-1} = P_{t-1} - 2500$ $\boxed{2}$ which means $P_{t-1} = U_{t-1} + 2500$

From $\boxed{1}$ $P_t = 1.04P_{t-1} - 100$ substitute $\boxed{2}$

$U_t + 2500 = 1.04(U_{t-1} + 2500) - 100$

$U_t = 1.04U_{t-1} + 2600 - 100 - 2500$

$\therefore U_t = 1.04U_{t-1} \quad \leftarrow b = 1.04 \quad U_t = P_t - 2500$

$subst \quad t = 0$

$P_o = 1000$

$U_o = P_o - 2500$

$U_o = 1000 - 2500 \ = \ -1500$

$U_t = U_o(b)^t$

$U_t = -1500(1.04)^t \ \boxed{4}$

$\therefore from\ \boxed{2} \quad U_t = P_t - 2500$

$P_t = U_t + 2500$

$\therefore P_t = -1500(1.04)^t + 2500 \quad from\,\boxed{4}$

C7. $P_t = b^t P_o$ and $P_o = 50$

a) $P_t = bP_{t-1} \ = 1.2P_{t-1}$

b) $P_t = (1.2)^t(50) = 50(1.2)^t$

C8.

a) $a = 1 + \beta - \gamma = 1 + 1.3 - 0.8 = 1.5$

$$P_t = 1.5P_{t-1} - 1500 \qquad harvested \quad \therefore negative$$

Short-Cut Method

$$P_t = \left(P_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}, a \neq 1$$

$$= \left(12\,500 - \frac{-1500}{1-1.5}\right)(1.5)^t + \frac{-1500}{1-1.5}$$

$$P_t = (12\,500 - 3000)(1.5)^t + 3000$$

$$P_t = 9500(1.5)^t + 3000 \qquad sub\ n = 5$$

C8. Long Method: Change of variables

$$p_t = 1.5p_{t-1} - 1500 \quad \boxed{1}, \qquad p_0 = 12500$$

Let $p_t = u_t - c$ and $p_{t-1} = u_{t-1} - c$

Substitute into $\boxed{p_t = 1.5p_{t-1} - 1500}$ $\boxed{1}$

$u_t - c = 1.5(u_{t-1} - c) - 1500$
$u_t = 1.5u_{t-1} - 1.5c - 1500 + c$
$-1.5c - 1500 + c = 0$
$-0.5c = 1500$
$\boxed{c = -3000}$

$\therefore p_t = u_t + 3000 \quad \boxed{2}$

Substitute $\boxed{2}$ and $p_{t-1} = u_{t-1} + 3000$ into $\boxed{1}$

$u_t + 3000 = 1.5(u_{t-1} + 3000) + 4500 - 1500$
$u_t + 3000 = 1.5u_{t-1} + 4500 - 1500$
$u_t = 1.5u_{t-1} + 4500 - 1500 - 3000$
$\boxed{u_t = 1.5u_{t-1}} \leftarrow b = 1.5$

$p_0 = u_0 + 3000, \quad p_0 = 12500$
$u_0 = p_0 - 3000$
$u_0 = 12500 - 3000 = \boxed{9500}$

$\therefore$ solution is $u_t = u_0 b^t$

$u_t = 9500(1.5)^t$

$\therefore$ from $\boxed{2}$
$\boxed{p_t = 9500(1.5)^t + 3000}$

Long Method #3: Recall, there are MANY methods you can use, so as long as you get the same answer, it is fine!

$P_o = 12\ 500$

$P_t = 1.5P_{t-1} - 1500\ \boxed{1}$ $\quad b = -1500$

$a = 1.5$

$P_o = 12\ 500$

$\Delta P_t = P_t - P_{t-1}$

$= 1.5P_{t-1} - 1500 - P_{t-1} \quad sub\ \boxed{1}$

$= P_{t-1} + 0.5P_{t-1} - 1500 - P_{t-1}$

$= \underbrace{P_{t-1} + 0.5P_{t-1} - 1500}_{\boxed{3}} - P_{t-1}$

$\Delta P_t = 0.5P_{t-1} - 1500$

Consider when $\Delta P_t = 0$

$0.5P_{t-1} - 1500 = 0$

$0.5P_{t-1} = 1500$

$P_{t-1} = 3000$

Define $u_t = P_t - 3000\ \boxed{2}$

$u_{t-1} = P_{t-1} - 3000 \quad sub\ \boxed{3}\ into\ \boxed{2}$

$u_t = P_{t-1} + 0.5P_{t-1} - 1500 - 3000$

$u_t = 1(P_{t-1} - 3000) + 0.5(P_{t-1} - 3000)$

$u_t = 1.05(P_{t-1} - 3000) \leftarrow u_{t-1}$

$u_t = 1.05u_{t-1} \quad b = 1.05$

From $\boxed{2}$ $u_t = P_t - 3000 \quad sub\ t = 0 \quad P_o = 12\ 500$

$U_o = P_o - 3000$

$U_o = 12\ 500 - 3000 = 9500$

$\therefore U_t = U_o(b)^t$

$U_t = 9500(1.05)^t$

$\therefore from\ \boxed{2} \quad U_t = P_t - 3000$

$P_t = U_t + 3000$

$\therefore P_t = 9500(1.05)^t + 3000$

b) $P_5 = 9500(1.5)^5 + 3000 = 75\ 140.6$

c) $P_t = 1.5P_{t-1} - 1500$ We want $P_t = P_{t-1}$ $; find\ b$

$P_t = 1.5P_{t-1} - b$

$12\ 500 = 1.5(12\ 500) - b$ $\quad since$ (12,500 is not changing)

$-6250 = -b$

$b = 6250$

C9. Verify that $\hat{b} = 5000$ is the equilibrium to $b_t = 1.04b_t - 200$.

Solution

Since $\hat{b} = 5000$ this means
$b_1 = 5000, b_2 = 5000, ect.$
$\boxed{b_t = 5000}$
i.e. 5000 is a constant and it isn't changing.

$\therefore\ b_t = 1.04b_t - 200$
$= b_t - (1.04b_t - 200)$
$= \hat{b} - (1.04\hat{b} - 200)$
$= 5000 - (1.04(5000) - 200)$
$= 5000 - (5200 - 200)$
$= 0$
$\therefore\ \hat{b} = 5000$ is an equilibrium solution.

C10. Replace $u_t + u_{t-1}\ by\ \hat{u}$
$u_t = \sqrt{7u_{t-1} - 12}$
$\hat{u} = \sqrt{7\hat{u} - 12}$
$(\hat{u})^2 = 7\hat{u} - 12$
$(\hat{u})^2 - 7\hat{u} + 12 = 0$
$(\hat{u} - 4)(\hat{u} - 3) = 0$
$\hat{u} = 4, 3$

So, there are two equilibrium values: 4 and 3.

C11. $p_t = \frac{-1}{4}p_{t-1} - \frac{1}{8}$ omit the square root in the question if you have the original printing as it shouldn't be there!

$\hat{p} = \frac{-1}{4}\hat{p} - \frac{1}{8}$
$1\hat{p} + \frac{1}{4}\hat{p} = -\frac{1}{8}$
$\frac{5}{4}\hat{p} = -\frac{1}{8}$
$\hat{p} = \frac{-1}{8} \times \frac{4}{5}$
$\hat{p} = \frac{-4}{40} = \boxed{\frac{-1}{10}}$

So, there is one equilibrium value, -1/10.

C12.

a) $P_t = 0.80P_{t-1} \qquad P_0 = 50$

b) $P_t = b^t P_0 = (0.8)^t(50)$

c) $P_5 = (0.8)^5(50) = 16.4$

D. Review of Substitution

Example 2. Integrate $\int \frac{cosx}{(1+sinx)^3}dx$

Substitution

$u = 1 + \sin x \qquad du = \cos x dx$

$\int \frac{1}{u^3}du = \int u^{-3}\, du = \frac{u^{-2}}{-2} + c = \frac{-1}{2u^2} + c = \frac{-1}{2(1+\sin x)^2} + c$

Practice Exam Questions on Substitution

D1. Substitution $u = e^x + 1 \quad du = e^x dx$

$$\int u^{-3}du = \frac{u^{-2}}{-2} + c = -\frac{1}{2}(e^x + 1)^{-2} + c$$

D2. Substitution Integrate: $\int \frac{\sec^2 x}{(1+\tan x)^3}dx$

Substitution

$u = 1 + \tan x \qquad du = \sec^2 x dx$

$= \int \frac{1}{u^3}du = \int u^{-3}du = \frac{u^{-2}}{-2} + c \quad or \quad \frac{-1}{2u^2} + c$

$= \frac{-1}{2(1+\tan x)^2} + c$

D3. Substitution

Integrate: $\int \frac{1}{x(ln\, x)^2}dx$

Substitution

$u = \ln x \qquad du = \frac{1}{x}dx$

$= \int \frac{1}{u^2}du = \int u^{-2}du = \frac{u^{-1}}{-1} + c = \frac{-1}{u} + c = \frac{-1}{\ln x} + c$

D4. Integrate $\int \frac{4sec^2x}{(1+tanx)^2}dx$ Substitution

$u = 1 + \tan x \qquad du = \sec^2 x dx$

$= \int \frac{4du}{u^2} = 4\int u^{-2}du = \frac{4u^{-1}}{-1} + c \quad or \quad \frac{-4}{u^1} + c \quad or \quad \frac{-4}{(1+\tan x)} + c$

D5. Integrate $\int \frac{5}{x(1+lnx)^3}dx$ Substitution

$u = 1 + \ln x \qquad du = \frac{1}{x}dx$

$= 5\int \frac{du}{u^3} = 5\int u^{-3}du = 5\frac{u^{-2}}{-2} + c = \frac{-5}{2u^2} + c = \frac{-5}{2(1+\ln x)^2} + c$

E. Separable Differential Equations

Example 1. a) $\frac{dy}{\sin y} = e^x dx \qquad Yes, it\ is\ separable$

*this one we can't solve

b) No, if we multiply by dx we get $dy = (x + 2y)dx$

$$\uparrow$$

Has both $x\ and\ y$ ∴ not separable

c) Yes

$\frac{dy}{dx} = e^x \cdot e^y \qquad using\ exponent\ rules$

$\frac{dy}{e^y} = e^x dx$

$\int e^{-y} dy = \int e^x dx$

$-e^{-y} = e^x + c$

$e^{-y} = -e^x - c$

$\ln e^{-y} = \ln|-e^x - c|$

$-y = \ln|-e^x - c|$

$y = -\ln|-e^x + c|$

Example 2.

Cross-multiply and get $\int 3u^2 du = \int 2t dt$ and integrate

$\frac{3u^3}{3} = t^2 + c$

$u^3 = t^2 + c$

and solve for u(t)

$u(t) = \sqrt[3]{t^2 + c}$

Example 3.

$\int (u + \cos u)du = \int (3t^4 - 2t)dx$

$\frac{u^2}{2} + \sin u = \frac{3t^5}{5} - t^2 + c$. We can't solve this for u(t) on the left!

Example 4.

$$\int \frac{1}{y+2}dy = \int \frac{1}{x-1}dx$$

$\ln|y+2| = \ln|x-1| + c$

$e^{\ln|y+2|} = e^{\ln|x-1|+c} = e^{\ln|x-1|}e^{c}$

$y + 2 = \pm|x-1|(e^{c})$

$y(x) = \pm e^{c}|x-1| - 2$ $\pm\ e^{c} = constant$= call it k or C

$y(x) = C|x-1| - 2$

Example 5. $\frac{du}{dx} = e^{-u(t)}(2t-4)$ $y(5) = 0$

$\frac{du}{e^{-u(t)}} = (2t-4)dx$

$\int e^{u(t)}du = \int(2t-4)dx$

$e^{u(t)} = t^2 - 4t + c$ $sub\ \ t = 5,\ u(t) = 0$

$e^{0} = 5^2 - 4(5) + c$ $c = -4$

$\ln e^{u(t)} = \ln|t^2 - 4t + c|$

$u(t) = \ln|t^2 - 4t - 4|$

Example 6.

$$\frac{dy}{dt} = \frac{4te^{t^2}}{2y+1} \qquad y(0) = 2$$

$$\int (2y+1)dy = \int 4te^{t^2}dt$$

Substitution:

$$let\, u = t^2$$
$$du = 2tdt$$
$$2du = 4tdt$$

$$\int (2y+1)\,dy = 2\int e^4 du$$

$y^2 + y = 2e^u + c$

$y^2 + y = 2e^{t^2} + c \qquad sub\ (0,2)$

$2^2 + 2 = 2e^0 + c$

$4 + 2 = 2 + c \qquad \therefore c = 4$

$\therefore y^2 + y = 2e^{t^2} + 4$ We can't solve for y on the left!

Example 7.

$\int \frac{du}{u(t)} = \int 5dt$

$\ln|u(t)| = 5t + c$

$u(t) = \pm e^{5t} \cdot e^c \qquad \pm e^c = constant$

$u(t) = ce^{5t}$

F. Linear Differential Equations

Example 1.

$x^2y' + 3xy = 2\sin x$

$y' + \frac{3}{x}y = \frac{2\sin x}{x^2}$

$V(x) = I = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$ (since x>0 $\ln|x| = lnx$)

The answer is $\boxed{C}$

Example 2.

$xy' - (2x+3)y = 2\sec x$

$y' - \frac{(2x+3)}{x}y = \frac{2\sec x}{x}$

$V(x) = I = e^{\int \frac{-2x-3}{x}dx} = e^{\int\left(-2-\frac{3}{x}\right)dx} = e^{-2x-3\ln x} = e^{-2x}e^{\ln x^{-3}} = e^{-2x} \cdot x^{-3} = x^{-3}e^{-2x}$

Example 3.

$xy' + 2y = x^{-3}$

$y' + \frac{2}{x}y = \frac{x^{-3}}{x}$

$y' + \frac{2}{x}y = x^{-4}$ $V(x) = I = e^{\int \frac{2}{x}dx} = e^{2|lnx|} = e^{2\ln x} = e^{\ln x^2} = x^2$ **since x>0, we can take off the absolute value of lnx and just say it is lnx

$V(x)y = \int V(x)Q(x)dx$

$x^2y = \int x^{-4}\,x^2dx$

$x^2y = \int x^{-2}\,dx$

$x^2y = \frac{x^{-1}}{-1} + c$

$x^2y = \frac{-1}{x} + c$

$y = \frac{-1}{x^3} + cx^{-2}$

Example 4.

$u'(t) + 3u(t) = e^{-t}$

$V(x) = I = e^{\int 3dx} = e^{3t}$

$$e^{3t}u(t) = \int e^{3t}e^{-t}\,dt$$

$$e^{3t}u(t) = \int e^{2t}dt$$

$$e^{3t}u(t) = \frac{e^{2t}}{2} + C$$

$$\therefore u(t) = \frac{e^{2t}}{2e^{3t}} + \frac{C}{e^{3t}}$$

$$u(t) = \frac{1}{2}e^{-t} + Ce^{-3t}$$

What if you have the initial condition (0,3)?

$e^{3t}u(t) = \frac{e^{2t}}{2} + C$ substitue t=0 and u(t) =3 to find c

$$e^{0}(3) = \frac{e^{0}}{2} + C$$

$$c = 3 - \frac{1}{2} = \frac{6}{2} - \frac{1}{2} = \frac{5}{2}$$

$e^{3t}u(t) = \frac{e^{2t}}{2} + \frac{5}{2}$ Divide by e^{3t}

$$u(t) = \frac{1}{2}e^{-t} + \frac{5}{2}e^{-3t}$$

Example 5. $I = V(x) = e^{\int P(x)dx} = e^{\int 3x^2dx} = e^{x^3}$

$$V(x)y = \int V(x)Q(x)dx$$

$Subst \ldots u = x^3 \quad du = 3x^2dx \quad 2du = 6x^2dx$

$$e^{x^3}y = \int e^{x^3}6x^2dx$$

$$e^{x^3}y = \int e^{u}(2du)$$

$$e^{x^3}y = 2e^{u} + c$$

$$e^{x^3}y = 2e^{x^3} + c$$

$$y = \frac{2e^{x^3}}{e^{x^3}} + \frac{c}{e^{x^3}}$$

$$y = 2 + ce^{-x^3}$$

Example 6.

Separable: $\frac{du}{dt} = 2u(t)$

$\int \frac{du}{u(t)} = \int 2dt$

$\ln|u(t)| = 2t + c$

$e^{\ln|u(t)|} = e^{2t+c}$

$e^{\ln|u(t)|} = e^{2t} \cdot e^{c}$

$u(t) = \pm e^{c} e^{2t}$

$\therefore u(t) = ke^{2t}$

Substitution: let $u(t) = Ae^{t}$

Example 7. If $y_1(x) and\ y_2(x)$ both solve the inhomogeneous D.E.

$$y'(x) = 6p(x)y(x) + 5q(x)$$

Find another solution:

$a = 6 \qquad 1 - a = 1 - 6 = -5$

$y(x) = ay_1(x) + (1 - a)y_2(x)$

$\therefore y(x) = 6y_1(x) - 5y_2(x)$ is also a solution

Example 8. Let $y''' - y'' - 2y' = 0$ be a differential equation.

Let $y = e^{2x}$

$y' = 4e^{2x}$

$y'' = 8e^{2x}$

$y''' - y'' - 2y'$ (substitute)

$= 8e^{2x} - 4e^{2x} - 2(2e^{2x})$

$= 4e^{2x} - 4e^{2x}$

$= 0$

$\therefore$ it is a solution

$y = 6e^{2x}$

$y' = 12e^{2x}$

$y'' = 24e^{2x}$

$y''' = 48e^{2x}$

$\underline{y''' - y'' - 2y'}$

$= 48e^{2x} - 24e^{2x} - 2(12e^{2x})$

$= 48e^{2x} - 24e^{2x} - 24e^{2x}$

$= 0$

$\therefore$ it is a solution

$y_1 - y_2 = 6e^{2x} - e^{2x}$

$y_1 - y_2 = 5e^{2x}$

Check

$(y_1 - y_2)' = 10e^{2x}$

$(y_1 - y_2)'' = 20e^{2x}$

$(y_1 - y_2)'' = 40e^{2x}$

$y''' - y'' - 2y'$
$= 40e^{2x} - 20e^{2x} - 2(10e^{2x})$
$= 0$
∴ it is a solution

→ Works for all answers!

Note: Not the same as the theorem for inhomogeneous we talked about as this equation in example 7 is a HOMOGENEOUS equation.

Example 9.

Solve: $u'(t) + \frac{1}{t+1}u(t) = t^2$

$$P(t) = \frac{1}{t+1} \qquad Q(t) = t^2$$

Since P(t) is discontinuous at $t = -1$

$D = (-\infty, -1) \cup ((-1, \infty)$

a) $u(o) = 1 \quad \therefore D = (-1, \infty)$

$$v(t) = e^{\int P(t)dt} = e^{\int \frac{1}{t+1}dt} = e^{\ln|t+1|} \quad *$$
$$= t + 1 \quad since\ t > -1$$

$$v(t)u(t) = \int v(t)Q(t)$$
$$(t+1)u(t) = \int (t+1)(t^2)dt$$
$$(t+1)u(t) = \int (t^3 + t^2)dt$$
$$(t+1)u(t) = \frac{t^4}{4} + \frac{t^3}{3} + c \quad sub\ (0,1) \quad t = 0, u(t) = 1$$
$$(0+1)(1) = 0 + 0 + c$$
$$c = 1$$
$$\therefore (t+1)u(t) = \frac{t^4}{4} + \frac{t^3}{3} + 1$$
$$(t+1)u(t) = \frac{3t^4}{12} + \frac{4t^3}{12} + \frac{12}{12}$$
$$(t+1)u(t) = \frac{3t^4+4t^3+12}{2}$$
$$u(t) = \frac{3t^4+4t^3+12}{2(t+1)}$$

b) Same equation as a)

$u(-2) = 1 \ \leftarrow \ D = (-\infty, -1) \ since \ t < -1$

from * in part a)

$v(t) = e^{\ln|t+1|} = -(t+1) \ since \ t < -1$

$\therefore \ -(t+1)u(t) = \int -(t+1)(t^2)dt$

$-(t+1)u(t) = \int(-t^3 - t^2)dt$

$-(t+1)u(t) = \frac{-t^4}{4} - \frac{t^3}{3} + c$

$sub \ \binom{-2,1}{t,u} \quad -(-2+1)(1) = \frac{-(-2)^4}{4} - \frac{(-2)^3}{3} + c$

$-(-1) = \frac{-16}{4} - \left(\frac{-8}{3}\right) + c$

$1 + \frac{16}{4} - \frac{8}{3} = c$

$c = \frac{12}{12} + \frac{48}{12} - \frac{32}{12}$

$= \frac{28}{12}$

$= \frac{7}{3}$

$\therefore -(t+1)u(t) = \frac{-t^4}{4} - \frac{t^3}{3} + \frac{7}{3}$

$-(t+1)u(t) = \frac{-3t^4 - 4t^3 + 28}{12}$

$u(t) = \frac{-(-3t^4 - 4t^3 + 28}{12(t+1)} \quad or \quad u(t) = \frac{3t^4 + 4t^3 - 28}{12t + 12}$

G. Bernoulli Differential Equations

Example 1.

$y' + \frac{4}{x}y = x^3y^2, y(2) = -1, x > 0$

$\uparrow n = 2 \; \frac{dy}{dx} + P(x)y = y^nQ(x) \; n = 2$

$Q(x) = x^3 \quad P(x) = \frac{4}{x}$

Use $\frac{du}{dx} + (1-n)up(x) = (1-n)Q(x) \quad u = y^{1-n} = y^{1-2} = y^{-1}$

$u' - \frac{4}{x}u = -x^3 \qquad V(x) = I = e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = e^{\ln x^{-4}} = \frac{1}{x^4}$ **since x>0, we can just take off the absolute value (since x>0 $\ln|x| = lnx$)

V(x) u= $\int V(x)Q(x)dx$

$\frac{1}{x^4}u = \int \frac{1}{x^4}(-x^3)\, dx$

$$\frac{1}{x^4}u = \int \frac{-1}{x}dx$$

$\frac{1}{x^4}u = -\ln|x| + c$

$u = x^4(-\ln|x| + c)$ *from substitution* $u = y^{-1}$

$y^{-1} = x^4(-\ln|x| + c)$

$y = \frac{1}{x^4(c - \ln|x|)} \; * \quad y(2) = -1$

$-1 = \frac{1}{2^4(c - \ln 2)}$

$-16(c - \ln 2) = 1$

$-16c + 16\ln 2 = 1$

$c = \frac{1 - 16\ln 2}{-16} \quad = \frac{-1}{16} + \ln 2$

$\therefore y = \frac{1}{x^4\left[\ln 2 - \frac{1}{16} - ln|x|\right]} \qquad from *$

Example 2. $x^3y'(x) = y(x)\big(1 + y(x)\big) \qquad y(1) = 1$

$$x^3y'(x) = y(x) + (y(x))^2$$
$$y'(x) = \frac{1}{x^3}y(x) + \frac{1}{x^3}(y(x))^2$$
$$y'(x) - \frac{1}{x^3}y(x) = \frac{1}{x^3}(y(x))^2$$
$$P(x) = -\frac{1}{x^3} \qquad Q(x) = \frac{1}{x^3}$$
$$y^n = y^2 \quad \therefore n = 2$$
$$u = y^{1-n} = y^{1-2} \ = y^{-1}$$

$$\frac{du}{dx} + (1-n)uP(x) = (1-n)Q(x)$$

$$\frac{du}{dx} + (-1)u\left(\frac{-1}{x^3}\right) = -1\left(\frac{1}{x^3}\right)$$
$$\frac{du}{dx} + x^{-3}u = -x^{-3} \quad linear$$

$$V(x) = e^{\int x^{-3}dx} = \ e^{x^{-2}/-2} = e^{-1/2x^2}$$
$$V(x)u = \int V(x)Q(x)dx$$
$$e^{-1/2x^2}u = \int e^{-1/2x^2}(-x^{-3})dx$$

Substitution $w = \frac{-1}{2x^2} = \frac{-1}{2}x^{-2}$

$$dw = -\frac{1}{2}(-2x^{-3})dx$$
$$dw = x^{-3}dx$$
$$e^{-1/2x^2}u = -\int e^w dw$$
$$e^{-1/2x^2}u = -e^w + c$$
$$e^{-1/2x^2}u = -e^{-1/2x^2} + c$$
$$u = -1 + ce^{1/2x^2}$$

Substitution $u = y^{-1}$

$$y^{-1} = -1 + ce^{1/2x^2}$$
$$y = \frac{1}{-1+ce^{1/2x^2}}$$

$$y(1) = 1$$
$$1 = \frac{1}{-1+ce^{1/2}}$$
$$-1 + ce^{1/2} = 1$$
$$ce^{1/2} = 2$$
$$c = 2\Big/e^{\frac{1}{2}} \qquad \therefore y(x) = \frac{1}{-1+\frac{2}{e^{\frac{1}{2}}}e^{1/2x^2}}$$

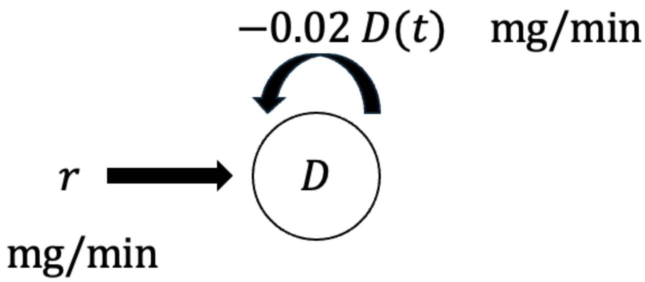
H. Applications

Rate of Change

Example 1.

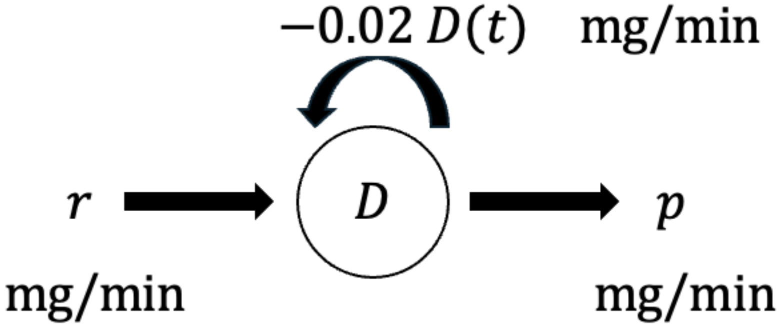
a) $D'(t) = -0.02\,D(t)$.

b) $D'(t) = \ -0.02\,D(t) + r$
mg/min mg/min mg/min

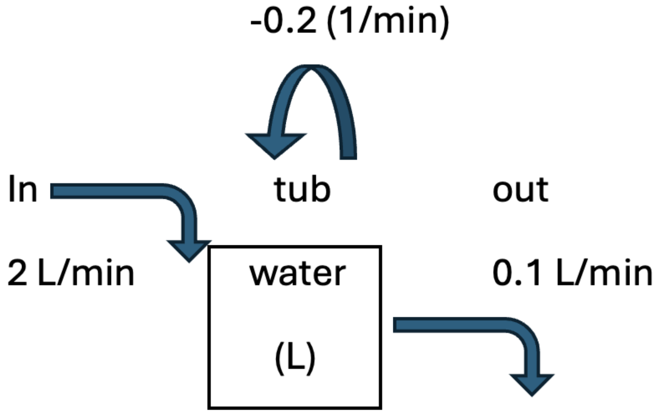


c) Blood loss means we should subtract "p" mg/min

$D'(t) = -0.02\,D(t) + r - p$
mg/min mg/min mg/min



Example 2.



$$V'(t) \text{ or } \frac{dV}{dt} = \overbrace{\underset{\frac{1}{min}\cdot mg}{-0.2V(t)}}^{\text{Internal}} + \overbrace{\underset{\frac{mg}{min}}{2} - \underset{\frac{mg}{min}}{0.1}}^{\text{External}}$$

Mixing Problems

Example 3. (0,0) Let y(t) be the amount of salt in grams
$\frac{dy}{dt} = 20(2) - \frac{1y}{100+t}$ so that means $\frac{dy}{dt} + \frac{1y}{100+t} = 40$

Where 20 g/L is the concentration of in flow at time t and 2L/min is the liquid inflow rate at time t and the liquid outflow rate is 1L/min and the concentration of outflow at time t is y(t).

Example 4. $(0,0)\quad t, salt \qquad V(t) = I = \ e^{\int \frac{1}{400}} = e^{\frac{1}{400}t}$
Let y(t) be the amount of salt in kg

Where 0.1kg/L is the concentration of in flow at time t and 5L/min is the liquid inflow rate at time t and the liquid outflow rate is 5L/min and the concentration of outflow at time t is y(t).

$$\frac{dy}{dt} = (in)(in) - \frac{y(out)}{volume}$$
$$\frac{dy}{dt} = (0.1)(5) - \frac{5y}{2000}$$
$$\frac{dy}{dt} = 0.5 - \frac{1}{400}y$$
$$\frac{dy}{dt} + \frac{1}{400}y = 0.5$$
$$V(t)y = \int V(t)Q(t)dt$$
$$e^{\frac{1}{400}t}y = \int e^{\frac{1}{400}t}0.5$$
$$e^{\frac{1}{400}t}y = \frac{0.5e^{\frac{1}{400}t}}{\frac{1}{400}} + c$$
$$e^{\frac{1}{400}t}y = 200e^{\frac{1}{400}t} + c$$
$$y = 200 + \frac{c}{e^{\frac{1}{400}t}}$$
$$solve\ for\ c\ \rightarrow sub(0,0) \qquad 0 = 200 + \frac{c}{e^0}$$
$\therefore c = -200 \qquad y = 200 - 200e^{\frac{-1}{400}t}$ and y(0)=100

NOTE: If they said you had 7kg/L at the start and 2000 L in the tank, then y(0) would be y(0)= 7 kg/L (2000L) = 14000 kg, or y(0)=14000

IV Drug Problems

Example 5. $Rate\ in\left(\frac{mg}{hr}\right) = concentration \times flow\ of\ drug$
$$= 150\frac{mg}{L} \times \frac{40mL}{h} \leftarrow units\ need\ to\ match$$
$$= 150\frac{mg}{L} \times 0.04\ L/hr$$
$$= 6\ mg/hr$$

Rate out = rate metabolized + rate exceeded

Rate metabolized= $k \cdot u(t)$

$$k = \ln\left(\frac{1}{1-p}\right) \leftarrow \%\ of\ drug\ metabolized$$
$$k = \ln\left(\frac{1}{1-0.8}\right)$$
$$k = \ln\left(\frac{1}{0.2}\right) = \ln 5$$
$$\therefore rate\ metabolized = \ln 5 \cdot u(t)$$

Rate excreted=concentration× flow

$$= \frac{u(t)mg}{5\ L} \times \frac{0.04L}{hr}$$
$$= \frac{1}{125} u(t) mg/hr$$

$$rate\ out = \ln 5 \cdot u(t) + \frac{1}{125} u(t)$$
$$= \left(\ln 5 + \frac{1}{125}\right) u(t)$$

$$\frac{du}{dt} = rate\ in - rate\ out$$
$$\frac{du}{dt} = 6 - \left(\ln 5 + \frac{1}{125}\right) u(t)$$

$$\frac{du}{dt} + \overbrace{\left(\ln 5 + \frac{1}{125}\right)}^{\swarrow P(t)} u(t) = 6 \quad \leftarrow Q(t)$$

$\searrow$ *let this be A*

$$V(t) = I = e^{\int A dt} = e^{At}$$

$$V(t)u = \int V(t)Q(t)dt$$
$$e^{At}u = \int 6e^{At} dt$$
$$e^{At}u = \frac{6e^{At}}{A} + c$$
$$u = \frac{6}{A} + ce^{-At}$$

$$\therefore u(t) = \frac{6}{A} + ce^{-At} \quad substitute\ \ (0,0)$$

$\uparrow (t, u)$

$$0 = \frac{6}{A} + ce^0$$
$$c = \frac{-6}{A}$$
$$\therefore u = \frac{6}{A} - \frac{6}{A} e^{-At}$$
$$u(t) = \frac{6}{A}(1 - e^{-At})$$
$$A = \left(\ln 5 + \frac{1}{125}\right)$$

$$u(t) = \frac{6}{(ln5 + \frac{1}{125})}(1 - e^{-\left(\ln 5 + \frac{1}{125}\right)t})$$

Population

Example 6. $N(t)\ is\ the\ number\ of\ animals\ at\ time\ \ t$

$$\frac{dN}{dt} = a\left(\frac{1}{time}\right) \times \frac{N}{pigs} + n\left(\frac{pigs}{time}\right) - b\left(\frac{1}{time}\right) \times \frac{N}{pigs}$$

$$\frac{dN}{dt} = aN - bN + n$$

$$= (a-b)N + n$$

To find long-term pop find $N(t)\ and\ let\ t \to \infty$

$\boxed{1}$ $N'(t) - (a-b)N = n$

$N'(t) + (b-a)N = n$ Q(t)=n P(t)= (b-a)N

$\boxed{2}$ $V(t) = I = e^{\int (b-a)dt} = e^{(b-a)t}$

$\boxed{3}$ $(V(t)N) = \int V(t)Q(t)dt$

$\left(e^{(b-a)t}N\right) = \int e^{(b-a)t} n\, dt$

$$e^{(b-a)t}N = \frac{ne^{(b-a)t}}{(b-a)} + c$$

$$N(t) = \frac{n}{b-a} + \frac{c}{e^{(b-a)t}}$$

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty}\left[\frac{n}{b-a} + \frac{c}{e^{(b-a)t}}\right]$$

$$= \boxed{\frac{n}{b-a}}\ pigs$$

I. Practice Exam Questions on Differential Equations

I1. $\int y dy = \int x^2 dx$

$\frac{y^2}{2} = \frac{x^3}{3} + c \quad \leftarrow sub\ (1,2)$

$\frac{2^2}{2} = \frac{1^3}{3} + c$

$2 = \frac{1}{3} + c$

$c = 2 - \frac{1}{3} = \frac{5}{3}$

$\therefore\ \frac{y^2}{2} = \frac{x^3}{3} + \frac{5}{3}$

$y^2 = \frac{2x^3}{3} + \frac{10}{3}$

$y = \pm\sqrt{\frac{2}{3}x^3 + \frac{10}{3}}$

I2. $\int y dy = \int 2x + \sec^2 x$

$\frac{y^2}{2} = x^2 + \tan x + c$

$y^2 = 2x^2 + 2\tan x + 2c$

$y = \pm\sqrt{2x^2 + 2\tan x + k}$

I3. $\int \frac{\boldsymbol{dy}}{\boldsymbol{y}} = \int \frac{\mathbf{1}}{\boldsymbol{x+1}} \boldsymbol{dx}$

$\ln|\mathrm{y}| = \ln|x+1| + c$

$e^{\ln y} = \pm e^{\ln|x+1|+c}$

$y = ke^{\ln|x+1|} \quad or \quad y = k|x+1|$

I4. $\frac{dy}{dx} = 3e^x e^y$

$\int \frac{dy}{e^y} = \int 3e^x dx$

$\frac{e^{-y}}{-1} = 3e^x + c$

$e^{-y} = -3e^x - c$ substitute x=0 and y=1

$e^{-1} = -3e^0 - c$

$c = -3 - \frac{1}{e}$

$\therefore e^{-y} = -3e^x + 3 + \frac{1}{e}$

$\ln e^{-y} = \ln\left|-3e^x + 3 + \frac{1}{e}\right|$

$-y = \ln\left|-3e^x + 3 + \frac{1}{e}\right|$

$y = -\ln\left|-3e^x + 3 + \frac{1}{e}\right|$

I5. $\int 2y dy = \int \frac{x}{x^2+3} dx$ *substitution* $u = x^2 + 3$ $\frac{du}{2} = \frac{2xdx}{2}$

$\int 2y dy = \frac{1}{2} \int \frac{1}{u} du$

$y^2 = \frac{1}{2} \ln|u| + c$

$y^2 = \frac{1}{2} \ln|x^2 + 3| + c$

$y = \pm\sqrt{\frac{1}{2} \ln|x^2 + 3| + c}$

I6. $\int y dy = \int \frac{t}{e^{t^2+1}} dt$ $u = t^2 + 1$ $\frac{du}{2} = \frac{2t}{2} dt$

$\frac{1}{2} du = t\, dt$

$\frac{y^2}{2} = \frac{1}{2} \int \frac{1}{e^u} du$

$\frac{y^2}{2} = \frac{1}{2} \frac{e^{-u}}{-1} + c$

$\frac{y^2}{2} = \frac{-1}{2e^u} + c$

$\frac{y^2}{2} = \frac{-1}{2e^{t^2+1}} + 2c$

$y^2 = \frac{-2}{2e^{t^2+1}} + k$

$y = \pm\sqrt{\frac{-1}{e^{t^2+1}} + k}$

or

$y = \pm\sqrt{\frac{-1}{e^{t^2+1}} + k}$ since k=2c is just a constant

I7. $\int e^y dy = \int \frac{\sin x}{\cos^2 x} dx$ $u = \cos x$ $-du = \sin x dx$

$e^y = -\int u^{-2} du$

$e^y = \frac{-u^{-1}}{-1} + c$

$e^y = \frac{1}{u} + c$

$e^y = \frac{1}{\cos x} + c$

$e^y = \sec x + c$

$y = \ln|\sec x + c|$

I8. $\int y^2 dy = \int \sin x dx$

$\frac{y^3}{3} = -\cos x + c$

$y^3 = -3\cos x + 3c$

$y = \sqrt[3]{-3\cos x + k}$ since k=3c is just a constant. Call it C or k, or whatever!

I9. $\frac{dy}{dt}(2y-2)=\frac{1}{t^2}$

$\int dy(2y-2)=\int t^{-2}dt$

$y^2-2y=\frac{t^{-1}}{-1}+c$

$y^2-2y=\frac{-1}{t}+c \quad sub\ t=1,\ y=0$

$0-0=\frac{-1}{1}+c \quad c=1$

$\therefore y^2-2y=\frac{-1}{t}+1$

I10. $\frac{dN}{dt}=t^3(N+3)$

$\int\frac{dN}{N+3}=\int t^3dt$

$\ln|N+3|=\frac{t^4}{4}+c \quad sub\ t=0, N=5 \quad to\ find\ c$

$\ln 8=0+c$

$c=\ln 8$

$\therefore \ln|N+3|=\frac{t^4}{4}+\ln 8 \quad sub\ \ t=2$

$\ln|N+3|=\frac{2^4}{4}+\ln 8$

$\ln|N+3|=4+\ln 8$

$N+3=e^{4+\ln 8}$

$N+3=e^4e^{\ln 8}$

$N+3=8e^4$

$\therefore N=8e^4-3$

I11. Linear

$P(x)=-1 \quad Q(x)=e^x \quad V(x)=I=e^{\int P(x)dx}=e^{\int -1dx}=e^{-x}$

$V(x)y=\int V(x)Q(x)dx$

$e^{-x}y=\int e^{-x}e^xdx$

$e^{-x}y=\int 1dx$

$e^{-x}y=x+c$

$y=xe^x+ce^x$

I12. Linear $\frac{dy}{dx} = \frac{x^{\frac{1}{2}}}{x} - \frac{y}{x}$

$$\frac{dy}{dx} + \frac{1}{x}y = x^{\frac{-1}{2}}$$

P(x)=1/x is not continuous at x=0. Since x>0 so we look at (0,∞) and we get: $|x| = x$. Remember, the domain can't include any negative numbers, as we have a square root of x.

$$V(x) = I = e^{\int \frac{1}{x}dx} = e^{ln|x|} = x$$

$$V(x)y = \int V(x)Q(x)dx$$

$$xy = \int xx^{\frac{-1}{2}} \text{ dx}$$

$$xy = \int x^{\frac{1}{2}dx}$$

$$xy = \frac{2}{3}x^{\frac{3}{2}} + c$$

$$y = \frac{2}{3}x^{\frac{1}{2}} + cx^{-1}$$

$$y = \frac{2}{3}\sqrt{x} + cx^{-1}$$

I13.

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{x^{-1}e^x}{x}$$

$\frac{dy}{dx} + \frac{2}{x}y = x^{-2}e^x$ P(x)=2/x and Q(x)= $x^{-2}e^x$

Linear V(x)= $I = e^{\int \frac{2}{x}dx} = e^{2ln|x|} = e^{ln|x|^2} = (-x)^2 = x^2$

P(x) is not continuous at x=0, so we look at (-∞, 0) and we get:

$|x| = -x$

So, V(x)= $e^{2ln|x|} = e^{ln|x|^2} = (-x)^2 = x^2$

If we look at (0, ∞)

$|x| = x$

So, V(x)= $e^{2ln|x|} = e^{ln|x|^2} = x^2$

Both answers are the same!

$$V(x)y = \int V(x)Q(x)dx$$

$$x^2y = \int x^2x^{-2}e^x \, dx$$

$$x^2y = \int e^x dx$$

$x^2y = e^x + c$ $y = \frac{e^x}{x^2} + \frac{c}{x^2}$

I14. Linear $V(x) = \quad I = e^{\int \frac{1}{x}dx} = e^{ln|x|} = e^{\ln x} = x$ and since the point is (1,2) x>0, so we just take off the absolute value and do nothing

$$\frac{dy}{dx} + \frac{x}{x^2}y = \frac{\ln x}{x^2}$$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\ln x}{x^2}$$

$$V(x)y = \int V(x)Q(x)dx$$

$$(xy) = \int x\,(\frac{\ln x}{x^2})dx$$

$$\int \frac{d}{dx}(xy) = \int \frac{\ln x}{x} \qquad u = \ln x \quad du = \frac{1}{x}dx$$

$$xy = \int udu = \frac{u^2}{2} + c = \frac{(\ln x)^2}{2} + c$$

$$sub\ (1,2) \quad (1)(2) = \frac{(\ln 1)^2}{2} + c \qquad c = 2$$

$$\therefore y = \frac{(\ln x)^2}{2x} + \frac{2}{x}$$

I15. Linear

P(x) is not continuous at x=0, if we look at x>0, we get $|x| = x$

$$V(x) = I = e^{\int \frac{-3}{x}dx} = e^{-3ln|x|} = e^{\ln|x|^{-3}} = |x|^{-3} = x^{-3}$$

$$\frac{dy}{dx} - \frac{3}{x}y = 2x^3$$

$$V(x)y = \int V(x)Q(x)dx$$

$$x^{-3}y = \int x^{-3}(2x^3)dx$$

$$\int \frac{d}{dx}(x^{-3}y) = \int 2dx$$

$$x^{-3}y = 2x + c$$

$$y = 2x^4 + cx^3$$

If $V(x) = I = e^{\int \frac{-3}{x} dx} = e^{-3ln|x|} = e^{\ln|x|^{-3}} = |x|^{-3}$=$(-x)^{-3}$=$-x^{-3}$
x<0, we get $|x| = -x$

$$-x^{-3}y = -\int x^{-3}(2x^3)dx$$

$\int \frac{d}{dx}(x^{-3}y) = \int 2dx$ divide both sides by -1

$$x^{-3}y = 2x + c$$

$$y = 2x^4 + cx^3$$

I16. Linear $\frac{dy}{dx} + \frac{1}{x^2}y = 3e^{\frac{1}{x}}$

$$V(X) = I = e^{\int \frac{1}{x^2} dx} = e^{\int x^{-2} dx} = e^{x^{-1}/-1} = e^{-\frac{1}{x}}$$

$$V(x)y = \int V(x)Q(x)dx$$

$$e^{-\frac{1}{x}}y = \int 3e^{\frac{1}{x}} e^{-\frac{1}{x}} dx$$

$$e^{-\frac{1}{x}}y = \int 3dx$$

$e^{-\frac{1}{x}}y = 3x + c$ divide every term by $e^{-\frac{1}{x}}$

$$y = 3xe^{\frac{1}{x}} + ce^{\frac{1}{x}}$$

I17. $V(x) = I = e^{\int P(x)dx} = e^{\int -e^x dx} = e^{-e^x}$

I18. Let y(t) be the amount of salt in kg
Where 0.25k/L is the concentration of in-flow at time t and 3L/min is the liquid inflow rate at time t and the liquid outflow rate is 3L/min and the concentration of outflow at time t is y(t).

$a)$ $(0,0)$ $\frac{dy}{dt} = (0.25)(3) - \frac{3y}{400}$

$$\frac{dy}{dt} = 0.75 - \frac{3}{400}y$$

$$b)\ \frac{dy}{dt} = 0.75 - \frac{4y}{400 - 1t}$$

I19. Let y(t) be the amount of salt in kg
Where 0.25kg/L is the concentration of in-flow at time t and 5L/min is the liquid inflow rate at time t and the liquid outflow rate is 5L/min and the concentration of outflow at time t is y(t).

$\frac{dy}{dt} = (in)(in) - \frac{y(out)}{volume\ at\ time\ t}$ $(0{,}25)$ $salt\ at\ start$

$$\frac{dy}{dt} = (0.25)(5) - \frac{y(5)}{1000} \quad OR \quad \frac{dy}{dt} = 1.25 - \frac{1}{200}y$$

I20. Linear a) $P(x) = \frac{-1}{x}$

$$V(x) = I = e^{\int f(x)dx} = e^{\frac{-1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$$

$$b)\ \frac{dy}{dx} - \frac{3x^2y}{x} = \frac{x^3}{x}$$

$$\frac{dy}{dx} - 3xy = x^2$$

$$V(x) = I = e^{\int P(x)dx} = e^{\int -3x} = e^{\frac{-3x^2}{2}}$$

$$c)\ \frac{dy}{dx} + y = x^2$$

$$P(x) = 1$$

$$V(x) = I = e^{\int 1dx} = e^x$$

I21. Linear $\frac{dy}{dx} - \frac{2y}{x} = \frac{x^2}{x}$ → $\frac{dy}{dx} - \frac{2y}{x} = x$ $P(x) is \ \frac{-2}{x}$ $x \ is \ Q(x)$

$$V(x) = I = e^{\int \frac{-2}{x}dx}$$

$$V(x) = I = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

** note: P(x)j is discontinuous at x=0, but here if you take x>0, you get V(x)= $x^{-2} = \frac{1}{x^2}$
As when you remove the absolute value, if it is a positive quantity under the absolute value, you just take off the absolute value and do nothing
If x<0, you get $V(x) = I = e^{-2ln|x|} = e^{\ln|x|^{-2}} = (-x)^{-2} = \frac{1}{x^2}$ so they are equal!!

$$V(x)y = \int V(x)Q(x)dx$$

$$x^{-2}y = \int x^{-2}xdx$$

$$x^{-2}y = \int \frac{1}{x}dx$$

$$x^{-2}y = \ln|x| + c$$

$$y = x^2\ln|x| + cx^2$$

I22. Linear $\frac{dy}{dx} + y = e^{-x}$ $P(x) = 1$ $Q(x) = e^{-x}$ y(0)=2

$$V(x) = I = e^{\int P(x)dx} = e^{\int 1 \, dx} = e^x$$

$$V(x)y = \int V(x)Q(x)dx$$

$$e^x y = \int e^{-x}e^x dx$$

$e^x y = \int$ 1dx

$$e^x y = x + c$$

$y = \frac{x}{e^x} + \frac{c}{e^x}$ OR $xe^{-x} + ce^{-x}$ sub (0,2) 2= $0e^0 + ce^0$ $c = 2$

$$y = xe^{-x} + 2e^{-x}$$

I23. Linear $y(\pi) = 0$. Since x=pi>0, when you take off the absolute value, you just get x:

$V(x) = I = e^{\int \frac{3}{x} dx} = e^{3ln|x|} = e^{\ln|x|^3} = x^3$ since $|x| = x$ as x>0

$\frac{dy}{dx} + \frac{3}{x} y = \frac{\cos x}{x^3}$

$$V(x)y = \int V(x)Q(x)dx$$

$$x^3 y = \int \frac{x^3 \cos x}{x^3} dx$$

$x^3 y = \int \cos x \, dx$

$x^3 y = \sin x + c \quad find\ c \quad sub\ \ x = \pi \quad y = 0$

$0 = \sin \pi + c$

$c = 0$

$y = x^{-3} \sin x + c x^{-3}$

$\therefore y = x^{-3} \sin x$

I24. Linear $3xy' + y = 12x$

$y' + \frac{1}{3x} y = \frac{12x}{3x}$

$y' + \frac{1}{3x} y = 4 \qquad V(x) = I = e^{\int \frac{1}{3x} dx} = e^{\frac{1}{3} \ln x} = e^{\ln x^{\frac{1}{3}}} = x^{\frac{1}{3}}$

$$V(x)y = \int V(x)Q(x)dx$$

$$x^{\frac{1}{3}} y = \int 4x^{\frac{1}{3}} dx$$

$x^{\frac{1}{3}} y = \frac{4x^{\frac{4}{3}}}{\frac{4}{3}} + c$

$x^{\frac{1}{3}} y = 3x^{\frac{4}{3}} + c$

$y = \frac{3x^{\frac{4}{3}}}{x^{\frac{1}{3}}} + \frac{c}{x^{\frac{1}{3}}}$

$\therefore y = 3x + cx^{-\frac{1}{3}}$

The answer is $\boxed{A}$

I25. Bernoulli

$\frac{dy}{dx} - \frac{y}{x} = y^9 \qquad n = 9 \quad P(x) = \frac{-1}{x} \qquad Q(x) = 1$

$\frac{du}{dx} + (1-n)u\,P(x) = (1-n)Q(x) \quad substitution \quad u = y^{1-n}$

$u = y^{1-9} = y^{-8}$

$\frac{du}{dx} + \frac{8}{x}u = -8 \qquad V(x) = I = e^{\int \frac{8}{x}dx} = e^{8\ln x} = x^8$

$V(x)u = \int V(x)Q(x)dx$

$x^8 u = \int x^8(-8)dx$

$x^8 u = \frac{-8x^9}{9} + c$

$\frac{x^8 u}{x^8} = \frac{-8x^9}{9x^8} + \frac{c}{x^8}$

$u = \frac{-8}{9}x + cx^{-8}$

$y^{-8} = \frac{-8}{9}x + cx^{-8}$

$y = \pm\left(\frac{-8}{9}x + cx^{-8}\right)^{\frac{-1}{8}}$

**Since we had 1-n= 1 - odd number, we end up with an even power and the 8th root of both sides is +/- our answer. If there was an initial condition such as y(0)=1, then 1 is positive and we would only give the + answer.

I26. Bernoulli

$$y' - y = y^2 e^x$$

$$\underset{P(x)}{\uparrow} \qquad \underset{Q(x)}{\uparrow} \qquad u = y^{1-n} = y^{1-2} = y^{-1}$$

$$\frac{du}{dx} + (1-n)uP(x) = (1-n)Q(x)$$

$$\frac{du}{dx} + (-1)u(-1) = -1e^x$$

$$\frac{du}{dx} + u = -e^x$$

$$V(x) = I = e^{\int 1dx} = e^x$$

$$V(x)u \int V(x)Q(x)dx$$

$$e^x u = \int e^x(-e^x)dx$$

$$e^x u = \int -e^{2x}dx$$

$$e^x u = \frac{-e^{2x}}{2} + c$$

$$u(x) = \frac{-e^{2x}}{2e^x} + \frac{c}{e^x}$$

$$u(x) = \frac{-e^x}{2} + ce^{-x}$$

$$y^{-1} = \frac{-e^x}{2} + ce^{-x}$$

$$\frac{1}{y} = \frac{-e^x}{2} + ce^{-x}$$

$$y = \frac{1}{\frac{-e^x}{2} + ce^{-x}}$$

I27. Bernoulli

$y' + \frac{1}{x}y = -xy^3 \qquad u = y^{1-n} = y^{1-3} = y^{-2}$

$\frac{du}{dx} + (1-n)uP(x) = (1-n)Q(x)$

$\frac{du}{dx} + (-2)u\left(\frac{1}{x}\right) = -2(-x)$

$\frac{du}{dx} - \frac{2}{x}u = 2x$

$V(x) = I = e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$

$I = \frac{1}{x^2}$

$$V(x)u = \int V(x)Q(x)dx$$

$$\frac{1}{x^2}u = \int \frac{1}{x^2} \cdot 2xdx$$

$\frac{1}{x^2}u = \int \frac{2}{x}dx$

$\frac{1}{x^2}u = 2\ln|x| + c$

$u(x) = x^2(2\ln|x| + c)$

$y^{-2} = x^2(2\ln|x| + c)$

$\frac{1}{y^2} = x^2(2\ln|x| + c)$

$$y^2 = \frac{1}{x^2(2\ln|x| + c)}$$

$$y = \pm\sqrt{\frac{1}{x^2(2\ln|x| + c)}}$$

**Since we had 1-n= 1- odd number, we end up with an even power and the square root of both sides is +/- our answer. If there was an initial condition such as y(0)=1, then 1 is positive and we would only give the + answer.

I28. Let y(t) be the amount of salt in kg
Where 0.1kg/L is the concentration of in-flow at time t and 5L/min is the liquid inflow rate at time t and the liquid outflow rate is 5L/min and the concentration of outflow at time t is y(t).

$(0,0) \quad t, salt \qquad V(t) = I = \ e^{\int \frac{1}{400}} = e^{\frac{1}{400}t}$

$\frac{dy}{dt} = (in)(in) - \frac{y(out)}{volume}$

$\frac{dy}{dt} = (0.1)(5) - \frac{5y}{2000}$

$\frac{dy}{dt} = 0.5 - \frac{1}{400}y$

$\frac{dy}{dt} + \frac{1}{400}y = 0.5$

$V(t)y = \int V(t)Q(t)dt$

$e^{\frac{1}{400}t}y = \int e^{\frac{1}{400}t}0.5$

$e^{\frac{1}{400}t}y = \frac{0.5e^{\frac{1}{400}t}}{\frac{1}{400}} + c$

$e^{\frac{1}{400}t}y = 200e^{\frac{1}{400}t} + c$

$y = 200 + \frac{c}{e^{\frac{1}{400}t}}$

$solve\ for\ c\ \rightarrow sub(0,0) \qquad 0 = 200 + \frac{c}{e^0}$

$\therefore c = -200 \qquad y = 200 - 200e^{\frac{-1}{400}t}$

b) $0.05(2000) = 100$

So, $100 = 200 - 200e^{-\frac{1}{400}t}$

$-100 = -200e^{-\frac{1}{400}t}$

$0.5 = e^{-\frac{1}{400}t}$

$ln0.5 = lne^{-\frac{1}{400}t}$

$ln0.5 = -\frac{1}{400}tlne$

t= -400ln0.5

I29. Bernoulli $\int xe^x dx = xe^x - \int e^x dx$

$6y' - 2y = xy^4, y(0) = -2$

$\downarrow n = 4$

$u = y^{1-n}$ $let\ u = y^{-3}$

Divide the original equation by 6 so it has the right form

$y' - \frac{1}{3}y = \frac{1}{6}xy^4$ where P(x)=-1/3 and Q(x)=1/6 x

$$\frac{du}{dx} + (1-n)u\,P(x) = (1-n)Q(x)$$

$$u' + (-3)\left(-\frac{1}{3}\right)u = -3\left(\frac{1}{6}x\right)$$

$$u' + u = \frac{-1}{2}x$$

$$V(x) = I = e^{\int 1\,dx} = e^x$$

$$V(x)u = \int V(x)Q(x)dx$$

$$e^x u = \int \frac{-1}{2}e^x \cdot xdx$$

$e^x u = \frac{-1}{2}[xe^x - \int e^x dx]$ *integral given in the question*

$$e^x u = \int \frac{-1}{2}xe^x + \frac{1}{2}e^x + c$$

$$e^x u = \frac{-1}{2}(x-1)e^x + c$$

$$u = \frac{\frac{-1}{2}(x-1)e^x + c}{e^x}$$

$u = \frac{-1}{2}(x-1) + ce^{-x}$ *substitution was* $u = y^{-3}$

$$\therefore y^{-3} = \frac{-1}{2}(x-1) + ce^{-x}$$

Subst $(0, -2)$ $(-2)^{-3} = \frac{-1}{2}(0-1) + ce^0$

$$\frac{-1}{8} = \frac{1}{2} + c$$

$$c = \frac{-1}{2} - \frac{4}{8} = \frac{-5}{8}$$

$$c = \frac{-1}{2} - \frac{u}{8} = \frac{-5}{8}$$

$$y(x) = \frac{1}{\sqrt[3]{\frac{-1}{2}(x-1)-\frac{5}{8}e^{-x}}}$$

$$y(x) = \frac{1}{\sqrt[3]{\frac{-4(x-1)-5e^{-x}}{8}}}$$

$$y(x) = \frac{1}{\sqrt[3]{\frac{-1}{8}}\sqrt[3]{+4(x-1)+5e^{-x}}}$$

$$= \frac{1}{\frac{-1}{2}\sqrt[3]{4x-4+5e^{-x}}}$$

$$= \frac{-2}{\sqrt[3]{4x-4+5e^{-x}}}$$

I30. Solve: $u'(t) = 10u(t)$

$\int \frac{du}{u(t)} = \int 10tdt$

$\ln|u(t)| = \frac{10t^2}{2} + c$
$\ln|u(t)| = 5t^2 + c$

$e^{ln|u(t)|} = e^{5t^2+c}$ ...take off the absolute value and add +/- and then the +/- combined with e^c is just some constant

$u(t) = \pm e^{5t^2} \cdot e^c$ multiplying with the same base, add the exponents, so this is working backwards

$u(t) = ce^{5t^2}$

I31.

$$\int \frac{1}{y+1}dy = \int \frac{1}{x-3}dx$$

$$\ln|y+1| = \ln|x-3| + c$$

$e^{\ln|y+1|} = e^{\ln|x-3|+c} = e^{\ln|x-3|}e^{c}$ multiplying with the same base, add the exponents, so this is working backwards

$$y + 1 = \pm|x-3|(e^{c})$$

$$y(x) = \pm e^{c}|x-3| - 1 \qquad \pm\, e^{c} = some\ constant$$

$$y(x) = k|x-1| - 2 \quad final\ constant\ can\ be\ k, C, or\ whatever!$$

I32.Which are separable?

A. $\frac{dy}{dx} = \cos y(x) + xy(x) \qquad NO$

B. $\frac{dy}{dx} = \cos y(x) - x^3 \qquad NO$

C. $\frac{dy}{dx} = (\cos x)y(x) + y(x) \qquad YES$

Factor $\frac{dy}{dx} = y(x)[\cos x + 1]$

D. $\frac{dy}{dx} = (\sin y)(x) - y(x) \qquad NO$

I33. a) Let y(t) be the amount of salt in kg

Where 0.1kg/L is the concentration of in-flow at time t and 5L/min is the liquid inflow rate at time t and the liquid outflow rate is 5L/min and the concentration of outflow at time t is y(t).

$$I = e^{\int \frac{1}{400}dt} = e^{\frac{1}{400}t}$$

$$(0,0) \quad \leftarrow pure\ water$$

$$\frac{dy}{dt} = (0.1)(5) - \frac{5y}{2000}$$

$$\frac{dy}{dt} + \frac{1}{400}y = 0.5$$

$$\int \frac{d}{dt}\left(e^{\frac{1}{400t}}y\right) = \int 0.5e^{\frac{1}{400t}}$$

$$e^{\frac{1}{400t}}y = \frac{0.5e^{\frac{1}{400t}}}{\frac{1}{400}} + c$$

$$y = 0.5\left(\frac{400}{1}\right) + ce^{\frac{-1}{400t}}$$

$y = 200 + ce^{\frac{-1}{400t}}$ sub (0,0)

$$0 = 200 + ce^{0}$$

$$c = -200$$

$$\therefore y = 200 - 200e^{\frac{-1}{400t}}$$

b) $(2000)0.05 = 200 - 200e^{\frac{-1}{400t}}$

$100 - 200 = -200e^{\frac{-1}{400t}}$

$\frac{-100}{-200} = e^{\frac{-1}{400t}}$

$0.5 = e^{\frac{-1}{400t}}$

$\text{Ln}\, 0.5 = \ln e^{\frac{-1}{400t}}$

$\text{Ln}\, 0.5 = \frac{-1}{400}t$

$t = -400 \ln 0.5$

I34. Let y(t) be the amount of salt in kg
Where 0.25kg/L is the concentration of in- flow at time t and 3L/min is the liquid inflow rate at time t and the liquid outflow rate is 3L/min and the concentration of outflow at time t is y(t).

$\frac{dy}{dt} = (0.25)(3) - \frac{3y}{400}$ $\quad (0,0)\ pure\ water$

$\frac{dy}{dt} = 0.75 - \frac{3y}{400}$ $\quad I = e^{\int \frac{3}{400}dt} = e^{\frac{3}{400}t}$

$\frac{dy}{dt} + \frac{3}{400}y = 0.75$

$\int \frac{d}{dt}\left(e^{\frac{3}{400}t}y\right) = \int 0.75e^{\frac{3}{400}t}$

$e^{\frac{3}{400}t}y = \frac{0.75e^{\frac{3}{400}t}}{\frac{3}{400}} + c$

$e^{\frac{3}{400}t}y = 100e^{\frac{3}{400}t} + c \quad sub\ (0,0) \ \rightarrow c = 400$

$y = 100 - \frac{100}{e^{\frac{3}{400}t}}$

b) $sub\ y = 0.2\ and\ solve\ for\ t$

$equation\ would\ be$ $\frac{dy}{dt} = 0.75 - \frac{3y}{400-1t}$ $\leftarrow lose\ 1L/min$. HERE, liquid outflow switches to 4L/min.

I35. Let y(t) be the amount of salt in kg
Where 0.25kg/L is the concentration of in-flow at time t and 5L/min is the liquid inflow rate at time t and the liquid outflow rate is 5L/min and the concentration of outflow at time t is y(t).

$$\frac{dy}{dt} = (0.25)(5) - \frac{y(5)}{1000}$$

$$(0{,}25) \leftarrow salt\ at\ start$$

$$\frac{dy}{dt} = 1.25 - \frac{1}{200}y$$

$$\frac{dy}{dt} + \frac{1}{200}y = 1.25 \qquad I = e^{\int \frac{1}{200}dt} \quad = e^{\frac{1}{200}t}$$

$$\int \frac{d}{dt}\left(e^{\frac{1}{200}t}y\right) = \int e^{\frac{1}{200}t}(1.25)$$

$$e^{\frac{1}{200}t}y = \frac{1.25e^{\frac{1}{200}t}}{\frac{1}{200}} + c$$

$$y = 250 + \frac{c}{e^{\frac{1}{200}t}} \qquad sub\ (0{,}25)\ to\ find\ c$$

$$25 = 250 + \frac{c}{e^0}$$

$$c = -225$$

$$\therefore y = 250 - \frac{225}{e^{\frac{1}{200}t}}$$

$$at\ 45\ minutes\ sub\ t = 45\ and\ find\ y$$

$$y = 250 - \frac{225}{e^{\frac{1}{200}(45)}}$$

I36. Where 0.2kg/L is the concentration of in-flow at time t and 5L/min is the liquid inflow rate at time t and the liquid outflow rate is 5L/min and the concentration of outflow at time t is y(t).

$$\frac{dy}{dt} = (0.2)(5) - \frac{y(5)}{200}$$

$$\frac{dy}{dt} = 1 - \frac{1}{40}y$$

$$\frac{dy}{dt} + \frac{1}{40}y = 1$$

$$I = e^{\int \frac{1}{40}dt} \quad = e^{\frac{1}{40}t}$$

$$\int \frac{d}{dt}(e^{\frac{1}{40}t}\ y) = \int e^{\frac{1}{40}t}$$

$$e^{\frac{1}{40}t}y = \frac{e^{\frac{1}{40}t}}{\frac{1}{40}} + c$$

$$y = 40 + \frac{c}{e^{\frac{1}{40}t}} \qquad sub\ (0{,}0)$$

$$c = -40$$

$$y = 40 - \frac{40}{e^{\frac{1}{40}t}} = 40 - 40e^{\frac{-1}{40}t}$$

$sub\ \ t = 8\ \ to\ find\ y$ $\qquad y = 40 - 40e^{\frac{-1}{40}(8)} = 40 - 40e^{-\frac{1}{5}}$kg is the mass and if they want concentration, divide by original volume of 200L

I37. $N(t)$ *is the number of animals at time t*

$$\frac{dN}{dt} = a\left(\frac{1}{time}\right) \times \frac{N}{pigs} + n\left(\frac{pigs}{time}\right) - b(\frac{1}{time}) \times \frac{N}{pigs}$$

$$\frac{dN}{dt} = aN - bN + n$$
$$= (a-b)N + n$$

To find long-term pop find $N(t)$ *and let* $t \to \infty$

$\boxed{1}$ $N'(t) - (a-b)N = n$

$N'(t) + (b-a)N = n$ Q(t)=n P(t)= (b-a)N

$\boxed{2}$ $V(t) = I = e^{\int (b-a)dt} = e^{(b-a)t}$

$\boxed{3}$ $(V(t)N) = \int V(t)Q(t)dt$

$\left(e^{(b-a)t}N\right) = \int e^{(b-a)t} n\, dt$

$$e^{(b-a)t}N = \frac{ne^{(b-a)t}}{(b-a)} + c$$
$$N(t) = \frac{n}{b-a} + \frac{c}{e^{(b-a)t}}$$
$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty}\left[\frac{n}{b-a} + \frac{c}{e^{(b-a)t}}\right]$$
$$= \boxed{\frac{n}{b-a}}\ pigs$$

I38. $Rate\ in\left(\frac{mg}{hr}\right) = concentration \times flow\ of\ drug$

$$= 200\frac{mg}{L} \times \frac{50mL}{h} \leftarrow units\ need\ to\ match$$
$$= 200\frac{mg}{L} \times 0.05\ L/hr$$
$$= 10\ mg/hr$$

Rate out = rate metabolized + rate exceeded

Rate metabolized= $k \cdot u(t)$

$$k = \ln\left(\frac{1}{1-p}\right) \leftarrow \%\ of\ drug\ metabolized$$
$$k = \ln\left(\frac{1}{1-0.8}\right)$$
$$k = \ln\left(\frac{1}{0.2}\right) = \ln 5$$

$\therefore rate\ metabolized = \ln 5 \cdot u(t)$

Rate excreted=concentration× flow

$$= \frac{u(t)mg}{5\ L} \times \frac{0.05L}{hr}$$
$$= \frac{1}{100}u(t)mg/hr$$

$$rate\ out = \ln 5 \cdot u(t) + \frac{1}{100}u(t)$$
$$= \left(\ln 5 + \frac{1}{100}\right)u(t)$$

$$\frac{du}{dt} = rate\ in - rate\ out$$
$$\frac{du}{dt} = 10 - \left(\ln 5 + \frac{1}{100}\right)u(t)$$

$$\frac{du}{dt}+\overbrace{\left(\ln 5+\frac{1}{100}\right)}^{\swarrow P(t)}u(t)=10 \quad \leftarrow Q(t)$$

$$\searrow \textit{let this be } A$$

$$V(t)=I=e^{\int A dt}=e^{At}$$

$$V(t)u=\int V(t)Q(t)dt$$

$$e^{At}u=\int 10e^{At}dt$$

$$e^{At}u=\frac{10e^{At}}{A}+c$$

$$u=\frac{10}{A}+ce^{-At}$$

$$\therefore u(t)=\frac{10}{A}+ce^{-At} \quad \textit{substitute } \underset{\uparrow(t,u)}{(0,0)}$$

$$0=\frac{10}{A}+ce^{0}$$

$$c=\frac{-10}{A}$$

$$\therefore u=\frac{10}{A}-\frac{10}{A}e^{-At}$$

$$u(t)=\frac{10}{A}(1-e^{-At})$$

$$A=\left(\ln 5+\frac{1}{100}\right)$$

$$u(t)=\frac{10}{(ln5+\frac{1}{100})}(1-e^{-\left(\ln 5+\frac{1}{100}\right)t}$$

Quiz 1: Practice on Sections A to I

1. $A = 4\pi(3r)^2 = 4\pi(9r^2) = 9(4\pi r^2)$

   $\therefore 9\ times$

   $V = \frac{4}{3}\pi(3r)^3 = \frac{4}{3}\pi(27r^3) = 27(\frac{4}{3}\pi r^3)$

   $\therefore 27\ times$

2. $speed = m/s$

   $\frac{D}{t} = \frac{m}{s} \quad \therefore time\ must\ be\ in\ seconds$

3. $= 2^{\frac{1}{10}}$

   $u(t) = c \times 2^{\frac{t}{10}} = c \times 2^{\frac{1}{10}t}$

4. a) homogeneous
   b) inhomogeneous
   c) inhomogeneous
   d) homogeneous

5. $P_{t+1} = 3000(1.08)^t - n \qquad a = 1.08 \quad P_0 = 3000 \quad b = -n$

   $Pt = \left(Po - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}$

$$P_t = \left(3000 - \frac{(-n)}{1-1.08}\right)(1.08)^t + \frac{-n}{1-1.08}$$

= $(3000 - 12.5n)(1.08)^t + 12.5n$

worthless after 10 years, so let t=10

$0 = (3000 - 12.5n)(1.08)^{10} + 12.5n$

$0 = 6476.77 - 26.98656247n + 12.5n$

$-6476.77 = -14.48656247n$

$n = \$447.09$

6. a) linear, 1st order
   b) non-linear, 1st order
   c) non-linear, 2nd order
   d) linear, 2nd order

7. $\frac{du}{dt} = t^2$

$$\int du = \int t^2 dt$$

$$u(t) = \frac{t^3}{3} + c$$

8. $\frac{du}{dt} = 2u(t)$

$$\frac{1}{u(t)} du = 2dt$$

$$\int \frac{1}{u(t)} du = \int 2dt$$

$$\ln|u(t)| = 2t + c$$

$$e^{\ln|u(t)|} = e^{2t+c}$$

$$e^{\ln|u(t)|} = e^{2t} \cdot e^c$$

$$u(t) = ae^{2t}$$

9. a) Linear $u'(t) - u(t) = \frac{t^3}{3} + t\cos t$

$$v(t) = e^{\int -1dt} = \boxed{e^{-t}}$$

b) $u'(t) + u(t) = t$

$$v(t) = e^{\int 1dt} = \boxed{e^t}$$

c) $u'(t) - (t+2)u(t) = t$

$$v(t) = e^{\int -(t+2)dt} = e^{\int (-t-2)dt} = \boxed{e^{\frac{-t^2}{2}-2t}}$$

$\therefore$ 1. $b)$
2. c)
3. a)

10. $Linear \ \ v(t) = e^{\int -\sin t \, dt} = e^{\cos t}$

$$V(t)u = \int V(t)Q(t)dt$$

$$e^{\cos t}u(t) = \int e^{\cos t} \sin t \, dt \text{ substitution } u = \cos t \quad -du = \sin t \, dt$$

$$e^{\cos t}u(t) = \int -e^u du$$

$$e^{\cos t}u(t) = -e^u + c$$

$$\frac{e^{\cos t}u(t)}{e^{\cos t}} = \frac{-e^{\cos t}}{e^{\cos t}} + \frac{c}{e^{\cos t}}$$

$$u(t) = -1 + ce^{-\cos t}$$

11. Linear $v(t) = e^{\int 1dt} = e^t$

$$V(t)u = \int V(t)Q(t)dt$$

$$e^t u(t) = \int 5e^t dt$$

$$e^t u(t) = 5e^t + c \quad \leftarrow sub \;\; u(t) = 3, \;\; t = 1$$

$$e^1(3) = 5e^1 + c$$

$$c = -2e$$

$$e^t u(t) = 5e^t - 2e$$

$$u(t) = 5 - \frac{2e^1}{e^t}$$

$$u(t) = 5 - 2e^{1-t}$$

12. $Linear\ y' - \frac{1}{x}y = 2\ln x + 1$ Since x=2

$$v(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

$$V(x)y = \int V(x)Q(x)dx$$

$$\frac{1}{x}y = \int \frac{1}{x}(2\ln x + 1)dx$$

$$\frac{1}{x}y = \int \frac{2\ln x\, dx}{x} + \int \frac{1}{x}dx$$

↘substitution $u = \ln x \qquad du = \frac{1}{x}dx$

$$\frac{1}{x}y = \int 2udu + \ln|x| + c$$

$$\frac{1}{x}y = \frac{2u^2}{2} + \ln|x| + c$$

$$\frac{1}{x}y = u^2 + \ln|x| + c$$

$$\frac{1}{x}y = (\ln x)^2 + \ln|x| + c \qquad \text{substitute } x = 2 \quad y = 1$$

$$\frac{1}{2}(1) = (\ln 2)^2 + \ln 2 + c$$

$$c = \frac{1}{2} - (\ln 2)^2 - \ln 2$$

$$\therefore \frac{1}{x}y = (\ln x)^2 + \ln|x| + \frac{1}{2} - (\ln 2)^2 - \ln 2$$

$$y = x\left[(\ln x)^2 + \ln|x| + \frac{1}{2} - (\ln 2)^2 - \ln 2\right]$$

J. Stability Analysis of Autonomous Differential Equations

* arrows to right indicate $f(y)$ is positive (above axis)
* arrows to left indicate $f(y)$ is negative (below axis)

Example 1. Find the equilibrium values of the differential equation

$y^2 + 2y - 8 = 0$

$(y+4)(y-2) = 0 \quad y = 2, -4$

Stable Unstable

$\rightarrow\leftarrow \quad \leftarrow\rightarrow$

$-4 \quad 2$

-4 2

$\boxed{-5}$	$\boxed{0}$	$\boxed{3}$
$(-5)^2 + 2(-5) - 8$		$3^2 + 2(3) - 8$
$\boxed{+}$	$\boxed{-}$	$\boxed{+}$
$\rightarrow \rightarrow \rightarrow \rightarrow$	$\leftarrow \leftarrow$	$\rightarrow \rightarrow \rightarrow \rightarrow$

$\therefore$ *equilibrium values* $\hat{y} = 2, \quad -4$

Unstable stable

Example 2.

$y^4 - 16y^2 = 0$

$y^2(y^2 - 16) = 0 \quad y = 0, 4, -4$

-4 0 4

-5	-1	1	5
$\boxed{+}$	$\boxed{-}$	$\boxed{-}$	$\boxed{+}$
↗	↘	↘	↗

$\longrightarrow\longleftarrow \quad \longleftarrow\longleftarrow \quad \longleftarrow\longrightarrow$

Stable semi-Stable unstable

Example 3. $y' = g(y) = y^3 - 4$

$y^3 - 4 = 0$

$y^3 = 4 \quad y = \sqrt[3]{4} = 1.59 \quad \therefore \hat{y} = 1.59$ *is an equilibrium point*

$y'' = g'(y) = 3y^2 + 0 = 3y^2$

$at\ \hat{y} = \sqrt[3]{4} \quad y'' = 3(\sqrt[3]{4})^2 > 0$

$\therefore \hat{y} = \sqrt[3]{4}$ *is unstable*

Logistic Growth Model

$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N = 0 \qquad note: r, K\ are\ \boxed{+}\ constants$

Example 4.

a) $r = 0\ \ (not\ possible\ since\ r > 0) \quad or \quad N = 0 \quad or \quad 1 - \frac{N}{K} = 0$

$$1 = \frac{N}{K} \qquad K = N \quad or\ N = K$$

$\therefore\ equilibria\ are\ \widehat{N} = 0, \widehat{N} = K$

$$g(N) = r\left(1 - \frac{N}{K}\right)N = r\left(N - \frac{N^2}{K}\right) = rN - \frac{N^2 r}{K} = rN - \frac{r}{K}N^2$$

$$g'(N) = r - \frac{r}{K}(2N) = r - \frac{2rN}{K}$$

$at\ \widehat{N} = 0 \quad g'(0) = r - 0 = r > 0 \qquad \therefore\ unstable\ at\ \widehat{N} = 0$

$at\ \widehat{N} = K \quad g'(k) = \frac{r-2r}{K}(K) = r - 2r = -r < 0$

$$\therefore\ stable\ at\ \widehat{N} = K$$

b) Solution:

$$\boldsymbol{N'(t) = \textcolor{red}{rN(t)}\left(1 - \frac{N(t)}{k}\right)}$$

$\boldsymbol{Since\ \ r > 0\ \ and\ \ N(t) > 0}$

And if you multiply two + quantities, it is also + $\boldsymbol{\textcolor{red}{\therefore rN(t) > 0}}$

And since $\boldsymbol{0 < N(t) < k}$, we know that $\boldsymbol{N(t) < k}$ and so the fraction $\boldsymbol{\frac{N(t)}{k} < 1}$

$$\boldsymbol{\therefore 1 - \frac{N(t)}{k} > 0}$$

$$\boldsymbol{N'(t) = (+)(+) = +}$$

$\boldsymbol{\therefore N'(t) > 0}$ and the population is growing

Example 5. In each case below, state how the population grows according to the model, remembering that r, k are positive constants.

$$\boldsymbol{N'(t) = rN(1-\frac{N}{K})}$$

a) If there is no crowding at time t, should N(t) be close to K or farther away from K?

Solution: If there is no crowding, then $N(t)$ will be farther away from K

Since $N(t)$ is farther away from K and there is no crowding, we expect N(t) to be less than K

$\therefore \quad \frac{N(t)}{K} \to 0$

$$N'(t) = rN(t)\underbrace{\left(1-\overbrace{\frac{N(t)}{K}}^{\text{Close to 0}}\right)}_{\text{Close to 1}}$$

$\therefore$ $\boldsymbol{N'(t) > 0}$ and the population will grow exponentially. Recall, the solution to the differential equation $N'(t) = rN(t)$ is an exponential function.

b) If there is crowding at time t, should N(t) be close to K or farther away from K?

Solution: If there is crowding, we expect $N(t)$ to be close to K, but slightly less than K.

$$N'(t) = rN(t)\underbrace{\left(1-\underbrace{\frac{N(t)}{K}}_{\text{Almost 1, but less than 1}}\right)}_{\text{Almost 0, grows }\underline{\textbf{slowly}}}$$

$N'(t)$ is positive but close to 0, $\therefore$ $N(t)$

$\therefore$ the population is **growing slowly**

c) If there is overcrowding at time t, should N(t) be larger or smaller than the constant K?

Solution:

If we have overcrowding, we expect $N(t)$ to be larger than K

$$\therefore N'(t) = \underbrace{rN(t)}_{\text{Greater than 0}} \underbrace{\left(1-\underbrace{\frac{N(t)}{k}}_{\text{Greater than 1}}\right)}_{\text{Less than 0}}$$

$\therefore N'(t) = (+)(-) = (-)$
$\therefore N'(t) < 0$

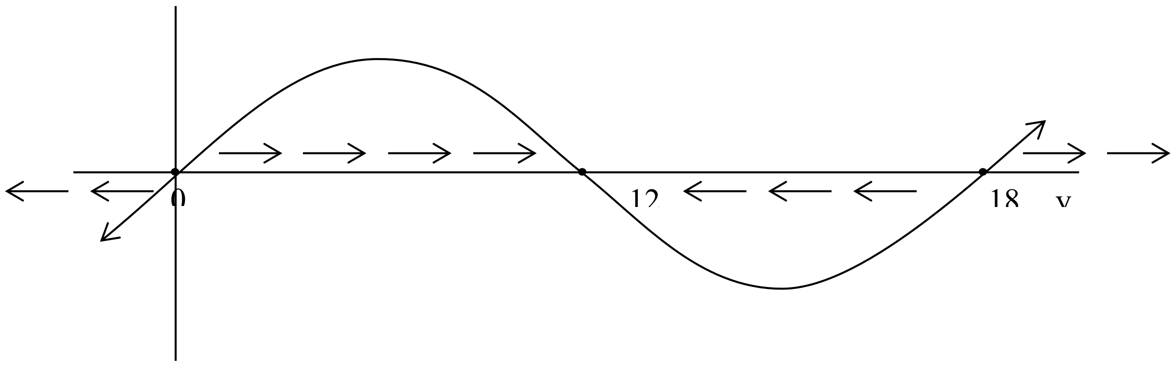
$\therefore$ the population is **not** growing, it is shrinking

d) Which part of the Logistic Equation accounts for crowding?

Solution:

We can say $\frac{N(t)}{k}$ or $\left(1-\frac{N(t)}{k}\right)$ account for crowding

Example 6. Answer the questions, given the phase plot below:



a) Identify where the graph is increasing and decreasing.
The graph is above the x-axis from 0 to 12, so it is increasing there, as well as from 18 onward
The graph is below the x-axis from 12 to 18, so it is decreasing there as well as from 0 backward
b) The equation is dy/dt = y (y-12)(y-18). Make sure you test a point in each region to make sure if it is positive, the graph is above the x-axis and negative is below the x-axis. In another scenario it might be dy/dt = -y (y-12)(y-18).

c) Find the equilibria and asses the stability of each of them.
The equilibria are the points where the graph crosses the x-axis, so they are at 0, 12 and 18
At x=0, we have an unstable equilibrium
At x=18, we have an unstable equilibrium (arrows point away from each other)
At x=12, we have a stable equilibrium (arrows point towards each other)

d) Given the following initial conditions, what will the value of y(t) tend towards?
If y(0) =2, y(t) will increase to 12
If y(0)=10, y(t) will increase to 12
If y(0)=15, y(t) will decrease to 12
If y(0)=18, y(t) will remain at 18 (equilibrium)

NOTE: If y(0)=22, y(t) will increase to infinity

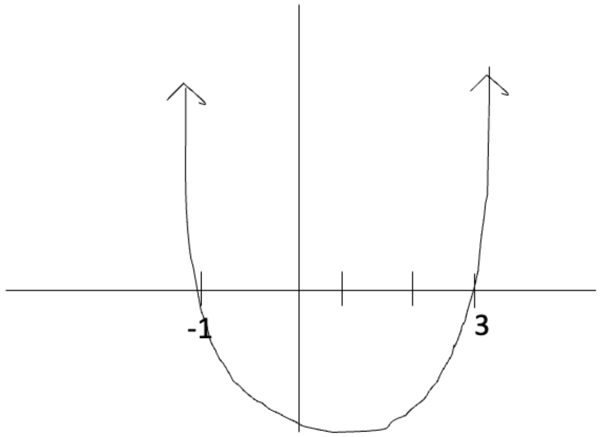
Example 7.

0 test point

-1	1
$(-1)^5$	$(+1)^5$
$\boxed{-}$	$\boxed{+}$
$\leftarrow\leftarrow\leftarrow$	$\rightarrow\rightarrow\rightarrow$

$\therefore\ unstable$

Example 8. The correct graph is:



It is increasing (above the x-axis from 3 to infinity and from negative infinity to -1. It is below the x-axis, or decreasing from -1 to 3. See the number line below: It is stable at -1 and unstable at 3.

Increasing from 3 to ∞ and from $-\infty$ to -1.
Decreasing from -1 to 3

→← ←→

-1 3
Stable unstable

Practice Exam Questions on Stability Analysis of Autonomous Differential Equations

J1. (a)

→←

$4 \quad stable$

(b) $y^3 - 3y^2 = 0$

$y^2(y-3) = 0$

y=0, 3

←← ←→

$0 \quad 3 \quad unstable\ at \ \ y = 3$

(c)

$2e^{2y} - 4e^y = 0$

$2e^y(e^y - 2) = 0$

$e^y = 0\ means\ no\ solution$ $e^y = 2$ take ln of both sides and y=ln2

←→

$\ln 2 \quad unstable\ at \ \ y = \ln 2$

(d) $0 = (y-3)(y+2)$ $y = 3 \ \ or \ \ y = -2$

-2 3

-3	0	4
$(-3)^2 + 3 - 6$	$\boxed{-}$	$4^2 - 4 - 6$
$\boxed{+}$ ↗	↘	$\boxed{+}$ ↗

⟶⟵ ⟵⟶

Stable at -2 unstable at 3

J2.

k is a constant k> 0

$k(A-H) = 0 \qquad H = A$

$\therefore equilibria \quad \hat{H} = A$

$$g(H) = k(A-H)$$

$$g'(H) = k(-1) = -k < 0 \qquad \therefore stable\ at\ \hat{H} = A$$

J3.

$kP = 0 \qquad P = 0 \quad since\ k\ is\ a\ constant \ > 0$

$\therefore equilibria \quad \hat{P} = 0$

J4.

$0.04(20 - H) = 0 \qquad \therefore H = 20$

Equilibrium $\hat{H} = 20$

$let\ g(H) = 0.04(20 - H) = 0.8 - 0.04H$

$g'(H) = -0.04 < 0 \quad \therefore \quad stable\ at\ \hat{H} = 20$

J5.

$H \geq 0 \quad and \quad H = 10$

$$\frac{dN}{dt} = 2N\left(1 - \frac{N}{100}\right) - H = 2N - \frac{2N^2}{100} - H$$

$$\frac{dN}{dt} = 0 \qquad 2N - \frac{2N^2}{100} - H = 0$$

$$2N - \frac{2N^2}{100} - 10 = 0 \quad (divide\ by - 2)$$

$$-N + \frac{1}{100}N^2 + 5 = 0 \qquad (\times 100)$$

$$N^2 - 100N + 500 = 0$$

$$N = \frac{100 \pm \sqrt{(-100)^2 - 4(1)(500)}}{2(1)} = \frac{100 \pm 89.44}{2}$$

$$N = 94.7 \quad or \;\; N = 5.28$$

$equilibria \;\; \widehat{N} = 5.28 \quad and\ 94.7$

J6. $a = unstable \quad b = stable \quad c = unstable$

$\leftarrow\rightarrow \qquad\qquad \rightarrow\leftarrow \qquad\qquad \leftarrow\rightarrow$

$- \;\; a \qquad + \;\; b \qquad - \;\; c \;\; +$

J7. -2 0 2

-3	-1	1	3
$\boxed{-}$	$\boxed{+}$	$\boxed{-}$	$\boxed{+}$

$\leftarrow -2 \rightarrow \qquad \rightarrow 0 \leftarrow \qquad \leftarrow 2 \rightarrow$

Unstable stable unstable

$y' = 0$

$\therefore y^5 - 16y = 0$

$y(y^4 - 16) = 0$

$y = 0, \;\; y^4 = 16$

$y = 2, -2 \quad \therefore \;\; A\ is\ the\ answer$

J8. $find\ equilibrium$

$$rV\left(1-\frac{V}{k}\right)=0$$

$$rV=0 \qquad or \qquad 1-\frac{V}{k}=0$$

$$V=0 \qquad\qquad 1=\frac{V}{k}$$

$$V=k$$

$classify\ equilibrium$

$V=0 \quad V=k$

-1	$k-1$	$k+1$
$\boxed{-}$	$\boxed{+}$	$\boxed{-}$

$$\frac{dV}{dt}=rV\left(1-\frac{V}{k}\right)=rV-\frac{rV^2}{k}$$

$$\frac{dV}{dt} \quad at\ V=-1,\ \frac{dV}{dt}=r(-1)-\frac{r(-1)^2}{k}=-r-\frac{r}{k}$$

$$=-r(1+\frac{1}{k})$$

$$<0 \quad since\ r,k\ \boxed{+}\ constants$$

$\leftarrow 0 \rightarrow \qquad \rightarrow k \leftarrow$

Unstable at $V=0$ stable $at\ V=k$

Method 1:

$$\frac{dv}{dt}=rV-\frac{r}{k}V^2$$

$$g(V)=rV-\frac{r}{k}V^2$$

$$g'(V)=r-\frac{r}{k}(2V)$$

$$g'(0)=r-\frac{r}{k}(2(0))$$

$$=r-0=r>0$$

<u>Unstable at 0</u>

$$g'^{(k)}=r-\frac{r}{\cancel{k}}(2\cancel{k})=r-2r$$

$$=-r<0$$

<u>Stable at k</u>

Method 2:

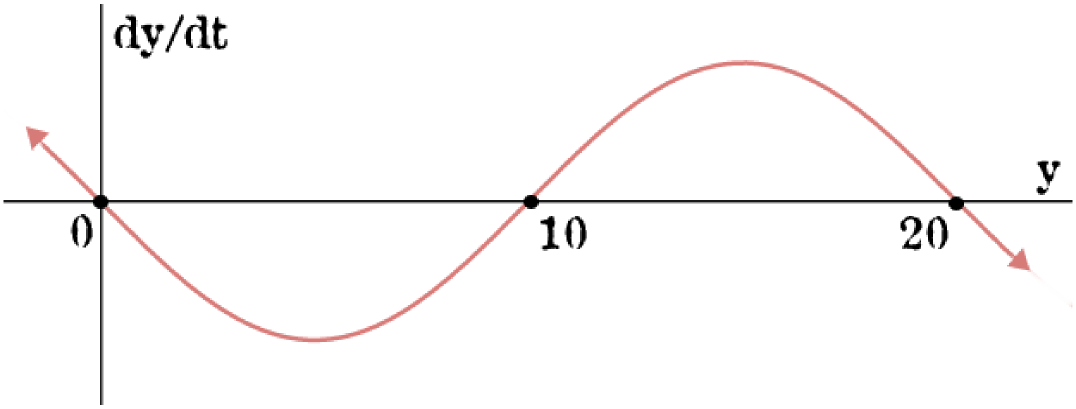
$at\ V = k + 1$

$$\frac{dV}{dt} = r(k+1) - r\frac{(k+1)^2}{k}$$
$$= rk + r - \frac{r(k^2+2k+1)}{k}$$
$$= rk + r - \frac{rk^2}{k} - \frac{2rk}{k} - \frac{r}{k}$$
$$= rk + r - rk - 2r - \frac{r}{k}$$
$$= -r - \frac{r}{k}$$
$$= -r\left(1 + \frac{1}{k}\right) < 0 \quad since\ r, k\ are\ \boxed{+}\ constants$$

$at\ V = k - 1$

$$\frac{dV}{dt} = r(k-1) - r\frac{(k-1)^2}{k}$$
$$= rk - r - \frac{r(k^2-2k+1)}{k}$$
$$= rk - r - \frac{rk^2}{k} + \frac{2rk}{k} - \frac{r}{k}$$
$$= rk - r - rk + 2r - \frac{r}{k}$$
$$= r(1 - \frac{1}{k}) > 0$$

J9. Answer the questions, given the phase plot below:



a) Identify where the graph is increasing and decreasing.
The graph is increasing from 10 to 20 and from 0 backwards and it is decreasing from 0 to 10 and from 20 onwards

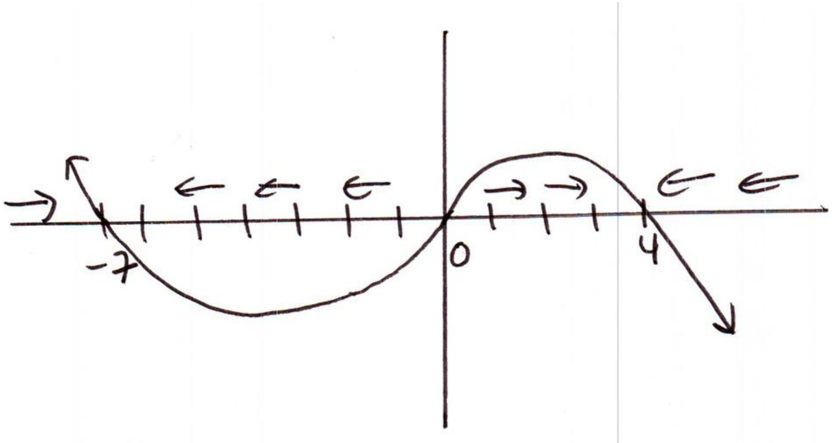
b) Find the equilibria and asses the stability of each of them.
The equilibria are 0, 10 and 20
At 0, and 20 we have stable equilibria and at 10, we have an unstable equilibrium

c) Given the following initial conditions, what will the value of y(t) tend towards?
If y(0) = 2, y(t) will decrease to 0.
If y(0)=10, y(t) will stay at 10 (it is an equilibria)

If y(0)=15, y(t) will increase to 20
If y(0)=18, y(t) will increase to 20
If y(0)=22, y(t) will increase to infinity

J10. The equilibrium are $\underline{0, 4, -7}$

b)



test point $y = 5$ $\frac{dy}{dx} = -y(y-4)(y+7)$
$= -5(5-4)(5+7)$
$< 0 \quad \therefore left\ (below\ axis)$

Test $y = 1$ $\frac{dy}{dx} = -1(1-4)(1+7) > 0 \ \therefore right\ (above\ x\ axis)$

Test $y = -1$ $\frac{dy}{dx} = -(-1)(-1-4)(-1+7) < 0 \quad \therefore left\ (below\ axis)$

$\therefore -7\ is\ stable$ (arrows pointing towards each other)

c)Start at 2, it will go to 4

d)$\lim_{x\to\infty} y(x) = 0\ \ (stays\ at\ equilibrium)$

J11. Stable at b and unstable at a and c.

Quiz 2: Practice on Sections I and J

1. a=175, b=375, n=8000
   a. $\frac{dN}{at} = (a-b)N + n$
      = (175 – 375)N + 8000
      = -200N + 8000

   b. $\frac{\lim}{\text{n}\to\infty} \quad \frac{n}{b-a} = \frac{8000}{375-175} = \frac{8000}{200} = 40$

2.
   a. V = 2000 + 5t – 8t
      V = 200 – 3t

   b. $\frac{dA}{dt} = (10)(5) - \frac{8A}{200-3t} = 50 - \frac{8A}{200-3t}$

3. 200 mg/L * 0.06L/hr = 12mg/hr

   $\text{K} = \text{In}\left(\frac{1}{1-0.6}\right) = In\left(\frac{1}{0.4}\right) = ln2.5$

   Rate out = In2.5 u(t) $+ \frac{u(t)}{5}$*0.06 L/hr

   = In2.5 u(f) + 0.012u(t)

   $\frac{du}{dt} = 12 - [In2.5 + 0.012\, u(t)]$

   $\frac{du}{dt} + [In2.5 + 0.012]u(t) = 12$

   Let P(t)= $[In2.5 + 0.012] = \text{A}$ $\qquad Q(t) = 12$

   $V(t) = e^{\int A dt} = e^{At}$

   $\int \frac{d}{dt} V(t)\, u(t) = \int V(t)Q(t)dt$

   $\int \frac{d}{dt} e^{At}\, u(t) = \int e^{At} * 12dt$

   $e^{At}\, u(t) \frac{12e^{At}}{A} + c$

$$U(t) = \frac{12}{A} + ce^{-At}$$

(0,0)

$$0 = \frac{12}{A} + ce^{0}$$

$$c = \frac{-12}{A}$$

$$u(t) = \frac{12}{A} - \frac{12}{A}e^{-At}$$

$$u(t) = \frac{12}{A}(1 - e^{-At})$$

$$u(t) = \frac{12}{In2.5 + 0.012}(1 - e^{-(In2.5+0.012)t})$$

Let t = 1

$$u(1) = \frac{12}{In2.5 + 0.012}(1 - e^{-(In2.5+0.012)(1)})$$

$= 7.82$

4. $y'(t) - \frac{1}{t}y(t) = 4\big(y(t)\big)^4$

$P(t) = -\frac{1}{t}$ $\qquad Q(t) = 4$

a) $u = y^{1-n} = y^{1-4} = y^{-3}$

b) $\frac{dy}{dt} + (1-n)\,P(t)\,u(t) = (1-n)\,Q(t)$

$$u'(t) - 3\left(-\frac{1}{t}\right)u(t) = -3(4)$$
$$u'(t) + \frac{3}{t}u(t) = -12$$

c) $V(t) = e^{\frac{3}{t}} = e^{3lnt} = e^{Int^3} = t^3$

$$\int \frac{d}{dt}\,(t^3u(t)) = \int t^3(-12)dt$$

$$t^3u(t) = -\frac{12t^4}{4} = c$$

$$t^3u(t) = -3t^4 + c$$

$$u(t) = -3t + \frac{c}{t^3}$$

$$1^{-3} = -3(1) = \frac{c}{1^3}$$

$$1 = -3 = c$$

$$c = 4$$

Therefore, $y^{-3} = 3t + \frac{4}{t^3}$

$$y^3 = \frac{1}{-\frac{3f}{1}+\frac{4}{t^3}}$$

$$y^3 = \frac{1}{\frac{-3t^4+4}{t^3}}$$

$$y^3 = \frac{t^3}{-3t^4+4}$$

$$y(t) = \left(\frac{t^3}{-3t^4+4}\right)^{\frac{1}{3}}$$

$$y(t) = \frac{t}{(-3t^4+4)^{(\frac{1}{3})}}$$

5. a)$y = u^{1-n} = u^{1-4} = u^{-3}$

b) $\frac{dy}{dt} + (1-n)\,P(t)\,y(t) = (1-n)\,Q(t)$

$\frac{dy}{dt} + (-3)(1)y(t) = -3(1)$ Q(t)= -3

$$\frac{dy}{dt} - 3y(t) = -3$$

$v(t) = e^{\int -3dt} = e^{-3t}$

$$\int \frac{d}{dt}\,(e^{-3t}y) = \int -3e^{-3t}dt$$

$e^{-3t}y = e^{-3t} + c$

$y = 1 + ce^{3t}$

$u^{-3} = 1 + ce^{3t}$

$$\left(\frac{1}{2}\right)^{-3} = 1 + ce^{0}$$

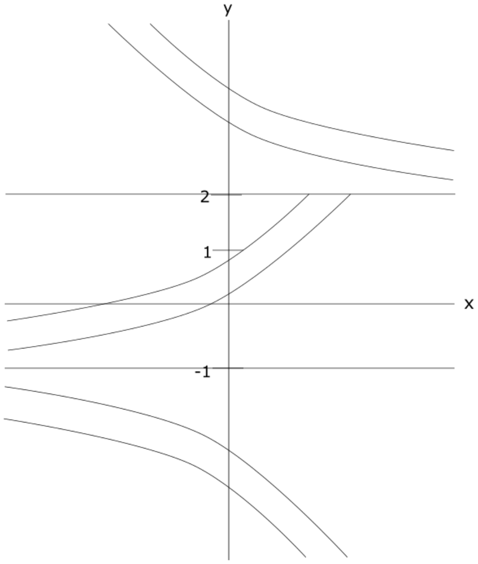
$(2)^3 = 1 + c$

$c = 7$

$u^{-3} = 1 + 7e^{3t}$

$u^3 = \frac{1}{1+7e^{3t}}$ We get:$u = \frac{1}{\sqrt[3]{(1+7e^{3t})}}$

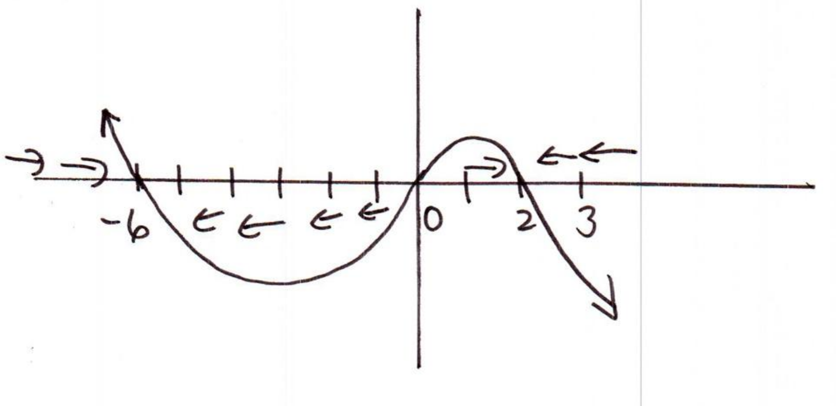
6. From -1 and below, it is below the x-axis, so it is decreasing and the lines go down and right. From -1 to 2, it is above the x-axis, so increasing and lines go up and right. From 2 onward, the graph is below the x-axis so it is decreasing and the lines go down and right.



7.a) The equilibrium are: 0, 2 and -6
b) Test Point y=3
dy/dx= -3(3-2)(3+6) < 0, so arrows point to the left

test point y=1
dy/dx = -1(1-2)(1+6) >0, so arrows point to the right

So, -6 is stable (arrows pointing towards each other)
c) start at 3, it would go down to 2
d) start at 0, it would stay at 0 (an equilibrium)



K. Practice Exam

K1. slope= $\frac{1}{2} \neq 1 \quad \therefore allometry$

K2. $\frac{LM}{T^2} = \frac{M}{L} \cdot v^2 \qquad \therefore v^2 = \frac{L^2}{T^2}$

K3. $P_{t+1} = 1.01P_t - 200$

K4. $I = x^2 \quad B$
$I = x^3 \quad A$
$I = x^{-3}e^{-2x} \quad C$

A) $y' + \frac{3xy}{x^2} = \frac{2\sin x}{x^2}$
$y' + \frac{3y}{x} = \frac{2\sin x}{x^2}$
$I = v(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

B) $y' + \frac{2}{x}y = \frac{x^{-3}}{x}$
$y' + \frac{2}{x}y = x^{-4}$
$I = e^{\int \frac{2}{x}} = e^{2\ln x} = e^{\ln x^2} = x^2$

C) $y' - \left(\frac{2x+3}{x}\right)y = \frac{2\sec x}{x}$
$I = e^{-\int\left(2+\frac{3}{x}\right)dx} = e^{-2x-3\ln x} = e^{-2x}e^{\ln x^{-3}} = e^{-2x}x^{-3}$

K5. $\frac{xy'}{x} + \frac{2y}{x} = \frac{x^{-3}}{x}$

$y' + \frac{2}{x}y = x^{-4}$

$v(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

$\int \frac{d}{dx}(v(x)y) = \int v(x)Q(x)$

$x^2y = \int x^2\, x^{-4}dx$

$x^2y = \int x^{-2}dx$

$x^2y = \frac{x^{-1}}{-1} + c$

$x^2y = \frac{-1}{x} + c$

Substitute $y(1) = 1$ to find c

$1^2(1) = \frac{-1}{1} + c$

$c = 2$

$x^2y = \frac{-1}{x} + 2$

$y = \frac{-1}{x^3} + \frac{2}{x^2}$

$y(2) = \frac{-1}{2^3} + \frac{2}{2^2}$

$= \frac{-1}{8} + \frac{2}{4}$

$= \frac{-1}{8} + \frac{4}{8}$

$= \frac{3}{8}$

K6. $v(t) = e^{\int \frac{1}{t-3dt}} = e^{\ln(t-3)} = t - 3$

$\frac{d}{dt}(v(t)u) = v(t)Q(t)$

$\int \frac{d}{dt}(t-3)u = \int (t-3)(1)dt$

$(t-3)u = \frac{t^2}{2} - 3t + c$

Substitute $u(\underset{\underset{t}{}}{\underset{\uparrow}{0}}) = \underset{\underset{u}{}}{\underset{\uparrow}{3}}$ $\quad (0-3)(3) = \frac{0^2}{2} - 2(0) + c$

$c = -9$

$\therefore (t-3)u = \frac{t^2}{2} - 3t - 9$

$u = \frac{1}{t-3}\left[\frac{t^2}{2} - 3t - 9\right]$

K7. a) $500 + 12t - 8t = 500 + 4t$

b) $\frac{dA}{dt} = (rate\ in)(rate\ in) - \frac{A(rate\ out)}{volume\ at\ time\ t}$

$\frac{dA}{dt} = (12)(12) - \frac{8A}{500+4t}$

$\frac{dA}{dt} = 144 - \frac{8A}{500+4t}$

K8. $\therefore n = 4$

a) $u = y^{1-n} = y^{1-4} = y^{-3}$ $\therefore p = -3$

b) $y'(t) - \frac{1}{t}y(t) = 4[y(t)]^4$

$P(t) = \frac{-1}{t}$ $Q(t) = 4$

c) $\frac{du}{dt} + (1-n)u\,p(t) = (1-n)Q(t)$

$\frac{du}{dt} + (-3)u\left(\frac{-1}{t}\right) = -3(4)$

$\frac{du}{dt} + \frac{3}{t}u = -12$

$\frac{d}{dt}(v(t)u) = v(t)Q(t)$

$v(t) = e^{\int \frac{3}{t}dt} = e^{3\ln t} = e^{\ln t^3} = t^3$

$\int \frac{d}{dt}(t^3u) = \int t^3 \cdot (-12)\text{dt}$

$t^3u = \frac{-12t^4}{4} + c$ $substitute\ u = y^{-3}$ $y(1) = 1$

$t^3y^{-3} = -3t^4 + c$

$1^3(1)^{-3} = -3(1)^4 + c$

$c = 4$

$t^3y^{-3} = -3t^4 + 4$

$y^{-3} = \frac{-3t^4}{t^3} + \frac{4}{t^3}$

$y^{-3} = \frac{-3t^4+4}{t^3}$

$y^3 = \frac{t^3}{-3t^4+4}$

$y = \sqrt[3]{\frac{t^3}{-3t^4+4}}$

K9. a) No, it will go to 20

b) 60 rats

c) 20 rats

K10. The first graph is correct. It is above the x-axis for greater than 0, so it is increasing there. It is below the x-axis between -1 and 0, so it is decreasing there and it is above the x-axis below -1, so it will be increasing below -1.

K11.

$$P_t = 1.04P_{t-1} - 50 \qquad P_o = 850 \qquad a = 1.04 \qquad b = -50$$

$$P_t = \left(P_o - \frac{b}{1-a}\right)(a^t) + \frac{b}{1-a}$$

$$P_t = \left(850 - \frac{-50}{1-1.04}\right)(1.04)^t + \frac{-50}{1-1.04}$$

$$P_t = (850 - 1250)(1.04)^t + 1250$$

$$P_t = -400(1.04)^t + 1250$$

$$\mathbf{50 = -400(1.04)^t + 1250}$$

$$\mathbf{-1200 = -400(1.04)^t}$$

$$\mathbf{3 = 1.04^t}$$

$$\boldsymbol{t} = \frac{\mathbf{\ln 3}}{\mathbf{\ln 1.04}}$$

Long Method:

$$P_t = 1.04P_{t-1} - 50 \quad \boxed{1}$$

$$\Delta P_t = P_t - P_{t-1} \quad sub\,\boxed{1}$$

$$= 1.04P_{t-1} - 50 - P_{t-1}$$

$$\Delta P_t = \underbrace{P_{t-1} + 0.04P_{t-1} - 50}_{\boxed{3}} - P_{t-1}$$

$$\Delta P_t = 0.04P_{t-1} - 50$$

Consider $\Delta P_t = 0$

$$0.04P_{t-1} - 50 = 0$$

$$0.04P_{t-1} = 50$$

$$P_{t-1} = 1250$$

Define $U_t = P_t - 1250$ $\boxed{2}$ which means $P_t = U_t + 1250$

$U_{t-1} = P_{t-1} - 1250$ $\boxed{2}$ which means $P_{t-1} = U_{t-1} + 1250$

From $\boxed{1}$ $P_t = 1.04P_{t-1} - 50$ substitute $\boxed{2}$

$$U_t + 1250 = 1.04(U_{t-1} + 1250) - 50$$

$$U_t = 1.04U_{t-1} + 1300 - 50 - 1250$$

$$\therefore U_t = 1.04U_{t-1} \quad \leftarrow b = 1.04 \quad U_t = P_t - 1250$$

$$subst \quad t = 0$$

$$P_o = 850$$

$$U_o = P_o - 1250$$

$$U_o = 850 - 1250 \;=\; -400$$

$$U_t = U_o(b)^t$$

$$U_t = -400(1.04)^t \quad \boxed{4}$$

$$\therefore from \;\boxed{2} \quad U_t = P_t - 1250$$

$$P_t = U_t + 1250$$

$\therefore P_t = -400(1.04)^t + 1250$ $\quad from\,\boxed{4}$...see purple above!!

L. Basic Probability

Example 1. a) $S = \{1,2,3,4,5,6\}$

b)$S = \{(1,1)(1,2)(1,3) \ldots (6,1)(6,2)(6,3)(6,4)(6,5)(6,6)\}$

c)$S = \{HHH, HTH, HTT, THH, THT, TTH, TTT, HHT\}$

Mutually Exclusive VS. Independence

Example 2. $E = \{2,4,6\} \quad F = \{1,2,3,4\}$

a) $= \{1,2,3,4,6\}$

b) $= \{2,4\}$

c) $= \{6\}$

d) $no\ since\ b)\ is\ not\ the\ empty\ set$

Example 3. $\Pr(at\ least\ 1\ tail) = 1 - \Pr\ (no\ tails)$

$$= 1 - \Pr\ (HHHHHH)$$

$$= 1 - \left(\frac{1}{2}\right)^6$$

$$= 1 - \frac{1}{64}$$

$$= \frac{63}{64}$$

Example 4.

Pr(sum seven)=Pr{(1,6)(2,5)(3,4)(4,3)(5,2)(6,1)}=6/36=1/6=0.167

Example 5.

a) $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr\ (A \cap B)$

$0.35 = 0.10 + \Pr(B) - 0.05$

$0.25 = \Pr(B) - 0.05$

$\Pr(B) = 0.30$

b) $\Pr(A \cup B^C) = \Pr(A) + \Pr(B^C) - \Pr\ (A \cap B^C)$

=0.10 + 0.70 – 0.05=0.75

Example 6. a) $\Pr(red) = 0.5$ b) $\Pr(face\ and\ heart) = \frac{3}{52} = 0.058$

c) $\Pr(face\ or\ heart) = \Pr(face) + \Pr(heart) - \Pr\ (face\ and\ heart)$

$$= \frac{12}{52} + \frac{13}{52} - \frac{3}{52} = \frac{22}{52} = 0.423$$

Example 7.

$\Pr(A \cap B) = \Pr(A) \times \Pr(B) = 0.2(0.5) = 0.10$ since A, B are independent

Pr(A ∪ B)=Pr(A) + Pr(B) - 0.06

=0.2 + 0.5 - 0.10

=0.60

So, a) is true since they are independent...b) is true

To check c)...$\Pr(\bar{A} \cap \bar{B}) = 1 - \Pr(A \cup B) = 1 - 0.6 = 0.4$

The answer is d). only a) and b) are true.

Example 8.

a) They are independent because the first flip being tails won't affect the second flip.

b) They are independent, since "ace" and "spades" don't affect each other. One is the type of suit and one is the denomination...i.e. we can get an ace of spades

c) These are disjoint, since one card can't be both a spade and a heart, i.e. prob. of both = 0

Example 9. a) $\frac{1}{4}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{64}$ b) $(\frac{13}{52})(\frac{12}{51})(\frac{11}{50})$

c)$\Pr\,(at\ least\ one\ club) = 1 - \Pr\,(no\ clubs)$

$= 1 - \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = 1 - \frac{27}{64} = \frac{37}{64}$

Example 10. $\Pr(A\ or\ B) = \Pr(A) + PR(B) - PR(A \cap B)$

$0.8 = 0.4 + 0.5 - \Pr\,(A \cap B)$

$0.8 = 0.9 - \Pr\,(A \cap B)$

$\Pr(A \cap B) = 0.1 \neq 0 \quad \therefore not\ mutually\ exclusive$

Independent

$check\ \Pr(A \cap B) = \Pr\,(A) \times \Pr\,(B)$

$0.1 \qquad 0.4 \times 0.5 = 0.2$

$\therefore no \quad The\ answer\ is\ (d).$

Example 11. $\Pr(E \cup F) = 0.3 + 0.4 = 0.7$

$\Pr(E^c \cap F^c) = 1 - 0.7 = 0.3$

Pr(E∩ F^c) = Pr (E) = 0.30 since all of E is outside of F (mutually exclusive)

Example 12. ***a***) $\Pr(swing \cup slide) = \Pr(swing) + \Pr(slide) - \Pr\ (both)$

$$= \frac{50}{80} + \frac{60}{80} - \frac{40}{80}$$
$$= \frac{110-40}{80} \quad = \frac{70}{80} = \frac{7}{8}$$

b)$\Pr(neither) = 1 - \frac{7}{8} = \frac{1}{8}$

Example 13.

a) $\Pr(5\ boys) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

b) Pr(at least 1 boy)=1-Pr(no boys) = 1 – 1/32 = 31/32

Example 14.

(a) Since $\Pr(A^C \cap B^C) = 0.20,$ we know that Pr(AUB)= 1 – 0.20=0.80

From the union formula,
$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
0.80=0.7+0.4-$\Pr(A \cap B)$
$\Pr(A \cap B)$=0.30

b) $Pr(B \cup A^C)^C = \Pr[B^C \cap (A^C)^C] = \Pr(B^C \cap A) = 0.7 - 0.3 = 0.4$

Example 15.

a) $A^c = \{7, 8\}$ $\quad A^C \cap B =$ in A^c and in $B = \{8\}$

$A^c \cup B =$ in $\mathrm{A^C}$ or in B or in both $= \{2, 4, 6, 7, 8\}$

$\Pr(A^c \cup B) = \frac{5 \leftarrow number\ element}{8 \leftarrow total\ elements\ in\ \Omega}$

Or use the formula

$$\Pr(A^c \cup B)$$
$$= \Pr(A^C) + \Pr(B) - \Pr(A^C \cap B)$$
$$= \frac{2}{8} + \frac{4}{8} - \frac{1}{8} = \boxed{\frac{5}{8}}$$

b) $A \cap B^C =$ in A and not in B

ie) in A and in B^C

$B^C = \{1,3,5,7\}$

$A = \{1,2,3,4,5,6\}$

$A \cap B^C = \{1,3,5\}$

$\therefore \Pr A \cap B^C = \boxed{\frac{3}{8}}$

c) $\Pr(B \cap A^C) =$ in B and not in A $=$ in B and in A^C

$B \cap A^C = \{8\}$

$\Pr(B \cap A^C) = \boxed{\frac{1}{8}}$

d) $\Pr\left(C^C \cap (A \cap B)\right)$

$A \cap B = \{2,4,6\}$

$C^C = \{1,2,6,8\}$

$C^C \cap (A \cap B) =$ in both $= \{2,6\}$

$\therefore \Pr\left(C^C \cap (A \cap B)\right) = \frac{2}{8} = \boxed{\frac{1}{4}}$

e) $\Pr\left(C^C \cup (A \cap B)\right)$

$A \cap B = \{2,4,6\}$

$C^C = \{1,2,6,8\}$

$C^C \cup (A \cap B) = \{1,2,4,6,8\}$ ← 5 elements

$\Pr\left(C^C \cup (A \cap B)\right) = \boxed{\frac{5}{8}}$

Or use union formula

$\Pr\left(C^C \cup (A \cap B)\right) = \Pr(C^C) + \Pr(A \cap B) - \Pr(C^C \cap (A \cap B))$

$= \frac{4}{8} + \frac{3}{8} - \frac{2}{8} = \boxed{\frac{5}{8}}$

Practice Exam Questions on Probability

L1. $\Pr(A) = 1 - \Pr(O) - \Pr(B) - \Pr(AB) = 1 - 0.50 - 0.20 - 0.05 = 0.25$. The answer is (c).

L2. Pr(both aces)=$\frac{4}{52} \times \frac{3}{51}$=0.00452

L3. Let E denote the event that at least one of the four mosquitoes was a carrier of the virus. Then $\overline{E}$ denotes the event that none of the four mosquitoes was a carrier of the virus. Since each mosquito has a 90% of not being a carrier of the virus,

$$\Pr(\overline{\mathrm{E}}) = (0.90)^4 = 0.6561 .$$

Therefore $\Pr(E) = 1 - \Pr(\overline{E}) = 1 - (0.90)^4 = 0.3439 = 34.39\%$.

L4. The probabilities of drawing 1 red ball, 1 green ball, or 1 yellow ball are

$$\Pr(R) = \frac{5}{10}, \qquad \Pr(G) = \frac{3}{10}, \qquad \Pr(Y) = \frac{2}{10},$$

respectively.

The probabilities of drawing 2 red balls, 2 green balls, or 2 yellow balls are

$$\Pr(RR) = \left(\frac{5}{10}\right)^2, \qquad \Pr(GG) = \left(\frac{3}{10}\right)^2, \qquad \Pr(YY) = \left(\frac{2}{10}\right)^2,$$

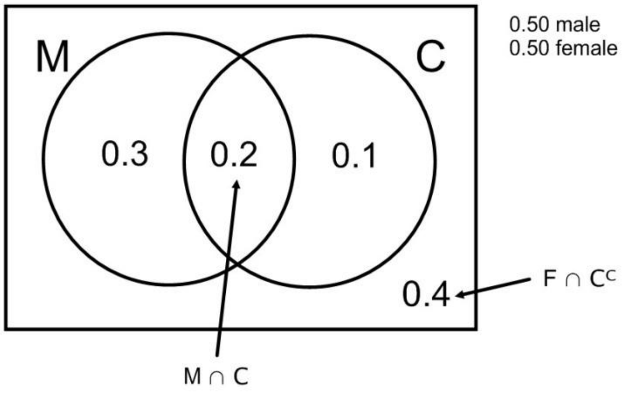
respectively.

The probability of drawing 2 balls of the same colour is therefore

$$\Pr(RRorGGorYY) = \left(\frac{5}{10}\right)^2 + \left(\frac{3}{10}\right)^2 + \left(\frac{2}{10}\right)^2 = 0.25 + 0.09 + 0.04 = 0.38 .$$

L5. If 30% have a college degree and 20% of men have a college degree, then 10% of the women have a college degree

Pr(female and college degree)= 0.10....female without college would be 0.4, if they asked!



L6.

The probability of *not* catching a fish each time you cast your line is $1-\frac{1}{4}=\frac{3}{4}$.

The probability of *not* catching a fish on the first two attempts is $(\frac{3}{4})^2=\frac{9}{16}$.

The probability of catching at least one fish within the first two attempts is thus $1-\frac{9}{16}=\frac{7}{16}$.

The answer is (b).

L7. Pr(F)=0.40 and Pr(N)=0.30, Pr(F and N)=0.20
Pr(F or N)= Pr(F) + P(N) – Pr(F and N)=0.40+0.30-0.20=0.50

L8. Using the table, find for a randomly selected individual from this population the probability that he or she:

a) Is in the age interval 40-49

Pr(40-49)=(10+15+50+70)/400
=145/400=0.3625

b) Is in the age interval 40-49 and weighs 170-189 lbs

=50/400

L9. There are 8 possible outcomes for three children and only 3 consist of exactly 2 girls...GGB, GBG and BGG...so, 3/8=0.375

L10. Pr(false)=5/90=0.056

L11. Pr(AB)= 5/100=0.05

b) Pr(O or Rh-)= Pr(O) + Pr(RH-) - Pr(both)
=45/100 + 14/100 - 6/100
=53/100=0.53

c) Pr(A and Rh+)= 35/100=0.35

L12. Pr(both A)=40/100 times 39/99

L13. Pr(all 4's)=1/6 x 1/6 x 1/6 = 1/216=0.158

b) Pr(no 4)=5/6 x 5/6 x 5/6 = 125/216=0.579

c) Pr(not all 4's) = 1 - Pr(all 4's) = 1 - 1/216 = 215/216=0.995

L14. a) Pr(2)= 3/12 since there are 3 number 2's out of 12 numbers

b) Pr(not a 1)= 1- Pr(get a 1) = 1 - 2/12 = 10/12

c) Pr(not a 2) = 1- 3/12 = 9/12

d) Pr(first 1, second 3) = 2/12 x 6/11 since you don't put them back, you would only have 11 left after removing the first "1"

e) Pr(first 1, second not a 3) = 2/12 x 5/11 = 10/132

L15. Pr(1st die 6 or second 5)= Pr(1st 6) + Pr(2nd 5) - Pr(both)

= 6/36 + 6/36 - 1/36
=11/36

L16. Pr(1st die 6 and 2nd 5 = Pr(getting (6,5) = 1/36 since this is one of 36 possible outcomes

L17. Pr(1st not a 5 or second not a 4)= to be in one set or the other or both...every element in the 36 either has 1st not a 5 OR second not a 4, except the outcome (5,4)...so,
1-1/36=35/36

L18.

a) $\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{64}$

b) $= 1 - \Pr\,(no\ clubs)$

$= 1 - \frac{3}{4}\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = 1 - \frac{27}{64} = \frac{37}{64}$

L19.

a) $\frac{1}{6}$

b) $5\ or\ 6\ \ \frac{2}{6}=\frac{1}{3}$

c) $4, 2, 6\ \ \frac{3}{6}=\frac{1}{2}$

d) $2, 4, 6\ \ \frac{3}{6}=\frac{1}{2}$

L20. A) 6/12 = ½

b) red or yellow = 8/12 = 2/3

c) (6+4)/12 = 10/12 = 5/ 6

L21.

$A-4\quad T-5\quad G-4\quad C-3\quad /16$

$\frac{4+4}{16}=\frac{8}{16}=\frac{1}{2}$

L22. $\frac{3}{10}\times\frac{3}{10}=\frac{9}{100}$ without would be $\frac{3}{10}\times\frac{2}{9}=\frac{6}{90}$

L23. A) Yes, they are disjoint because you can't belong to more than one weight category

b) Pr(D)=1-0.02-0.39-0.35=0.24

L24. Pr(A and B)=Pr(A)Pr(B)= 0.2(0.5)=0.10

L25. $\Pr(heterozygous)=\frac{2}{4}=\frac{1}{2}$

$\Pr(\text{at least one B})=\frac{3}{4}$

L26. a) $=\Pr(E)\times\Pr(F)$

$=0.3\times0.4=0.12$

b)$=\Pr(E)+\Pr(F^c)-\Pr(E)\times\Pr(F^c)$

$=0.3+0.6-0.3\times0.6$

$=0.9-0.18=0.72$

c)$=\Pr(E)\times\Pr(F^c)$

$=0.3\times0.6=0.18$

L27. Pr(sum greater than 10)={(5,6)(6,5)(6,6)}=3/36=1/12=0.083

L28. P(A and B)= 1/36 only one outcome since it would be only {(6,1)}
P(AUB)=P(A or B)=P(A)+P(B)- P(A and B)
=6/36 + 6/36 - 1/36
=11/36

L29. $\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$
$0.86 = 0.50 + 0.35 - \Pr(E \cap F)$
$\Pr(E \cap F) = 0.01 \neq 0$
$\therefore E, F$ *are NOT mutually exclusive*
Check to see if E and F are independent
$\Pr(E) \times \Pr(F)$
$= 0.5 \times 0.35$
$= 0.175$
$\Pr(E \cap F) = 0.01$
$\therefore \Pr(E \cap F) \neq \Pr(E) \times \Pr(F)$
$\therefore E, F$ *are not independent*

L30. BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG are all the possibilities.
Pr(exactly 2 girls)=3/8=0.375

L31. $\Pr(A) = \frac{1}{2}$ (*half the rolls are even*)
$\Pr(B) = \frac{6}{36} = \frac{1}{6}$ [(1,6)(2,6)(3,6)(4,6)(5,6)(6,6)]
$\Pr(A \cap B) = \frac{3}{36} = \frac{1}{12}$ [(2,6)(4,6)(6,6)]
$\Pr(A) \times \Pr(B) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$
$\therefore \Pr(A \cap B) = \Pr(A) \times \Pr(B)$ $\therefore$ *they are independent*

L32. Suppose $\Omega = \{1, 2, 3, \ldots, 10\}$ ← 10 elements
$A = \{1, 2, 3\}$ $B = \{2, 6, 8\}$ $C = \{3, 4, 5, 7\}$
Find

a) $\Pr\left(C^C \cup (A \cap B)\right)$

$A \cap B = \{2\}$ $C^C \cap (A \cap B) = \{2\}$
$C^C = \{1, 2, 6, 8, 9, 10\}$
$C^C \cup (A \cap B) = \{1, 2, 6, 8, 9, 10\}$ ← 6 elements

$\therefore \Pr\left(C^C \cap (A \cap B)\right) = \boxed{\frac{6}{10}}$

Or use union formula

$\Pr\big(C^C \cup (A \cap B)\big) = \Pr(C^C) + \Pr(A \cap B) - \Pr(C^C \cap (A \cap B))$

$= \frac{6}{10} + \frac{1}{10} - \frac{1}{10} = \frac{6}{10} = \boxed{\frac{3}{5}}$

b) $A^C = \{4, 5, 6, 7, 8, 9, 10\}$

$A \cap B = \{2\}$

$(A \cap B)^C = \{1, 3, 4, \dots, 10\}$

$B^C = \{1, 3, 4, 5, 7, 9, 10\}$

$A^C \cap B^C = \{4, 5, 7, 9, 10\}$

$\Pr(A \cap B)^C$

$= \Pr(A^C \cup B^C)$

$= \Pr(A^C) + \Pr(B^C) - \Pr(A^C \cap B^C)$

$= \frac{7}{10} + \frac{7}{10} - \frac{5}{10} = \frac{14}{10} - \frac{5}{10} = \boxed{\frac{9}{10}}$

Or since $(A \cap B)^C = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$ ← 9 elements

$\Pr(A \cap B)^C = \frac{9}{10}$

L33. Suppose $\Omega = \{1, 2, 3, 4, 5, 6\}$

$E = \{2, 3, 5\}$ $F = \{1, 3, 5\}$ $G = \{3, 4\}$

Find

a) $\Pr[(E \cap F^C) \cap (E^C \cup G^C)]$

$F^C = \{2, 4, 5, 6\}$ $E^C = \{1, 4, 6\}$ $G^C = \{1, 2, 5, 6\}$

$E \cap F^C =$ in E and not in F $= \{2, 5\}$

$E^C \cup G^C =$ in E^{C} or G^C or both $= \{1, 2, 4, 5, 6\}$

$= \{1, 2, 4, 5, 6\}$

$(E \cap F^C) \cap (E^C \cup G^C) =$ in both $= \{2, 5\}$

$\therefore \Pr[(E \cap F^C) \cap (E^C \cup G^C)] = \frac{2 \leftarrow 2 \textit{ elements in both}}{6 \leftarrow \Omega \text{ has 6 elements}} = \frac{2}{6} = \frac{1}{3}$

b) $\Pr[(E \cap F) \cup (E \cap F^C)]$

$E \cap F = \{3, 5\}$

$E \cap F^C = \{2, 5\}$

$\therefore (E \cap F) \cup (E \cap F^C) = \{2, 3, 5\}$

$\therefore \Pr[(E \cap F) \cup (E \cap F^C)] = \frac{3}{6} = \frac{1}{2}$

M. Conditional Probability

Example 1.

$$\Pr(F/E) = \frac{\Pr(F \cap E)}{\Pr(E)} = \frac{0.30}{0.40} = \frac{3}{4} = 0.75$$

Example 2. $\Pr(N) = \frac{35}{100}$ $\Pr(R) = \frac{50}{100}$ $\Pr(N \cup R) = \frac{80}{100}$

$\Pr(R/\mathrm{N}) = \frac{\Pr(R \cap N)}{\Pr(N)}$

Find $\Pr(R \cap N)$

$\Pr(N \cup R) = \Pr(N) + \Pr(R) - \Pr(N \cap R)$

$\frac{80}{100} = \frac{35}{100} + \frac{50}{100} - \Pr(N \cap R)$

$\Pr(N \cap R) = \frac{5}{100}$

$\therefore \Pr(R/\mathrm{N}) = \frac{5/100}{35/100} = \frac{5}{35} = \frac{1}{7}$

Example 3. $\Pr(E/\mathrm{F}) = \frac{\Pr(E \cap F)}{\Pr(F)}$

$\frac{1}{3} = \frac{\Pr(E \cap F)}{1/4}$

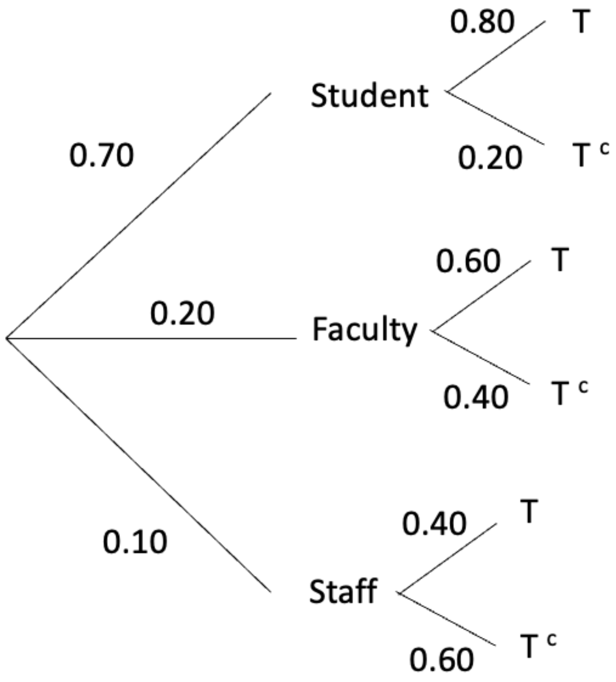
$\Pr(E \cap F) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$

$\Pr(F/\mathrm{E}) = \frac{\Pr(E \cap F)}{\Pr(E)}$

$= \frac{1/12}{2/3}$

$= \frac{1}{12} \times \frac{3}{2} = \frac{3}{24} = \frac{1}{8}$

Example 4.a)



$$\Pr(T) = \Pr(student \cap T) + \Pr(faculty \cap T) + \Pr\,(staff \cap T)$$
$$= 0.70(0.8) + 0.2(0.6) + 0.1(0.4)$$
$$= 0.72$$

b)$\Pr\,(student/\mathrm{T}) = \frac{\Pr\,(student \cap T)}{\Pr\,(T)}$

$$= \frac{0.70 \times 0.80}{0.72} = \frac{56}{72} = \frac{7}{9}$$

Example 5.

a) $S\ Reduced = \{(4,1)(4,2)(4,3)(4,4)(4,5)(4,6)\}$

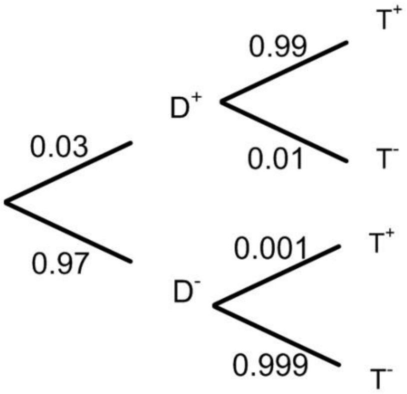
$\nwarrow \frac{1}{6}$ Since the reduced sample space is what is given, ie. 1st die is a 4 and then we circle how many of these 6 outcomes have a sum greater than 9 and there is only 1 outcome

b)$SReduced = \{(1,1)(2,1)(3,1)(4,1)(5,1)(6,1)\} = \frac{0}{6} = 0$

Example 6. $\Pr(T+) = \Pr(D^+ \cap T^+) + \Pr(D^- \cap T^+)$

$= 0.03(0.99) + 0.97(0.001)$

$= 0.0307$



Or (3/100)(99/100)+(97/100)(1/1000)
=(30/1000)(99/100)+(97/100)(1/1000)
=(2970 + 97)/100000
=3067/10000

Example 7.

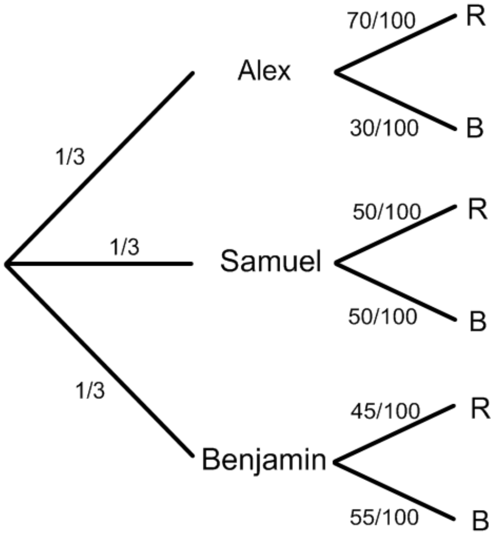
a) $\Pr(B) = \Pr(Alex \cap B) + \Pr(Samuel \cap B) + \Pr(Ben \cap B)$

$= \Pr(A) \times \Pr(B/A) + \Pr(S) \times \Pr(B/S) + \Pr(B) \times \Pr(B/B)$

$= \frac{1}{3}\left(\frac{30}{100}\right) + \frac{1}{3}\left(\frac{50}{100}\right) + \frac{1}{3}\left(\frac{55}{100}\right)$

$= \frac{30}{300} + \frac{50}{300} + \frac{55}{300}$

$= \frac{135}{300} = \frac{9}{20} = 0.45$

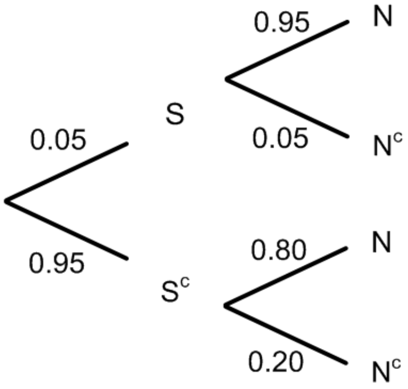


b) $\Pr(Samuel/B) = \frac{\Pr(Samuel \cap B)}{\Pr(B)}$

$$= \frac{(\frac{1}{3})(\frac{50}{100})}{\frac{9}{20}} = \frac{1}{6}\left(\frac{20}{9}\right) = \frac{20}{54} = \frac{10}{27}$$

Example 8.

$\Pr(N^c) = \Pr(S \cap N^c) + \Pr(S^c \cap N^c)$
$= \Pr(S) \times \Pr(N^c/S) + \Pr(S^c) \times \Pr(N^c/S^c)$
$= 0.05(0.05) + 0.95(0.20)$
$= 0.1925$



Or (5/100)(5/100)+(95/100)(20/100)
=(25+1900)/10000=1925/10000=0.1925 or 77/400

$\Pr(S^c/N) = 1 - \Pr(S/N)$
$= 1 - \frac{\Pr(S \cap N)}{\Pr(N)}$
$= 1 - \frac{0.05(0.95)}{1 - 0.1925}$
$= 0.941$

Example 9.

Yy $yellow$
yy $green$
Yy $yellow$
yy $green$
$\Pr(green) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Practice Exam Questions on Conditional Probability

M1.

a) $\Pr(T) = Tr(Box\ 1\ and\ T) + \Pr(Box\ 2\ and\ T) + \Pr(Box\ 3\ and\ T)$

$$= \frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{2}{4}$$

$$= \frac{1}{9}+\frac{1}{6}+\frac{1}{6} = \frac{4}{36}+\frac{6}{36}+\frac{6}{36} = \frac{16}{36} = \frac{8}{18} = \frac{4}{9}$$

b) $\Pr(2nd|T) = \frac{\Pr(2nd\cap T)}{\Pr(T)} = \frac{1/3\times 1/2}{4/9} = \frac{1/6}{4/9} = \frac{1}{6}\left(\frac{9}{4}\right) = \frac{9}{24} = \frac{3}{8}$

M2.

a) $\Pr(B) = \Pr(W\cap B) + \Pr(W^C\cap B^c)$

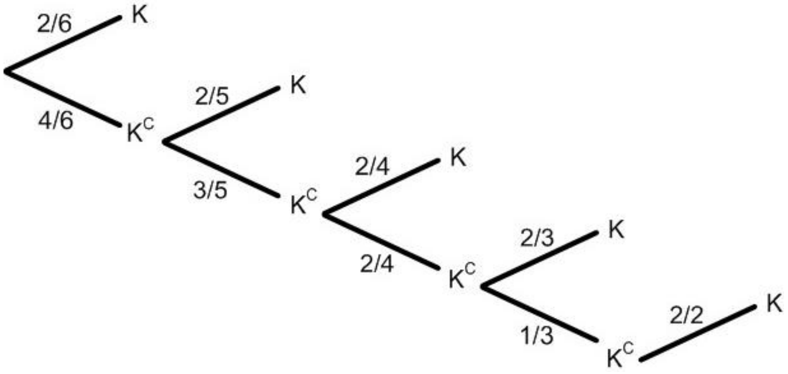
$$= 0.60\times 0.50 + 0.40\times 0.30$$
$$= 0.30 + 0.12$$
$$= 0.42$$

$b)\ \Pr(M|B) = \frac{\Pr\,(M\cap B)}{\Pr\,(B)} = \frac{0.40\times 0.30}{0.42} = \frac{0.12}{0.42} = \frac{12}{42} = \frac{2}{7}$

M3. $\Pr(k) + \Pr(k^ck) + \Pr(k^ck^ck) + \Pr(k^ck^ck^ck)\ OR$

$1 - \Pr(k^ck^ck^ck^ck)$

$$= 1 - \frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\times 1$$
$$= 1 - \frac{4}{5\times 4\times 3}$$
$$= 1 - \frac{1}{15}$$
$$= \frac{14}{15}$$



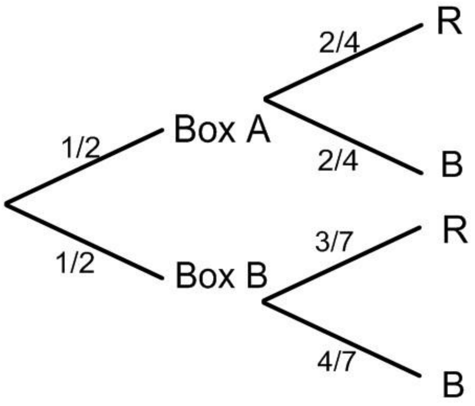
M4.

a)

$$\Pr(B) = \Pr(Box\ A\ and\ B) + \Pr(Box\ B\ and\ B)$$
$$= \frac{1}{2} \times \frac{2}{4} + \frac{1}{2} \times \frac{4}{7}$$
$$= \frac{1}{4} + \frac{2}{7}$$
$$= \frac{7}{28} + \frac{8}{28} = \frac{15}{28}$$

b)

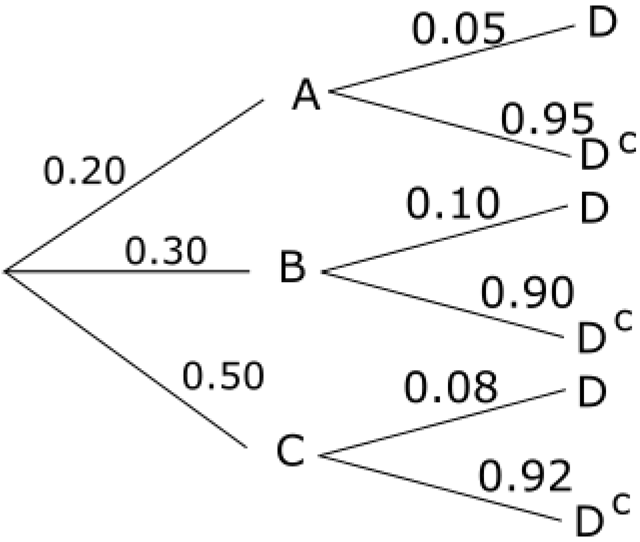
$$\Pr(BoxA|R) = \frac{\Pr\ (BoxA\ and\ R)}{\Pr\ (R)} = \frac{{}^{1}/_{2}\left(\frac{2}{4}\right)}{{}^{1}/_{2}\,({}^{2}/_{4}) + {}^{1}/_{2}\,({}^{3}/_{7})} = \frac{1/4}{{}^{1}/_{4} + {}^{3}/_{14}}$$



M5.

$$\Pr(A|D) = \frac{\Pr\text{ (D and A)}}{\Pr(D)}$$

$$= \frac{(0.05)(0.20)}{0.2(0.05) + 0.3(0.10) + 0.5(0.08)}$$



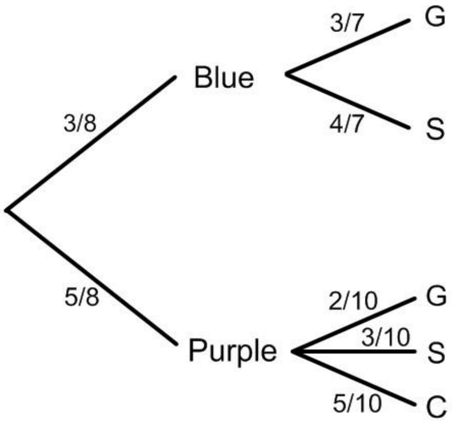
M6.

a)

$\Pr(G) = \Pr(B \cap G) + \Pr(P \cap G)$

$= \frac{3}{8} \times \frac{3}{7} + \frac{5}{8} \times \frac{2}{10}$

$= \frac{9}{56} + \frac{10}{80}$



b)

$\Pr(P|S) = \frac{\Pr\text{ (P and S)}}{\Pr(S)}$

$= \frac{{}^{3}/_{10}\left({}^{5}/_{8}\right)}{{}^{3}/_{8}\left({}^{4}/_{7}\right) + {}^{5}/_{8}\left({}^{3}/_{10}\right)} = \frac{{}^{15}/_{80}}{{}^{12}/_{56} + {}^{15}/_{80}}$

M7.

Pr(F/E)=0.20

Pr(F/E^{C})=0.10

Pr(E)=0.40

Find $\Pr(E \cap F)$

Pr(F/E)=$\frac{\Pr(F \cap E)}{\Pr(E)}$

0.20=$\frac{\Pr(F \cap E)}{0.40}$

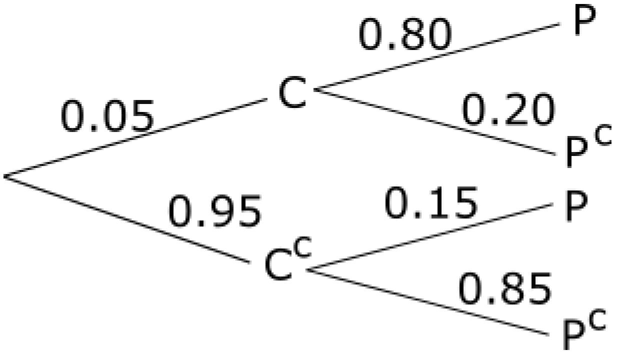
Pr $(F \cap E)$=(0.20)(0.40)=0.08

The answer is C.

M8.

Draw a Tree diagram

$\Pr(C/P) = \frac{\Pr(C \cap P)}{\Pr(P)} = \frac{0.05(0.80)}{0.05(0.80)+0.95(0.15)}$



M9.

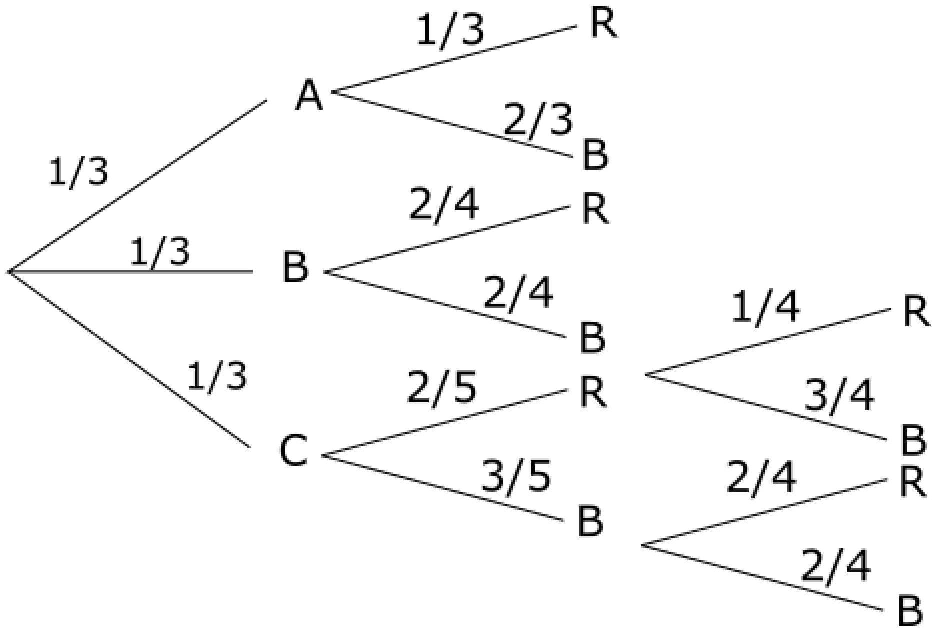
Pr(red) = Pr(red from Box A) + Pr(red from Box B) + Pr(red from Box C)

$=\frac{1}{3}\left(\frac{1}{3}\right)+\frac{1}{3}\left(\frac{2}{4}\right)+\frac{1}{3}\left(\frac{2}{5}\right)=\frac{1}{9}+\frac{1}{6}+\frac{2}{15}$

M10.

Pr(C and 2nd red) = Pr(C and BR) + Pr(C and RR)

$=\frac{1}{3}\times\frac{3}{5}\times\frac{2}{4}+\frac{1}{3}\times\frac{2}{5}\times\frac{1}{4}=\frac{1}{10}+\frac{1}{30}=\frac{3}{30}+\frac{1}{30}=\frac{4}{30}=\frac{2}{15}$



M11. **(a)** What is the probability that a randomly selected individual is experiencing hypertension?

$$\text{Pr(hypertension)} = \frac{\text{\# with hypertension}}{\text{total \#}} = \frac{21+36+30}{180} = \frac{87}{180} \approx 0.48$$

(b) Given that a heavy smoker is selected at random from this group, what is the probability that the person is experiencing hypertension?

$$\text{Pr(hypertension | heavy smoker)} = \frac{\text{Pr(hypertension} \cap \text{heavy smoker)}}{\text{Pr(heavy smoker)}} = \frac{30}{30+19} \approx 0.61$$

(c) Are the events "hypertension" and "heavy smoker" independent? Give supporting calculations.

Since $\text{Pr(hypertension | heavy smoker)} = \frac{30}{49} \neq \frac{87}{180} = \text{Pr(hypertension)}$, the two events are *not* independent.

M12. Consider the following events:

$A =$ "adult selected has a college level education";
$B =$ "adult selected is a male with the highest level of education being secondary";
$C =$ "adult selected is a female".

(a) Are the events *A* and *B* disjoint? Explain.

Yes. They are disjoint because an adult cannot have a college level education and have his highest level of education be secondary.

(b) Are the events *A* and *C* disjoint? Explain.

No. They are not disjoint since females can have a college level education.

(c) What is the probability that an adult selected at random either has a college level education or is female?

Pr(college or female) = Pr(college) + Pr(female) - Pr(college and female)

$$= \frac{22+17}{200} + \frac{45+50+17}{200} - \frac{17}{200} = \frac{39}{200} + \frac{112}{200} - \frac{17}{200} = \frac{134}{200} = 0.67.$$

(d) What is the probability that an adult selected at random has a college level education given that the adult is a female?

$$\text{Pr(college | female)} = \frac{\text{Pr(college and female)}}{\text{Pr(female)}} = \frac{\frac{17}{200}}{\frac{112}{200}} = \frac{17}{112} \approx 0.15$$

(e) Are the events *A* and *C* independent?

$\Pr(A) = \frac{22+17}{200} = \frac{39}{200}$; $\Pr(C) = \frac{45+50+17}{200} = \frac{112}{200} = \frac{14}{25}$;

$\Pr(A \cap C) = \frac{17}{200} = 0.085$; $\Pr(A)\Pr(C) = \frac{39}{200} \cdot \frac{14}{25} = \frac{273}{2500} = 0.1092$;

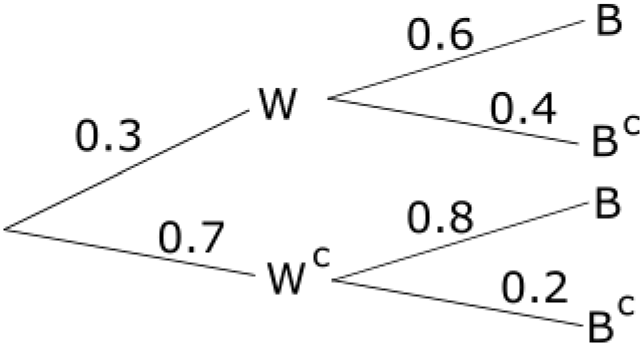
Since $\Pr(A \cap C) \neq \Pr(A)\Pr(C)$, the events *A* and *C* are not independent.

M13.

$\Pr(B) = \Pr(W \cap B) + \Pr(W^C \cap B) = 0.3(0.6) + (0.70)(0.80) = 0.18 + 0.56 = 0.74$

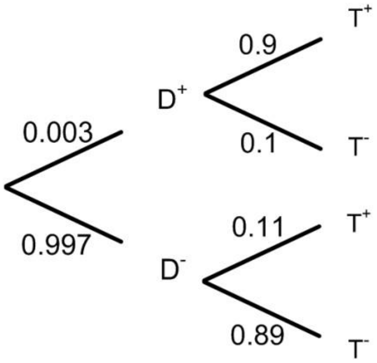
M14.

$\Pr(W^C/B) = \frac{\Pr(W^C \cap B)}{\Pr(B)} = \frac{0.7(0.8)}{0.84} = \frac{56}{84}$



M15.

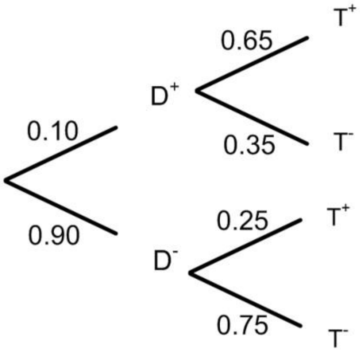
$$\Pr(D^+/T^+) = \frac{\Pr(D^+ \text{ and } T^+)}{\Pr(T^+)}$$
$$= \frac{(0.90)(0.003)}{0.003(0.9)+0.997(0.11)} = 0.024$$



M16.

$\Pr(T^+/D^+) = 0.65$
$\Pr(T^-/D^-) = 0.75$

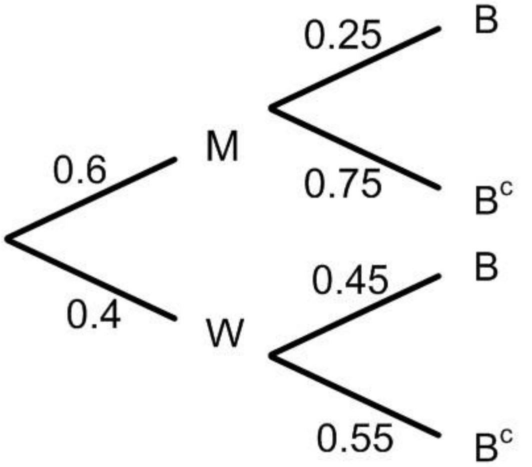
$$\Pr(D^+/T^+) = \frac{\Pr(T^+ and\ D^+)(}{Pr(T^+)} = \frac{0.65(0.10)}{0.10(0.65)+0.9(0.25)} = 0.224$$



M17.

$$\Pr(W|B) = \frac{\Pr(B|W)\Pr(W)}{\Pr(B)}$$

$$= \frac{0.45(0.4)}{0.6(0.25) + 0.4(0.45)}$$

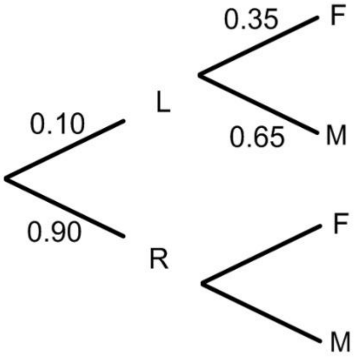


M18. a) $\Pr(C) = 0.5(0.10) + 0.5(0.01) = 0.055\ or\ 5.5\%$
b) $\Pr(M/C^c) = \frac{\Pr(C^c/M)\Pr(M)}{\Pr(C^c)} = \frac{0.90(0.5)}{1-0.055} = 0.476$

M19. $\Pr(L) = 0.10$ $\Pr(M) = 0.60$

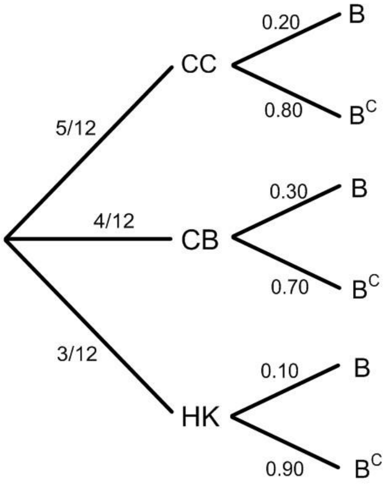
a) $\Pr(L \cap M) = \Pr(L) \times \Pr(M) = 0.10 \times 0.60 = 0.06\ or\ 6\%$
b) $\Pr(F/L) = 0.35$

$\Pr(L \cap M) = \Pr(L) \times \Pr(M/L)$
$= 0.10 \times 0.65 = 0.065$



M20.

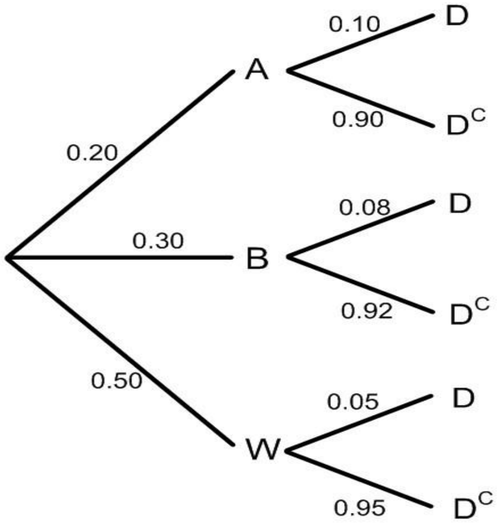
$$\Pr(CC/B^C) = \frac{\Pr\,(CC\ and B^C)}{\Pr(B^C)} = \frac{\frac{5}{12}(0.8)}{\frac{5}{12}(0.8)+\frac{4}{12}(0.7)+\frac{3}{12}(0.9)}$$



M21.

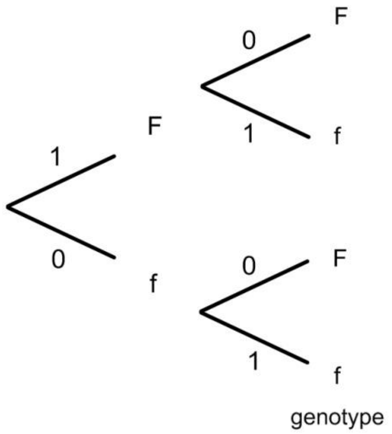
$$\Pr(A|D) = \frac{\Pr\,(\mathrm{A}\cap D)}{\Pr(D)}$$

$$= \frac{(0.10)(0.20)}{0.2(0.1)+0.3(0.08)+0.5(0.05)} = 0.29$$

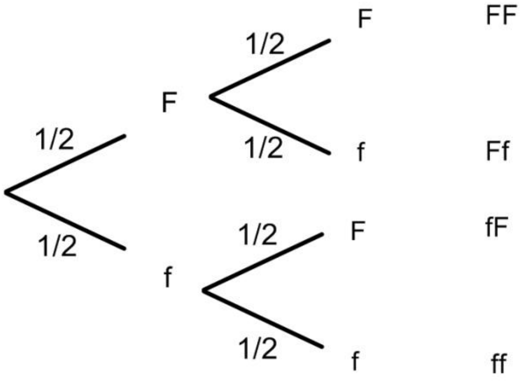


M22. $a)$ $\Pr(FF) = (1)(0) = 0$

b)$\Pr(ff) = 0(1) = 0$
$\Pr(Ff) = 1(1) + a(a) = 1$
c)$Purple = Ff \quad \therefore 1$

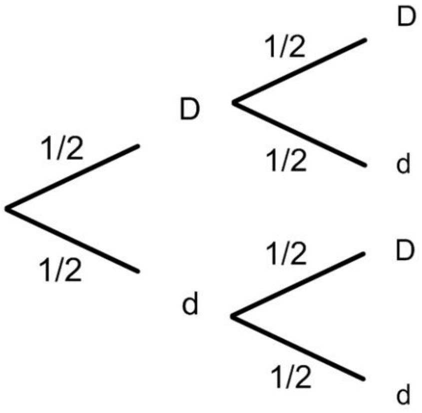


M23. a)
b) Pr(white)=ff = 1/4
c)$\Pr(Ff) = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{2}$



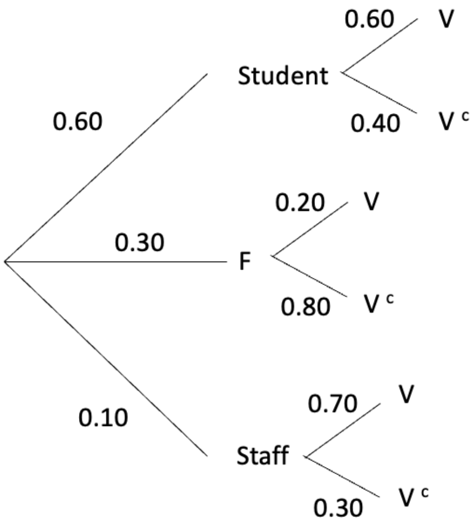
M24.
$Ff \quad - purple$
$Ff \quad - purple$
$ff \quad - white$
$ff \quad - white$
$\Pr(ff) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

M25. $\Pr(Dd) = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$



Quiz 3: Practice on Sections L and M

1.a)



$$\begin{aligned}\Pr(V) &= \Pr(student \cap V) + \Pr(F \cap V) + \Pr\,(staff \cap V)\\ &= 0.6(0.6) + 0.3(0.2) + 0.10(0.70)\\ &= 0.49\end{aligned}$$

b)$\Pr\,(staff/\text{V}) = \frac{\Pr\,(staff \cap V)}{\Pr\,(V)}$

$= \frac{0.10(0.70)}{0.49}$

$= 0.143 \quad or \;\; 14.3\%$

2. $\Pr(at\ least\ 1\ hand) = 1 - \Pr\,(no\ hands)$

$= 1 - \Pr\,(TTTT)$

$= 1 - \left(\frac{1}{2}\right)^4$

$= 1 - \frac{1}{16}$

$= \frac{15}{16}$

3. The answer is *independent.*

4. $\Pr(A) = \frac{1}{2} \quad (\frac{1}{2}\ rolls\ are\ even)$

$\Pr(B) = \frac{6}{36} = \frac{1}{6} \quad ((1,1)(1,2)(1,3)(1,4)(1,5)(1,6))$

$\Pr(A \cap B) = \frac{3}{36} = \frac{1}{12} \quad ((1,1)(1,3)(1,5))$

$\Pr(A \cap B) = \Pr\,(A) \times \Pr\,(B)$

$\therefore they\ are\ independent$

5. a) $\Pr(S \cup C) = \Pr(S) + \Pr(C) - \Pr(S \cap C)$

$$= \frac{40}{75} + \frac{30}{75} - \frac{20}{75}$$

$$= \frac{50}{75} = 0.667$$

b) $\Pr(S^c \cap C^c) = 1 - \Pr(S \cup C) = 1 - 0.667 = 0.333$

6. $\Pr(F/\mathrm{E}) = \frac{\Pr(E \cap F)}{\Pr(E)}$

$$0.6 = \frac{\Pr(\mathrm{E} \cap \mathrm{F})}{0.2}$$

$\Pr(E \cap F) = 0.6 \times 0.2 = 0.12$

7. a) $E^c = there\ are\ no\ queens\ among\ the\ 5\ cards\ drawn$

b) $E \cap F = there\ is\ at\ least\ one\ queen\ and\ all\ 5\ cards\ are\ red$

∴ You must have either the Queen of Hearts or the Queen of Diamonds (or both) among the 5 cards chosen

8. $\Pr(E \cap F) = 0$

9. $\Pr(E \cap F) = \Pr(E) \times \Pr(F)$

N. Practice Exam Questions on Probability

N1. $\frac{8+3+1}{8+3+1+2} = \frac{12}{14} = 0.857$

N2. $1 - \Pr(no\ girls) = 1 - \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= 1 - \frac{1}{8} = \frac{7}{8}$

N3. Pr $(roll\ 4$ /even)= $\frac{1}{3}$ $\nwarrow roll\ 4$

$S_{reduced} = \Omega\, reduced = \{only\ even\} = \{2, 4, 6\}$

↓

Denominator

N4. Pr (roll 5/even)= 0 ← impossible since 5 is not even

N5. Pr(1T, 2H) = Pr(THH, HHT, HTH) = $\frac{3}{8}$ $(draw\ a\ tree)$

N6. a) $\frac{1}{4}\left(\frac{1}{4}\right) = \frac{1}{16}$

b)$\frac{4}{52}$) $\left(\frac{4}{52}\right) = 0.005917$

c)$\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4} = 0.25$

N7. $B^C = \{2, 4, 6\}$

a) $= \{2, 4\}$

b) $= \{1, 2, 3, 4, 5\} = A$

N8. $B^C = \{4, 5, c, d\}$

a) $A \cap B = \{1, 3, a\} \neq C^c = \{1, 5, a\}$

b) $A \cap B^C = \{5\} \neq C^C = \{1, 5, a\}$

c) Always true

d) $A \cap C = \{3\} \neq C^C = \{1, 5, a\}$

e) False

∴ C is the answer.

N9.

a) $= \{1, 2, 3, 4, 5, 6\}$

b) $= \{1, 2, 3, 4\}$

c) $= \{1, 2, 3, 4\}$

d) $= \{\emptyset\}$ $\therefore D\ is\ the\ answer$

N10. Pr(sum 7)

$= \Pr((1,6),(6,1),(2,5),(5,2),(3,4),(4,3))$

N11. $\Pr(BBG) = \frac{3}{7} \times \frac{2}{6} \times \frac{4}{5} = \frac{4}{35} = 0.114$

N12. $3x + x = 1$

$4x = 1 \quad x = \frac{1}{4}$

$\text{Pr(T)=Pr}(A \cap T) + \Pr(B \cap T)$

$= \frac{3}{4}(0.6) + \frac{1}{4}(0.75)$

$= 0.6375$

N13. a) NO should be $(E \cup F)^C = E^c \cap F^c$

b) YES

c)NO

d)YES $\therefore B\ and\ D\ is\ the\ answer$

N14. $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$= 0.20 + 0.50 - 0.15$

$= 0.70 - 0.15 \quad = 0.55$

N15. D is the answer.

N16. D = has disease D^c = doesn't have disease

$P = test\ is +\quad P^c = test\ is -$

$\Pr(D/\text{P})= \frac{\Pr(D \cap P)}{\Pr(P)} = \frac{0.03 \times 1}{0.03 \times 1 + 0.97 \times 0.15} = 0.17$

Material Since the Midterm

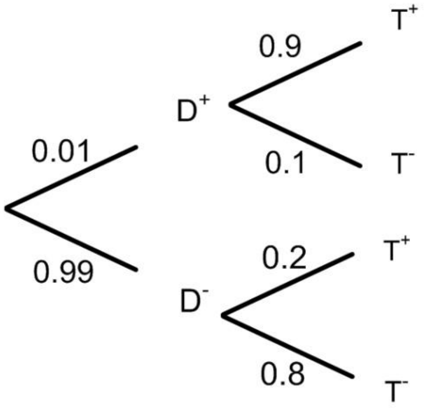
O. Sensitivity and Specificity and Type I and II Errors

Example 1. $\Pr(T^{+}/D^{+}) = sensitivity = 0.9$

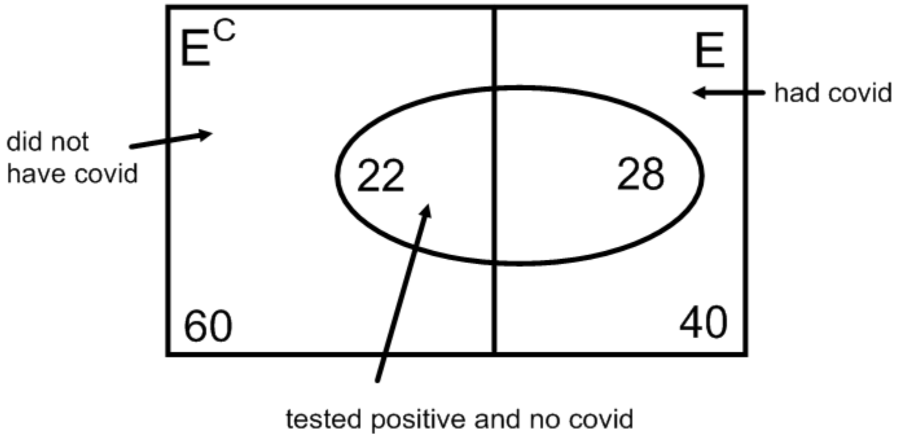
$\Pr(T^{-}/D^{-}) = specificity = 0.8$

$\Pr(D^{-}/T^{-}) = \frac{\Pr(D^{-} and\ T^{-})}{\Pr(T^{-})} = \frac{0.8(0.99)}{0.01(0.1)+0.99(0.8)}$=0.999

$\uparrow given$



Example 2.



$\therefore 60 - 22 = 38$ *tested negative and no covid*

Sensitivity

$$\Pr(T^{+}/D^{+}) = \frac{\Pr(T^{+} \cap D^{+})}{\Pr(D^{+})}$$

$$= \frac{\frac{28}{100}}{\frac{40}{100}}$$

$$= \frac{28}{40} = 0.70$$

$\Pr(type\ II\ error) = 1 - 0.70 = 0.30$

Specificity

$$\Pr(T^{-}/D^{-}) = \frac{\Pr(T^{-} \cap D^{-})}{\Pr(D^{-})}$$

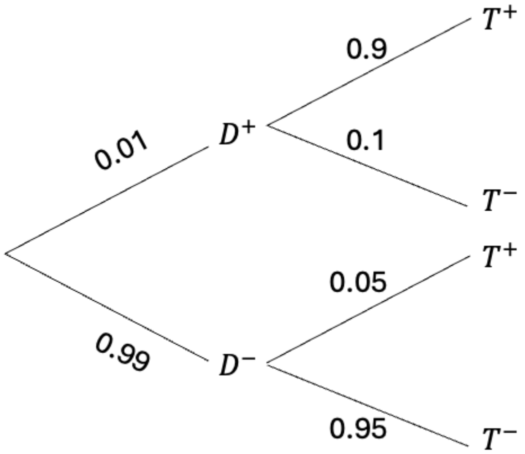
$$= \frac{\frac{38}{100}}{\frac{60}{100}}$$

$$= \frac{38}{60} = 0.63$$

$\Pr(type\ I\ error) = 1 - 0.63 = 0.37$

Example 3.

$\Pr\,(T+/D+) = 0.9$ (Sensitivity)
$\Pr\,(T-/D-) = 0.9$ (Specificity)



a) $E(x) = \text{np} = 1000(0.01) = 10\,people$

b) false +
$\Pr\,(T+/D-) = 0.05$
Number false $+= 0.05\ \times \text{n}$
Where n= number <u>without</u> disease

$= 0.05(1000 - 10)$
$= 0.05(990)$
<u>= 49.5 people</u>

* false positives only occur if the person doesn't have the disease

c) $\Pr(\text{false} -) = \Pr(T-/D+)$
$= 0.10$

$= 0.10\ \times \text{n}$ where n =# number with the disease
<u># false$-$ = 0.10(10) = 1 *person*</u>

d) <u>true +</u>
$\Pr(T+/D+) = 0.9$
\# of true+ $= 0.9 \times 10 =$<u>9 people</u>

<u>true −</u>
$\Pr(T-/D-) = 0.95$
\# of true negatives
$= 0.95 \times 990$
<u>$= 940.5$ people</u>

NOTE: 49.5 + 1 +9 +940.5 = 1000 people!!

e) We see from b), c), and d) that there are 49.5 people with a false + and only 9 people with a true positive, so it is much more likely it was an error in the test than they actually have the disease

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Practice Exam Questions on Sensitivity and Specificity and Type I and II Errors

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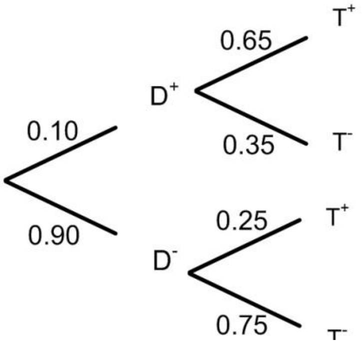
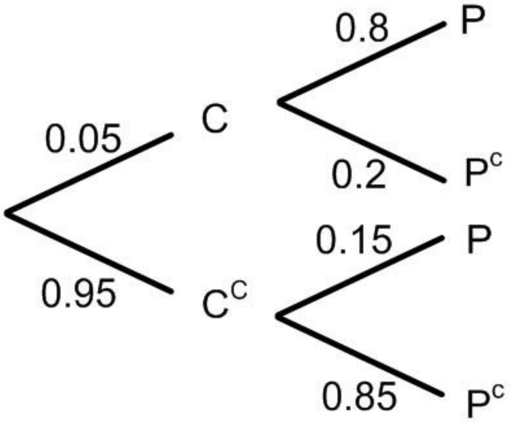
O1.
$\Pr(T^+/D^+) = 0.65$
$\Pr(T^-/D^-) = 0.75$

$$\Pr(D^+/T^+) = \frac{\Pr(T^+ and\ D^+)(}{Pr(T^+)} = \frac{0.65(0.10)}{0.10(0.65)+0.9(0.25)}=0.224$$



O2.
Draw a Tree diagram

$$\Pr(C^C/P)=\frac{\Pr(C^C\ and\ P)}{\Pr(P)} = \frac{0.95(0.15)}{0.95(0.15)+0.05(0.80)}=0.781$$



b) Sensitivity=0.80 (test positive/have disease)
c) Specificity=0.85 (test negative/don't have disease)

d) $\Pr(type\ II\ error) = 1 - 0.80 = 0.20$
$\Pr(type\ I\ error) = 1 - 0.85 = 0.15$

03.

$\therefore 70 - 25 = 45 \ tested \ negative \ and \ no \ covid$

Sensitivity $\Pr\left(\frac{T^+}{D^+}\right) = \frac{\Pr\,(T^+ \cap D^+)}{\Pr\,(D^+)}$

$= \frac{\frac{25}{100}}{\frac{30}{100}} = \frac{25}{30} = 0.83$

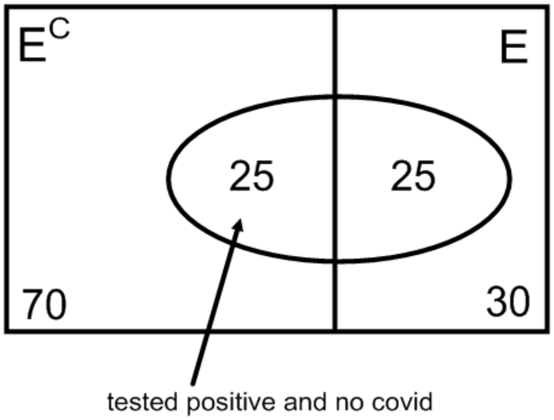
$\Pr(type \ II \ error) = 1 - 0.83 = 0.17$

Specificity $\Pr\left(\frac{T^-}{D^-}\right) = \frac{\Pr\,(T^- \cap D^-)}{Pr(D^-)}$

$= \frac{\frac{45}{100}}{\frac{70}{100}}$

$= \frac{45}{70} = 0.64$

$\Pr(type \ I \ error) = 1 - 0.64 = 0.36$



A. Discrete and Continuous Random Variables

Discrete Random Variables

Example 1. If you roll a die and you let random variable X be the number on the up face, then:

What is $f_X(2)$? This is the probability of rolling a 2, which is Pr(X=2) = 1/6.

Example 2. Suppose a discrete random variable X takes on values 0, 1, and 2. Which of the following could be a valid PMF for X?

a) $f_X(0) = 1, f_X(1) = \frac{1}{2} and \ f_X(2) = 2$

No, since 1 + ½ + 2 is NOT 1.

b) $f_X(0) = -1/4, f_X(1) = \frac{1}{2} and \ f_X(2) = 1/4$

No, since probabilities cannot be negative and this says Pr(X=0) = -1/4.

c) $f_X(0) = \frac{1}{4}, f_X(1) = \frac{1}{8} and \ f_X(2) = \frac{5}{8}$

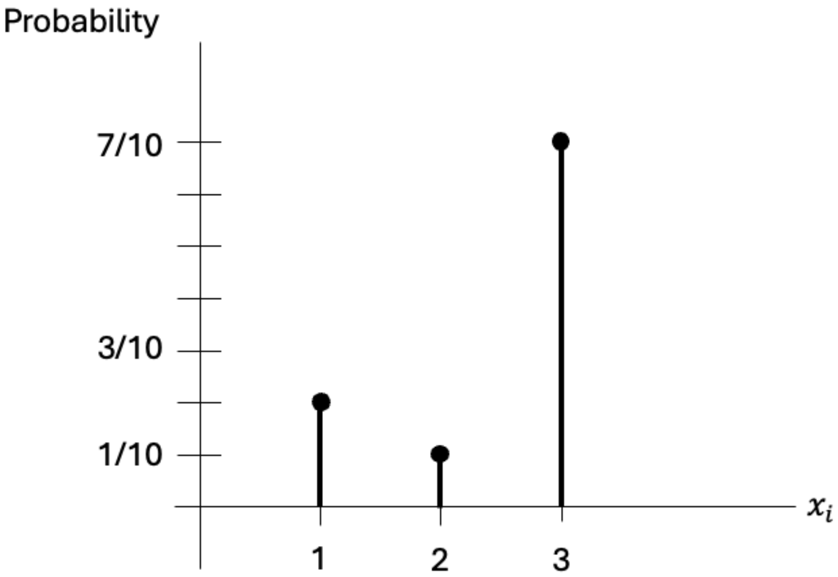
Yes, all probabilities are between 0 and 1 and the total probability adds to 1.

Example 3. Let x be a discrete random variable that takes on values 1, 2, and 3. Draw the graph for the PMF and CDF.

We can find the missing probability for Pr(X=3) or $f_X(3)$ by remembering that all of the probabilities must add up to 1.

i.e. $f_X(3) = 1 - \frac{1}{5} - \frac{1}{10} = \frac{10}{10} - \frac{2}{10} - \frac{1}{10} = \frac{7}{10}$

Graph of the PMF



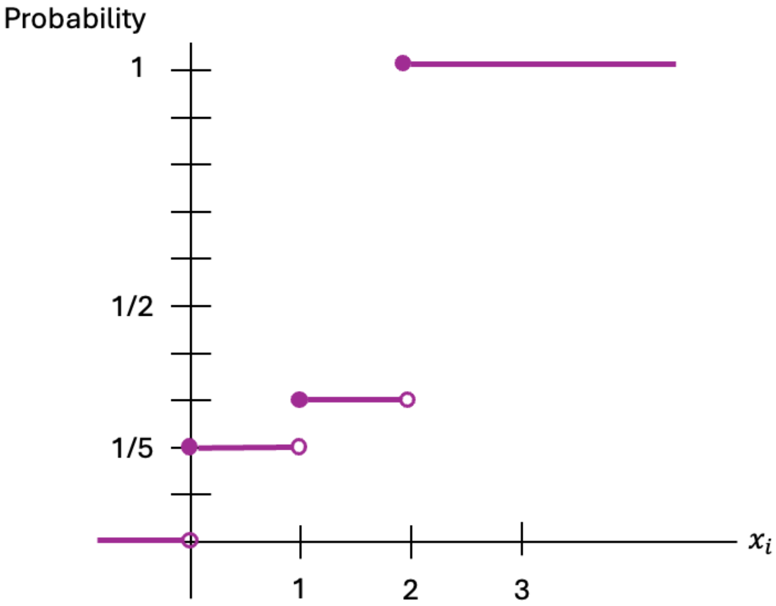
x	Pr (x)= $\boldsymbol{f_X(x)}$	$\boldsymbol{F_X}(x) = F(x)$
1	1/5	1/5 = 2/10
2	1/10	3/10
3	7/10	1

$$\boldsymbol{F_X(1) = F(1) = \frac{1}{5}}$$

$$\boldsymbol{F_X(2) = F(2) = \frac{1}{5} + \frac{1}{10} = \frac{2}{10} + \frac{1}{10} = \frac{3}{10}}$$

$\boldsymbol{F_X(3) = \frac{1}{5} + \frac{1}{10} + \frac{7}{10} = 1}$ **the last entry MUST be 1, since we are adding up all of the probabilities**

Graph of the CDF



Note: We can find any probability using the cumulative function

Pr(X=3) = F(3) – F(2) = 1 – 3/10 = 7/10

Example 4. net winnings, so you take away the $1 you pay from your winnings

X	Pr[X=x]
2-1=1	3/6 (even #)
-2-1 = -3	2/6 (roll 1 or 3)
4-1 = 3	1/6 (roll a 5)

E(X)=1(3/6)+(-3)(2/6)+3(1/6)=$ 0 So, it is a fair game!

Example 5.

X	Pr(X)	X^2
-1	0	1
0	$1/2$	0
1	1/2	1

$$E(X) = -1 \times 0 + 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$$
$$E(X^2) = 1(0) + 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = \frac{1}{2}$$
$$V(X) = E(X^2) - (E(X))^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$
$$\sigma = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Continuous Random Variables

Which of the following are probability density functions?

A) no, the area under the graph isn't 1
B) yes, it is non-negative and the area is 1
C) yes, it is non-negative and the area is 1
D) no, the graph is negative from -1 to 2

Which of the following depicts a cumulative distribution function?

A) Yes, it is non-decreasing, right continuous and $\lim_{x\to\infty} F(x) = 1$ *and* $\lim_{x\to-\infty} F(x) = 0$
B) No, it is decreasing!
C) Yes, it is non-decreasing, right continuous and $\lim_{x\to\infty} F(x) = 1$ *and* $\lim_{x\to-\infty} F(x) = 0$
D) No, it is not right continuous.
E) Yes, it is non-decreasing, right continuous and $\lim_{x\to\infty} F(x) = 1$ *and* $\lim_{x\to-\infty} F(x) = 0$
F) No, $\lim_{x\to\infty} F(x) \neq 1$

Example 1.

$$\begin{aligned} F(x) &= \int_0^x f(t)dt = \int_0^x \frac{3t}{c^4} dt \\ &= \left[\frac{3t^2}{2(c^4)}\right]_0^x \\ &= \left[\frac{3t^2}{2c^4}\right]_0^x \\ &= \frac{3x^2}{2c^4} \end{aligned}$$

Example 2.

$$\int_{-a}^{a}(a^4 - x^4)dx = 1$$

$$\left[a^4x - \frac{x^5}{5}\right]_{-a}^{a} = 1$$

$$\left[a^4(a) - \frac{a^5}{5}\right] - \left[a^4(-a) - \frac{(-a)^5}{5}\right] = 1$$

$$a^5 - \frac{a^5}{5} + a^5 - \frac{a^5}{5} = 1$$

$$\frac{2a^5}{1} - \frac{2a^5}{5} = 1$$

$$\frac{10a^5}{5} - \frac{2a^5}{5} = 1$$

$$\frac{8a^5}{5} = 1$$

$$a^5 = \frac{5}{8}$$

$$a = \sqrt[5]{\frac{5}{8}}$$

Example 3.

$$f(x) = \begin{cases} x, 0 \le x \le 1 \\ 2 - x, 1 \le x \le 2 \\ 0, otherwise \end{cases}$$

$$E(x) = \int_0^1 x \cdot x\, dx + \int_1^2 x(2 - x)dx$$

$$= \int_0^1 x^2 dx + \int_1^2 (2x - x^2)dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$= \left(\frac{1}{3} - 0\right) + \left[\left(2^2 - \frac{2^3}{3}\right) - \left(1^2 - \frac{1^3}{3}\right)\right]$$

$$= \frac{1}{3} + \left[4 - \frac{8}{3} - 1 + \frac{1}{3}\right]$$

$$= \frac{1}{3} + \left[3 - \frac{7}{3}\right]$$

$$= \frac{1}{3} + \left[\frac{9}{3} - \frac{7}{3}\right]$$

$$= \frac{1}{3} + \frac{2}{3}$$

$$= 1$$

(b) $E(x^2) = \int_0^1 (x^2)(x)\,dx + \int_1^2 x^2(2-x)dx$

$= \int_0^1 x^3 + \int_1^2 (2x^2 - x^3)dx$

$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$

$= \left(\frac{1}{4} - 0\right) + \left[\left(\frac{2(2)^3}{3} - \frac{2^4}{4}\right) - \left(\frac{2}{3}(1)^3 - \frac{1^4}{4}\right)\right]$

$= \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}$

$= \frac{1}{2} + \frac{14}{3} - \frac{4}{1}$

$= \frac{3}{6} + \frac{28}{6} - \frac{24}{6}$

$= \frac{7}{6}$

$Var(x) = E(x^2) - \mu^2$

$= \frac{7}{6} - 1^2$

$= \frac{7}{6} - 1$

$= \frac{7}{6} - \frac{6}{6} \;\; = \frac{1}{6}$

Example 4. a) $\int_0^1 k(x^2 + x) = 1$ *since it is a probability function*

$k\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = 1$

$k\left[\frac{1}{3} + \frac{1}{2}\right] = 1$

$k\left[\frac{2}{6} + \frac{3}{6}\right] = 1$

$k\left(\frac{5}{6}\right) = 1$

$k = \frac{6}{5}$

b) $F(x) = \int_{-\infty}^{x} f(t)dt$

$F(x) = \int_0^x \frac{6}{5}(t^2 + t)dt$

$= \frac{6}{5}\left[\frac{t^3}{3} + \frac{t^2}{2}\right]_0^x$

$= \frac{6}{5}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]$

c) $F(x) = \frac{6}{5}(\frac{x^3}{3} + \frac{x^2}{2})$

$f(x) = F'(x) = \frac{6}{5}(\frac{3x^2}{3} + \frac{2x}{2}) \;= \frac{6}{5}(x^2 + x)$

d) $E(x) = \int_{-\infty}^{\infty} x\,f(x)dx$

$= \frac{6}{5}\int_0^1 x(x^2 + x)dx$

$= \frac{6}{5}\int_0^1 (x^3 + x^2)dx$

$= \frac{6}{5}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1$

$= \frac{6}{5}\left[\frac{1}{4} + \frac{1}{3}\right]$

$= \frac{6}{5}\left[\frac{3}{12} + \frac{4}{12}\right]$

$= \frac{6}{5}\left(\frac{7}{12}\right) = \frac{7}{5(2)} = \frac{7}{10}$

e) $Var(x) = \int_{-\infty}^{\infty} x^2 f(x)dx - [E(x)]^2$

$= \frac{6}{5}\int_0^1 x^2(x^2 + x)dx - \left(\frac{7}{10}\right)^2$

$= \frac{6}{5}\int_0^1 (x^4 + x^3)dx - \frac{49}{100}$

$= \frac{6}{5}\left[\frac{x^5}{5} + \frac{x^4}{4}\right]_0^1 - \frac{49}{100}$

$= \frac{6}{5}\left(\frac{1}{5} + \frac{1}{4}\right) - \frac{49}{100}$

$= \frac{6}{5}\left(\frac{4+5}{20}\right) - \frac{49}{100}$

$= \frac{6}{5}\left(\frac{9}{20}\right) - \frac{49}{100}$

$= \frac{54-49}{100} = \frac{5}{100}$

$= \frac{1}{20}$

Example 5.

a) $F(x) = \begin{cases} 0, x < 0 \\ x^2/4 \quad 0 \le x < 2 \end{cases}$

$\quad 1 \qquad x \ge 2$

$\quad 2 \qquad x \ge 2$

f(x)= $F'(x) = \frac{2x}{4} = \frac{x}{2}$ *derivative* to find the pdf from cdf

b) $E(x) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^2 x\left(\frac{x}{2}\right)dx$

$= \int_0^2 \frac{x^2}{2}dx$

$= \left[\frac{x^3}{6}\right]_0^2$

$= \frac{2^3}{6} - 0 = \frac{8}{6} = \frac{4}{3}$

Example 6. $\Pr\left(|x| < \frac{3}{2}\right) = \Pr\left(\frac{-3}{2} \le x \le \frac{3}{2}\right)$

$$= \Pr\left(0 \le x \le \frac{3}{2}\right) \quad since\ x > 0$$

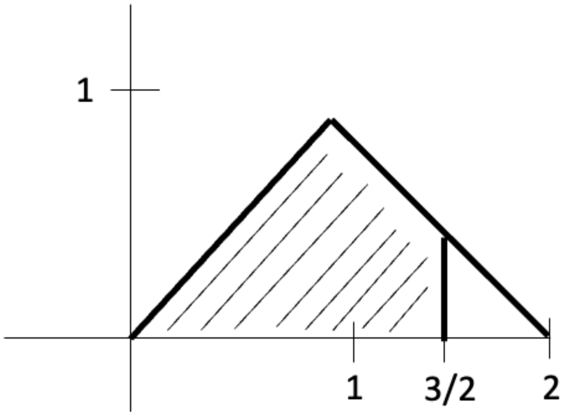
$$= 1 - \Pr\left(x > \frac{3}{2}\right)$$

$$= 1 - \frac{bh}{2} = 1 - \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{2}$$

$$= 1 - \frac{\frac{1}{4}}{2}$$

$$= 1 - \frac{1}{8}$$

$$= \frac{7}{8}$$



Method 2: $\Pr\left(|x| < \frac{3}{2}\right)$

$$= \Pr\left(0 < x < \frac{3}{2}\right) = \Pr(0 < x < 1) + \Pr\left(1 < x < \frac{3}{2}\right)$$

$$= \int_0^1 x\,dx + \int_1^{3/2} (2 - x)dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^{3/2}$$

$$= \left(\frac{1}{2} - 0\right) + \left(2\left(\frac{3}{2}\right) - \frac{\left(\frac{3}{2}\right)^2}{2}\right) - \left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{2} + 3 - \frac{9}{8} - 2 + \frac{1}{2}$$

$$= \frac{2}{1} - \frac{9}{8} \quad = \frac{16}{8} - \frac{9}{8} \quad = \frac{7}{8}$$

Practice Exam Questions on Continuous Random Variables

A1.

A. $\int_0^{\frac{5\pi}{2}} \sin x dx = [-\cos x]_0^{\frac{5\pi}{2}}$

$= -\cos\frac{5\pi}{2} + \cos 0$

$= 0 + 1 = 1$ But, y=sinx is negative from pi to 2pi, so it is not a probability density function.

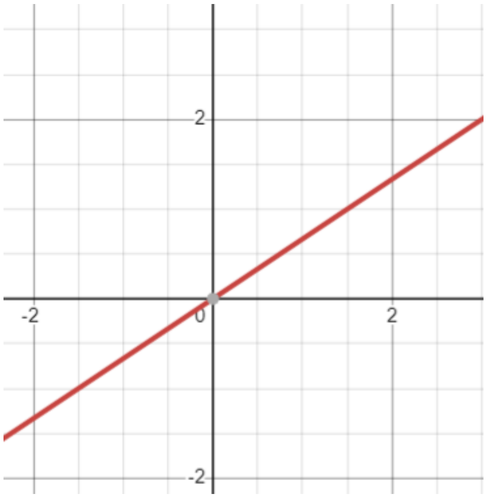
B. $\int_{-1}^{2} \frac{2}{3} x dx = \left[\frac{2}{3}\frac{x^2}{2}\right]_{-1}^{2}$

$= \left[\frac{1}{3}x^2\right]_{-1}^{2}$

$= \frac{1}{3}(2)^2 - \frac{1}{3}(-1)^2$

$= \frac{3}{3} = 1$

But, the graph is below the x-axis from -1 to 0, so it is NOT a probability density function.



A2. $f(x) = 2(x+1)^{-3},\ x > 0$

$F(x) = \int_0^x 2(t+1)^{-3} dt \quad since\ x > 0$

$= \left[\frac{2(t+1)^{-2}}{-2}\right]_o^x$

$= [-(t+1)^{-2}]_o^x$

$= [-(x+1)^{-2} + (0+1)^{-2}]$

$= \frac{-1}{(x+1)^2} + \frac{1}{1^2}$

$= 1 - \frac{1}{(x+1)^2}$

A2b).

$f(x) = 2(x+1)^{-3}; x > 0$

$\Pr(x > 3) = 1 - \Pr(x \le 3)$

$= 1 - \int_0^3 2(x+1)^{-3} dx$

$= 1 - \left[\frac{2(x+1)^{-2}}{-2}\right]_0^3$

$= 1 - \left[\frac{-1}{(x+1)^2}\right]_0^3$

$= 1 - \left[\frac{-1}{(3+1)^2} + \frac{1}{(0+1)^2}\right]$

$= 1 - \left[-\frac{1}{16} + 1\right]$

$= \frac{1}{16}$

The answer is a).

A3 a).

$\mu = 1(0.3) + 2(0.2) + 0(0.5) = 0.3 + 0.4 = 0.7$

X	Pr(X)	X^2
1	0.3	1
2	0.2	4
0	0.5	0

E(X)= $\mu = 1(0.3) + 2(0.2) + 0(0.5) = 0.3 + 0.4 = 0.7$
$E(X^2) = 0.3(1) + 4(0.2) + 0(0.5) = 0.3 + 0.8 = 1.1$

b) $V(X)=E(X^2) - (E(X))^2$
$=1.1 - 0.7^2$
$=1.1-0.49$
$=0.61$

A4. $\int_0^{10} \frac{x^3}{5000}(10-x) = \left[\frac{1}{500}\frac{x^4}{4} - \frac{1}{5000}\frac{x^5}{5}\right]_0^{10}$

$$= \frac{10^4}{4(500)} - \frac{1}{25\,000}10^5 = 5 - 4 = 1$$

a) $= \int_1^4 \frac{x^3}{5000}(10-x)dx = \left[\frac{10x^4}{4(5000)} - \frac{x^5}{5(5000)}\right]_1^4$

$$= \left(\frac{10(4)^4}{20\,000} - \frac{4^5}{25\,000}\right) - \left(\frac{10(1)^4}{20\,000} - \frac{1^5}{25\,000}\right)$$

$$= \left(\frac{2560}{20\,000} - \frac{1024}{25\,000}\right) - \left(\frac{10}{20\,000} - \frac{1}{25\,000}\right)$$

$$= \left(\frac{2550}{20\,000} - \frac{1023}{25\,000}\right) = 0.1275 - 0.04092 = 0.08658$$

b) $\int_6^{10} f(x)dx = \left[\frac{10x^4}{4(5000)} - \frac{x^5}{5(5000)}\right]_6^{10} = \left[\frac{10(10)^4}{4(5000)} - \frac{(10)^5}{5(5000)}\right] - \left[\frac{10(6)^4}{4(5000)} - \frac{(6)^5}{5(5000)}\right]$

$$= \left[\frac{10(10)^4}{4(5000)} - \frac{(10)^5}{5(5000)}\right] - \left[\frac{10(6)^4}{4(5000)} - \frac{(6)^5}{5(5000)}\right]$$

$$= [\,5 - 4] - [0.648 - 0.31104] = 0.66304$$

A5.

$CDF = F(x) = \int_0^x 3t^2 dt = \left[\frac{3t^3}{3}\right]_0^x = [t^3]_0^x = x^3 - 0 = x^3$

A6.

$CDF = F(x) = \int_0^x f(t)dt = \int_0^x \frac{t^3}{4}dt = \left[\frac{t^4}{16}\right]_0^x = \frac{x^4}{16}$

A7.

Pr $\left(0 < x < \frac{1}{4}\right) = \int_0^{0.25} 4x^3dx = \left[\frac{4x^4}{4}\right]_0^{0.25} = [x^4]_0^{0.25}$

$= (0.25)^4 - 0^4 = 0.003906$

A8.

$PDF =$ do the derivative

$f(x) = F'(x) = \frac{2x(1+x^2)-2x(x^2)}{(1+x^2)^2}$

$= \frac{2x+2x^3-2x^3}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$

A9.

a) $\mu = \int_{-\infty}^{\infty} xf(x)dx$

$$\mu = \int_0^2 x\left(\frac{1}{2}x\right)dx = \int_0^2 \frac{1}{2}x^2 = \left[\frac{x^3}{6}\right]_0^2 = \frac{2^3}{6} - 0 = \frac{8}{6} = \frac{4}{3}$$

$Var(x) = \int_{-\infty}^{\infty} x^2 f(x) - [E(x)]^2$

$= \int_0^2 x^2\left(\frac{1}{2}x\right)dx - \left(\frac{4}{3}\right)^2 = \int_0^2 \frac{1}{2}x^3dx - \frac{16}{9}$

$= \left[\frac{1}{2}\frac{x^4}{4}\right]_0^2 - \frac{16}{9} = \left(\frac{1}{8}(2)^4 - 0\right) - \frac{16}{9}$

$= \frac{16}{8} - \frac{16}{9} \quad = \frac{2}{1} - \frac{16}{9} \quad = \frac{18}{9} - \frac{16}{9} \quad = \frac{2}{9}$

b)

a) $E(x) = \int_0^1 x(4x^3)dx = \int_0^1 4x^4dx = \left[\frac{4x^5}{5}\right]_0^1$

$= \frac{4}{5}(1)^5 - 0 = \frac{4}{5}$

$$b)\ Var(x) = \int_0^1 x^2(4x^3)dx - (\frac{4}{5})^2 = \int_0^1 4x^5dx - \frac{16}{25}$$

$= \left[\frac{4x^6}{6}\right]_0^1 - \frac{16}{25} = \frac{2}{3}(1)^6 - 0 - \frac{16}{25} \quad = 0.02\overline{6}$

A10.

Consider the probability density function

$f(x) = \frac{c}{(x+1)^2} \quad x \geq 0$

Find the CDF

CDF$= F(x) = \int_{-\infty}^x c(t+1)^{-2}dt$ since $x \geq 0$

$= \int_0^x c(t+1)^{-2}ds$

$= \left[\frac{c(t+1)^{-1}}{-1}\right]_0^x = \left[\frac{-c}{(t+1)^1}\right]_0^x = \frac{-c}{x+1} + \frac{c}{0+1} = \frac{-c}{x+1} + c$

A11.

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{0}^{x}\left(\frac{6t^3}{15} + \frac{6t^2}{10}\right)dx$$
$$= \left[\frac{6x^4}{4(15)} + \frac{6x^3}{3(10)}\right]_0^x$$
$$= \frac{x^4}{10} + \frac{x^3}{5}$$

A12. a)

Let $f(x) = \begin{cases} 2(1-x) \\ 0 \end{cases}$ $if\ 0 \le x \le 1 \quad otherwise$

Find $E(x) + Var(x)$

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)dx \quad = \int_0^1 x(2)(1-x)dx$$
$$= 2\int_0^1 (x - x^2)dx \quad = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \quad = 2\left[\left(\frac{1}{2} - \frac{1}{3}\right) - (0-0)\right]$$
$$= 2\left(\frac{3}{6} - \frac{2}{6}\right) \quad = \frac{1}{3}$$
$$E(x^2) = \int_0^1 x^2 2(1-x)dx \quad = \int_0^1 (2x^2 - 2x^3)dx$$
$$= \left[\frac{2x^3}{3} - \frac{2x^4}{4}\right]_0^1 \quad = \left(\frac{2}{3} - \frac{2}{4}\right) - (0-0) \quad = \frac{8}{12} - \frac{6}{12}$$
$$= \frac{2}{12} \quad = \frac{1}{6}$$
$$Var(x) = E(x^2) - \mu^2$$
$$= \frac{1}{6} - \left(\frac{1}{3}\right)^2 \quad = \frac{1}{6} - \frac{1}{9} \quad = \frac{3}{18} - \frac{2}{18} \quad = \frac{1}{18}$$

b) This is a discrete random variable that only takes on values -1, 0 and 1.

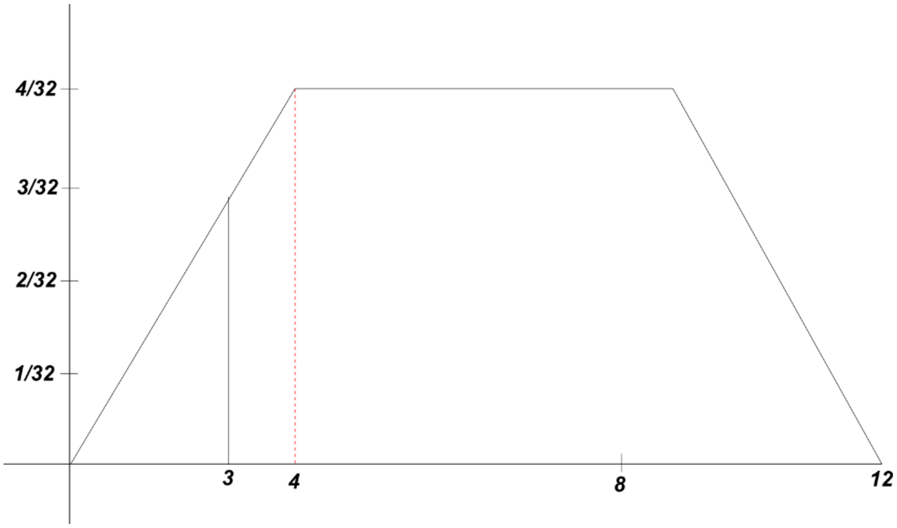
X	Pr(X)	X^2
-1	$1/2$	1
0	$1/4$	0
1	$1/4$	1

$$E(X) = -1 \times \frac{1}{2} + 0 \times \frac{1}{4} + 1 \times \frac{1}{4} = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$$
$$E(X^2) = 1\left(\frac{1}{2}\right) + 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{4}\right) = \frac{3}{4}$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{3}{4} - \left(\frac{-1}{4}\right)^2 = \frac{12}{16} - \frac{1}{16} = \frac{11}{16}$$
$$\sigma(X) = \sqrt{\frac{11}{16}}$$

A13. $A1 + A2 + A3 = 1$

$$\frac{4(4x)}{2} + 4(4x) + \frac{4(4x)}{2} = 1$$
$$8x + 16x + 8x = 1$$
$$32x = 1$$
$$x = \frac{1}{32}$$



$$\Pr(3 < x < 5) = \Pr(0 < x < 5) - \Pr(0 < x < 3)$$
$$= \left[\frac{4(\frac{4}{32})}{2} + 1(\frac{4}{32})\right] - \frac{(3)\left(\frac{3}{32}\right)}{2}$$
$$= \frac{16}{64} + \frac{4}{32} - \frac{9}{64}$$
$$= \frac{16}{64} + \frac{8}{64} - \frac{9}{64} = \frac{15}{64}$$

Or do the area from 3 to 5 directly by adding the rectangle and triangle from 3 to 4 and then the rectangle from 4 to 5

$$(1)\frac{3}{32} + \frac{1\left(\frac{1}{32}\right)}{2} + 1\left(\frac{4}{32}\right) = \frac{6}{64} + \frac{1}{64} + \frac{8}{64} = \frac{15}{64}$$

A14.

$$a)\ \int_2^3 \frac{x^3}{5000}(10 - x)ds = \frac{1}{5000}\ \int_2^3 (10x^3 - x^4)dx$$
$$= \frac{1}{5000}\left[\frac{10x^4}{4} - \frac{x^5}{5}\right]_2^3$$
$$= \frac{1}{5000}\left[\frac{10(3)^4}{4} - \frac{3^5}{5} - (\frac{10(2)^4}{4} - \frac{2^5}{5})\right]$$
$$= \frac{1}{5000}\ [202.5 - 48.6 - 40 + 6.4]$$
$$= 0.02406$$

b) $E(x) = \int_0^{10} x\, f(x)dx = \int_0^{10} \frac{x \cdot x^3}{5000}(10 - x)dx$

$= \int_0^{10} \frac{x^4(10-x)dx}{5000}$

$= \frac{1}{5000} \int_0^{10}(10x^4 - x^5)dx$

$= \frac{1}{5000} \left[\frac{10x^5}{5} - \frac{x^6}{6}\right]_0^{10}$

$= \frac{1}{5000}\left[2(10)^5 - \frac{10^6}{6} - (0 - 0)\right]$

$=\frac{1}{5000}\left[200\ 000 - \frac{1000\ 000}{6}\right]$

$=\frac{1}{5000}\left[\frac{1\ 200000}{6} - \frac{1000\ 000}{6}\right]$

$= \frac{1}{5000}\left[\frac{200\ 000}{6}\right]$

$=\frac{1}{5000}\left[\frac{200\ 000}{6}\right]$

$= \frac{40}{6} = \frac{20}{3}$

c) $Var(x) = \int_{-\infty}^{\infty} x^2 f(x) - [E(x)]^2$

$= \int_0^{10} x^2 f(x) - (\frac{20}{3})^2$

$= \int_0^{10} \frac{x^2 \cdot x^3(10-x)}{5000} dx - (\frac{20}{3})^2$

$= \frac{1}{5000} \int_0^{10}(10x^5 - x^6)ds - (\frac{20}{3})^2$

$= \frac{1}{5000}\left[\frac{10x^6}{6} - \frac{x^7}{7}\right]_0^{10} - (\frac{20}{3})^2$

$= \frac{1}{5000}\left[\frac{10(10)^6}{6} - \frac{10^7}{7}\right] - (\frac{20}{3})^2$

$$= \frac{1}{5000}\left[\frac{10\ 000\ 000}{6} - \frac{10\ 000\ 000}{7}\right] - \left(\frac{20}{3}\right)^2$$

$= \left[\frac{2000}{6} - \frac{2000}{7}\right] - \left(\frac{20}{3}\right)^2 = \left[\frac{14000}{42} - \frac{12000}{42}\right] - \left(\frac{20}{3}\right)^2$

$$= \left[\frac{2000}{42}\right] - \left(\frac{20}{3}\right)^2 = \frac{1000}{21} - \frac{400}{9}$$

=47.619 -44.444=3.18

A15. $f(x) = derivative\ of\ \frac{x+1}{4}$

$f(x) = \frac{1}{4}$

$E(x) = \int_a^b x\,f(x)dx = \int_0^1 \frac{1}{4}xdx$

$= \left[\frac{1}{4}\frac{x^2}{2}\right]_0^1$

$= \left[\frac{x^2}{8}\right]_0^1$

$= \frac{1}{8} - 0 \quad = \frac{1}{8}$

A16. $\mathrm{E}(x) = \int_1^2 x(2x^{-2})dx$

$= \int_1^2 2x^{-1}dx = \int_1^2 \frac{2}{x}dx$

$= 2[\ln x]_1^2$

$= 2[\ln 2 - \ln 1]$

$= 2[\ln 2 - 0]$

$= 2\ln 2 \quad or \quad \ln 2^2 \quad = \ln 4$

$V(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \int_1^2 x^2(2x^{-2})dx$

$= \int_1^2 2dx = [2x]_1^2$

$= 2(2) - 2(1) = \quad \boxed{2}$

$Var(x) = 2 - (\ln 4)^2$

A17. Given $f(x) = \frac{x^3}{4} \quad for \quad 0 < x < 2$

a) Find $F(x)$ $\quad F(x) = \int_o^x \frac{t^3}{4}dt = \left[\frac{t^4}{16}\right]_o^x$

$= \frac{x^4}{16} - 0$

$= \frac{x^4}{16}$

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^4}{16}, & 0 < x < 2 \\ 1 & if \quad x \ge 2 \end{cases}$$

$0 < x < 2$

b) $Find\ E(x)$ $E(x) = \int_0^2 x\left(\frac{x^3}{4}\right)dx$

$$= \int_0^2 \frac{x^4}{4}dx$$

$$= \left[\frac{x^5}{5\cdot 4}\right]_0^2$$

$$= \left[\frac{x^5}{20}\right]_0^2$$

$$= \frac{2^5}{20}$$

$$= \frac{32}{20}$$

$$= \frac{8}{5}$$

A18. Given $f(x) = \{6x - 6x^2, \quad 0 \leq x \leq 1$

Find $E(x) + Var(x)$

$$E(x) = \int_0^1 x(6x - 6x^2)dx$$

$$= \int_0^1 (6x^2 - 6x^3)dx$$

$$= \left[\frac{6x^3}{3} - \frac{6x^4}{4}\right]_0^1$$

$$= \left(\frac{6}{3} - \frac{6}{4}\right) - (0 - 0)$$

$$= 2 - 1.5$$

$$= \frac{1}{2}$$

$$E(x^2) = \int_0^1 x^2(6x - 6x^2)dx$$

$$= \int_0^1 (6x^3 - 6x^4)dx$$

$$= \left[\frac{6x^4}{4} - \frac{6x^5}{5}\right]_0^1$$

$$= \left[\frac{3}{2}(1) - \frac{6}{5}(1)\right] - [0 - 0]$$

$$= \frac{3}{2} - \frac{6}{5}$$

$$= \frac{15}{10} - \frac{12}{10}$$

$$= \frac{3}{10}$$

$$Var(x) = E(x^2) - \mu^2$$

$$= \frac{3}{10} - (\frac{1}{2})^2$$

$$= \frac{3}{10} - \frac{1}{4}$$

$$= \frac{6}{20} - \frac{5}{20}$$

$$= \frac{1}{20}$$

A19. $Let\ f(x) = \frac{3}{8}x^2 \quad 0 \le x \le 2$

Find $E(x)$

$$E(x) = \int_0^2 x\, f(x)dx$$
$$= \int_0^2 \frac{3}{8}x^3 dx$$
$$= \frac{3}{8}\left[\frac{x^4}{4}\right]_0^2$$
$$= \frac{3}{8}\left[\frac{2^4}{4} - 0\right]$$
$$= \frac{3}{8}(4)$$
$$= \frac{12}{8}$$
$$= \frac{3}{2}$$

A20.

$V(X) = E(X^2) - \left(E(X)\right)^2 = 20 - 4^2 = 20 - 16 = 4$

$\sigma(X) = \sqrt{4} = 2$

A21. $\sigma(X) = 9,\ so\ V(X) = 81$

$V(X) = E(X^2) - \left(E(X)\right)^2$

$81 = E(X^2) - (5)^2$

$E(X^2) = 81 + 25 = 106$

A22.

a) $\int_0^{10} \frac{x^3}{5000}((10 - x)dx = \frac{1}{5000}\int_0^{10}(10x^3 - x^4)dx$

$$= \frac{1}{5000}\left[\frac{10x^4}{4} - \frac{x^5}{5}\right]_0^{10}$$
$$= \frac{1}{5000}\left[\frac{10(10)^4}{4} - \frac{10^5}{5} - (0 - 0)\right] = 1$$

b) $CDF = F(x) = \int_{-\infty}^{x} \frac{t^3}{5000}(10 - t)dt \quad but\ \ 0 \le x \le 10$

$$\therefore \frac{1}{5000}\int_0^x (10t^3 - t^4)dt = \frac{1}{5000}\left[\frac{10t^4}{4} - \frac{t^5}{5}\right]_0^x$$
$$= \frac{1}{5000}\left[\frac{10x^4}{4} - \frac{x^5}{5} - (0 - 0)\right]$$
$$= \frac{1}{5000}\left[\frac{5}{2}x^4 - \frac{x^5}{5}\right]$$

$c)\ \int_2^3 \frac{t^3}{5000}(10-t)dt = \frac{1}{5000}\ \int_2^3 (10t^3 - t^4)ds$

$= \frac{1}{5000}\left[\frac{10t^4}{4} - \frac{t^5}{5}\right]_2^3$

$= \frac{1}{5000}\left[\frac{10(3)^4}{4} - \frac{3^5}{5} - (\frac{10(2)^4}{4} - \frac{2^5}{5})\right]$

$= \frac{1}{5000}\ [202.5 - 48.6 - 40 + 6.4]\ = 0.02406$

$d)\ E(x) = \int_0^{10} x\, f(x)dx = \int_0^{10} \frac{x \cdot x^3}{5000}(10-x)dx$

$= \int_0^{10} \frac{x^4(10-x)dx}{5000}$

$= \frac{1}{5000}\ \int_0^{10}(10x^4 - x^5)dx$

$= \frac{1}{5000}\ \left[\frac{10x^5}{5} - \frac{x^6}{6}\right]_0^{10}$

$= \frac{1}{5000}\left[2(10)^5 - \frac{10^6}{6} - (0-0)\right]$

$= 6.67$

e) $Var(x) = \int_{-\infty}^{\infty} x^2 f(x) - [E(x)]^2$

$= \int_0^{10} x^2 f(x) - 6.67^2$

$= \int_0^{10} \frac{x^2 \cdot x^3(10-x)}{5000}dx - 6.67^2$

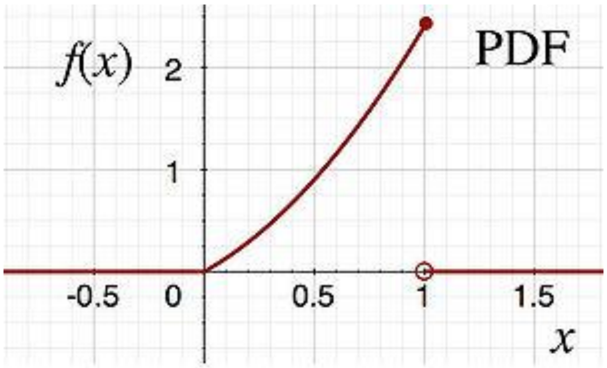
$= \frac{1}{5000}\int_0^{10}(10x^5 - x^6)dx - 6.67^2$

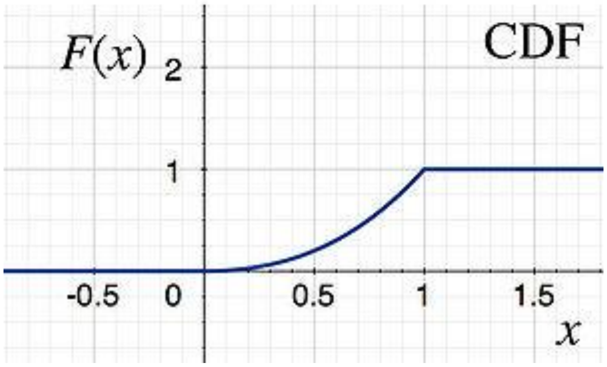
$= \frac{1}{5000}\left[\frac{10x^6}{6} - \frac{x^7}{7}\right]_0^{10} - 6.67^2$

$= \frac{1}{5000}\left[\frac{10(10)^6}{6} - \frac{10^7}{7}\right] - 6.67^2$

$= 47.619 - 6.67^2 = 3.1$

A23. Find the value of k. The integral from 0 to 1 of the function must be =1 since it represents probability.





$$\int_0^1 k(x^2+x)dx = 1$$

$$k\left[\frac{x^3}{3}+\frac{x^2}{2}\right]\begin{matrix}1\\0\end{matrix} = 1$$

$$k\left[\frac{1^3}{3}+\frac{1^2}{2}-(0-0\right] = 1$$

$$k\left(\frac{2}{6}+\frac{3}{6}\right) = 1$$

$$k\left(\frac{5}{6}\right) = 1 \; so \; k = 6/5$$

To get the graph of the pdf

Find f(x)

$$f(x) = \frac{6}{5}(t^2+t)\;dt$$

$$F(x) = \int_0^x \frac{6}{5}(t^2+t)\;dt$$

$$F(x) = \frac{6}{5}\left[\frac{t^3}{3}+\frac{t^2}{2}\right]_0^x$$

$$F(x) = \frac{6}{5}\left[\frac{x^3}{3}+\frac{x^2}{2}-(0+0)\right]$$

$$\therefore f(x) = \frac{6}{5}\left(\frac{x^3}{3}+\frac{x^2}{2}\right)$$

Do a table of values for 0, 0.5, 1 and substitute into f(x)

x	f(x)=6/5(x^2+x)
0	0
0.5	0.9
1	2.4

To get the CDF, type values into F(X)=$\frac{6}{5}(\frac{x^3}{3}+\frac{x^2}{2})$

x	F(X)=$\frac{6}{5}(\frac{x^3}{3}+\frac{x^2}{2})$
0	0
0.5	6/5(0.166)=0.2
1	6/5(1/3+1/2)=6/5(5/6)=1

Last entry has to be 1 for a cumulative graph as probabilities add up to 1

$F(0.5)=\frac{6}{5}(\frac{0.5^3}{3}+\frac{0.5^2}{2})$ (in the next half hour, so less than or equal to 0.5 hr, which is F(0.5))

$=\frac{6}{5}(\frac{1}{24}+\frac{1}{8})$

$=\frac{6}{5}(\frac{1}{24}+\frac{3}{24})$

$=\frac{6}{5}(\frac{4}{24})$

$=\frac{1}{5}$

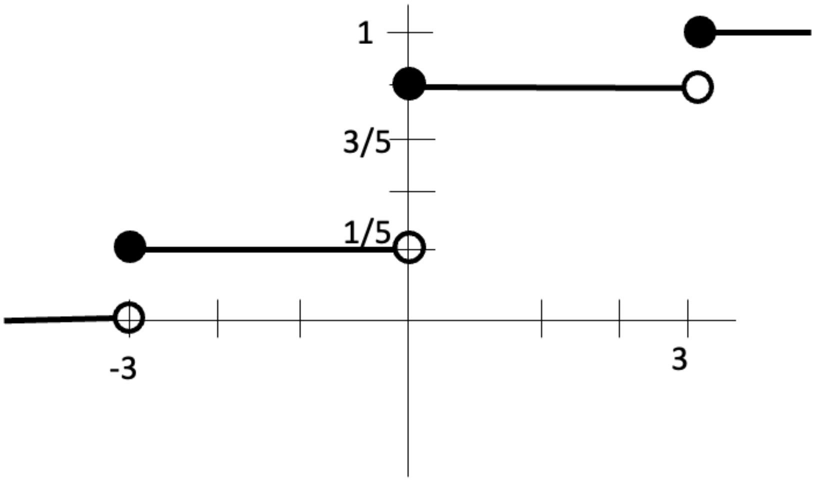
A24. Let x be a discrete random variable that takes on values $-3, 0$ and 3 $where$

$$\Pr(X = x) = \begin{cases} \frac{1}{5} & if\ x = \pm 3 \\ \frac{3}{5} & otherwise \end{cases}$$

Graph the cumulative distribution function.

x	Pr (x)= $\boldsymbol{f_X(x)}$	$F(x)$
-3	1/5	1/5
0	3/5	4/5
3	1/5	1

Since this is a discrete graph, not a continuous one with area, we can add up the probabilities going down the chart to get the cumulative values. i.e. F(0)=1/5 +3/5 =4/5



A25. To find the value of a, find the area of each of the shapes and set it equal to 1, since the total area is the same as the total probability

A=$\frac{(2)\left(\frac{1}{2}a\right)}{2} + (6)\frac{1}{2}a = 1$

$$1 = \frac{1}{2}a + 3a$$

$$1 = \frac{7}{2}a$$

$$a = \frac{2}{7}$$

B. Pseudorandom Variables and Simulation

Example 1.

a) $9 \div 4 = 2\ R1 \quad \therefore 9\ mod\ 4 = 1$
b) $18 \div 7 = 2\ R4 \quad \therefore 18\ mod\ 7 = 4$
c) $= 4\ since\ 10\ can't \div into\ 4$
d) $= 45 \div 7 = 6\ R3 \quad \therefore 45\ mod\ 7 = 3$
e) $= 5\ since\ 6\ can't \div into\ 5$

Example 2.

a) $U_t = 2U_{t-1} + 1 \quad mod\ 5 \quad U_0 = 1$
$U_1 = 2U_0 + 1 = 2(1) + 1 = 3\ mod\ 5 = 3$
$U_2 = 2U_1 + 1 = 2(3) + 1 = 7\ mod\ 5 = 2$
$U_3 = 2U_2 + 1 = 2(2) + 1 = 5\ mod\ 5 = 0$

b) $U_t = 4U_{t-1} + 4\ \ mod\ 3\ \ U_0 = 2$
$U_1 = 4U_0 + 4 = 4(2) + 4 = 12\ mod\ 3 = 0$
$U_2 = 4U_1 + 4 = 4(0) + 4 = 4\ mod\ 3 = 1$
$U_3 = 4U_2 + 4 = 4(1) + 4 = 8\ mod\ 3 = 2$

Example 3.

Step 1. $F(x) = \int_{-\infty}^{-2} f(s)ds + \int_{-2}^{x} f(s)ds = 0 + \int_{-2}^{x} f(s)ds$

$= \int_{-2}^{x} \frac{1}{2} s\ ds \ = \left[\frac{1}{2}\ \frac{s^2}{2}\right]_{-2}^{x} \ = \left[\frac{s^2}{4}\right]_{-2}^{x}$

$= \frac{x^2}{4} - \frac{(-2)^2}{4} \ = \frac{x^2}{4} - 1$

Let $y = \frac{x^2}{4} - 1$

Step 2. $x = \frac{y^2}{4} - 1 \quad solve\ for\ y$

$x + 1 = \frac{y^2}{4}$

$4(x + 1) = y^2$

$y = \underset{\uparrow}{-}\sqrt{4(x + 1)} \qquad \therefore F^{-1}(x) = -2\sqrt{x + 1}$

Since the domain is negative

Practice Exam Questions on Pseudorandom Variables and Simulation

B1.

a) $9 \div 2 = 4\,R1 \quad \therefore 9\,mod\,2 = 1$
b) $18 \div 5 = 3\,R3 \quad \therefore 18\,mod\,5 = 3$
c) $= 4\,since\,8\,can't \div into\,4$
d) $45 \div 8 = 5\,R5 \quad \therefore 45\,mod\,8 = 5$
e) $= 5\,since\,7\,can't \div into\,5$

B2.

a) $U_t = 2U_{t-1} + 2\ \ mod\,4 \quad U_0 = 1$

$U_1 = 2U_0 + 2 = 2(1) + 2 = 4\,mod\,4 = 0$
$U_2 = 2U_1 + 2 = 2(0) + 2 = 2\,mod\,4 = 2$
$U_3 = 2U_2 + 2 = 2(2) + 2 = 6\,mod\,4 = 2$

B3.

$F(x) = \int_{-\infty}^{-10} f(s)ds + \int_{-10}^{x} f(s)ds = 0 + \int_{-10}^{x} f(s)ds$

$= \int_{-10}^{x} \frac{1}{50} ds \quad = \left[\frac{1}{50}\frac{s^2}{2}\right]_{-10}^{x} \quad = \left[\frac{s^2}{100}\right]_{-10}^{x} \quad = \frac{x^2}{100} - \frac{(-10)^2}{100} \quad = \frac{x^2}{100} - 1$

Let $y = \frac{x^2}{100} - 1$

$x \leftrightarrow y \qquad x = \frac{y^2}{100} - 1$

$x + 1 = \frac{y^2}{100}$

$100(x + 1) = y^2$

$y = -\sqrt{100x + 100}$ (negative since the domain is negative)

$\therefore F^{-1}(x) = -\sqrt{100x + 100}$

B4. From the graph $m = -\frac{1}{4}$

$\therefore y = -\frac{1}{4}x$ $(it\ is\ at\ 1\ at\ x = -4 \quad \therefore start\ integral\ at - 4)$

$\boxed{1}$ $F(x) = \int_{-4}^{x} -\frac{1}{4}s\, ds$

$= \left[-\frac{1}{4}\frac{s^2}{2}\right]_{-4}^{x}$

$= \left[\frac{-s^2}{8}\right]_{-4}^{x}$

$y = \frac{-x^2}{8} + \frac{(-4)^2}{8}$

$y = \frac{-x^2}{8} + \frac{16}{8}$

$y = \frac{-x^2+16}{8}$

$\boxed{2}$ $Find\ the\ inverse$ (switch x and y first)

$x = \frac{-y^2+16}{8}$

$8x = -y^2 + 16$

$16 - 8x = y^2$

$y^2 = 16 - 8x$

$y = -\sqrt{16 - 8x}$

$\therefore Use\ the\ function\ \ y = -\sqrt{16 - 8x}$(take the negative square root since x is negative in the original graph.)

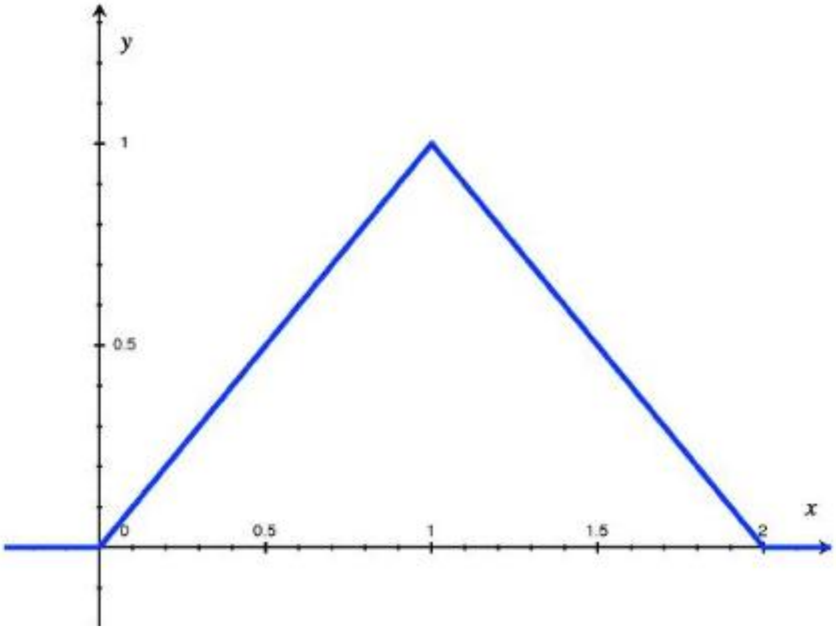
Quiz 4: Practice on Sections A and B

1. $\Pr\left(|x| < \frac{1}{2}\right) = \Pr\left(-\frac{1}{2} \leq x \leq \frac{1}{2}\right)$

$= \Pr\left(0 \leq x \leq \frac{1}{2}\right)$ *since* $x \geq 0$

$= \frac{b \times h}{2}$

$= \frac{\frac{1}{2}(\frac{1}{2})}{2} = \frac{1}{8}$ *using the graph*



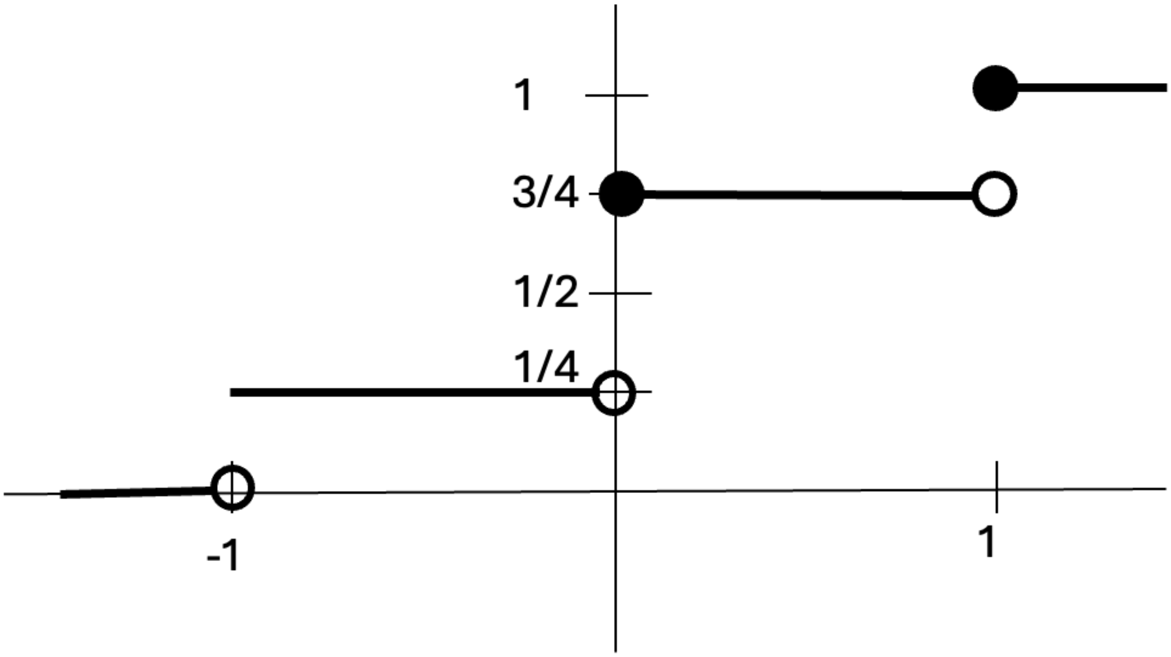
2. $F(x) = \int_0^x f(s)ds = \int_0^x \frac{2s}{c^3} ds$

$= \left[\frac{2}{c^3}\left(\frac{s^2}{2}\right)\right]_0^x$

$= \left[\frac{s^2}{c^3}\right]_0^x$

$= \frac{x^2}{c^3}$

3.



x	$\Pr(x)$	$F(x)$
-1	$\frac{1}{4}$	$\frac{1}{4}$
0	$\frac{2}{4}$	$\frac{3}{4}$
1	$\frac{1}{4}$	1

4. $\displaystyle\int_{-a}^{a}(a^3 - x^3)dx = 1$

$$\left[a^3x - \frac{x^4}{4}\right]_{-a}^{a} = 1$$

$$\left(a^3(a) - \frac{a^4}{4}\right) - \left(a^3(-a) - \frac{(-a)^4}{4}\right) = 1$$

$$a^4 - \frac{a^4}{4} + a^4 + \frac{a^4}{4} = 1$$

$$2a^4 = 1$$

$$a^4 = \frac{1}{2}$$

$$a = \sqrt[4]{\frac{1}{2}}$$

5. a) $\underline{3\ ,}$ $\underline{2\ }$

b) $\underline{2\ }$

6. $\underline{0\ }$

7. a) $u_1 = (3u_0 + 3) mod\ 5$
$= (3(2) + 3) mod\ 5$
$= 9\ mod\ 5$
$= 4 \quad since\ 9 \div 5 = 1\ \ R4$

b) $u_2 = (3u_1 + 3) mod\ 5$
$= (3(4) + 3) mod\ 5$
$= 15 \ \ mod\ 5$
$= 0\ \ since\ 15 \div 5 = 3\ \ R\ 0$

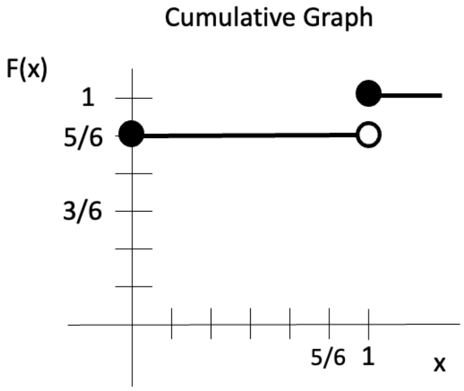
c) $u_3 = (3u_2 + 3) mod\ 5$
$= (3(0) + 3) mod\ 5$
$= 3\ mod\ 5$
$= 3\ since\ 5\ can't\ divide\ into\ the\ 3$

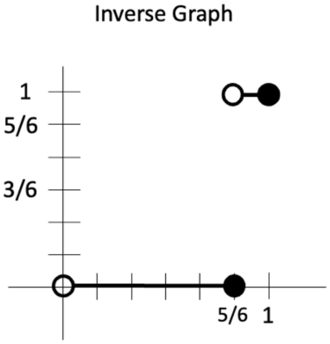
d) $\underline{divide\ by\ m \quad \therefore since\ it\ is\ mod\ 5 \quad \div 5}$

8.

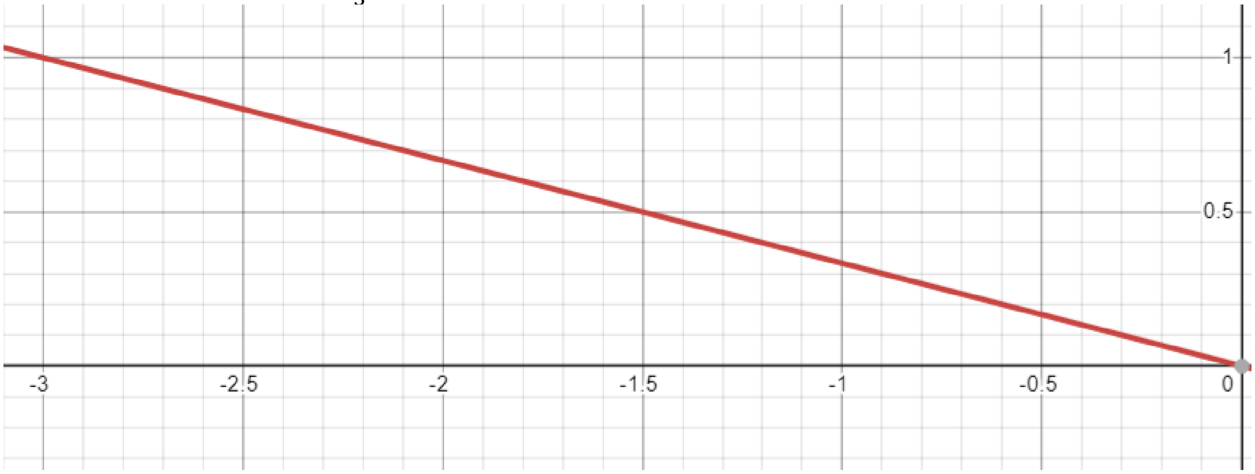
x	$\Pr(x)$	$F(x)$
0	$5x = \frac{5}{6}$	$\frac{5}{6}$
1	$x = \frac{1}{6}$	1
	$\overline{1}$	

$5x + x = 1$
$6x = 1$
$x = \frac{1}{6}$


Cumulative Graph


Inverse Graph

9.From the graph $m = -\frac{1}{3}$



$\therefore y = -\frac{1}{3}x \quad (it\ is\ 1\ at\ x = -3 \quad \therefore start\ integral\ at -3$

$\boxed{1}$ $F(x) = \int_{-3}^{x} -\frac{1}{3}s\ \ ds$

$= \left[\frac{-1}{3}\frac{s^2}{2}\right]_{-3}^{x}$

$= \left[\frac{-s^2}{6}\right]_{-3}^{x}$

$y = \left[\frac{-x^2}{6} - (-\frac{(-3)^2}{6})\right]$

$y = \left[\frac{-x^2}{6} + \frac{9}{6}\right]$

$y = \frac{-x^9+9}{6}$

$\boxed{2}$ $Find\ the\ inverse$ (switch x and y)

$x = \frac{-y^2+9}{6}$

$6x = -y^2 + 9$

$6x - 9 = -y^2$

$y^2 = -6x + 9$

$y = -\sqrt{-6x+9}$ ($take$ $-$ square root since x is negative in original graph)

$\therefore use\ the\ function\ \ y = -\sqrt{-6x+9}$

C. Working with Data

$$\bar{x} = \frac{80+60+70+70+20}{5} = \frac{300}{5} = 60$$

$$s = \sqrt{\frac{(80-60)^2+(60-60)^2+(70-60)^2+(70-60)^2+(20-60)^2}{5-1}}$$

$$s = \sqrt{\frac{(20)^2+(0)^2+(10)^2+(10)^2+(-40)^2}{4}}$$

$$= \sqrt{\frac{400+0+100+100+1600}{4}}$$

$$= \sqrt{\frac{2200}{4}}$$

$$= \sqrt{550}$$

Z score of mean= $\mathbf{\frac{60}{\sqrt{550}}}$

Example 2.

a) $z_1 = \frac{x-\bar{x}}{s} = \frac{60-67}{10} = -0.7$

b) $z_2 = \frac{x-\bar{x}}{s} = \frac{80-75}{5} = 1$

c) $z_3 = \frac{x-\bar{x}}{s} = \frac{75-80}{3} = -1.67$

Largest to smallest is Z3, Z2, Z1

Example 3. $z = \frac{x-\bar{x}}{s}$

$$z = \frac{90-58}{16} = 2$$

$$2 = \frac{x-62}{8}$$

$$x = 78$$

Example 4. $t = \sqrt{n} \cdot T$

$$= \sqrt{n} \cdot \left|\frac{\bar{x}}{s}\right| = \sqrt{10}\left|\frac{5}{3}\right| = \frac{5\sqrt{10}}{3}$$

Practice Exam Questions on Working with Data

C1.

a) $\bar{x} = \frac{80+90+72+60}{4} = \frac{302}{4} = 75.5$

$$s = \sqrt{\frac{(80-75.5)^2+(90.75.5)^2+(72-75.5)^2+(60-75.5)^2}{3}}$$

$$s = \sqrt{\frac{4.5^2 + 14.5^2 + (-3.5)^2 + (-15.5)^2}{3}}$$

$$= \sqrt{\frac{483}{3}} = \sqrt{161} = 12.7$$

b) Student 1

$z_1 = \frac{x-\bar{x}}{s} = \frac{80-75.5}{12.7}$

$= 0.35$

Student 4

$z_4 = \frac{60-75.5}{12.7}$

$= -1.22$

Student 4 is more extreme (1.22 standard deviations away from mean)

C2.

$t = \sqrt{n} \cdot T$

$= \sqrt{n}\left|\frac{\bar{x}}{s}\right| = \sqrt{25}\left|\frac{4}{4}\right| = 5$

D. Practice Test on Probability

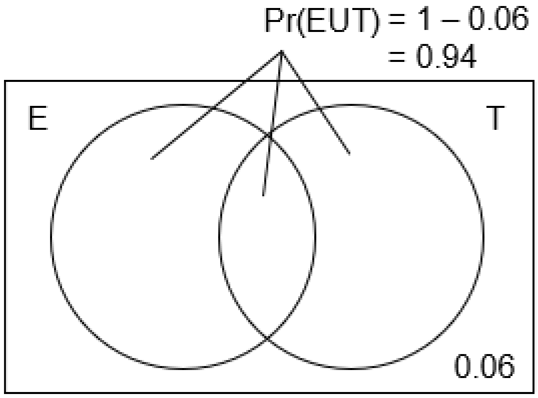
D1.a) $T \cap E^c$ $\qquad$ $\Pr(E \cup T) = 1 - 0.06$

$= 0.94$

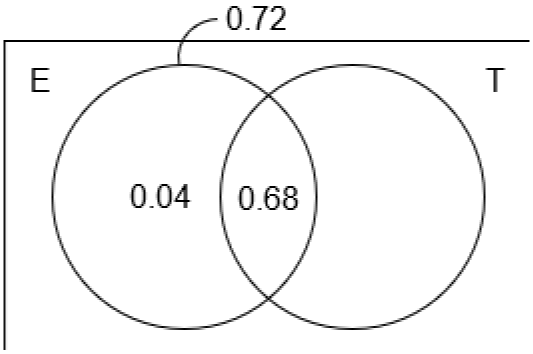
b) $\Pr(E \cup T) = \Pr(E) + \Pr(T) - \Pr\,(E \cap T)$

$0.94 = 0.72 + 0.90 - \Pr\,(E \cap T)$

$\Pr(E \cap T) = 0.68$



c)

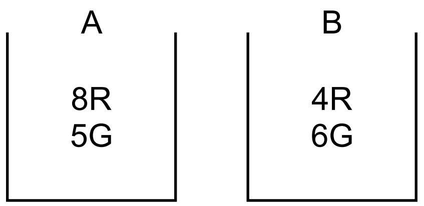


$\Pr(E \cap T^c) = \Pr(E) - \Pr\,(E \cap T)$

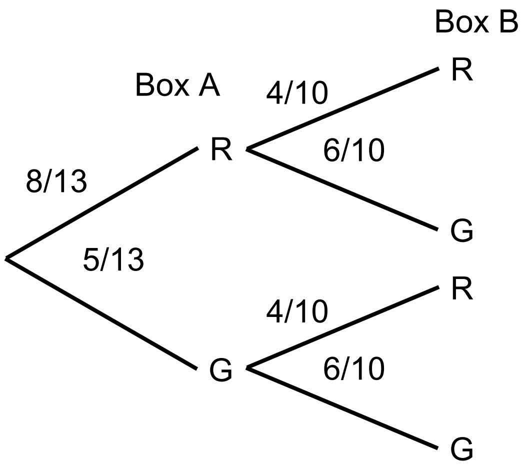
$= 0.72 - 0.68$

$= 0.04$

D2.a)



$\Pr(GG) = \frac{5}{13} \times \frac{6}{10} = \frac{30}{130} = \frac{3}{13}$



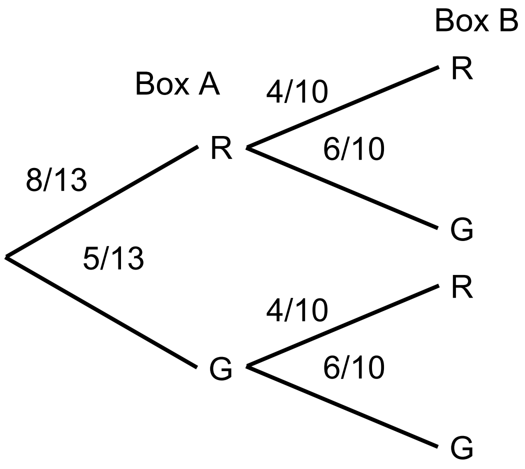
b) $\Pr\,(one\ green\ \cap one\ red)$

$= \Pr(RG) + \Pr\,(GR)$

$= \left(\frac{8}{13}\right)\left(\frac{6}{10}\right) + \left(\frac{5}{13}\right)\left(\frac{4}{10}\right)$

$= \frac{48}{130} + \frac{20}{130}$

$= \frac{68}{130} \quad = \boxed{\frac{34}{65}}$



c) $\Pr\left(box\ {}^{B}/_{R}\right) = \frac{\Pr\ (box\ B \cap R)}{\Pr\ (R)}$

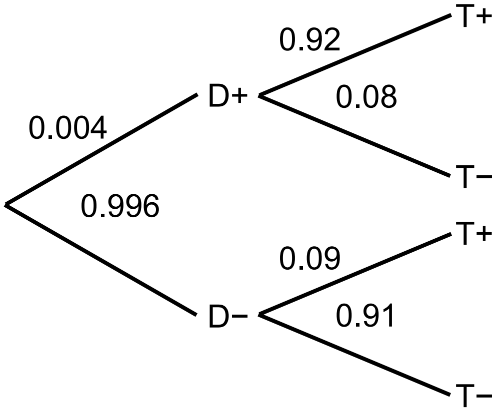
$$= \frac{\frac{1}{3}(\frac{4}{10})}{\frac{1}{3}\left(\frac{4}{10}\right)+\frac{2}{3}(\frac{8}{13})}$$

$$= \frac{\frac{4}{30}}{\frac{4}{30}+\frac{16}{39}}$$

$$= \frac{0.1333333}{0.543589743}$$

$$= 0.245$$

D3. $\frac{4}{1000} = 0.004$



sensitivity$= 0.92 = \Pr\ ({}^{T\ +}/_{D\ +})$

specificity$= 0.91 = \Pr\ ({}^{T\ -}/_{D\ -})$

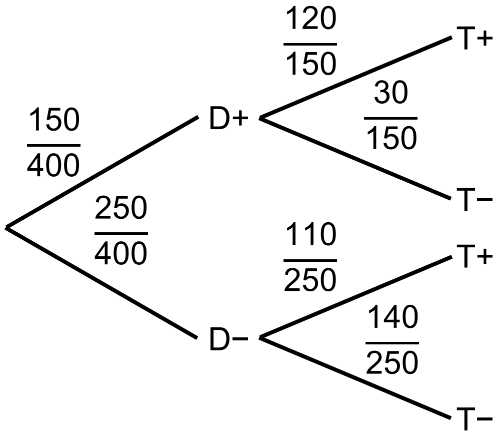
$\Pr(T\ +) = \Pr(D\ + \cap\ T\ +) + \Pr\ (D\ - \cap\ T+)$

$= 0.004(0.92) + 0.996(0.09)$

$= 0.00368 + 0.08964$

$= 0.09332$

D4.a)

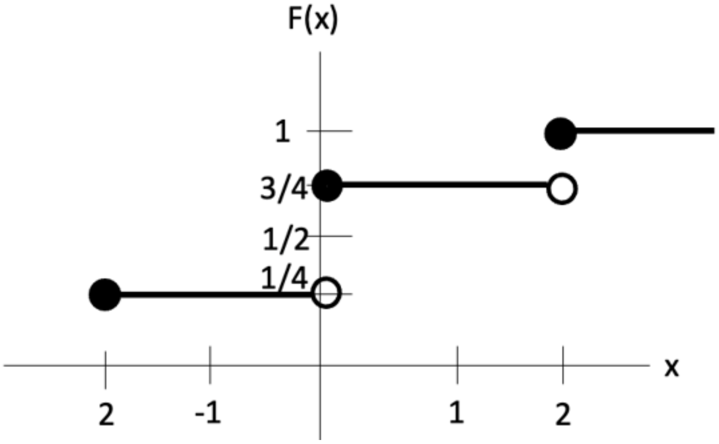


sensitivity= $\text{Pr}\left({}^{T\,+}/_{D\,+}\right) = \frac{120}{150} = 0.8$

b) specificity= $\text{Pr}\left({}^{T\,-}/_{D\,-}\right) = \frac{140}{250} = \frac{14}{25} = 0.56$

D5.

x	Pr (x)	$F(x)$
-2	$\frac{1}{4}$	$\frac{1}{4}$
0	$\frac{1}{2}$	$\frac{3}{4}$
2	$\frac{1}{4}$	1



D6. Probability = Area under graph

$= A_{triangle} + A_{rectangle} + 0.5\ (Area\ from - 1\ to\ 0)$

$= \frac{1}{2}(0.5)(0.5) + (0.5)(0.5) + 0.5$

$= 0.125 + 0.25 + 0.5$

$= 0.875$

D7. $u_{t+1} = 4u_t + 8 \quad mod\ 10$

a) $u_1 = 4u_0 + 8\ \ mod\ 10$

$u_1 = 4(6) + 8\ \ mod\ 10$

$u_1 = 32\ \ mod\ 10 \qquad 32 \div 10 = 3R2$

$\therefore u_1 = 2$

b) $u_2 = 4u_1 + 8 \quad mod\ 10$

$u_2 = 4(2) + 8 \quad mod\ 10$

$u_2 = 16 \quad mod\ 10 \quad 16 \div 10 = 1R6$

$\therefore u_2 = 6$

D8. a) $\int_0^k e^{-\frac{x}{4}} dx = 1$

$\left[\frac{e^{\frac{x}{4}}}{-\frac{1}{4}}\right]_0^k = 1$

$\left[-4e^{-\frac{x}{4}}\right]_0^k = 1$

$-4e^{-\frac{k}{4}} + 4e^0 = 1$

$-4e^{-\frac{k}{4}} = 1 - 4$

$-4 = e^{-\frac{k}{4}} = -3$

$e^{-\frac{k}{4}} = \frac{3}{4}$

$\ln e^{-\frac{k}{4}} = \ln(\frac{3}{4})$

$\frac{-k}{4} \ln e = \ln(\frac{3}{4})$

$\frac{-k}{4} = \ln(\frac{3}{4})$

$k = -4\ln(\frac{3}{4})$

b) $F(x) = \int_{-\infty}^{x} f(s)ds$

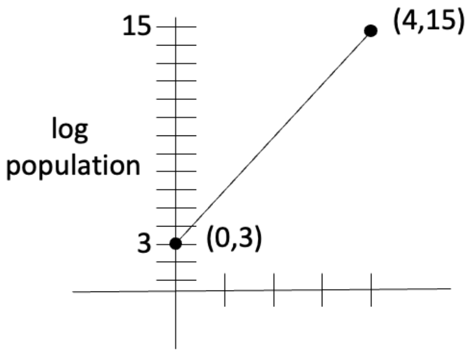
$= \int_0^x e^{-\frac{s}{4}} ds$

$= \left[\frac{e^{-\frac{s}{4}}}{-\frac{1}{4}}\right]_0^x$

$= \frac{e^{-\frac{x}{4}}}{-\frac{1}{4}} - \frac{e^0}{-\frac{1}{4}}$

$= -4e^{-\frac{x}{4}} + 4 \quad or \quad 4 - 4e^{-\frac{x}{4}}$

D9. The value of r is equal to the slope of the line in log-time



slope$= \frac{15-3}{4-0} = \frac{12}{4} = 3$

$$r = slope\ of\ line = \frac{rise}{run} = \frac{12}{4} = 3$$

D10. a) $F(x) = \sqrt{\frac{x}{3}} = \frac{\sqrt{x}}{\sqrt{3}} = \frac{1}{\sqrt{3}}x^{\frac{1}{2}}$

$CDF \rightarrow PDF\ is\ do\ the\ derivative$

$f(x) = F'(x) = \frac{1}{\sqrt{3}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{1}{2\sqrt{3}}x^{-\frac{1}{2}}$

$\therefore E(x) = \int_{-\infty}^{\infty} x\, f(x)dx$

$E(x) = \int_0^3 x(\frac{1}{2\sqrt{3}})x^{-\frac{1}{2}}dx$

$= \int_0^3 \frac{1}{2\sqrt{3}}x^{\frac{1}{2}}dx$

$= \frac{1}{2\sqrt{3}}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$

$= \frac{1}{2\sqrt{3}}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^3$

$= \frac{1}{2\sqrt{3}}\left[\frac{2}{3}(3)^{\frac{3}{2}} - 0\right]$

$= \frac{1}{2\sqrt{3}}\left[\frac{2}{3}(\sqrt{3})^3\right]$

$= \frac{1}{2\sqrt{3}}(\frac{2}{3}(\sqrt{3})(\sqrt{3})(\sqrt{3}))$

$= \frac{3}{3} = 1$

b) $Var\ (x) = \int_{-\infty}^{\infty} x^2 f(x) - [E(x)]^2$

$= \int_0^3 x^2\left(\frac{1}{2\sqrt{3}}x^{-\frac{1}{2}}\right)dx - 1^2$

$= \int_0^3 \frac{1}{2\sqrt{3}}x^{\frac{3}{2}} - 1$

$= \left[\frac{1}{2\sqrt{3}}\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^3 - 1$

$= \left[\frac{1}{2\sqrt{3}}\left(\frac{2}{5}x^{\frac{5}{2}}\right)\right]_0^3 - 1$

$= \left[\frac{1}{5\sqrt{3}}(3)^{\frac{5}{2}} - 0\right] - 1$

$= \left[\frac{1}{5\sqrt{3}} \times \sqrt{3}\sqrt{3}\sqrt{3}\sqrt{3}\sqrt{3}\right] - 1$

$= \frac{3\times 3}{5} - 1$

$= \frac{9}{5} - \frac{5}{5}$

$= \frac{4}{5}$

D11. a) Cheetah Lion

$z = \frac{x-\bar{x}}{s}$ $z = \frac{75-80}{3}$

$z = \frac{98-102}{2}$ $= -1.67$

$= -2$

∴ Lion runs faster relative to its breed (closer to its mean) If it was +2 and +1.67, the Cheetah would be faster!

b) $z = \frac{105-102}{2} = 1.5$

$\therefore 1.5 = \frac{x-80}{3}$

$x = 1.5(3) + 80 = 84.5$

∴ 84.5 km/hr is equivalent to the cheetah's speed.

E. Vectors

Example 1. b) $= \sqrt{(-2)^2 + (-1)^2 + 1^2}$

$= \sqrt{6}$

c) $= \sqrt{5^2 + (-1)^2 + (-2)^2}$

$= \sqrt{25 + 1 + 4}$

$= \sqrt{30}$

Example 2.

a) $= 5(-2,-1,1) = \ (-10,-5,5)$

$b) = -2(-1,-2,3) = (2,4,-6)$

c) $= \|-3(5,-1,-2)\|$

$= \|(-15,3,6)\| = \sqrt{(-15)^2 + 3^2 + 6^2}$

$= \sqrt{225 + 9 + 36} \quad = \sqrt{270}$

Example 3.

(3,-4)-2(-2,4)=(7, -12)

$\|(7,-12)\| = |(7,-12)|=\sqrt{7^2 + (-12)^2} = \sqrt{49 + 144} = \sqrt{193}$

The answer is C.

Example 4.

b) $= (-2,-1,1) + (5,-1,-2) = (3,-2,-1)$

c) $= (2,-1,-2) + (-1,-2,3) = (1,-3,1)$

d) $= 3(-1,-2,3) - 5(-2,-1,1)$

$= (-3,-6,9) + (10,5,-5) = (7,-1,4)$

e) $= -(-2,-1,1) + 3(5,-1,-2)$

$= (2,1,-1) + (15,-3,-6) = (17,-2,-7)$

f) $= 3(-1,-2,3) - (-2,-1,1) + 2(5,-1,-2)$

$= (-3,-6,9) + (2,1,-1) + (10,-2,-4)$

$= (9,-7,4)$

Example 5.

$$\boxed{1}\quad c_1 + 2c_2 = 17 \quad \times(-2) \qquad -2c_1 - 4c_2 = -34$$

$$\boxed{2}\quad 3c_1 + 4c_2 = 39 \qquad\qquad \underline{\quad 3c_1 + 4c_2 = 39}$$

$$ADD \quad c_1 = 5$$

Substitution $c_1 = 5 \ into \ \boxed{1}$

$$c_1 + 2c_2 = 17$$

$$5 + 2c_2 = 17$$

$$2c_2 = 17 - 5$$

$$2c_2 = 12$$

$$c_2 = 6$$

Practice Exam Questions

E1. $\overrightarrow{AB} = B - A = (5,3) - (2,-3) = (3,6)$

$\overrightarrow{CB} = B - C = (5,3) - (-2,1) = (7,2)$

$\left\|\overrightarrow{AB} - \overrightarrow{CB}\right\| = \left|\overrightarrow{AB} - \overrightarrow{CB}\right| = |(3,6) - (7,2)| = |(-4,4)|$

$= \sqrt{(-4)^2 + 4^2}$

$= \sqrt{32} \quad = \sqrt{16}\sqrt{2} \quad = 4\sqrt{2}$

$\therefore \textit{The answer is A}$

E2. $3(1,2,1) - (1,3,-1)$

$= (3,6,3) - (1,3,-1) = (2,3,4)$

Therefore, the answer is C).

E3. $2(3,6,1) - 3(0,-1,3)$

$= (6,12,2) - (0,-3,9) = (6,15,-7)$

Therefore, the answer is B).

E4. $\|\vec{u}\| = \sqrt{3^2 + 1^2 + 5^2}$

$= \sqrt{9 + 1 + 25} = \sqrt{35}$

Therefore, the answer is C).

E5. $= \|(1,3,1) + (2,5,1)\|$

$= \|(3,8,2)\|$

$= \sqrt{3^2 + 8^2 + 2^2}$

$= \sqrt{9 + 64 + 4} = \sqrt{77}$

Therefore, the answer is D).

E6.

$c_1 + 2c_2 = 11 \; \boxed{1}$

$2c_1 + 3c_2 = 18 \; \boxed{2}$

$$\begin{array}{lr} \boxed{1} \times (-2) & -2c_1 - 4c_2 = -22 \\ & 2c_1 + 3c_2 = 18 \\ \hline ADD & -c_2 = -4 \\ & c_2 = 4 \end{array}$$

Substitution $c_2 = 4 \quad into \; \boxed{1}$

$c_1 + 2(4) = 11$

$c_1 + 8 = 11$

$c_1 = 3$

F. Vector Fields

Example 1. a) Find unknown constants a and b such that $\vec{x} = \boldsymbol{a\vec{u} + b\vec{v}}$ where matrix $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$ where $\vec{u} = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} -24 \\ -6 \end{bmatrix}$.

$$\begin{bmatrix} -24 \\ -6 \end{bmatrix} = a\begin{bmatrix} -8 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$-24 = -8a + 2b$$
$$-6 = a - b \quad \times 2$$

$$-24 = -8a + 2b$$
$$-12 = 2a - 2b$$

Add $-36 = -6a$

$a = 6$

$-6 = a - b$

Substitute $a = 6$

$-6 = 6 - b$

$-12 = -b$

$b = 12$

$\therefore a = 6, b = 12$

b) Find $A^n\vec{u}$, $A^n\vec{v}$

$$A\vec{u} = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} -8 \\ 1 \end{bmatrix} = \begin{bmatrix} -48+16 \\ 8-4 \end{bmatrix} = \begin{bmatrix} 32 \\ 4 \end{bmatrix}$$

$\begin{bmatrix} -8 \\ 1 \end{bmatrix}$ **is an eigenvector associated with eigenvalue $\lambda = 4$**

$$A\vec{v} = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 12-16 \\ -2+4 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **is an eigenvector associated with eigenvalue $\lambda = -2$**

$$4\vec{u} = 4\begin{bmatrix} -8 \\ 1 \end{bmatrix} = \begin{bmatrix} -32 \\ 4 \end{bmatrix} = A\vec{u}$$

$$-2\vec{v} = -2\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = A\vec{v}$$

We will learn shortly $A\vec{v} = \lambda\vec{v}$ for any eigenvalue λ and eigenvector $\vec{v}$ and $A^n\vec{v} = \lambda^n\vec{v}$ as well!!

Example 3.

a) $F\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 3+(-3) \\ 3(3)-(-3) \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$

The first element is 0econd element is 12.

b) $F\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} u+v \\ 3u-v \end{bmatrix} = \begin{bmatrix} 0+3 \\ 3(0)-3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

The first element is 3.

c) The second element is -3.

Example 4. Let $\vec{z} = -80\vec{x} + 70\vec{y}$ Find the first and second elements of $\vec{F}(\vec{z})$

$\vec{F}(\vec{x}) = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$ $\vec{F}(\vec{y}) = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

$\mathrm{F}(\vec{z}) = -80F(\vec{x}) + 70F(\vec{y}\ )$

The first element is:

$$\vec{F}(\vec{z}) = -80(0) + 70(3) = 210$$

The second element is

$$\vec{F}(\vec{z}) = -80(12) + 70(-3) = -1170$$

Example 5. $a)\, F(\vec{x}) = f\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = -2(3) + 5(3) = -6 + 15 = 9$

$b)\ F(\vec{y}) = f\left(\begin{bmatrix} -1 \\ 4 \end{bmatrix}\right) = -2(-1) + 5(4) = 2 + 20 = 22$

$c)\ F(\vec{z}) = -4F(\vec{x}) + 6F(\vec{y}) = -4(9) + 6(22) = -36 + 132 = 96$

Example 6. $F = \begin{bmatrix} 2x + y \\ x \end{bmatrix}$

$(-5,5)$

$x, y \qquad F = \begin{bmatrix} 2(-5) + 5 \\ -5 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \end{bmatrix} \begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$

$\therefore the\ vector\ goes\ down\ and\ left$

$(5,2)$

$x, y \qquad F = \begin{bmatrix} 2(5) + 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \end{bmatrix} \begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$

$\therefore the\ vector\ goes\ up\ and\ right$

$(-4,-2)$

$x, y \qquad F = \begin{bmatrix} 2(-4) + (-2) \\ -4 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix} \begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$

$\therefore the\ vector\ goes\ down\ and\ left$

$(3,-2)$

$x, y \qquad F = \begin{bmatrix} 2(3) + (-2) \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$

$\therefore the\ vector\ goes\ up\ and\ right$

$F = \begin{bmatrix} y \\ 2x - y \end{bmatrix}$

$(-5,5)$

$x, y \qquad F = \begin{bmatrix} 5 \\ 2(-5) - 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \end{bmatrix} \begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$

$\therefore$the vector goes down and right

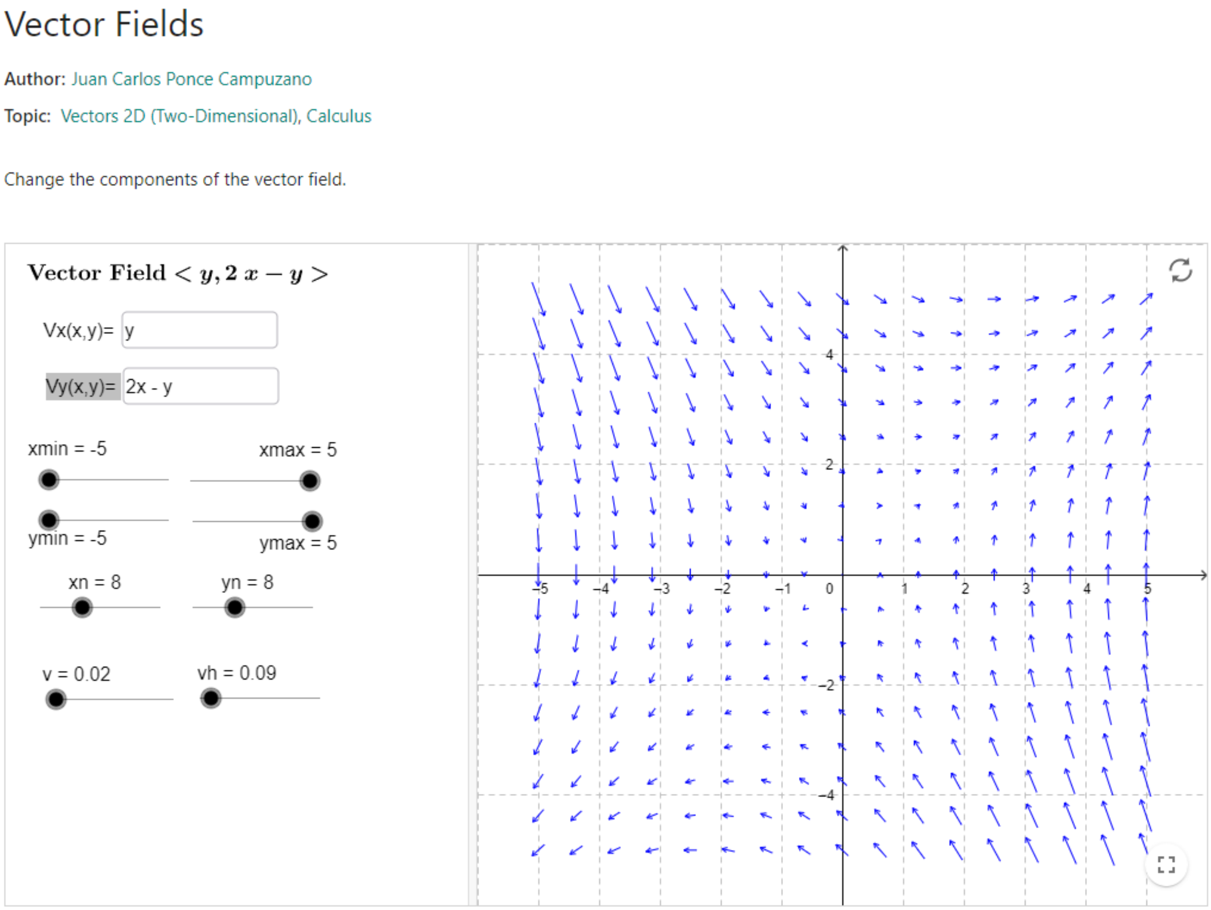
Example 7.

The graphs in order are 2,1,3

$\overrightarrow{F}(\begin{bmatrix}u\\v\end{bmatrix}) = \begin{bmatrix}x\\y\end{bmatrix}$ *Graph* 2

$\overrightarrow{F}(\begin{bmatrix}u\\v\end{bmatrix}) = \begin{bmatrix}2x+y\\x\end{bmatrix}$ *Graph* 1

$\overrightarrow{F}(\begin{bmatrix}u\\v\end{bmatrix}) = \begin{bmatrix}y\\2x-y\end{bmatrix}$ *Graph* 3



Vector Fields

Author: Juan Carlos Ponce Campuzano

Topic: Vectors 2D (Two-Dimensional), Calculus

Change the components of the vector field.

Vector Field $< y, 2\,x - y >$

Vx(x,y)= y

Vy(x,y)= 2x - y

xmin = -5

xmax = 5

ymin = -5

ymax = 5

xn = 8

yn = 8

v = 0.02

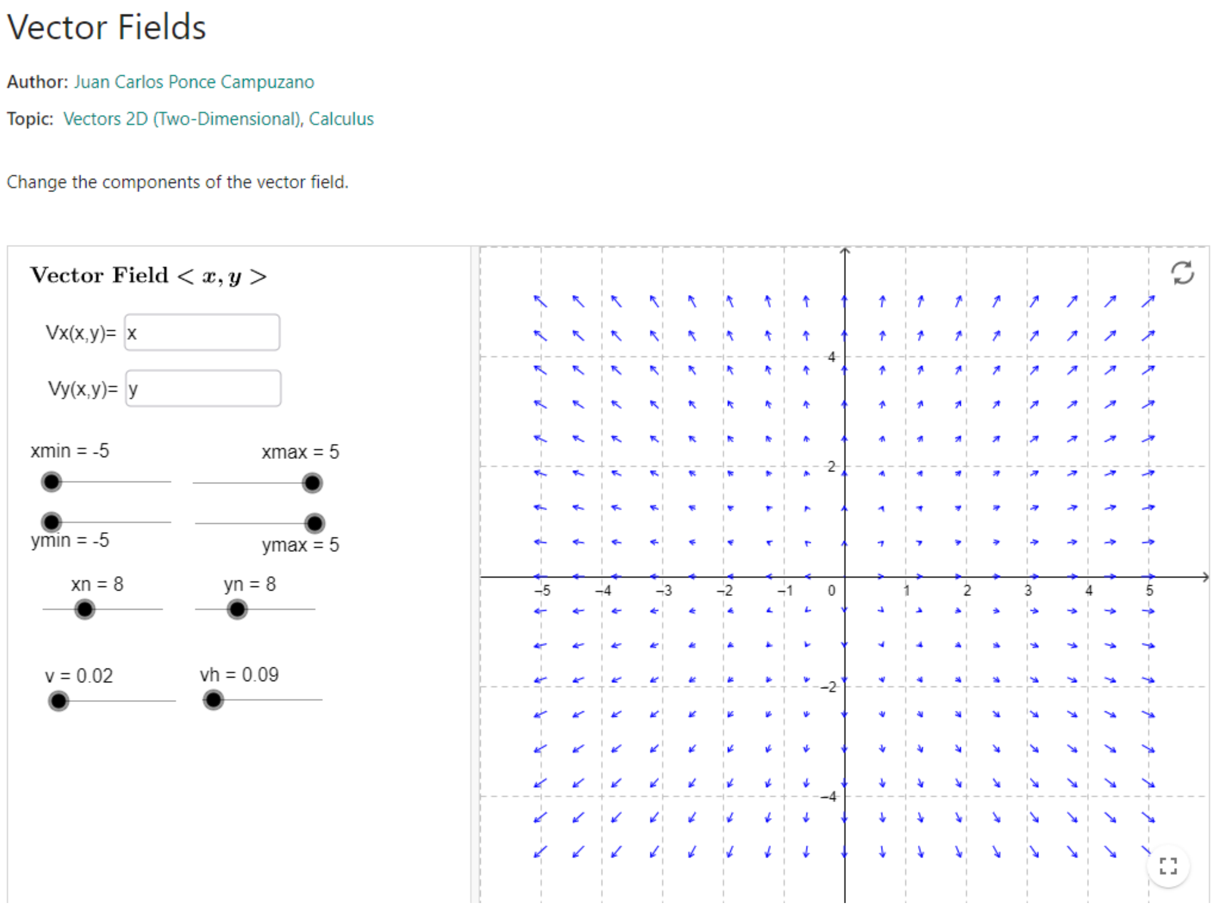
vh = 0.09

$\overrightarrow{F}(\begin{bmatrix}u\\v\end{bmatrix}) = \begin{bmatrix}y\\2x-y\end{bmatrix}$ graph 3

Test point (-4,4)

$$\overrightarrow{F}(\begin{bmatrix}-4\\4\end{bmatrix}) = \begin{bmatrix}4\\2(-4)-4\end{bmatrix} = \begin{bmatrix}4\\-12\end{bmatrix}$$

Top number is 4 (run)
Bottom number is -12 (rise)
So, the vector goes down and right



Vector Fields

Author: Juan Carlos Ponce Campuzano

Topic: Vectors 2D (Two-Dimensional), Calculus

Change the components of the vector field.

Vector Field $< x, y >$

Vx(x,y)= x

Vy(x,y)= y

xmin = -5

xmax = 5

ymin = -5

ymax = 5

xn = 8

yn = 8

v = 0.02

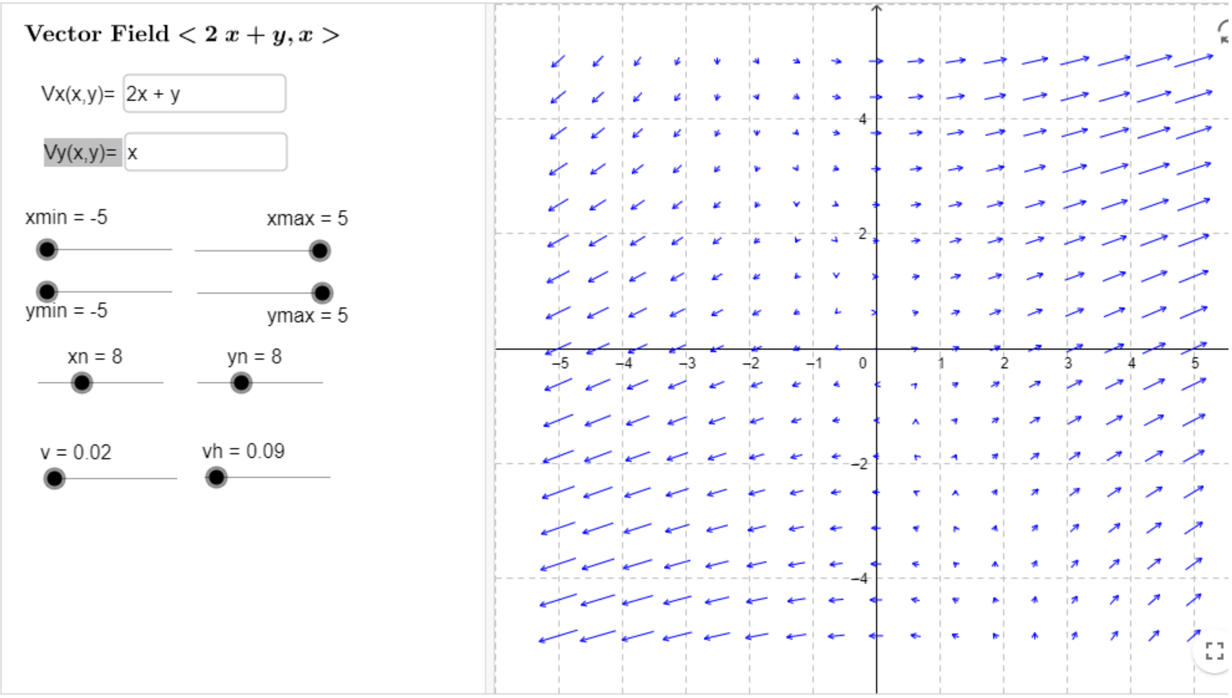
vh = 0.09

$$\vec{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} = \text{graph } 2$$

Test point (-4,4)

$$\vec{F}\left(\begin{bmatrix} -4 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$

Top number is -4 (run)
Bottom number is 4 (rise)
So, the vector goes up and left



Vector Field $< 2x + y, x >$

Vx(x,y)= 2x + y

Vy(x,y)= x

xmin = -5

xmax = 5

ymin = -5

ymax = 5

xn = 8

yn = 8

v = 0.02

vh = 0.09

$\overrightarrow{F}(\begin{bmatrix}u\\v\end{bmatrix}) = \begin{bmatrix}2x+y\\x\end{bmatrix}$ graph 1

Test point (-4,4)

$$\overrightarrow{F}(\begin{bmatrix}-4\\4\end{bmatrix}) = \begin{bmatrix}2(-4)+4\\-4\end{bmatrix} = \begin{bmatrix}-4\\-4\end{bmatrix}$$

Top number is -4 (run)
Bottom number is -4 (rise)
So, the vector goes down and left

Practice Exam Questions

F1.

$$F=\begin{bmatrix} y \\ 2x+y \end{bmatrix} \quad check\ \left(\genfrac{}{}{0pt}{}{-4,4}{x,y}\right)$$

$$F=\begin{bmatrix} 4 \\ 2(-4)+4 \end{bmatrix}=\begin{bmatrix} 4 \\ -4 \end{bmatrix}\begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$$

∴the vector goes down and right

$$\therefore\ the\ 2nd\ graph\ is\ \ F=\begin{bmatrix} y \\ 2x+y \end{bmatrix}$$

$$F\begin{bmatrix} x \\ 2x+y \end{bmatrix} \quad check\ \left(\genfrac{}{}{0pt}{}{-4,4}{x,y}\right)$$

$$F=\begin{bmatrix} -4 \\ 2(-4)+4 \end{bmatrix}=\begin{bmatrix} -4 \\ -4 \end{bmatrix}\begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$$

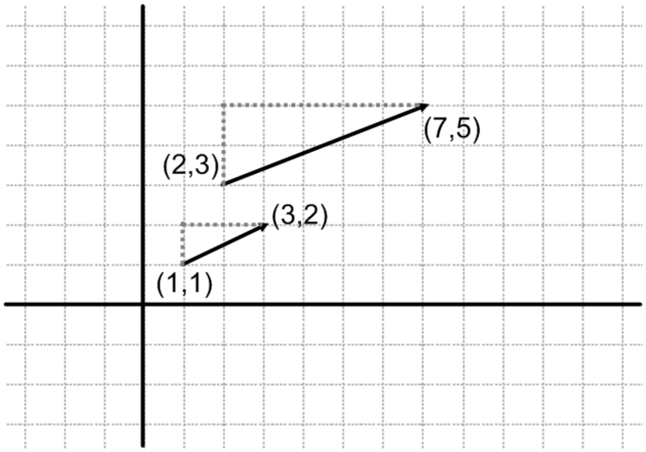
∴vector goes down and left

$$\therefore\ the\ 1st\ graph\ is\ f=\begin{bmatrix} x \\ 2x+y \end{bmatrix}$$

F2.

$$F\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$$

$$F\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}=\begin{bmatrix} 5 \\ 2 \end{bmatrix}\begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$$



F3. $F\begin{bmatrix} 2 \\ -2 \end{bmatrix}=\begin{bmatrix} 2+3(-2) \\ 2-2(-2) \end{bmatrix}=\begin{bmatrix} -4 \\ 6 \end{bmatrix}$

$\therefore\ 1st\ element\ is-4\ and\ 2nd\ is\ 6.$

a) $F\begin{bmatrix} 1 \\ 2 \end{bmatrix}=\begin{bmatrix} u+3v \\ u-2v \end{bmatrix}=\begin{bmatrix} 1+3(2) \\ 1-2(2) \end{bmatrix}=\begin{bmatrix} 7 \\ -3 \end{bmatrix}$

$\therefore\ 1st\ element\ is\ 7\ and\ 2nd\ is-3.$

b) $F(\vec{z})=-50F(\vec{x})+40F(\vec{y})$

$$=-50\begin{bmatrix} -4 \\ 6 \end{bmatrix}+40\begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

$\therefore\ the\ 1st\ element\ is-50(-4)+40(7)=480$ and

$the\ 2nd\ element\ is-50(6)+40(-3)=-300-120=-420$

F4.

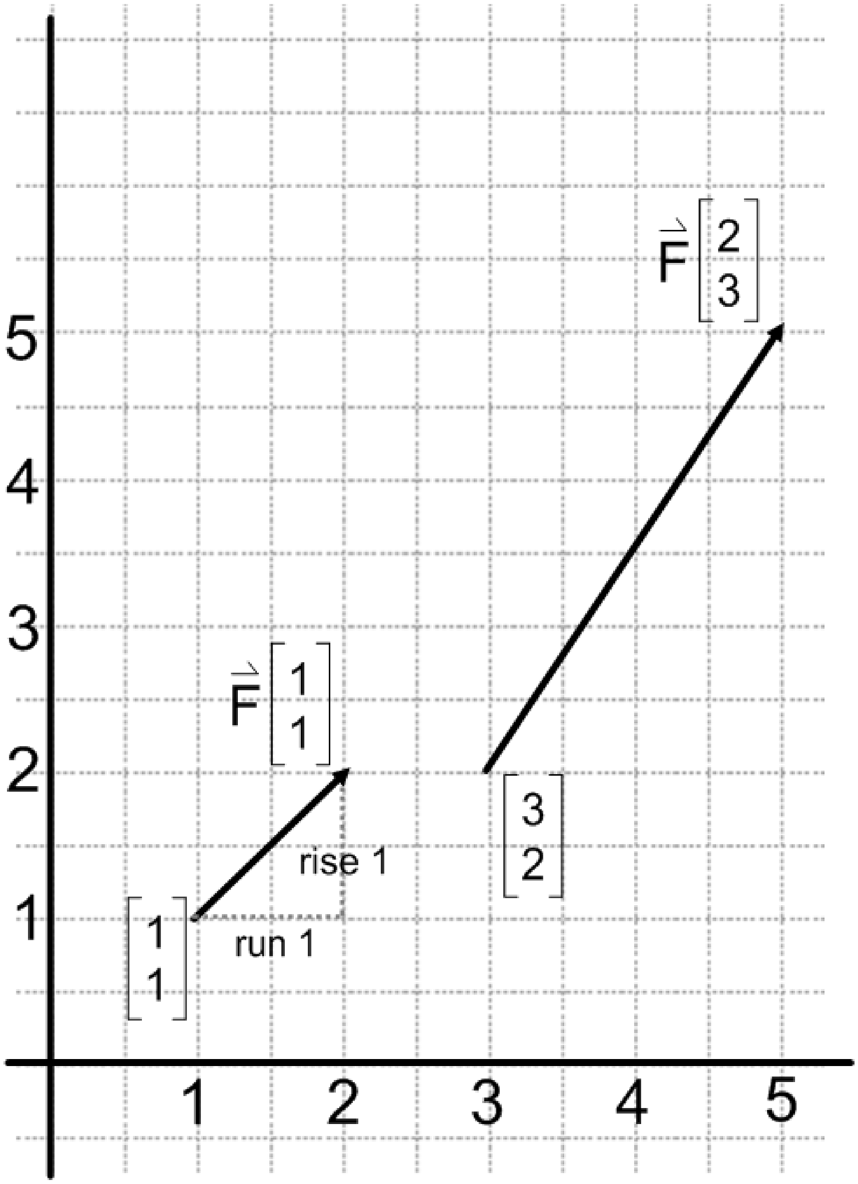
a) $F(\vec{x}) = f\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = -3(3) + 7(2) = -9 + 14 = 5$

b) $F(\vec{y}) = f\begin{bmatrix}-1\\3\end{bmatrix} = -3(-1) + 7(3) = 24$

c) $F(\vec{z}) = -5f(\vec{x}) + 7(\vec{y}) = -5(5) + 7(24) = -25 + 168 = 143$

F5. $\vec{F}\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}\begin{matrix}\leftarrow run\\\leftarrow rise\end{matrix}$

$\vec{F}\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}\begin{matrix}\leftarrow run\\\leftarrow rise\end{matrix}$



F6. Show that $F(\vec{x})$=$3x_1 + 2x_2$ is linear.

$\vec{x} = (x_1, x_2) \quad \vec{y} = (y_1, y_2)$

We want $F(c_1\vec{x} + c_2\vec{y}) = c_1 f(\vec{x}) + c_2 f(\vec{y})$

$$c_1\vec{x} + c_2\vec{y} = c_1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + c_2\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1x_1 + c_2y_1 \\ c_1x_2 + c_2y_2 \end{bmatrix}$$

$$\begin{aligned} F(c_1\vec{x} + c_2\vec{y}) &= 3(c_1x_1 + c_2y_1) + 2(c_1x_2 + c_2y_2) \\ &= c_1(3x_1 + 2x_2) + c_2(3y_1 + 2y_2) \\ &= c_1 f(\vec{x}) + c_2 f(\vec{y}) \end{aligned}$$

$\therefore linear$

F7. $\vec{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 3x + y \\ x \end{bmatrix}$ $graph\ 3$ $\quad \vec{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} y \\ 2x + y \end{bmatrix}$ $graph\ 1$

$\vec{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} x \\ x + y \end{bmatrix}$ graph 2

Do test points

G. Matrices, Inverses and Determinants

Addition and Subtraction

Example 1.
b) dim is 2x3
c) dim is 1x3
d) dim is 3x3
e) dim is 4x3

Example 4.

$$\begin{bmatrix} 2+3 & -1+2 \\ 1+1 & 3+4 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 7 \end{bmatrix}$$

Example 5.

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & -3 & 4 \\ 0 & -1 & 0 \end{bmatrix}$$

Multiplication by a Scalar

Example 6.

$$-2A = -2\begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ -10 & -2 \end{bmatrix}$$

Transpose of a Matrix

Example 7.

a) $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

b) $B^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

c) $C^T = \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}$

Matrix Multiplication

8a) $\begin{bmatrix} \mathbf{5} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}$

b) $= \begin{bmatrix} 1(3)+3(2) \\ 2(3)+2(2) \end{bmatrix} = \begin{bmatrix} 3+6 \\ 6+4 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \end{bmatrix}$

c) $= \begin{bmatrix} 1(2)+(-1)(3)+3(1) \\ 2(2)+2(3)+2(1) \\ 1(2)+1(3)+1(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 6 \end{bmatrix}$

Powers of a Matrix

Example 9.
omit

Determinant

Example 11.

detA=48

Since the determinant is not 0, the vector field is invertible.

Example 12.

For which value(s) of k is the matrix invertible?

Not invertible if $detA = 0$

$3k + 6 = 0$

$3k = -6 \quad k = -2$

So, it is invertible as long as k is not equal to -2.

Example 13.

$$\begin{bmatrix} 2 & 1 & 1 \\ -2 & -2 & 2 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_2 + x_3 = 0 \ \boxed{1}$

$-2x_1 - 2x_2 + 2x_3 = 0 \quad \boxed{2}$

$x_1 + 2x_3 = 0 \quad \rightarrow \boxed{3} \quad from \boxed{3} \ x_3 = \frac{-1}{2}x_1$

Substitute into $\boxed{2}$

$-2x_1 - 2x_2 + 2\left(\frac{-1}{2}x_1\right) = 0$

$-2x_1 - 2x_2 - x_1 = 0$

$-3x_1 - 2x_2 = 0$

$\therefore -3x_1 = 2x_2$

$x_2 = \frac{-3}{2}x_1$

$$\therefore vector\ is\ \vec{x} = \begin{bmatrix} x_1 \\ \frac{-3}{2}x_1 \\ \frac{-1}{2}x_1 \end{bmatrix}$$

Example 14.

Let $A = \begin{bmatrix} -3 & 5 \\ 1 & 2 \end{bmatrix}$ $and\ \vec{x} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $\quad Find\ A\vec{x}$

$$\begin{bmatrix} -3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -3-15 \\ 1-6 \end{bmatrix} = \begin{bmatrix} -18 \\ -5 \end{bmatrix}$$

Let $B = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 5 \\ 0 & 4 & 2 \end{bmatrix}$ $and\ \vec{y} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $Find\ the\ 3rd\ entry\ of\ B\vec{y}.$

$= [0, 4, 2] \cdot [1, 0, -1]$
$= 0 + 0 - 2$
$= -2$

Practice Exam Questions

G1. Find the determinant of each of the following:
a) det A=7
b) detB=ad - bc = (2)(-7) - (-1)(3) = -14 + 3 = -11
c) C=$\begin{bmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

Det A=8, so the vector field is invertible since det A is NOT 0

G2. Determine the value of x so that matrix A=$\begin{bmatrix} 2 & 0 & 4 \\ x & 1 & 3 \\ 2 & 0 & x \end{bmatrix}$ has an inverse if the determinant is 2x-8. It has an inverse if detA≠0

If det A=0, 2x-8 = 0 so x=4 and this would be the value for which there is NO inverse
The answer is c).

G3. Not invertible, $ie) detA = 0$
$k(k+5) - (-6)(1) = 0$
$k^2 + 5k + 6 = 0$
$(k+2)(k+3) = 0 \quad k = -2, -3$

H: Quiz 5: Practice on Sections A to G

1. a) $E(x) = \int_0^4 x\left(\frac{1}{2}x\right)dx$

$= \int_0^4 \frac{1}{2}x^2 dx$

$= \left[\frac{1}{2}\frac{x^3}{3}\right]_0^4$

$= \left[\frac{x^3}{6}\right]_0^4$

$= \frac{4^3}{6}$

$= \frac{64}{6}$

$= \frac{32}{3}$

b) $Var(x) = ?$

$E(x^2) = \int_0^4 x^2\left(\frac{1}{2}x\right)dx$

$= \frac{1}{2}\int_0^4 x^3 dx$

$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^4$

$= \frac{1}{8}[x^4]_0^4$

$= \frac{1}{8}[4^4 - 0]$

$= \frac{4^4}{8}$

$Var(x) = E(x^2) - [E(x)]^2$

$= \frac{4^4}{8} - (\frac{32^2}{3})^2$

$= \frac{256}{8} - \frac{1024}{9}$

$= -81\frac{7}{9}$

2. a) $E(x) = \frac{-2+0-0.8-0.4-0.5}{5} = \frac{-3.7}{5}$ note: there are no probabilities given, so it is just the mean which is to add up all the numbers and divide by how many there are

$= -0.74$

b) $Var(x) = \sum\frac{(x-\bar{x})^2}{n-1}$ (again, no probabilities given) n=5 numbers or 5 data

$$Var(x) = \frac{(-2+0.74)^2 + (0+0.74)^2 + (-0.8+0.74)^2 + (-0.4+0.74)^2 + (-0.5+0.74)^2}{5-1}$$

$= \frac{(-1.26)^2+0.74^2+(-0.06)^2+(0.34)^2+(0.24)^2}{4}$

$\frac{2.31}{4} = 0.58$

3. $Z \;=\dfrac{95-62}{12}=2.75$

$2.75=\frac{x-50}{9}$

$x=74.75$

4. $F\begin{bmatrix}3\\-3\end{bmatrix}=\begin{bmatrix}3+(-3)\\3-2(-3)\end{bmatrix}=\begin{bmatrix}0\\9\end{bmatrix}$

a) The first element is 0.

b) The second element is 9

c)$\vec{F}\begin{bmatrix}0\\4\end{bmatrix}=\begin{bmatrix}u+v\\u-2v\end{bmatrix}=\begin{bmatrix}0+4\\0-2(4)\end{bmatrix}=\begin{bmatrix}4\\-8\end{bmatrix}$

The first element is 4.

d)The second element is -8.

e)$\vec{F}(\vec{x})=\begin{bmatrix}0\\9\end{bmatrix}\qquad \vec{F}(\vec{y})=\begin{bmatrix}4\\-8\end{bmatrix}$

$\vec{F}(z)=-90.5(0)+85.5(4)=342$

The first element is 342.

f)$\vec{F}(z)=-90.5(9)+85.5(-8)=-1498.5$

The second element is – 1498.5.

5. a)$f(\vec{x})=\vec{f}\begin{bmatrix}u\\v\end{bmatrix}=-3u+4v$

$=f\left(\begin{bmatrix}4\\4\end{bmatrix}\right)=-3(4)+4(4)=-12+16=4$

b)$f(\vec{y})=f(\begin{bmatrix}-2\\2\end{bmatrix}=-3(-2)+4(2)=6+8\;=14$

c)$\vec{z}=-3\vec{x}+5\vec{y}$

$f(\vec{z})=-3f(\vec{x})+5\vec{f}(y)$

$=-3(4)+5(14)=-12+70=58$

6. $A)\;\vec{F}\left(\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}2x+y\\x\end{bmatrix}$

$\vec{F}\left(\begin{bmatrix}4\\-4\end{bmatrix}\right)=\begin{bmatrix}2(4)-4\\4\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix}$ (right, up) So, graph 3.

$B)\;\vec{F}\left(\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}y\\2x-y\end{bmatrix}\quad \vec{F}\left(\begin{bmatrix}4\\-4\end{bmatrix}\right)=\begin{bmatrix}-4\\2(4)-(-4)\end{bmatrix}=\begin{bmatrix}-4\\12\end{bmatrix}$ (left and up) So, graph 1.

$C)\;\vec{F}\left(\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}x\\y\end{bmatrix}\quad \vec{F}\left(\begin{bmatrix}4\\-4\end{bmatrix}\right)=\begin{bmatrix}4\\-4\end{bmatrix}$ (right and down) So, graph 2.

7. a) $\begin{bmatrix} 0 & 2 \\ 0.4 & 0.2 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0+6 \\ 2+0.6 \end{bmatrix} = \begin{bmatrix} 6 \\ 2.6 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 \\ 0.4 & 0.2 \end{bmatrix} \begin{bmatrix} 6 \\ 2.6 \end{bmatrix} = \begin{bmatrix} 5.2 \\ 2.92 \end{bmatrix}$$

$\therefore$ 5.2 *will be* in the first category

b) $\therefore$ 2.92 will be in the second category

8. $A\vec{x} = \begin{bmatrix} -2 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2-8 \\ 1-6 \end{bmatrix} = \begin{bmatrix} -10 \\ -5 \end{bmatrix}$

$\therefore the\ first\ element\ is - 10\ and\ the\ second\ element\ is - 5.$

9. 3rd entry $= \begin{bmatrix} & & \\ & & \\ 5 & -1 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} = [-5+0+4] = -1$

10. $\det A = ad - bc = 1(6) - 2(3) = 0$

$\therefore No$

I. Matrix Models and Leslie Matrices

Example 2.

$$\begin{aligned}\vec{x}\cdot\vec{y} &= [\beta,\gamma]\cdot[0,7]\\ &= \beta(0)+7\gamma\\ &= 7\gamma\end{aligned}$$ The answer is C.

Example 3.

a)

$$G = \begin{array}{c|cc} & H & A \\ \hline H & 1.2 & 1.5 \\ A & 0.5 & 0.3 \end{array}$$

b) $\begin{bmatrix}1.2 & 1.5\\0.5 & 0.3\end{bmatrix}\begin{bmatrix}100\\50\end{bmatrix} = \begin{bmatrix}195\\65\end{bmatrix}$

There would be 195 hatchlings and 65 adults in year 1.

Example 4.

a)

$$G = \begin{array}{c|ccc} & H & J & A \\ \hline H & 0 & 0.2 & 1 \\ J & 0.5 & 0 & 0 \\ A & 0 & 0.4 & 0 \end{array}$$

b) $\begin{bmatrix}0 & 0.2 & 1\\0.5 & 0 & 0\\0 & 0.4 & 0\end{bmatrix}\begin{bmatrix}30\\10\\10\end{bmatrix} = \begin{bmatrix}0+2+10\\15+0+0\\0+4+0\end{bmatrix} = \begin{bmatrix}12\\15\\4\end{bmatrix}$

$\therefore$ 12 *hatchlings*, 15 *juveniles*, 4 *adults* in year 2

$\begin{bmatrix}0 & 0.2 & 1\\0.5 & 0 & 0\\0 & 0.4 & 0\end{bmatrix}\begin{bmatrix}12\\15\\4\end{bmatrix} = \begin{bmatrix}7\\6\\6\end{bmatrix}$ $\therefore$ 7 *hatchlings*, 6 *juveniles*, 6 *adults* in year 3

c) $\begin{bmatrix}0 & 0.2 & 1\\0.5 & 0 & 0\\0 & 0.4 & 0\end{bmatrix}\begin{bmatrix}12\\15\\4\end{bmatrix} = \begin{bmatrix}7\\6\\6\end{bmatrix}$

$$\begin{bmatrix}0 & 0.2 & 1\\0.5 & 0 & 0\\0 & 0.4 & 0\end{bmatrix}\begin{bmatrix}7\\6\\6\end{bmatrix} = \begin{bmatrix}7.2\\3.5\\2.4\end{bmatrix}$$

$$\begin{bmatrix}0 & 0.2 & 1\\0.5 & 0 & 0\\0 & 0.4 & 0\end{bmatrix}\begin{bmatrix}7.2\\3.5\\2.4\end{bmatrix} = \begin{bmatrix}3.1\\3.6\\1.4\end{bmatrix}$$

$$\begin{bmatrix}0 & 0.2 & 1\\0.5 & 0 & 0\\0 & 0.4 & 0\end{bmatrix}\begin{bmatrix}3.1\\3.6\\1.4\end{bmatrix} = \begin{bmatrix}2.12\\1.55\\1.44\end{bmatrix}$$

$$\begin{bmatrix}0 & 0.2 & 1\\0.5 & 0 & 0\\0 & 0.4 & 0\end{bmatrix}\begin{bmatrix}2.12\\1.55\\1.44\end{bmatrix} = \begin{bmatrix}1.75\\1.06\\0.62\end{bmatrix}$$

Etc.

$\therefore$ total population decreases to zero.

The answer is A.

Practice Exam Questions

I1.

$$G = \begin{array}{c|cc} & \text{H} & \text{A} \\ H & 0.8 & 1.2 \\ A & 0.6 & 0.4 \end{array}$$

b) $\begin{bmatrix}0.8 & 1.2\\0.6 & 0.4\end{bmatrix}\begin{bmatrix}60\\50\end{bmatrix} = \begin{bmatrix}108\\56\end{bmatrix}$

c) $\begin{bmatrix}0.8 & 1.2\\0.6 & 0.4\end{bmatrix}\begin{bmatrix}108\\56\end{bmatrix} = \begin{bmatrix}153.6\\87.2\end{bmatrix} \therefore 87\ adults$

I2.

$$G = \begin{array}{c|cc} & \text{H} & \text{A} \\ H & 1.2 & 1.5 \\ A & 0.5 & 0.3 \end{array} \quad top\ is\ A\ to\ H \quad and\ bottom\ is\ A\ to\ A$$

a)

b) $\begin{bmatrix}1.2 & 1.5\\0.5 & 0.3\end{bmatrix}\begin{bmatrix}40\\20\end{bmatrix} = \begin{bmatrix}78\\26\end{bmatrix}$

c) $\begin{bmatrix}1.2 & 1.5\\0.5 & 0.3\end{bmatrix}\begin{bmatrix}78\\26\end{bmatrix} = \begin{bmatrix}132.6\\46.8\end{bmatrix} \therefore 133\ hatchlings\ and\ 47\ adults\ in\ year\ 3.$

I3.

$$\begin{array}{c|ccc} & A & B & C \\ A & 0 & 0 & 5 \\ B & 3 & 4 & 0 \\ C & 0 & 2 & 0 \end{array} \qquad \begin{bmatrix}a_t\\b_t\\c_t\end{bmatrix}$$

I4.

a)

$$\begin{array}{c|cc} & H & A \\ H & 1.3 & 1.5 \\ A & 0.6 & 0.5 \end{array}$$

b) $\begin{bmatrix}1.3 & 1.5\\0.6 & 0.5\end{bmatrix}\begin{bmatrix}100\\30\end{bmatrix} = \begin{bmatrix}175\\75\end{bmatrix}$

c)

$$\begin{bmatrix}1.3 & 1.5\\0.6 & 0.5\end{bmatrix}\begin{bmatrix}175\\75\end{bmatrix} = \begin{bmatrix}340\\142.5\end{bmatrix}$$

So, we expect 340 hatchlings and 143 adults in year 3.

I5.

a)

$$G = \begin{array}{c|ccc} & H & J & A \\ \hline H & 0 & 0.3 & 1 \\ J & 0.6 & 0 & 0 \\ A & 0 & 0.5 & 0 \end{array}$$

b) Estimate the population of hatchlings and adults in year 2 if there are 120 birds in the population: 60 hatchlings, 40 juveniles and 20 adults.

$$\begin{bmatrix} 0 & 0.3 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}\begin{bmatrix} 60 \\ 40 \\ 20 \end{bmatrix}\begin{matrix} H \\ J \\ A \end{matrix} = \begin{bmatrix} 0+12+20 \\ 36+0+0 \\ 0+20+0 \end{bmatrix} = \begin{bmatrix} 32 \\ 36 \\ 20 \end{bmatrix}$$

$\therefore 32 \ hatchlings \ in \ year \ 2$

I6.

a) $C_{t+1} = 0.4C_t$

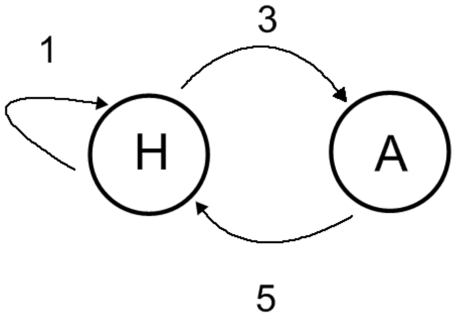
$a_{t+1} = 0.098C_t + 0.75a_t + 0.77r_t$

$r_{t+1} = 0.27a_t$

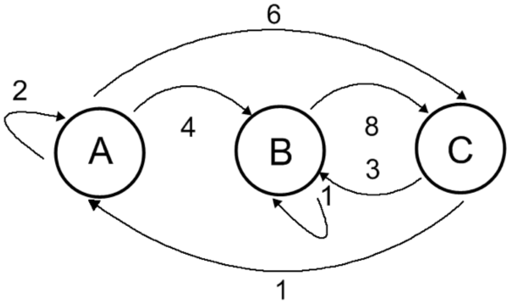
$$\begin{array}{c|ccc} & C_1 & a_1 & r_1 \\ \hline C_1 & 0 & 0 & 0.4 \\ a_1 & 0.098 & 0.75 & 0.77 \\ r_1 & 0 & 0.27 & 0 \end{array}\begin{bmatrix} C_t \\ a_t \\ r_t \end{bmatrix}$$

I7.

a)



b)



I8. a)

Construct the corresponding matrix…

$$\begin{array}{c|c} & \begin{matrix} x & y \end{matrix} \\ \begin{matrix} x \\ y \end{matrix} & \begin{bmatrix} 4 & 2 \\ 0 & 0.8 \end{bmatrix} \end{array} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$x_i + 1 = 4x_i + 2y_i$$
$$y_i + 1 = 0x_i + 0.8y_i$$

b) $$\begin{array}{c|c} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

I9. a)

$$G = \begin{array}{c|c} & \begin{matrix} H & A \end{matrix} \\ \begin{matrix} H \\ A \end{matrix} & \begin{bmatrix} 1.5 & 1.8 \\ 0.5 & 0.4 \end{bmatrix} \end{array}$$

b) $$\begin{bmatrix} 1.5 & 1.8 \\ 0.5 & 0.4 \end{bmatrix} \begin{bmatrix} 50 \\ 20 \end{bmatrix}_{A}^{H} = \begin{bmatrix} 75 + 36 \\ 25 + 8 \end{bmatrix} = \begin{bmatrix} 111 \\ 33 \end{bmatrix}$$

c) $$\begin{bmatrix} 1.5 & 1.8 \\ 0.5 & 0.4 \end{bmatrix} \begin{bmatrix} 111 \\ 33 \end{bmatrix} = \begin{bmatrix} 166.5 + 59.4 \\ 55.5 + 13.2 \end{bmatrix} = \begin{bmatrix} 225.9 \\ 68.70 \end{bmatrix}$$

$\therefore$ *approx* 226 *hatchlings* and 69 adults in year 3.

I10.

$$\begin{array}{c|c} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} A \\ B \end{matrix} & \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \end{array}$$

I11. a)

$$G = \begin{array}{c|c} & \begin{matrix} H & A \end{matrix} \\ \begin{matrix} H \\ A \end{matrix} & \begin{bmatrix} 0.70 & 1.3 \\ 0.5 & 0.6 \end{bmatrix} \end{array}$$

b) $$\begin{bmatrix} 0.70 & 1.3 \\ 0.5 & 0.6 \end{bmatrix} \begin{bmatrix} 60 \\ 30 \end{bmatrix} = \begin{bmatrix} 42 + 39 \\ 30 + 18 \end{bmatrix} = \begin{bmatrix} 81 \\ 48 \end{bmatrix}$$

c) $$\begin{bmatrix} 0.7 & 1.3 \\ 0.5 & 0.6 \end{bmatrix} \begin{bmatrix} 81 \\ 48 \end{bmatrix} = \begin{bmatrix} 119.1 \\ 69.3 \end{bmatrix}$$ *approx* 69 *adults* and 119 hatchlings in year 3.

I12.

a) $$\begin{bmatrix} x \\ y \end{bmatrix}_1 = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

b) $$\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad doesn't\ move$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad changes$$

The solution is A.

I13. $\begin{bmatrix} 0 & 2 \\ 0.5 & 0.3 \end{bmatrix}\begin{bmatrix} 10 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5.6 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 \\ 0.5 & 0.3 \end{bmatrix}\begin{bmatrix} 4 \\ 5.6 \end{bmatrix} = \begin{bmatrix} 11.2 \\ 3.68 \end{bmatrix}$$

Therefore, two years into the future, there will be approximately 11 and 4 individuals.

I14.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_t = \begin{array}{c} \\ x \\ y \\ z \end{array}\begin{bmatrix} x & y & z \\ 0.85 & 0 & 0 \\ 0.15 & 0.80 & 0 \\ 0 & 0.20 & 1 \end{bmatrix}$$

J. Eigenvalues and Eigenvectors

Example 2. $\det(A-\lambda I)=0$

$$A-\lambda I$$

$$=\begin{bmatrix}2 & 3\\ -2 & -1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2-\lambda & 3\\ -2 & -1-\lambda\end{bmatrix}$$

$\det(A-\lambda I)=0 \quad \therefore (2-\lambda)(-1-\lambda)-(3)(-2)=0$

$-2-2\lambda+\lambda+\lambda^2+6$

$\lambda^2-\lambda+4=0$ OR

Short-cut: (Multiple Choice only)

use tr(A) =1 and det A = -2 + 6 = 4

$\lambda^2-\lambda+4=0$

Example 3. Express $\frac{-2+3i}{3+7i}$ in the form $a+bi$

Solution:

Multiply by the conjugate of the denominator!

$$\frac{(-2+3i)}{(3+7i)}\,\frac{(3-7i)}{(3-7i)}=\frac{-6+14i+9i-21i^2}{9-21i+21i-49i^2}$$

$$=\frac{-6+23i-21(-1)}{9-49(-1)}$$

$$=\frac{15+23i}{58}$$

$$=\frac{15}{58}+\frac{23}{58}i$$

Example 4. Find the sum, product, and difference of $5-3i$ and $-2+8i$

Solution:

$(5-3i)+(-2+8i)=3+5i$

$(5-3i)-(-2+8i)=5+2-3i-8i=7-11i$

$$\begin{aligned}(5-3i)(-2+8i)&=-10+40i+6i-24i^2\\&=-10+40i-24(-1)\\&=14+46i\end{aligned}$$

Example 5. Find the absolute value of 4+5i.

Solution:

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

Here,

$a = 4, b = 5$

$$\begin{aligned} \therefore |z| &= \sqrt{4^2 + 5^2} \\ &= \sqrt{16 + 25} \\ &= \sqrt{41} \end{aligned}$$

Example 6.

$a = 1 \quad b = -1 \quad c = 4$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4(1)(4)}}{2(1)} = \frac{1 \pm \sqrt{-15}}{2} = \frac{1 \pm \sqrt{15}i}{2}$$

Example 7.

Find the eigenvalues and eigenvectors for A.

$$\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \\ \begin{array}{cc} c & d \end{array} \end{array}$$

Short-cut: (Multiple Choice only)

$trA = 2 + 3 = 5$

$det\ A = 6 - 0 = 6$

$\lambda^2 - tr(A)\lambda + detA = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0 \qquad \lambda = 2, 3$ eigenvalues

$\lambda = 2$

Short-cut: (Multiple Choice only and only for distinct eigenvalues)

$eigenvector\ multiple\ of$

$$\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 0 \\ 2 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ \therefore must\ use\ the\ other\ form$$

$$\begin{bmatrix} \lambda - d \\ c \end{bmatrix} = \begin{bmatrix} 2 - 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \ \therefore non - zero\ multiples\ of\ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{or} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 3 \qquad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 0 \\ 3 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ \therefore non - zero\ multiples\ of\ \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So, the eigenvalue that corresponds to the eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is 2.

The answer is A.

Long Method:

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{bmatrix} = 0$$

$ad - bc = 0$
$(2-\lambda)(3-\lambda) - 0 = 0$
$(2-\lambda)(3-\lambda) = 0$
$\lambda = 2, 3$

$(A - \lambda I)\vec{v} = \vec{0}$ or you can also use A$\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0v_1 + v_2 = 0$ $\boxed{1}$
$v_1 + v_2 = 0$
$v_2 = -v_1$

$$\therefore \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \; or \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda = 3$

$$\begin{bmatrix} 2-3 & 0 \\ 1 & 3-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-v_1 + 0v_2 = 0$
$-v_1 = 0$
$v_1 = 0$

$\therefore$ let $v_2 = t$

Eigenvector is $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} \; or \; t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (any multiples). This is called the family of eigenvectors!

Example 8. **Short-cut: (Multiple Choice only)**

$tr(A) = 2 + 2 = 4$

$detA = ad - bc = 2(2) - (-1)(3) = 4 + 3 = 7$

characteristic polynomial $\lambda^2 - tr(A)\lambda + detA = 0$

$$\lambda^2 - 4\lambda + 7 = 0$$

$a = 1, \quad b = -4, \quad c = 7 \qquad \lambda = \frac{4\pm\sqrt{16-4(7)}}{2} \quad \therefore \lambda = \frac{4\pm\sqrt{-12}}{2}$

$$\lambda = \frac{4\pm\sqrt{4}\sqrt{-3}}{2} = \frac{4\pm2\sqrt{3}i}{2}$$

$$\lambda = 2 + \sqrt{3}i, \ 2 - \sqrt{3}i$$

Short-cut: (Multiple Choice only and only for distinct eigenvalues)

eigenvector $\lambda = 2 + \sqrt{3}i \qquad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -1 \\ 2 + \sqrt{3}i - 2 \end{bmatrix} = \begin{bmatrix} -1 \\ \sqrt{3}i \end{bmatrix}$

$\lambda = 2 - \sqrt{3}i \quad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -1 \\ 2 - \sqrt{3}i - 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -\sqrt{3}i \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3}i \end{bmatrix}$

Long-Method 1:

$\lambda = 2 + \sqrt{3}i$

$A\vec{v} = \lambda\vec{v}$

$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (2 + \sqrt{3}i)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$2v_1 - v_2 = (2 + \sqrt{3}i)v_1$

$3v_1 - 2v_2 = (2 + \sqrt{3}i)v_2$

$2v_1 - v_2 = 2v_1 + \sqrt{3}i\, v_1$

$2v_1 - 2v_1 - \sqrt{3}iv_1 = v_2$

$v_2 = -\sqrt{3}iv_1$

Let $v_1 = 1$

$v_2 = -\sqrt{3}i$

OR

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -\sqrt{3}v_1 \end{bmatrix} = v_1\begin{bmatrix} 1 \\ -\sqrt{3}i \end{bmatrix}$

The eigenvector is $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\sqrt{3}i \end{bmatrix}$ or in general, $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} t \\ -\sqrt{3}t \end{bmatrix}$

Long-Method 2:

$(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 2-\lambda & -1 \\ 3 & 2-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = 2 + \sqrt{3}i$

$$\begin{bmatrix} 2-(2+\sqrt{3}i) & -1 \\ 3 & 2-(2+\sqrt{3}i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\sqrt{3}i & -1 \\ 3 & -\sqrt{3}i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-\sqrt{3}i\, v_1 - v_2 = 0 \ \boxed{1}$

From $\boxed{1}$ $\quad v_2 = -\sqrt{3}i\, v_1$

Let $v_1 = 1$

$v_2 = -\sqrt{3}i$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\sqrt{3}i \end{bmatrix}$$

QUESTION: $\begin{bmatrix} i \\ \sqrt{3} \end{bmatrix} \times i = \begin{bmatrix} i^2 \\ \sqrt{3}i \end{bmatrix} = \begin{bmatrix} -1 \\ \sqrt{3}i \end{bmatrix} = \vec{v}$

$\therefore$ $\vec{v}$ *and* $\vec{w}$ are equal

$$A\overrightarrow{W} = A\vec{v} = \lambda\vec{v} = [2+\sqrt{3}i]\begin{bmatrix} i \\ \sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2i + \sqrt{3}i^2 \\ 2\sqrt{3} + 3i \end{bmatrix} = \begin{bmatrix} 2i - \sqrt{3} \\ 2\sqrt{3} + 3i \end{bmatrix}$$

Example 9.

$$A = \begin{bmatrix} \overbrace{2}^{a} & \overbrace{-4}^{b} \\ \underbrace{5}_{c} & \underbrace{5}_{d} \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2 & -4 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 2-\lambda & -4 \\ 5 & 5-\lambda \end{bmatrix} = 0$$
$$ad - bc = 0$$
$$(2-\lambda)(5-\lambda) + 20 = 0$$

$$10 - 7\lambda + \lambda^2 + 20 = 0$$

$$\lambda^2 - 7\lambda + 30 = 0$$

$$\lambda = \frac{7 \pm \sqrt{49 - 4(1)(30)}}{2(1)}$$

$$\lambda = \frac{7 \pm \sqrt{71}i}{2}$$

$$\lambda = \frac{7 + \sqrt{71}i}{2}, \lambda = \frac{7 - \sqrt{71}i}{2}$$

Long Method 1:

$$A = \begin{bmatrix} 2 & -4 \\ 5 & 5 \end{bmatrix}$$
Eigenvalues $\lambda = \frac{7}{2} \pm \frac{\sqrt{71}}{2}i$

$2v_1 - 4v_2 = \lambda v_1$

$5v_1 + 5v_2 = \lambda v_2$

Substitute $\lambda = \frac{7}{2} + \frac{\sqrt{71}}{2}i$

$2v_1 - \lambda v_1 = 4v_2$

$\boxed{4v_2 = (2-\lambda)v_1}$

$\therefore 4v_2 = [2 - \left(\frac{7}{2} + \frac{\sqrt{71}}{2}i\right) v_1$ substitute

Let $v_1 = 1$

$4v_2 = \left[\frac{10}{2} - \frac{7}{2} - \frac{\sqrt{71}}{2}i\right]$ which means $4v_2 = \left[\frac{3}{2} - \frac{\sqrt{71}}{2}i\right]$

$v_2 = \frac{-3}{8} - \frac{\sqrt{71}}{8}i$ ****Note: In prep, when I divided by 4, I forgot to include the 2 originally on the bottom, so it should be over 8, not 4 in the final answer!**

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{8} - \frac{\sqrt{71}}{8}i \end{bmatrix}$$

Long Method 2:

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$\left(\begin{bmatrix} 2 & -4 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & -4 \\ 5 & 5-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda = \frac{7 + \sqrt{71}i}{2}$$

$$\begin{bmatrix} 2 - \left(\frac{7 + \sqrt{71}i}{2}\right) & -4 \\ 5 & 5 - \left(\frac{7 + \sqrt{71}i}{2}\right) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-3 - \sqrt{71}i}{2} & -4 \\ 5 & \frac{3 - \sqrt{71}i}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$5v_1 + \frac{3 - \sqrt{71}i}{2}v_2 = 0$$

$$5v_1 = -\frac{\left(3 - \sqrt{71}i\right)}{2}v_2$$

$$5v_1 = \frac{-3 + \sqrt{71}i}{2}v_2$$

Let $v_2 = 2, v_1 = -\frac{3}{5} + \frac{\sqrt{71}}{5}i$

∴ the eigenvector is $\begin{bmatrix} -\frac{3}{5} + \frac{\sqrt{71}}{5}i \\ 2 \end{bmatrix}$

NOTE: These eigenvectors are the same in both methods, but it takes some algebra to convert between them

They are equivalent:

$$\begin{bmatrix} \frac{-3}{5} + \frac{\sqrt{71}i}{5} \\ 2 \end{bmatrix} \div 2 = \begin{bmatrix} \frac{-3}{10} + \frac{\sqrt{71}i}{10} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-3}{10} + \frac{\sqrt{71}i}{10} \\ 1 \end{bmatrix} \begin{matrix} \times \left(\frac{-3}{8} - \frac{\sqrt{71}i}{8}\right) \\ \times \left(\frac{-3}{8} - \frac{\sqrt{71}i}{8}\right) \end{matrix}$$

$$= \begin{bmatrix} \frac{9}{80} + \frac{3\sqrt{71}i}{80} - \frac{3\sqrt{71}i}{80} - \frac{71i^2}{80} \\ \frac{-3}{8} - \frac{\sqrt{71}i}{8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{80} + \frac{71}{80} \\ \frac{-3}{8} - \frac{\sqrt{71}i}{8} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{8} - \frac{\sqrt{71}i}{8} \end{bmatrix} = \textbf{same as long method 1}$$

<u>**Example 10.**</u> $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = 0.$

$ad - bc = 0$
$(1-\lambda)^2 - 0 = 0$
$(1-\lambda)^2 = 0$
$\lambda = 1, 1$

(only one eigenvalue)

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$v_1 + 0 = v_1$
$0 + v_2 = v_2$

$v_1 = v_1$
$v_2 = v_2$

This means we can choose $v_1$and v_2 to be any numbers as long as the vector isn't the zero vector

$v_1 = s, \;\; v_2 = t$
$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$ where s, t ∈ R.

Practice Exam Questions

J1.

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ A = & \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} \\ & \begin{array}{cc} c & d \end{array} \end{array}$$

Short-cut: (Multiple Choice only)

$tr(A) = 6 + (-4) = 2$

$detA = 6(-4) - 16(-1) = -24 + 16 \qquad detA = -8$

$\lambda^2 - 2\lambda - 8 = 0$

$(\lambda - 4)(\lambda + 2) = 0 \qquad \lambda = 4, -2 \ eigenvalues$

$\lambda = 4$ eigenvector multiple of $\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 16 \\ 4 - 6 \end{bmatrix} = \begin{bmatrix} 16 \\ -2 \end{bmatrix}$

Short-cut: (Multiple Choice only and only for distinct eigenvalues)

$$\lambda = 4 \ eigenvector = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 16 \\ 4 - 6 \end{bmatrix} = \begin{bmatrix} 16 \\ -2 \end{bmatrix} \ or \ any \ multiple$$

Long method

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 6 - \lambda & 16 \\ -1 & -4 - \lambda \end{bmatrix} = 0$$

$ad - bc = 0$

$(6 - \lambda)(-4 - \lambda) - 16(-1) = 0$

$-24 - 6\lambda + 4\lambda + \lambda^2 + 16 = 0$

$\lambda^2 - 2\lambda - 8 = 0$

$(\lambda - 4)(\lambda + 2) = 0$

$\lambda = 4, -2$

One way…

$A\vec{v} = \lambda\vec{v}$

$\lambda = 4$

$$\begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$6v_1 + 16v_2 = 4v_1 \quad \boxed{1}$

$-v_1 - 4v_2 = 4v_2 \quad \boxed{2}$

From $\boxed{1}$ $\quad 16v_2 = -2v_1$

$v_1 = -8v_2 \quad$ Let $v_2 = 1$

$\vec{v} = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$ same as $\begin{bmatrix} 16 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$

Eigenvector
$\lambda = 4$
Another way…
$(A - \lambda I)(\vec{v}) = \vec{0}$

$$\begin{bmatrix} 6-\lambda & 16 \\ -1 & -4-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6-4 & 16 \\ -1 & -4-4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 16 \\ -1 & -8 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2v_1 + 16v_2 \quad \boxed{1}$
$-v_1 - 8v_2 = 0 \, \boxed{2} \rightarrow \; v_1 = 8v_2$

Let $v_2 = 1$
$v_1 = -8$

$\therefore \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$ or any multiple. The family of eigenvectors is t(-8,1), i.e. any multiple of this vector.

J2.

$$A = \begin{array}{cc} a & b \\ \left[\begin{matrix} -4 & 2 \\ 3 & -5 \end{matrix}\right. & \left.\right] \\ c & d \end{array}$$

Short-cut: (Multiple Choice only)

$tr(A) =$ -4+(-5)= -9

$Det(A) = -4(-5) - 2(3) = 20 - 6 = 14$

$\lambda^2 + 9\lambda + 14 = 0$

$(\lambda + 7)(\lambda + 2) = 0 \qquad \lambda = -7, -2 \; eigenvalues$

$\lambda = -2$

Short-cut: (Multiple Choice only and only for distinct eigenvalues)

$$eigenvector\ multiple\ of\ \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ -2-(-4) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \text{ or} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Long Method: $\lambda = -2$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} -4 & 2 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -2\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$-4v_1 + 2v_2 = -2v_1 \quad \boxed{1}$

$3v_1 - 5v_2 = -2v_2 \quad \boxed{2}$

From $\boxed{1}$ $2v_2 = -2v_1 + 4v_1$

$2v_2 = 2v_1$

$v_2 = v_1$

$\therefore$ let $v_2 = 1$

$\therefore \ \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Another way…

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} -4 & 2 \\ 3 & -5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} -4-\lambda & 2 \\ 3 & -5-\lambda \end{bmatrix} = 0$$

$ad - bc = 0$

$(-4-\lambda)(-5-\lambda) - 2(3) = 0$

$+20 + 4\lambda + 5\lambda + \lambda^2 - 6 = 0$

$\lambda^2 - 9\lambda + 14 = 0$

$(\lambda + 7)(\lambda + 2) = 0$

$\lambda = -7, -2$

Eigenvector

$\lambda = -2$

$$\begin{bmatrix} -4+2 & 2 \\ 3 & -5+2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2v_1 + 2v_2 = 0$

$2v_2 = 2v_1$

$v_2 = v_1$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector. The family of this eigenvector is all multiples of this vector, ie. t(1,1).

J3. $\det(A-\lambda I)=0$ $\quad\therefore \det\left(\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)=0$. In this question, we need the factor theorem.

$$det\begin{bmatrix}-\lambda&1&1\\1&-\lambda&1\\1&1&-\lambda\end{bmatrix}=0 \qquad right=-\lambda^3+1+1$$

$$left=-3\lambda$$

$$detA=right-left$$

$$0=-\lambda^3+2-(-3\lambda)$$

$$0=-\lambda^3+2+3\lambda \quad or \quad \lambda^3-3\lambda-2=0$$

$$let\ f(\lambda)=\lambda^3-3\lambda-2 \qquad f(-1)=-1+3-2=0$$

$$(\lambda+1) is\ a\ factor \qquad \begin{array}{c|cccc} -1 & 1 & 0 & -3 & -2 \\ & \downarrow & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & 0 \ \ R \end{array}$$

$$\therefore (\lambda+1)(\lambda^2-\lambda-2)=0$$

$$(\lambda+1)(\lambda-2)(\lambda+1)=0 \qquad \lambda=-1,-1,2$$

$\therefore \lambda=-1$ $\ find\ eigenvector$

$$A\vec{v}=\lambda\vec{v}$$

$$\left(\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=-1\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

$$\left.\begin{aligned}v_2+v_3&=-v_1\\v_1+v_3&=-v_2\\v_1+v_2&=-v_3\end{aligned}\right\} solve\ system \text{ (all equations are identical)}$$

$From\ equation\ (1) \qquad -v_1=v_2+v_3$

$$v_1=-v_2-v_3$$

The eigenvector is $\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}-v_2-v_3\\v_2\\v_3\end{bmatrix}=v_2\begin{bmatrix}-1\\1\\0\end{bmatrix}+\ v_3\begin{bmatrix}-1\\0\\1\end{bmatrix}$

↘ ↙

eigenvectors

If you want to find the other one:

$$\lambda = 2 \qquad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 2\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$v_2 + v_3 = 2v_1$ $\boxed{1}$ which is the same as $2v_1 - v_3 = v_2$

$v_1 + v_3 = 2v_2$ $\boxed{2}$

$v_1 + v_2 = 2v_3$ $\boxed{3}$

$\boxed{1} + \boxed{2}$ $3v_1 = 3v_2$ and we get: $v_1 = v_2$

From $\boxed{3}$ $v_1 + v_2 = 2v_3$ becomes $v_1 + v_1 = 2v_3$ or $2v_1 = 2v_3$ and $v_1 = v_3$

The vector is $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1 \\ v_1 \end{bmatrix} = v_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and let $v_1 = 1$

$$\therefore (1,1,1) is\ a\ vector$$

J4.

$$A = \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix} \\ \begin{array}{cc} c & d \end{array} \end{array}$$

Short-cut: (Multiple Choice only)

$tr(A) = 7 + 3 = 10$

$detA = ad - bc$

$= 7(3) - (-1)(4)$

$= 21 + 4 \quad = 25$

$\lambda^2 - tr(A)\lambda + detA = 0$

$\lambda^2 - 10\lambda + 25 = 0$

$(\lambda - 5)(\lambda - 5) = 0 \qquad \lambda = 5,\ 5$ **** Don't use short cut to find your eigenvectors, as you will only obtain one eigenvector, even if there are really 2!!**

Long method $\lambda = 5$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 5\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$7v_1 - v_2 = 5v_1 \quad \boxed{1}$

$4v_1 + 3v_2 = 5v_2 \quad \boxed{2}$

From $\boxed{1}$ $2v_1 = v_2$

$v_2 = 2v_1$

let $v_1 = 1$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the eigenvector. The family of this eigenvector is all multiples of it, .ie. t(1,2).

J5. $A\vec{v} = \lambda\vec{v}$ *to find eigenvectors*

$$\lambda = 2 \quad \begin{bmatrix} 2 & 1 & -2 \\ -3 & 0 & 4 \\ -2 & -1 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 2\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$\boxed{1}$ $2v_1 + v_2 - 2v_3 = 2v_1 \quad \therefore v_2 = 2v_3$

$\boxed{2}$ $-3v_1 + 4v_3 = 2v_2$

Subst. from $\boxed{1}$ $-3v_1 + 4v_3 = 2(2v_3)$

$-3v_1 + 4v_3 = 4v_3$

$-3v_1 = 0 \quad v_1 = 0$

$$let\ t = v_3 \quad \therefore vector \begin{bmatrix} 0 \\ 2t \\ t \end{bmatrix} or\ any\ multiple$$

$$let\ t = 1, you\ get \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ *B is the solution*

NOTE: [0 0 0] is NEVER an eigenvector

J6. **Short-cut: (Multiple Choice only)**

$tr(A) = 2 + 1 = 3$

$detA = ad - bc = 2(1) - 2(1) = 0$

$\lambda^2 - tr(A) + detA = 0$

$\lambda^2 - 3\lambda = 0$

$\lambda(\lambda - 3) = 0$

$\lambda = 0{,}3 \quad \therefore$ *E is the answer*

Long Method:

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 2-\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix} = 0$$

$ad - bc = 0$

$(2 - \lambda)(1 - \lambda) - 2(1) = 0$

$\lambda^2 - 3\lambda = 0$

$\lambda(\lambda - 3) = 0$

$\lambda = 0, 3$

J7. **Short-cut: (Multiple Choice only)**

$tr(A) = 1 + 3 = 4$

$detA = 1(3) - 0(2) = 3$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 3)(\lambda - 1) = 0$

$\lambda = 1,3$

$\therefore E\ is\ the\ solution$

Long method

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{bmatrix} = 0$$

$ad - bc = 0$

$(1 - \lambda)(3 - \lambda) - 0 = 0$

$(1 - \lambda)(3 - \lambda) = 0$

$\lambda = 1, 3$

J8. $A = \lambda^2 - 8\lambda + 12 = 0$

$(\lambda - 2)(\lambda - 6) = 0$

$\lambda = 2{,}6$

Short-cut: (Multiple Choice only and only for distinct eigenvalues)

$eigenvector \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 3 \\ 2 - 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ or any multiple

$\therefore v_1$

$B = \lambda^2 - 8\lambda + 12 = 0$

$\lambda = 2{,}6\ eigenvector \quad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 3 \\ 2 - 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$\therefore v_2$

$C = \begin{bmatrix} 4 & 2 \\ -2 & 0 \end{bmatrix}$

$\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ 2 - 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} or\ any\ multiple$

$\therefore v_3$

Long method

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 2 - \lambda & 3 \\ 0 & 6 - \lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(2 - \lambda)(6 - \lambda) - 0 = 0$

$\lambda = 2, 6$

$\det(B - \lambda I) = 0$

$\det\left(\begin{bmatrix} 6 & 3 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 6 - \lambda & 3 \\ 0 & 2 - \lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(6 - \lambda)(2 - \lambda) - 0 = 0$

$(6 - \lambda)(2 - \lambda) = 0$

$\lambda = 2, 6$

J9. **Short-cut: (Multiple Choice only)**

$$detA = ad - bc = 0(4) - (-1)(0) = 0$$
$$\lambda^2 - 4\lambda = 0$$
$$\lambda(\lambda - 4) = 0$$
$$\lambda = 0{,}4$$
$$\therefore A \text{ is the answer}$$

J10. a) **Short-cut: (Multiple Choice only)**

$$tr(A) = a + d = 1 + 1 = 2$$
$$detA = ad - bc = 1(1) - 2(-1) = 3$$
$$\lambda^2 - tr(A)\lambda + detA = 0$$
$$\lambda^2 - 2\lambda + 3 = 0 \qquad a = 1 \quad b = -2 \quad c = 3$$

won't factor

$$\lambda = -\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{2\pm\sqrt{(-2)^2-4(1)(3)}}{2(1)} = \frac{2\pm\sqrt{4-12}}{2} = \frac{2\pm\sqrt{-8}}{2} = \frac{2\pm\sqrt{4}\sqrt{2}i}{2}$$

$$\lambda = \frac{2+2\sqrt{2}i}{2}, \frac{2-2\sqrt{2}i}{2} \quad or \quad 1+\sqrt{2}\ i \ \ 1-\sqrt{2}i$$

$$\lambda = 1+\sqrt{2}i \qquad\qquad A = \begin{array}{c} \begin{array}{cc} a & \ \ b \end{array} \\ \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ c & d \end{bmatrix} \end{array}$$

b) **Short-cut: (Multiple Choice only and only for distinct eigenvalues)**

Eigenvector $\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ 1+\sqrt{2}i - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2}i \end{bmatrix}$

Long method:

If $\lambda = 1+\sqrt{2}i \qquad \lambda = 1-\sqrt{2}i$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left(1+\sqrt{2}i\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$v_1 + 2v_2 = \left(1+\sqrt{2}i\right)v_1 \quad \boxed{1}$$

$$-v_1 + v_2 = \left(1+\sqrt{2}i\right)v_2 \quad \boxed{2}$$

$$\cancel{v_1} + 2v_2 = \cancel{v_1} + \sqrt{2}i\ v_1$$
$$-v_1 + \cancel{v_2} = \cancel{v_2} + \sqrt{2}i\ v_2$$

From $\boxed{1}$ $\quad 2v_2 = \sqrt{2}i\ v_1$

$$v_2 = \frac{\sqrt{2}i}{2}\ v_1$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2}i \end{bmatrix}$$

$\lambda = 1-\sqrt{2}i$

Eigenvector$=\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ 1-\sqrt{2}i - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -\sqrt{2}i \end{bmatrix}$

Long Method 1:

$\lambda = 1 - \sqrt{2}i$

$A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = 0$

$$\begin{bmatrix} 1-\lambda & 2 \\ -1 & 1-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\left(1-\sqrt{2}i\right) & 2 \\ -1 & 1-\left(1-\sqrt{2}i\right) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2i} & 2 \\ -1 & \sqrt{2i} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\sqrt{2i}\, v_1 + 2v_2 = 0$

$2v_2 = -\sqrt{2i}\, v_1$

$2v_2 = -\dfrac{\sqrt{2i}}{2}\, v_1$

Let $v_1 = 2$

$v_2 = -\sqrt{2i}$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -\sqrt{2i} \end{bmatrix}$$

Long Method 2:

If $\lambda = 1 + \sqrt{2}i \qquad \lambda = 1 - \sqrt{2}i$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left(1-\sqrt{2}i\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$v_1 + 2v_2 = \left(1-\sqrt{2}i\right)v_1 \quad \boxed{1}$

$-v_1 + v_2 = \left(1-\sqrt{2}i\right)v_2 \quad \boxed{2}$

From $\boxed{1}$

$v_1 + 2v_2 = v_1 - \sqrt{2}i\, v_1$

$2v_2 = -\sqrt{2}i\, v_1$

$v_2 = -\frac{\sqrt{2}i}{2}\, v_1$

Let $v_1 = 2$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -\sqrt{2}i \end{bmatrix}$$

J11. A= $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$,

Short-cut: (Multiple Choice only)

$tr(A) = 8$

$\det A = ad - bc = 5(3) - (-2)(1)$
$= 17$

$\lambda^2 - tr(A)\lambda + \det A = 0$
$\lambda^2 - 8\lambda + 17 = 0$

Doesn't factor
$\lambda^2 - 8\lambda + 16 = -17 + 16$
$(\lambda - 4)^2 = -1$
$\lambda - 4 = \pm\sqrt{-1}$
$\lambda - 4 = i, \quad \lambda - 4 = -i$

$\lambda = 4 + i, 4 - i$

Find the eigenvector:(Long Method)

$\lambda = 4 + i$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (4 + i)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$5v_1 - 2v_2 = 4v_1 + iv_1 \quad \boxed{1}$
$v_1 - iv_1 = 2v_2$
$v_1(1 - i) = 2v_2$
$v_2 = \frac{v_1(1 - i)}{2}$

Let $v_1 = 2 \;\; v_2 = 1 - i$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 - i \end{bmatrix}$$

Note: it could also be
$\begin{bmatrix} 2 \\ 1 - i \end{bmatrix} \begin{matrix} \times 1 + i \\ \times 1 + i \end{matrix}$

$\begin{bmatrix} 2 + 2i \\ 1 + i - i - i^2 \end{bmatrix}$
$= \begin{bmatrix} 2 + 2i \\ 1 + 1 \end{bmatrix} = \begin{bmatrix} 2 + 2i \\ 2 \end{bmatrix}$
$= 2\begin{bmatrix} 1 + i \\ 1 \end{bmatrix}$

$(A - \lambda I)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 4 + i$

$\begin{bmatrix} 5-(4+i) & -2 \\ 1 & 3-(4+i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(1-i)v_1 - 2v_2 = 0$

$(1-i)v_1 = 2v_2$

$v_2 = \frac{(1-i)}{2}v_1$

Let $v_1 = 2$
$v_2 = 1 - i$
$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$

Note: it could also be

$\begin{bmatrix} 2 \\ 1-i \end{bmatrix}\begin{matrix} \times 1+i \\ \times 1+i \end{matrix}$
$\begin{bmatrix} 2+2i \\ 1+i-i-i^2 \end{bmatrix}$
$= \begin{bmatrix} 2+2i \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2+2i \\ 2 \end{bmatrix}$

$= 2\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$

J12. $A = \begin{bmatrix} 5 & -4 \\ 1 & 3 \end{bmatrix}$ Find eigenvalues and the e-vector for $a - bi$

Long method

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 5 & -4 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 5 - \lambda & -4 \\ 1 & 3 - \lambda \end{bmatrix} = 0$
$ad - bc = 0$
$(5 - \lambda)(3 - \lambda) + 4 = 0$

$15 - 5\lambda - 3\lambda + \lambda^2 + 4 = 0$

$\lambda^2 - 8\lambda + 19 = 0$
$\lambda^2 - 8\lambda = -19$
$\lambda^2 - 8\lambda + 16 = -19 + 16$
$(\lambda - 4)^2 = -3$
$\lambda - 4 = \pm\sqrt{-3}$
$\lambda = 4 \pm \sqrt{-3}$

$(A - \lambda I)\vec{v} = 0$

$\begin{bmatrix} 5 - \lambda & -4 \\ 1 & 3 - \lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 4 + \sqrt{-3}$
$\begin{bmatrix} 5 - (4 + \sqrt{-3}) & -4 \\ 1 & 3 - (4 + \sqrt{-3}) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 - \sqrt{-3}i & -4 \\ 1 & -1 - \sqrt{-3}i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_1 + (-1 - \sqrt{3}i)v_2 = 0$
$v_1 = -(-1 - \sqrt{3}i)v_2$
$v_1 = (1 + \sqrt{3}i)v_2$

Let $v_2 = 1$
$\therefore \begin{bmatrix} 1 + \sqrt{3}i \\ 1 \end{bmatrix}$ is the eigenvector

Complex Numbers

J13.

1. $(3+4i)+(10-2i)$

$= 13+2i$

2. $(4+7i)(-2-3i)$

$= -8-12i-14i-21i^2$

$= -8-26i-21(-1)$

$= 13-26i$

J14. Evaluate and express in form $a+bi$

$$\frac{4-2i}{1+3i}$$

$$\frac{(4-2i)}{(1+3i)}\frac{(1-3i)}{(1-3i)} = \frac{4-12i-2i+6i^2}{1-3i+3i-9i^2}$$

$$= \frac{4-14i+6(-1)}{1-9(-1)}$$

$$= \frac{10-14i}{10} = 1-\frac{7}{2}i$$

J15. Find the absolute value of $2+3\sqrt{2i}$

$|z| = \sqrt{a^2+b^2} \qquad a=2 \quad b=3\sqrt{2}$

$$= \sqrt{2^2+\left(3\sqrt{2}\right)^2}$$

$$= \sqrt{4+9(2)} = \sqrt{22}$$

K. Solving Systems of Recursion Models (Recurrence Equations)

Example 1

a) $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ $x_t = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t$

$$= c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}(2)^t + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}(3)^t$$

Find the unknowns

At $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $find\ c_1\ and\ c_2 \quad (t = 0)$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}(2)^0 + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}(3)^0$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \end{bmatrix}$$

$1 = -c_1 \quad \therefore c_1 = -1$

$1 = c_1 + c_2$

$1 = -1 + c_2 \quad c_2 = 2$

$x_t = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t$

$$\therefore \overrightarrow{x_t} = \left[-1\begin{bmatrix} -1 \\ 1 \end{bmatrix}(2)^t + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}(3)^t\right]$$

b) $let\ t = 3$

$$\overrightarrow{x_3} = \left(-1\begin{bmatrix} -1 \\ 1 \end{bmatrix}(2)^3 + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}(3)^3\right)$$

$$= -8\begin{bmatrix} -1 \\ 1 \end{bmatrix}(2)^3 + 54\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 54 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 8 \\ 46 \end{bmatrix}$$

c) as x approaches infinity, $\overrightarrow{x_t} = \left[-1\begin{bmatrix} -1 \\ 1 \end{bmatrix}(2)^t + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}(3)^t\right]$approaches infinity as both $(2)^t\ and (3)^t$will approach infinity.

System of Recursion Models

Example 3. Consider the system of recursion equations:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

If $x_1(0) = 3, x_2(0) = -5$, then find the solution for $x(t)$.

$tr(A) = 4 + (-7) = -3$

$detA = ad - bc = 4(-7) - (-3)(6) = -10$

$\lambda^2 + 3\lambda - 10 = 0$

$(\lambda + 5)(\lambda - 2) = 0$

$\lambda = -5,\ \ 2$

$$\begin{array}{cc} a & b \\ \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \\ c & d \end{array}$$

$$\vec{v}_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -3 \\ -5-4 \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -3 \\ 2-4 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$\overrightarrow{x_t} = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t$

$\overrightarrow{x_t} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} (-5)^t + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} (2)^t$

At $t = 0$ $\quad c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$1c_1 + 3c_2 = 3 \ \boxed{1} \quad (x - 3)$

$3c_1 + 2c_2 = -5 \ \ \boxed{2}$

$-3c_1 - 9c_2 = -9$

$3c_1 + 2c_2 = -5$

$Add \quad -7c_2 = -14$

$c_2 = 2$

$c_1 + 3(2) = 3$

$c_1 = 3 - 6$

$c_1 = -3$

$c_1 = -3, \quad c_2 = 2$

$\therefore \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t$

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} (-5)^t + 2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} (2)^t$

$x_1(t) = -3(1)(-5)^t + 2(3)(2)^t = -3(-5)^t + 6(2)^t$

$x_2(t) = -3(3)(-5)^t + 2(2)(2)^t = -9(-5)^t + 4(2)^t$

b) Find $\vec{x}(20)$

$\boldsymbol{\vec{x}(t) = -3\begin{bmatrix}1\\3\end{bmatrix}(-5)^t + 2\begin{bmatrix}3\\2\end{bmatrix}(2)^t}$ **substitute t=20 into this equation:**

$$\boldsymbol{\vec{x}(20) = -3\begin{bmatrix}1\\3\end{bmatrix}(-5)^{20} + 2\begin{bmatrix}3\\2\end{bmatrix}(2)^{20}}$$

You leave it like this without the use of a calculator!

c) $\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = -3\begin{bmatrix}1\\3\end{bmatrix}(-5)^t + 2\begin{bmatrix}3\\2\end{bmatrix}(2)^t$

The geometric rate of increase of the population is 5 since that is the largest eigenvalue in absolute value. Since 5 >1, it is an increasing population, rather than a shrinking one.

d) The eigenvector associated with the eigenvalue of 5 is $\begin{bmatrix}1\\3\end{bmatrix}$, in the long-run we would expect 1 juvenile for every 3 adults.

e) Since the dominant eigenvector that corresponds to the dominant eigenvalue is $\vec{v}_1 = \begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$, the long-term behaviour as a ratio will be $\frac{v_1}{v_2} = \frac{1}{3} = 0.33$.

Practice Exam Questions

K1. $find\ eigenvalues\ \ trA = -9$

$$detA = 20 - 6 = 14$$

$$\lambda^2 + 9\lambda + 14 = 0$$

$$(\lambda + 2)(\lambda + 7) = 0$$

$$\lambda = -2, -7$$

$$\lambda = -2\ eigenvector \quad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ -2 + 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}\ or\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -7\ eigenvector \quad \begin{bmatrix} 2 \\ -7 + 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

a)

$$\overrightarrow{x_t} = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t \qquad \therefore t = 0$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (-2)^0 + c_2 \begin{bmatrix} 2 \\ -3 \end{bmatrix} (-7)^0]$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ -3c_2 \end{bmatrix}$$

$$1 = c_1 + 2c_2$$

$$-2 = c_1 - 3c_2$$

$$-1 = 5c_2 \qquad c_2 = -\frac{1}{5}$$

$$1 = c_1 + 2(-\frac{1}{5})$$

$$1 = c_1 - \frac{2}{5} \qquad c_1 = \frac{7}{5}$$

$$\overrightarrow{x_t} = \left(\frac{7}{5}\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} (-2)^t + \left(\frac{-1}{5}\right) \begin{bmatrix} 2 \\ -3 \end{bmatrix} (-7)^t$$

b) $t = 3$

$$\overrightarrow{x_3} = \left(\frac{7}{5}\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} (-2)^3 + \left(\frac{-1}{5}\right) \begin{bmatrix} 2 \\ -3 \end{bmatrix} (-7)^3$$

$$= -\frac{56}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{343}{5} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} -\dfrac{56}{5} \\ -\dfrac{56}{5} \end{bmatrix} + \begin{bmatrix} \dfrac{686}{5} \\ -\dfrac{1029}{5} \end{bmatrix}.\ So, \overrightarrow{x_3} = \begin{bmatrix} 126 \\ -217 \end{bmatrix}$$

K2.

$$A = \begin{array}{c} \begin{matrix} a & \;\; b \end{matrix} \\ \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ c & d \end{bmatrix} \end{array} \qquad trA = 2 \qquad detA = 1 - 4 = -3$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0 \quad \lambda = 3, -1$$

$$\lambda = 3\ eigenvector \quad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ 3 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -1\ eigenvector \quad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ -1 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\overrightarrow{x_t} = c_1 \overrightarrow{v_1} (\lambda_1)^t + c_2 \overrightarrow{v_2} (\lambda_2)^t \qquad \therefore \overrightarrow{x_t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (3)^t + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} (-1)^t$$

$$x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad at\ t = 0 \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (3)^0 + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} (-1)^0$$

$$\begin{array}{rcl} 1 & = & c_1 + c_2 \quad \boxed{1} \\ 2 & = & c_1 - c_2 \quad \boxed{2} \\ \hline 3 & = & 2c_1 \end{array} \qquad c_1 = \frac{3}{2} \qquad 1 = \frac{3}{2} + c_2$$

$$c_2 = 1 - \frac{3}{2} = -\frac{1}{2}$$

$$\overrightarrow{x_t} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} (3)^t + \left(\frac{-1}{2}\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} (-1)^t$$

$$\overrightarrow{x_2} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} (3)^2 + \left(\frac{-1}{2}\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} (-1)^2$$

$$= \frac{3}{2}(9) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \pm \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{27}{2} \\ \frac{27}{2} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 13 \\ 14 \end{bmatrix}$$

K3. Since A is primitive (all entries are positive), we know there is an eigenvalue that is real and positive and that this eigenvalue is greater in magnitude than all other eigenvalues

$$\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ c & d \end{bmatrix} \end{array} \qquad tr(A) = 4 \qquad detA = 3$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda = 3,1$$

Since 3 is the largest eigenvalue in magnitude i.e. absolute value, the growth will be by a factor of 3

Eigenvectors are $\lambda = 1$ eigenvector $\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 1 \\ 1 - 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

And $\lambda = 3$ eigenvector $\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 1 \\ 3 - 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

a) as t tends to infinity, n_t will eventually grow in magnitude by a factor of $\lambda_1 = 3$, each time step, regardless of the initial vector n_0 (the dominant eigenvector in absolute value)

b) as t tends to infinity, n_t will eventually point in the direction of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, regardless of the initial vector n_0

K4. A) 0.5 is the number of new offspring produced by young ducks

0.25 is the survival of young ducks to become mature ducks

1.5 is the mature ducks producing new young ducks

0.90 is the mature ducks surviving to the next year

$$B)\ A = \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 0.5 & 1.5 \\ 0.25 & 0.90 \end{bmatrix} \\ \begin{array}{cc} c & d \end{array} \end{array} \begin{array}{l} tr(A) = 1.4 \\ \det(A) = 0.075 \end{array}$$

$$\lambda^2 - tr(A)\lambda + detA = 0$$

$$\lambda^2 - 1.4\lambda + 0.075 = 0$$

$$\lambda = \frac{1.4 \pm \sqrt{1.4^2 - 4(1)(0.075)}}{2(1)}$$

$$= \frac{1.4 \pm \sqrt{1.66}}{2}$$

$$= \frac{1.4 + 1.29}{2}, \frac{1.4 - 1.29}{2} = 0.055$$

$$\uparrow 1.35$$

Dominant eigenvector (largest in absolute value)

$$v_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.35 - 0.5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.85 \end{bmatrix}$$

$$\text{Total } 1.5\ to\ .85 = 2.35$$

$$\frac{1.5}{2.35} = 0.638 \text{ or } 63.8\% \text{ young ducks}$$

$$\frac{0.85}{2.35} = 0.361 \text{ or } 36.1\% \text{ mature ducks}$$

C) $0.361x = 1000\ mature$

$x = 2770\ total\ ducks$

$2770 \times 0.638 = 1767\ young\ ducks$

K5. $\lambda_1 = 0.4$ (dominant e-value)

$$V_1 = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix} \begin{array}{l} {}^{2}/_{12} \times 1800 = 300 \\ {}^{7}/_{12} \times 1800 = 1050 \\ {}^{3}/_{12} \times 1800 = 450 \end{array}$$

Total 12

At step 400

Long term growth $\begin{bmatrix} {}^{2}/_{12} \\ {}^{7}/_{12} \\ {}^{3}/_{12} \end{bmatrix}$

It grows by a factor of λ_1 on each step

$$\therefore\ step\ 401\ would\ be\ \begin{bmatrix} 300 \\ 1050 \\ 450 \end{bmatrix} \times 0.4$$

$$= \begin{bmatrix} 120 \\ 420 \\ 180 \end{bmatrix} \begin{array}{l} \rightarrow stage\ 1 \\ \rightarrow stage\ 2 \\ \rightarrow stage\ 3 \end{array}$$

K6. Consider the system of recursion equations:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

If $x_1(0) = 6,\ x_2(0) = -8$, Find the solution for x(t).

$$tr(A) = 5 + (-2) = 3$$
$$detA = 5(-2) - (-3)(2) = -10 + 6 = -4$$
$$\lambda^2 - 3\lambda - 4 = 0$$
$$(\lambda - 4)(\lambda + 1) = 0$$
$$\lambda = 4, \quad -1$$
$$\begin{array}{cc} a & b \\ \left[\begin{matrix} 5 \\ 2 \end{matrix}\right. & \left.\begin{matrix} -3 \\ -2 \end{matrix}\right] \\ c & d \end{array}$$

$$\vec{v}_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -3 \\ 4-5 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -3 \\ -1-5 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\overrightarrow{x_t} = c_1 \overrightarrow{v_1} (\lambda_1)^t + c_2 \overrightarrow{v_2} (\lambda_2)^t$$
$$\overrightarrow{x_t} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} (4)^t + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} (-1)^t$$

At $t = 0$ $\begin{bmatrix} 6 \\ -8 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$6 = 3c_1 + c_2 \ (x - 2)$$
$$-8 = c_1 + 2c_2$$

$$-12 = -6c_1 - 2c_2$$
$$\underline{-8 = c_1 + 2c_2}$$
$$Add \ -20 = -5c_1$$
$$c_1 = 4$$
$$-8 = 4 + 2c_2$$
$$-12 = 2c_2$$
$$c_2 = -6$$
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 4 \begin{bmatrix} 3 \\ 1 \end{bmatrix} (4)^t - 6 \begin{bmatrix} 1 \\ 2 \end{bmatrix} (-1)^t$$

b) The dominant eigenvalue in absolute value is 4, so the geometric rate of increase of the population is 4. Since 4>1, the population is growing.

c)The eigenvector associate with eigenvalue 4 is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ so we would expect to count 3 juveniles for every 1 adult in the long-run.

d) Since the dominant eigenvector that corresponds to the dominant eigenvalue is $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, the long-term behaviour as a ratio will be $\frac{v_1}{v_2} = \frac{3}{12} = 3$.

K7. Consider the system of recursion equations:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$If\ x_1(0) = 4,\ \ x_2(0) = 7,$ then find the solution for $x(t)$.

$$trA = 1 + 4 = 5$$

$$detA = 1(4) - 1(-2) = 4 + 2 = 6$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2, \quad 3$$

$$\begin{array}{cc} a & b \\ \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \\ c & d \end{array}$$

$$\vec{v}_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 1 \\ 2 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\overrightarrow{x_t} = c_1 \overrightarrow{v_1} (\lambda_1)^t + c_2 \overrightarrow{v_2} (\lambda_2)^t$$

$$\overrightarrow{x_t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (2)^t + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} (3)^t$$

At $t = 0$ $\begin{bmatrix} 4 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$4 = c_1 + c_2$$

$$7 = c_1 + 2c_2$$

$$Subtract \quad -3 = -c_2$$

$$c_2 = 3$$

$$4 = c_1 + 3$$

$$c_1 = 1$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (2)^t + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} (3)^t$$

K8. Find the long-term behaviour if $\vec{x}[t] = 2\begin{bmatrix}1\\1\end{bmatrix}\left(\frac{1}{2}\right)^t + 3\begin{bmatrix}1\\3\end{bmatrix}(4)^t$
as $\vec{t} \rightarrow \ \infty$

$\vec{x}[t] = 2\begin{bmatrix}1\\1\end{bmatrix}\left(\frac{1}{2}\right)^\infty + 3\begin{bmatrix}1\\3\end{bmatrix}(4)^\infty$
$\therefore \vec{x}[t] = 0 + \infty \rightarrow \ \infty$

K9. Find the long-term behaviour if $\vec{x}[t] = 2\begin{bmatrix}1\\1\end{bmatrix}\left(\frac{1}{2}\right)^t + 3\begin{bmatrix}1\\3\end{bmatrix}\left(\frac{1}{4}\right)^t$
As t→ ∞

both $\left(\frac{1}{2}\right)^t$ and $\left(\frac{1}{4}\right)^t \rightarrow 0$

$\therefore \vec{x}[t] \rightarrow 0$ as well

L. Systems of Differential Equations

Example 1.
Rewrite the second equation:

$u_1'(t) = -5u_1(t) + u_2(t)$
$u_2'(t) = \ u_1(t) + -2u_2(t) + 3$

$\vec{\boldsymbol{u}}(\boldsymbol{t}) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ $\vec{\boldsymbol{\alpha}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ **and** $A = \begin{bmatrix} -5 & 1 \\ 1 & -2 \end{bmatrix}$

$\overrightarrow{\boldsymbol{u'}}(\boldsymbol{t}) = \boldsymbol{A}\vec{\boldsymbol{u}}(\boldsymbol{t}) + \vec{\boldsymbol{\alpha}}$

$\overrightarrow{\boldsymbol{u'}}(\boldsymbol{t}) = \begin{bmatrix} -5 & 1 \\ 1 & -2 \end{bmatrix} \vec{\boldsymbol{u}}(\boldsymbol{t}) + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

Example 2.

$$\frac{\boldsymbol{d\vec{x}}}{\boldsymbol{dt}} = \boldsymbol{A\vec{x}(t)} + \vec{\boldsymbol{b}}$$

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{-0.2} & \mathbf{3} \\ \mathbf{5} & \mathbf{2} & \mathbf{-4} \\ \mathbf{-4} & \mathbf{6} & \mathbf{0} \end{bmatrix} \quad \vec{\boldsymbol{b}} = \begin{bmatrix} \mathbf{0.4} \\ \mathbf{0} \\ \mathbf{0.8} \end{bmatrix}$$

$$\therefore \frac{\boldsymbol{d\vec{x}}}{\boldsymbol{dt}} = \begin{bmatrix} \mathbf{0} & \mathbf{-0.2} & \mathbf{3} \\ \mathbf{5} & \mathbf{-4} & \mathbf{2} \\ \mathbf{-4} & \mathbf{6} & \mathbf{0} \end{bmatrix} \boldsymbol{\vec{x}(t)} + \begin{bmatrix} \mathbf{0.4} \\ \mathbf{0} \\ \mathbf{0.8} \end{bmatrix} \quad \textbf{with } \boldsymbol{\vec{x}(0)} = \begin{bmatrix} \mathbf{4} \\ \mathbf{3} \\ \mathbf{-2} \end{bmatrix}$$

Example 3.

Verify $\vec{x}(t) = \begin{bmatrix} -2e^{4t} + 3e^{t} \\ 2e^{4t} + 6e^{t} \end{bmatrix}$ is a solution for $\frac{d\vec{x}}{dt} = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \vec{x}(t)$.

LS=$\frac{d\vec{x}}{dt} = \frac{d}{dt}\begin{bmatrix} -2e^{4t} + 3e^{t} \\ 2e^{4t} + 6e^{t} \end{bmatrix} = \begin{bmatrix} -8e^{4t} + 3e^{t} \\ 8e^{4t} + 6e^{t} \end{bmatrix}$

RS= $\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \vec{x}(t) = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -2e^{4t} + 3e^{t} \\ 2e^{4t} + 6e^{t} \end{bmatrix}$

$= \begin{bmatrix} -6e^{4t} + 9e^{t} - 2e^{4t} - 6e^{t} \\ 4e^{4t} - 6e^{t} + \ 4e^{4t} + 12e^{t} \end{bmatrix} = \begin{bmatrix} -8e^{4t} + 3e^{t} \\ 8e^{4t} + 6e^{t} \end{bmatrix}$=LS

Case 1. Real and Distinct Eigenvalues

Example 4. Determine the solution to $\overrightarrow{u'}(t) = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ with initial condition $\vec{u}(0) = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$.

The solution is:

$u(t) = c_1 e^{\lambda_1 t}\overrightarrow{v_1} + c_2 e^{\lambda_2 t}\overrightarrow{v_2}$

Find $c_1 \& c_2$

$$\vec{u}(0) = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

Let $t = 0$ $\begin{bmatrix} 1 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^0 + c_2 e^0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$1 = -c_1 + c_2 \quad \boxed{1}$

$8 = c_1 + 2c_2 \quad \boxed{2}$

ADD $\overline{9 = 3c_2}$

$c_2 = 3 \quad \textit{subst into} \boxed{2}$

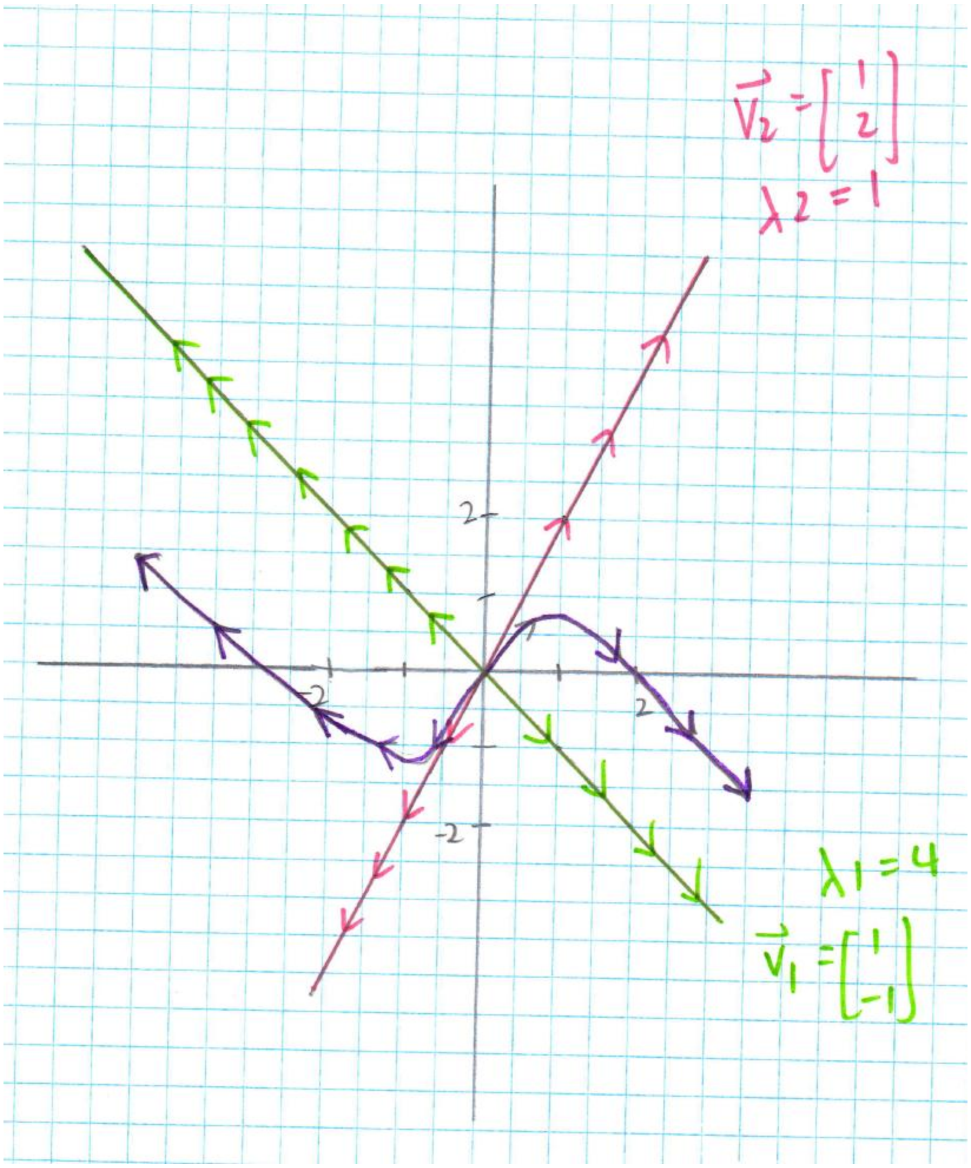
$8 = c_1 + 2(3)$

$8 = c_1 + 6$

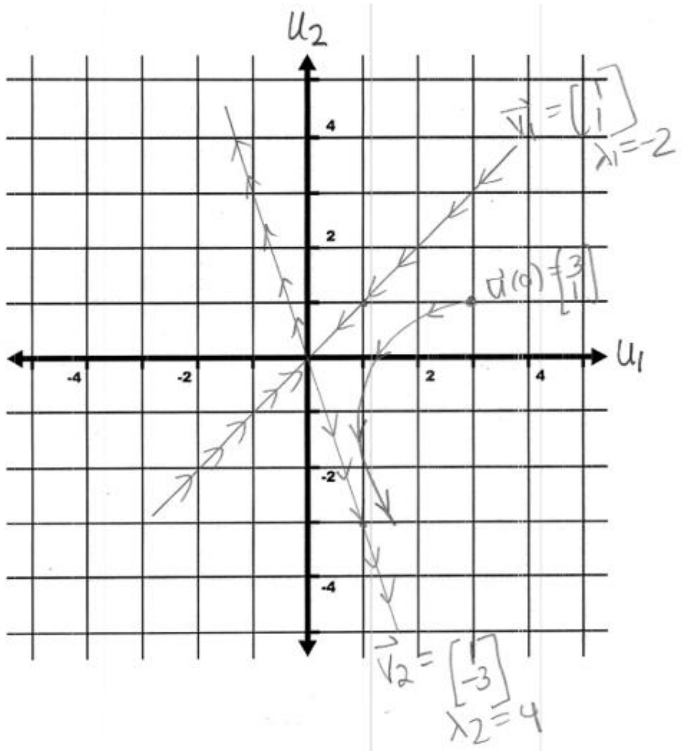
$c_1 = 2$

$\therefore \vec{u}(t) = 2e^{4t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3e^{t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Since $\lambda_1 = 4$ is the dominant eigenvalue, all solutions look like it's eigenvector vector $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as t→ ∞. Since both eigenvalues are positive, the arrows point away from the origin along BOTH eigenvectors. All solutions will start according to the smaller eigenvalue's eigenvector, $\overrightarrow{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and tend toward the dominant eigenvalue's eigenvector as t→ ∞.


$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$\lambda_2 = 1$
$\lambda_1 = 4$
$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
2
-2
2
-2

Example 5.



Case 2. Complex Eigenvalues

Example 6.

Solve $$\overrightarrow{u'}(t) = \begin{array}{c} \begin{matrix} a & b \end{matrix} \\ \begin{bmatrix}9 & -1\\5 & 7\end{bmatrix} \\ \begin{matrix} c & d \end{matrix} \end{array} \vec{u}(t) \; with \; \vec{u}(0) = \begin{bmatrix}5\\5\end{bmatrix}$$

1. Find eigenvalues

$$tr(A) = 9 + 7 = 16$$
$$detA = 63 + 5 = 68$$
$$\lambda^2 - 16\lambda + 68 = 0$$
$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\lambda = \frac{16 \pm \sqrt{256 - 4(1)(68)}}{2(1)}$$
$$\lambda = \frac{16 \pm \sqrt{-16}}{2}$$
$$= \frac{16 \pm 4i}{2}$$
$$= 8 \pm 2i$$

2. Find the corresponding eigenvectors

$$\begin{bmatrix}\lambda - d\\c\end{bmatrix} \; gives \; a \; vector \; with \text{ i } on \; top$$

$\lambda_1 = 8 + 2i$ $(positive \; imaginary \; part)$ $\quad \lambda_2 = 8 - 2i$

$$\begin{bmatrix}8 + 2i - 7\\5\end{bmatrix} \qquad \begin{bmatrix}8 - 2i - 7\\5\end{bmatrix}$$

$$\therefore \overrightarrow{v_1} = \begin{bmatrix}1 + 2i\\5\end{bmatrix} \qquad \overrightarrow{v_2} = \begin{bmatrix}1 - 2i\\5\end{bmatrix}$$

($\overrightarrow{v_1}$ has the <u>positive</u> imaginary part) (the conjugate of $\overrightarrow{v_1}$)

$$\overrightarrow{v_1} = \begin{bmatrix}1 + 2i\\5\end{bmatrix} = \underset{\vec{x}}{\underset{\uparrow}{\begin{bmatrix}1\\5\end{bmatrix}}} + i \underset{\vec{y}}{\underset{\uparrow}{\begin{bmatrix}2\\0\end{bmatrix}}}$$

3. $\vec{u}(t) = c_1 e^{at}(\cos(bt)\,\vec{x} - \sin(bt)\,\vec{y}) + c_2 e^{at}(\sin(bt)\,\vec{x} + \cos(bt)\,\vec{y})$

$\lambda_1 = 8 + 2i \qquad a = 8,\ b = 2$

$$\vec{u}(t) = c_1 e^{8t}\left(\cos(2t)\begin{bmatrix}1\\5\end{bmatrix} - \sin(2t)\begin{bmatrix}2\\0\end{bmatrix}\right) + c_2 e^{8t}(\sin(2t)\begin{bmatrix}1\\5\end{bmatrix} + \cos(2t)\begin{bmatrix}2\\0\end{bmatrix})$$

Substitute in initial condition:

$\vec{u}(0)=\begin{bmatrix}5\\5\end{bmatrix}$

$$\begin{bmatrix}5\\5\end{bmatrix} = c_1 e^0\left(1\begin{bmatrix}1\\5\end{bmatrix} - 0\right) + c_2 e^0(0 + 1\begin{bmatrix}2\\0\end{bmatrix})$$

$$\begin{bmatrix}5\\5\end{bmatrix} = c_1\begin{bmatrix}1\\5\end{bmatrix} + c_2\begin{bmatrix}2\\0\end{bmatrix}$$

$$\begin{bmatrix}5\\5\end{bmatrix} = c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2}$$

$$\begin{bmatrix}5\\5\end{bmatrix} = c_1\begin{bmatrix}1\\5\end{bmatrix} + c_2\begin{bmatrix}8\\0\end{bmatrix}$$

$5 = c_1 + 2c_2 \quad \boxed{1}$

$5 = 5c_1 + 0 \qquad c_1 = 1 \quad \boxed{2}$

Substitute $c = 1\ into$ $\boxed{1}$

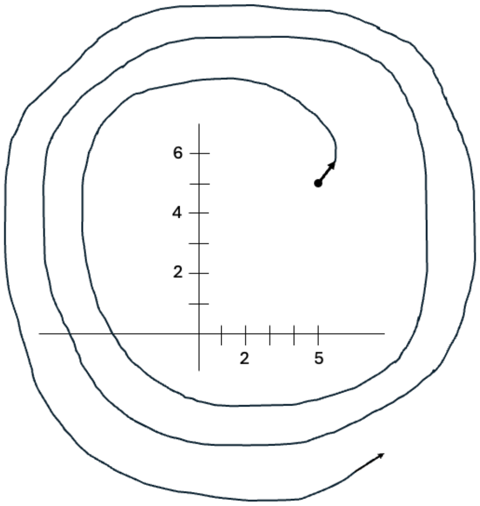
$5 = 1 + 2c_2$

$4 = 2c_2$

$c_2 = 2$

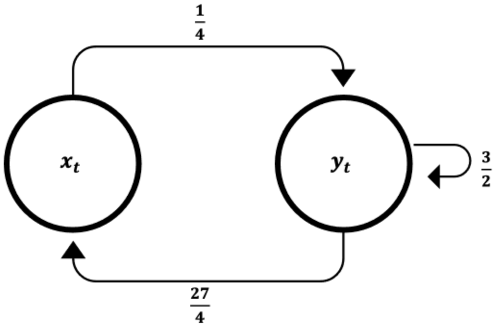
Final Answer:

$$\boldsymbol{\vec{u}(t) = e^{8t}\left(\cos(2t)\begin{bmatrix}1\\5\end{bmatrix} - \sin(2t)\begin{bmatrix}2\\0\end{bmatrix}\right) + 2e^{8t}(\sin(2t)\begin{bmatrix}1\\5\end{bmatrix} + \cos(2t)\begin{bmatrix}2\\0\end{bmatrix})}$$



Example 7.

a)



b)

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{array}{c|cc} & x_t & y_t \\ x_t & 0 & 27/4 \\ y_t & 1/4 & 3/2 \end{array} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix}$$

And $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 24 \\ 8 \end{bmatrix}$

Find eigenvalues A= $\begin{bmatrix} 0 & 27/4 \\ 1/4 & 3/2 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 0 & \frac{27}{4} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} -\lambda & \frac{27}{4} \\ \frac{1}{4} & \frac{3}{2} - \lambda \end{bmatrix} = 0$$

ad-bc=0

$$-\lambda\left(\frac{3}{2} - \lambda\right) - \frac{27}{4}\left(\frac{1}{4}\right) = 0$$

$\lambda^2 - \frac{3}{2}\lambda - \frac{27}{16} = 0$ x16

$$16\lambda^2 - 24\lambda - 27 = 0$$

$$(4\lambda - 9)(4\lambda + 3) = 0$$

$4\lambda = 9$ $4\lambda = -3$

$\lambda_1 = \frac{9}{4}$ $\lambda_2 = -\frac{3}{4}$

Eigenvectors

$$\begin{bmatrix} -\lambda & \frac{27}{4} \\ \frac{1}{4} & \frac{3}{2}-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda_1 = \frac{9}{4}$

$\frac{3}{2} - \frac{9}{4} = \frac{6}{4} - \frac{9}{4} = -\frac{3}{4}$

$$\begin{bmatrix} -\frac{9}{4} & \frac{27}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\frac{1}{4}v_1 - \frac{3}{4}v_2 = 0$
$\frac{1}{4}v_1 = \frac{3}{4}v_2$
$3v_2 = v_1$
Let $v_1 = 3$
$v_2 = 1$
$\therefore \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is the eigenvector

$\lambda_2 = -\frac{3}{4}$
$\frac{3}{2} - \left(-\frac{3}{4}\right) = \frac{6}{4} + \frac{3}{4} = \frac{9}{4}$

$$\begin{bmatrix} \frac{3}{4} & \frac{27}{4} \\ \frac{1}{4} & \frac{9}{4} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\frac{1}{4}v_1 + \frac{9}{4}v_2 = 0$
$\frac{1}{4}v_1 = \frac{-9}{4}v_2$
$v_1 = -9v_2$
$v_2 = -\frac{1}{9}-9v_1$
let $v_1 = 9$
$\therefore \begin{bmatrix} 9 \\ -1 \end{bmatrix}$ is the eigenvector

c) The eigenvalues are $\frac{-3}{4}, \frac{9}{4}$

$\therefore$ the largest one in absolute value is $\frac{9}{4}$. Therefore, the geometric rate of increase is $\frac{9}{4}$ because $\frac{9}{4} > 1$ we know the population is growing.

d) $\lambda_1 = \frac{9}{4}$ has eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\therefore$ in the long run we have 3 juveniles for every adult

Example 8.

a) $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix} = 0$

ad-bc=0

$(4-\lambda)(1-\lambda) - (-1)(2) = 0$

$4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\lambda = 2, 3$

$\lambda_1 = 2$

$[A - \lambda I]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 4-2 & -1 \\ 2 & 1-2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2v_1 - v_2 = 0$

$v_2 = 2v_1$

let $v_1 = 1$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the eigenvector

$\lambda_2 = 3$

$\begin{bmatrix} 4-3 & -1 \\ 2 & 1-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_1 - v_2 = 0$

$v_2 = v_1$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector

$u_1(0) = 0$
$u_2(0) = 3$

$\vec{u}(t) = C_1 v_1 e^{\lambda t} + C_2 v_2 e^{\lambda t}$
$\vec{u}(t) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$
$\begin{bmatrix} 0 \\ 3 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{0} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{0}$
$0 = C_1(1) + C_2(1) \rightarrow \; C_1 = -C_2$
$3 = 2C_1 + C_2$

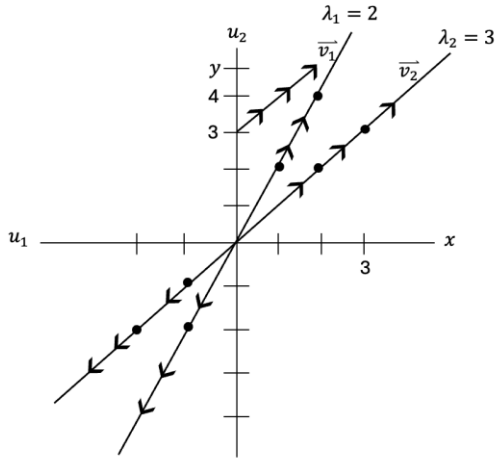
$\therefore 3 = 2(-C_2) + C_2$
$3 = -2C_2 + C_2$
$3 = -C_2$

$C_2 = -3$
$\therefore C_1 = -3$

$\vec{u}(t) = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} - 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$
$\overrightarrow{u_1}(t) = 3e^{2t} - 3e^{3t}$ and $\overrightarrow{u_2}(t) = 6e^{2t} - 3e^{3t}$

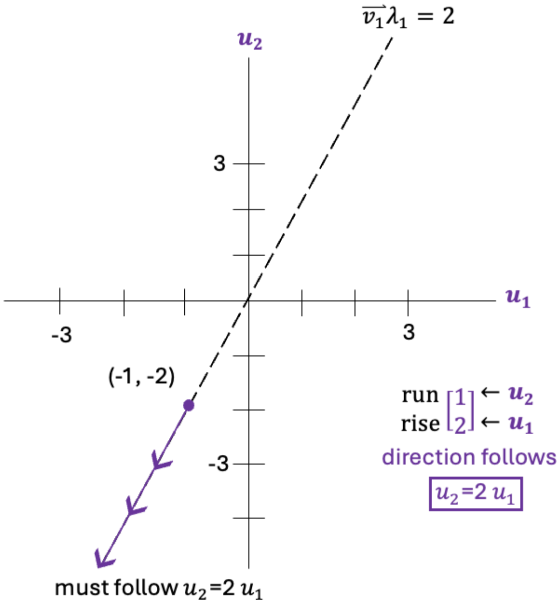
b) $\lambda_1 = 2$ (+=away from origin) $\lambda_2 = 3 = +=$ (away from origin)

$\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\overrightarrow{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

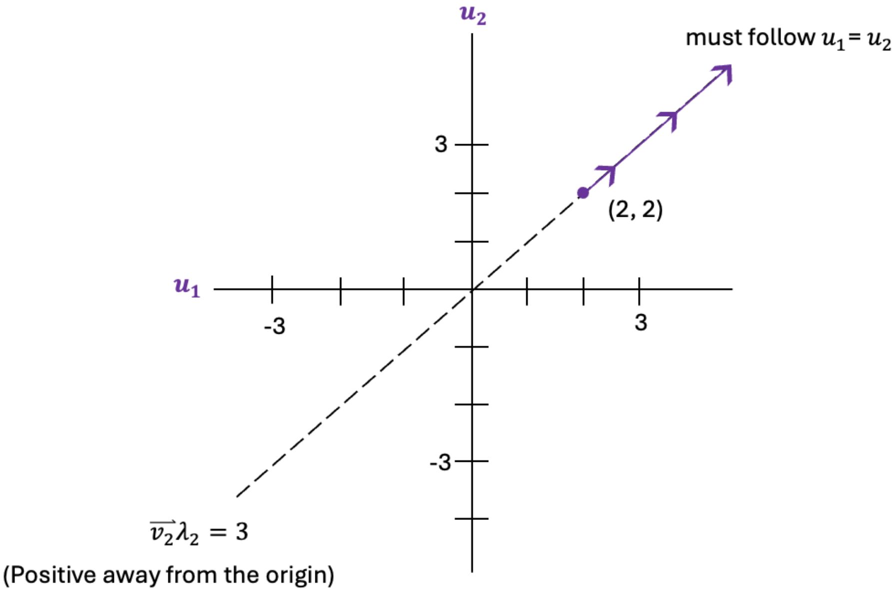


The trajectory of (0,3) can't cross $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
i.e. the line $u_2 = 2u_1$

c)
$u_1(0) = -1$
$u_2(0) = -2$
$\vec{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$



d)
$u_1(0) = 2$
$u_2(0) = 2$
$\vec{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$



Practice Exam Questions

L1. Determine the solution to $\vec{u}'(t) = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}$ with initial condition

$\vec{u}(0) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$tr(A) = 4 + (-7) = -3$

$detA = ad - bc = 4(-7) - (-3)(6)$

$= -28 + 18 = -10$

$\lambda^2 - tr(A)\lambda + detA = 0$

$\lambda^2 + 3\lambda - 10 = 0$

$(\lambda + 5)(\lambda - 2) = 0$

$\lambda = -5, 2$

$$\begin{array}{cc} a & b \\ \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \\ c & d \end{array}$$

$$v_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -3 \\ -5 - 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\overrightarrow{v_2} = \begin{bmatrix} -3 \\ 2 - 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The solution is

$$\vec{u}(t) = c_1 e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

t=0

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 e^0 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$3 = c_1 + 3c_2 \ \boxed{1} \ (x - 3)$

$-5 = 3c_1 + 2c_2 \ \boxed{2}$

$-9 = -3c_1 - 9c_2$

$-5 = 3c_1 + 2c_2$

$Add \ -14 = -7c_2$

$c_2 = 2 \quad into \boxed{1}$

$3 = c_1 + 3(2)$

$c_1 = 3 - 6 = -3$

$$\vec{u}(t) = -3e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The eigenvalues are real, one positive and one negative, so it is a saddle point.

L2. Determine the solution to $\vec{u}'(t) = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}$ with initial condition

$$\vec{u}(0) = \begin{bmatrix} 6 \\ -8 \end{bmatrix}$$

$$tr(A) = 5 + (-2) = 3$$

$$detA = 5(-2) - (-3)(2) = -10 + 6 = -4$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$$\lambda = 4, -1$$

$$\begin{array}{c} \begin{array}{cc} \text{a} & \text{b} \end{array} \\ \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix} \\ \begin{array}{cc} \text{c} & \text{d} \end{array} \end{array}$$

$$\overrightarrow{v_1} = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -3 \\ 4 - 5 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -3 \\ -1 - 5 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The solution is $u(t) = c_1 e^{4t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$t = 0 \quad \begin{bmatrix} 6 \\ -8 \end{bmatrix} = c_1 e^0 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$6 = 3c_1 + c_2 \quad \boxed{1} \quad \times -2$$

$$-8 = c_1 + 2c_2 \quad \boxed{2}$$

$$\begin{array}{r} -12 = -6c_1 - 2c_2 \\ -8 = c_1 + 2c_2 \\ \hline ADD \quad -20 = -5c_1 \\ c_1 = 4 \end{array}$$

Sub into $\boxed{1}$ $\quad 6 = 3(4) + c_2$

$$c_2 = 6 - 12 = -6$$

$$\vec{u}(t) = 4e^{4t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} - 6e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

L3. Determine the solution to $\vec{u}'(t) = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$ with initial condition $\vec{u}(0) = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$

$$tr(A) = 2 + (-1) = 1$$

$$detA = 2(-1) - 2(5) = -2 - 10 = -12$$

$$\lambda^2 - \lambda - 12 = 0$$

$$(\lambda - 4)(\lambda + 3) = 0$$

$$\lambda = 4, \qquad \qquad \lambda = -3$$

$$\begin{array}{cc} \text{a} & \text{b} \\ \left[\begin{matrix} 2 \\ 5 \end{matrix}\right. & \left.\begin{matrix} 2 \\ -1 \end{matrix}\right] \\ \text{c} & \text{d} \end{array}$$

$$v_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ 4 - 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ -3 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

The solution is $\vec{u}(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

$$t = 0 \quad \begin{bmatrix} 10 \\ 3 \end{bmatrix} = c_1 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$10 = c_1 + 2c_2 \quad \boxed{1}$$

$$3 = c_1 - 5c_2 \quad \boxed{2}$$

$$\overline{SUBTRACT \qquad 7 = 7c_2}$$

$$c_2 = 1$$

$$sub\ c_2 = 1 \ \ into \ \boxed{1}$$

$$10 = c_1 + 2(1)$$

$$c_1 = 8$$

$$\vec{u}(t) = 8e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

L4. Determine the solution to $\vec{u}'(t) = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ with initial condition $\vec{u}(0) = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

$$tr(A) = 1 + 4 = 5$$
$$detA = 1(4) - 1(-2) = 4 + 2 = 6$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 2)(\lambda - 3) = 0$$
$$\lambda = 2\,, \qquad \lambda = 3$$

$$\begin{matrix} \text{a} \quad \text{b} \\ \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \\ \text{c} \quad \text{d} \end{matrix}$$

$$\vec{v}_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 1 \\ 2 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The solution is $\vec{u}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$t = 0 \quad \begin{bmatrix} 4 \\ 7 \end{bmatrix} = c_1 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$4 = c_1 + c_2 \quad \boxed{1}$$
$$7 = c_1 + 2c_2 \quad \boxed{2}$$
$$Subtract \quad -3 = -c_2$$
$$c_2 = 3$$

Sub into $\boxed{1}$ $\quad 4 = c_1 + 3$

$$c_1 = 1$$

The solution is:

$$\vec{u}(t) = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

L5. Solve $\vec{u}'(t) = \begin{bmatrix} 2 & 5 \\ -2 & 4 \end{bmatrix} \; \vec{u}(t) \; with \; \vec{u}(0) = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

$$tr(A) = 2 + 4 = 6$$
$$detA = 2(4) - 5(-2) = 8 + 10 = 18$$
$$\lambda^2 - 6\lambda + 18 = 0$$
$$\lambda = \frac{-b \pm \sqrt{6^2 - 4ac}}{2a}$$
$$= \frac{6 \pm \sqrt{36 - 4(1)(18)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{-36}}{2}$$
$$= \frac{6 \pm 6i}{2}$$
$$= 3 \pm 3i$$

$a = 3 > 0$ ∴spiral away from origin

$$\begin{array}{c} \begin{matrix} a & b \end{matrix} \\ \begin{bmatrix} 2 & 5 \\ -2 & 4 \end{bmatrix} \\ \begin{matrix} c & d \end{matrix} \end{array}$$

$\lambda_1 = 3 + 3i$ $\lambda_2 = 3 - 3i$

$a = 3$

$b = 3$

$$\vec{v}_1 = \begin{bmatrix} \lambda - d \\ c \end{bmatrix} = \begin{bmatrix} 3 + 3i - 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 + 3i \\ -2 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} -1 + 3i \\ -2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ -2 \end{bmatrix}}_{\vec{x}} + i \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\vec{y}}$$

$$\vec{u}(t) = c_1 e^{at}(\cos(bt)\vec{x} - \sin(bt)\vec{y}) + c_2 e^{at}(\sin(bt)\vec{y}) + c_2 e^{at}(\sin(bt)\vec{x} + \cos(bt)\vec{y})$$

$$\therefore \vec{u}(t) = c_1 e^{3t}\left(\cos(3t)\begin{bmatrix} -1 \\ -2 \end{bmatrix} - \sin(3t)\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right) + c_2 e^{3t}(\sin(3t)\begin{bmatrix} -1 \\ -2 \end{bmatrix} + \cos(3t)\begin{bmatrix} 3 \\ 0 \end{bmatrix})$$

Substitute $\vec{u}(0) = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

$t = 0$

$$\begin{bmatrix} 7 \\ 4 \end{bmatrix} = c_1 e^0 \left[\cos 0 \begin{bmatrix} -1 \\ -2 \end{bmatrix} - \sin(0)\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right] + c_2 e^0 (\sin 0 \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \cos 0 \begin{bmatrix} 3 \\ 0 \end{bmatrix})$$

$$\begin{bmatrix} 8 \\ 4 \end{bmatrix} = c_1\left(\begin{bmatrix} -1 \\ -2 \end{bmatrix} - 0\right) + c_2(0 + \begin{bmatrix} 3 \\ 0 \end{bmatrix})$$

$$\begin{bmatrix} 8 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$8 = -c_1 + 3c_2$ $\boxed{1}$

$4 = -2c_1 + 0$ $\boxed{2}$

$c_1 = -2$ sub into $\boxed{1}$

$8 = -(-2) + 3c_2$

$8 - 2 = 3c_2$

$3c_2 = 6$

$c_2 = 2$

$$\vec{u}(t) = -2e^{3t}(\cos(3t)\begin{bmatrix} -1 \\ -2 \end{bmatrix} - \sin(3t)\begin{bmatrix} 3 \\ 0 \end{bmatrix}) + 2e^{3t}(\sin(3t)\begin{bmatrix} -1 \\ -2 \end{bmatrix} + \cos(3t)\begin{bmatrix} 3 \\ 0 \end{bmatrix})$$

A= $\begin{bmatrix} 2 & 5 \\ -2 & 4 \end{bmatrix}$

Since in the complex eigenvalues, the real part a=3>0, it is a spiral away from the origin.
Since the value of $a_{21} = -2 < 0$, it goes in a clockwise motion.

Long Method

$\lambda = 3 + 3i$

$$A = \begin{bmatrix} 2 & 5 \\ -2 & 4 \end{bmatrix}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$\left(\begin{bmatrix} 2 & 5 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = 3 + 3i$

$$\begin{bmatrix} 2 - (3 + 3i) & 5 \\ -2 & 4 - (3 + 3i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 - 3i & 5 \\ -2 & 1 - 3i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2v_1 + v_2 - 3iv_2 = 0$$

$$-2v_1 = -v_2 + 3iv_2$$

$$v_1 = \frac{v_2 - 3iv_2}{2}$$

Let $v_2 = 2$

$$v_1 = \frac{2 - 3i(2)}{2}$$

$$v_1 = 1 - 3i$$

$\therefore \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 - 3i \\ 2 \end{bmatrix}$ or × -1

Or $\begin{bmatrix} -1 + 3i \\ -2 \end{bmatrix}$

L6. Solve $\overrightarrow{u'}(t) = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix} \vec{u}(t)\ \ with\ \vec{u}(0) = \begin{bmatrix} 18 \\ 10 \end{bmatrix}$

$tr(A) = 3 + 1 = 4$

$detA = 3 + 13(5) = 68$

$\lambda^2 - 4\lambda + 68 = 0$

$$\lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(68)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-256}}{2}$$

$$= \frac{4 \pm 16i}{2}$$

$\lambda = 2 \pm 8i$

$$\begin{array}{c} \begin{matrix} a & b \end{matrix} \\ \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix} \\ \begin{matrix} \mathrm{c} & \mathrm{d} \end{matrix} \end{array}$$

$\lambda_1 = 2 + 8i \qquad \lambda_2 = 2 - 8i$

$a = 2 \ \ b = 8$

$a = 2 > 0$ spiral away from origin

$$\vec{v}_1 = \begin{bmatrix} \lambda - d \\ c \end{bmatrix} = \begin{bmatrix} 2 + 8i - 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 + 8i \\ 5 \end{bmatrix}$$

$$= \underset{\vec{x}}{\begin{bmatrix} 1 \\ 5 \end{bmatrix}} + i \underset{\vec{y}}{\begin{bmatrix} 8 \\ 0 \end{bmatrix}}$$

$$\vec{u}(t) = c_1 e^{at}(\cos(bt)\vec{x} - \sin(bt)\vec{y}) + c_2 e^{at}(\sin(bt)\vec{x} + \cos(bt)\vec{y})$$

$$\vec{u}(t) = c_1 e^{2t}(\cos(8t)\begin{bmatrix} 1 \\ 5 \end{bmatrix} - \sin(8t)\begin{bmatrix} 8 \\ 0 \end{bmatrix}) + c_2 e^{2t}(\sin(8t)\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \cos(8t)\begin{bmatrix} 8 \\ 0 \end{bmatrix})$$

Substitute $\vec{u}(0) = \begin{bmatrix} 18 \\ 10 \end{bmatrix}$

$t = 0$

$$\begin{bmatrix} 18 \\ 10 \end{bmatrix} = c_1 e^0(\cos 0 \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \sin 0 \begin{bmatrix} 8 \\ 0 \end{bmatrix}) + c_2 e^0(\sin 0 + \cos 0 \begin{bmatrix} 8 \\ 0 \end{bmatrix})$$

$$\begin{bmatrix} 18 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

$18 = c_1 + 8c_2 \quad \boxed{1}$

$10 = 5c_1 \quad \boxed{c_1 = 2} \quad sub\ into\ \boxed{1}$

$18 = 2 + 8c_2$

$\therefore 16 = 8c_2 \quad \boxed{c_2 = 2}$

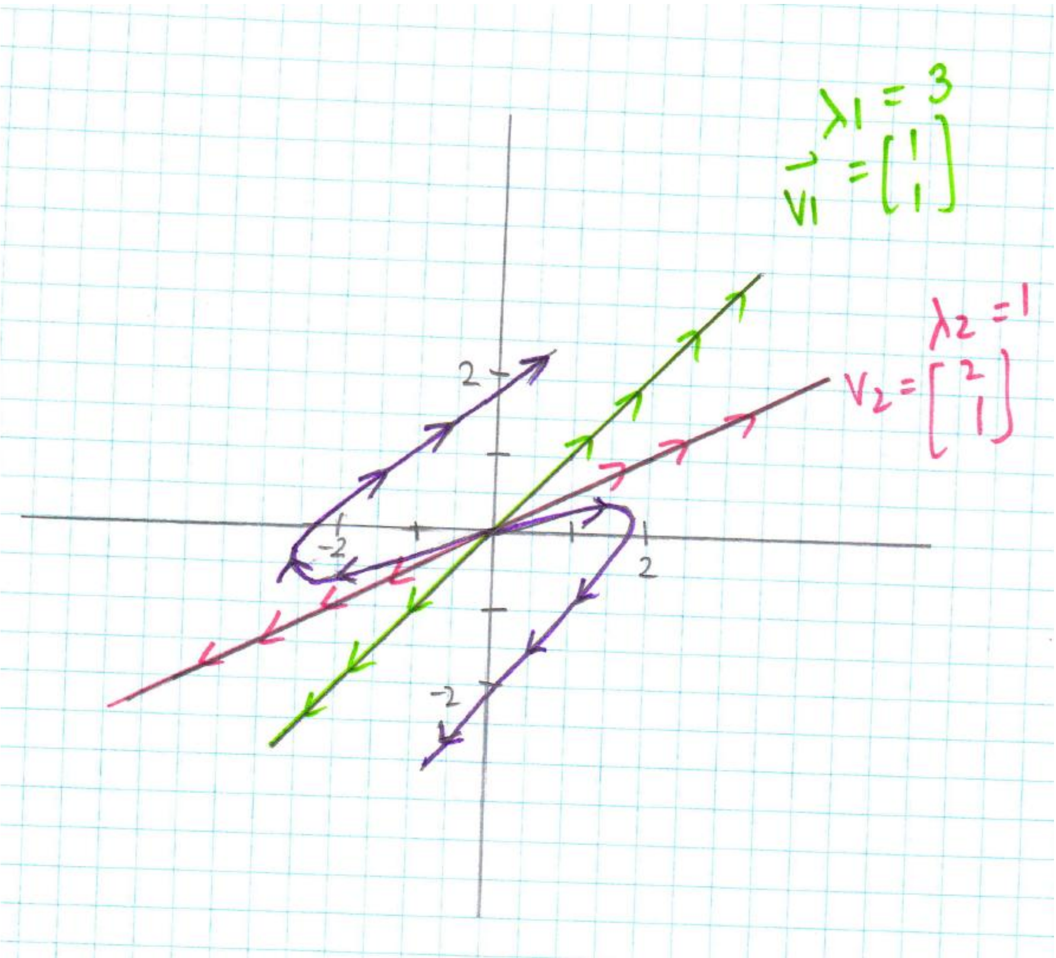
$$\vec{u}(t) = 2e^{2t}(\cos(8t)\begin{bmatrix} 1 \\ 5 \end{bmatrix} - \sin(8t)\begin{bmatrix} 8 \\ 0 \end{bmatrix}) + 2e^{2t}(\sin(8t)\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \cos(8t)\begin{bmatrix} 8 \\ 0 \end{bmatrix})$$

Does the spiral move in a clockwise or counter clockwise direction?

$A = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix}$ **$a = 2 > 0$ spiral away from origin. And, $a_{21} = 5 > 0$, so it is moving counterclockwise.**

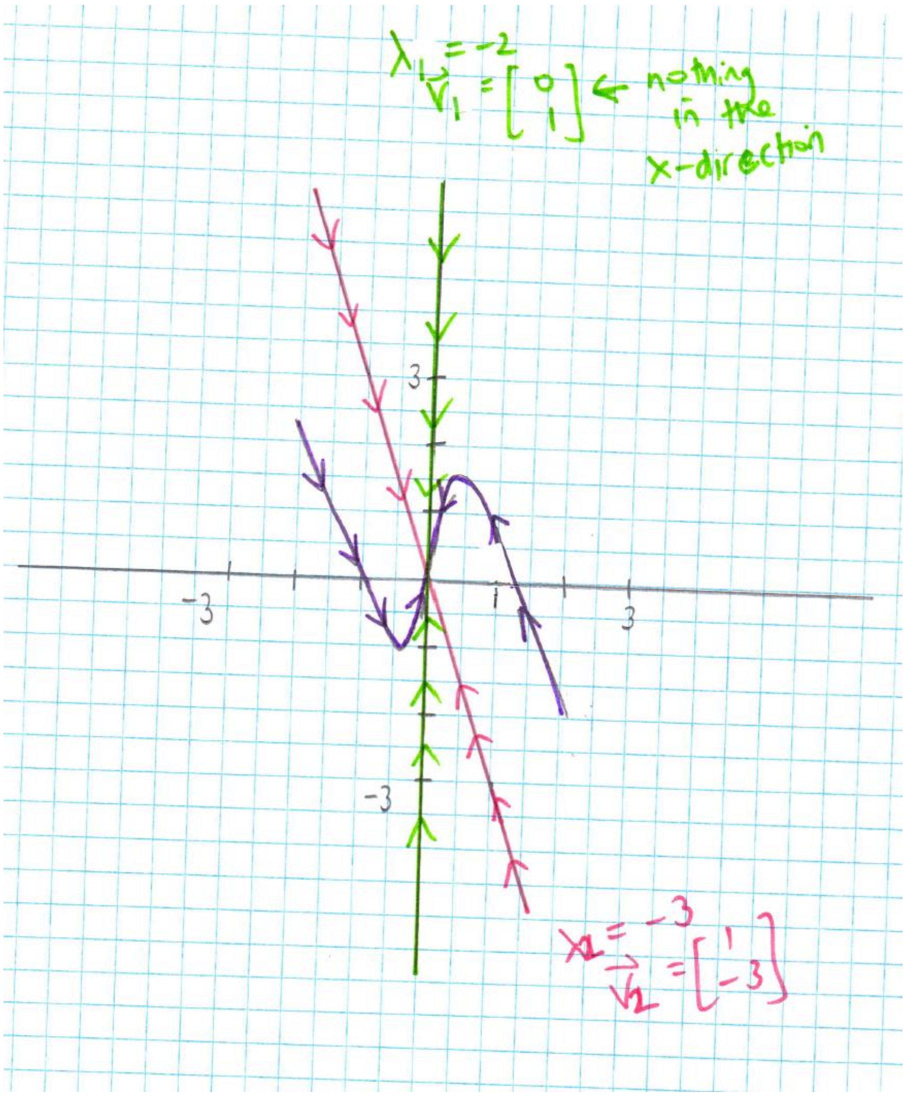
L7. Eigenvalues 3, 1 with corresponding eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ *and* $\begin{bmatrix}2\\1\end{bmatrix}$.

Since $\lambda_1 = 3$ is the dominant eigenvalue, all solutions look like it's eigenvector vector $\overrightarrow{v_1} = \begin{bmatrix}1\\1\end{bmatrix}$ as t→ ∞. Since both eigenvalues are positive, the arrows point away from the origin along BOTH eigenvectors. All solutions will start according to the smaller eigenvalue's eigenvector, $\overrightarrow{v_2} = \begin{bmatrix}2\\1\end{bmatrix}$ and tend toward the dominant eigenvalue's eigenvector as t→ ∞.



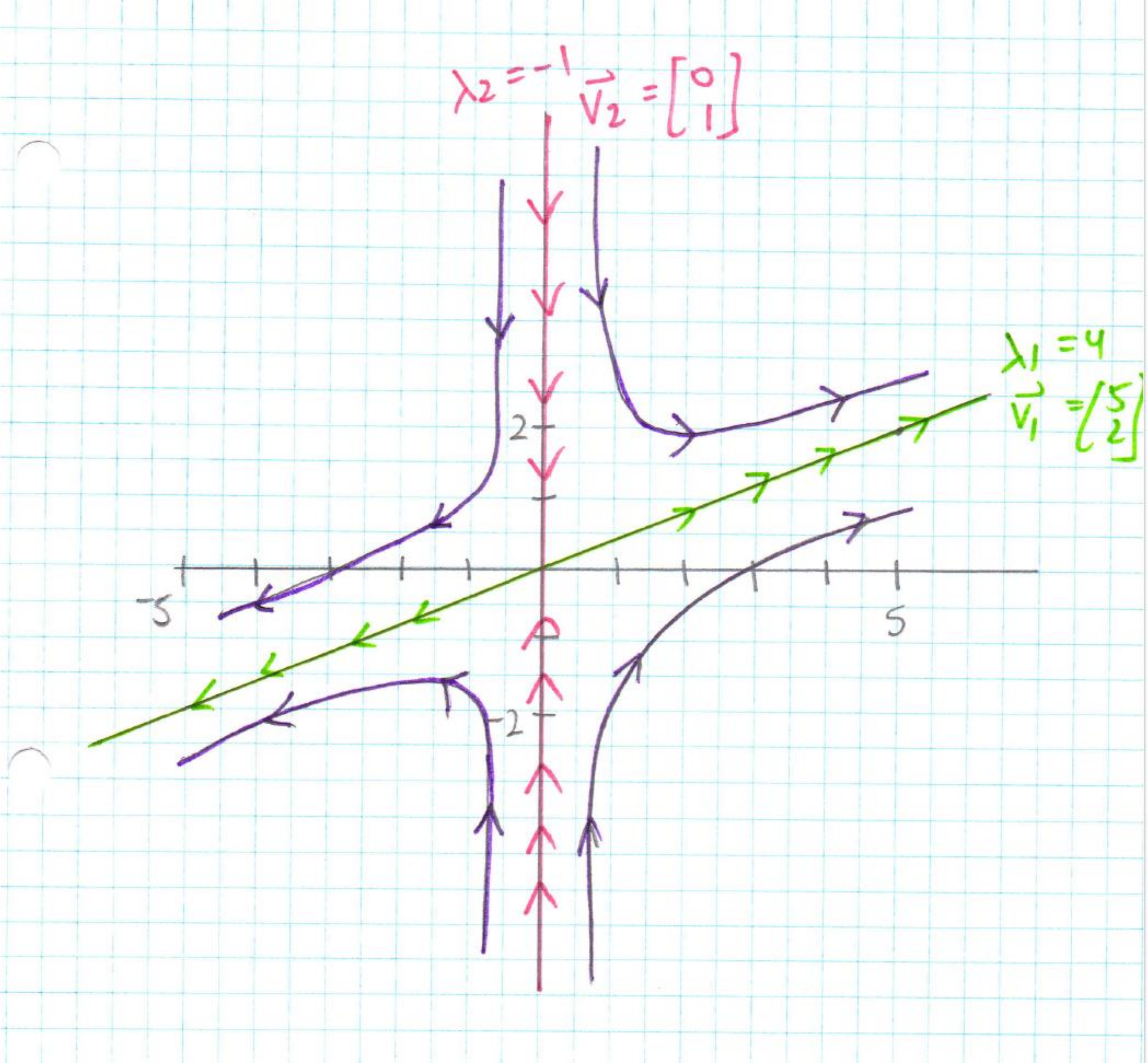
L8. Eigenvalues -3, -2 with corresponding eigenvectors $\begin{bmatrix}1\\-3\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

Since both eigenvalues are negative, arrows along both eigenvectors point toward the origin. All solutions end at the origin and start away from the origin and they start away parallel to the smaller eigenvalue (-3) and get closer to the larger eigenvalue (-2). So, here the solutions start parallel to the eigenvector $\begin{bmatrix}1\\-3\end{bmatrix}$ that corresponds to eigenvalue -3 and become parallel to the eigenvector $\begin{bmatrix}0\\1\end{bmatrix}$.



L9. Eigenvalues 4, -1 with corresponding eigenvectors $\begin{bmatrix}5\\2\end{bmatrix}$ *and* $\begin{bmatrix}0\\1\end{bmatrix}$.

Here, we have one positive and one negative eigenvalue, so the solutions don't tend toward or away from the origin. This is a saddle point. Along the eigenvector corresponding to the positive eigenvalue 4, the arrows point away from the origin and along the eigenvector corresponding to the negative eigenvalue -1, the arrows point toward the origin.



M. Quiz 6: Practice on Sections H to L

1. $A\vec{x} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 6-8 \\ 18+16 \end{bmatrix} = \begin{bmatrix} -2 \\ 34 \end{bmatrix}$

   First element is = -2
   Second element is =34

2. $M\vec{y} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 2 & 4 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0(2) - 1(3) + 2(4) \\ 1(2) + 2(3) + 4(4) \\ 1(2) + (-2)(3) + 3(4) \end{bmatrix}$

   $M\vec{y} = \begin{bmatrix} 0-3+8 \\ 2+6+16 \\ 2-6+12 \end{bmatrix} = \begin{bmatrix} 5 \\ 24 \\ 8 \end{bmatrix}$

   The first element is 5.

3. $A = \begin{bmatrix} 2 & 2 \\ 0 & 5 \end{bmatrix} \quad trA = 7 \qquad detA = 10$

   $\lambda^2 - 7\lambda + 10 = 0$

   $(\lambda - 2)(\lambda - 5) = 0$

   $\lambda = 2 \qquad\qquad \lambda = 5$

   $v_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} \qquad\qquad v_2 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix}$

   $= \begin{bmatrix} 2 \\ 2-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 2 \\ 5-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\vec{x_t} = c_1 v_1 {\lambda_1}^t + c_2 v_2 {\lambda_2}^t$$

Let t=0

$$\vec{x_t} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (2)^t + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} (5)^t$$

$$\begin{bmatrix} 8 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (2)^0 + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} (5)^0$$

$$8 = c_1 + 2c_2$$

$$\underline{6 = 3c_2}$$

$$6 = 3c_2$$

$$c_2 = 2$$

$8 = c_1 + 2(2)$

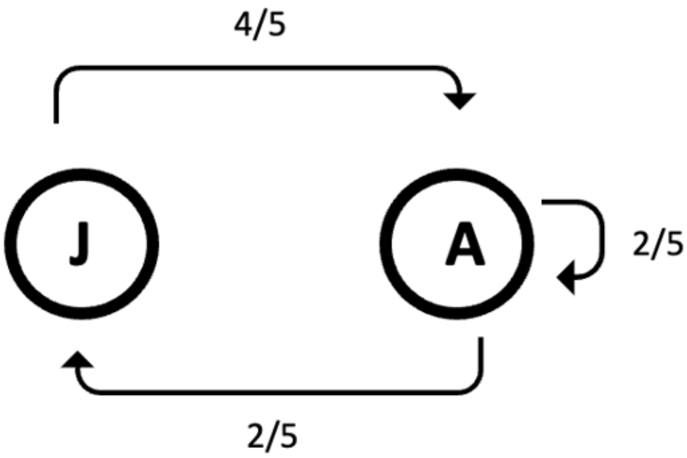
$8 = 4 + c_1$

$c_1 = 4$

$\vec{x_t} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (2)^t + 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} (5)^t$

$\therefore x(t) = 4(2)^t + 4(5)^t$

$y(t) = 6(5)^t$



4.

$$\begin{array}{c|cc} & J & A \\ \hline J & 0 & {}^{2}/_{5} \\ A & {}^{4}/_{5} & {}^{2}/_{5} \end{array}$$

$trA = \frac{2}{5} \qquad detA = 0 - \frac{8}{25} = -\frac{8}{25}$

$\lambda^2 - \frac{2}{5} - \frac{8}{25} = 0$

$\left(\lambda + \frac{2}{5}\lambda\right)\left(\lambda - \frac{4}{5}\right) = 0$

$\lambda = -\frac{2}{5}, \qquad \lambda = \frac{4}{5}$

$$\overrightarrow{v_1} = \begin{bmatrix} \frac{2}{5} \\ -\frac{2}{5} - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\overrightarrow{v_2} = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} - 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x(\vec{t}) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} (-\tfrac{2}{5})^t + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} (\tfrac{4}{5})^t$$

At $t = 0$ $\quad \begin{bmatrix} 4 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{array}{r} 4 = c_1 + c_2 \\ \underline{7 = -c_1 + 2c_2} \\ 11 = 3c_2 \end{array}$$

$c_2 = \frac{11}{3}$

$4 = c_1 + \frac{11}{3}$

$c_1 = \frac{12}{3} - \frac{11}{3} = \frac{1}{3}$

$\therefore \vec{x}(t) = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} (-\frac{2}{5})^t + \frac{11}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} (\frac{4}{5})^t$

a) $Juvenile\ x1(t) = \frac{1}{3}(1)(-\frac{2}{5})^t + \frac{11}{3}(1)(\frac{4}{5})^t = \frac{1}{3}(-\frac{2}{5})^t + \frac{11}{3}(\frac{4}{5})^t$

b) $Adults\ x2(t) = \frac{1}{3}(-1)(-\frac{2}{5})^t + \frac{11}{3}(2)(\frac{4}{5})^t = \frac{-1}{3}(-\frac{2}{5})^t + \frac{22}{3}(\frac{4}{5})^t$

c) the population is shrinking, since as t approaches infinity, both (-2/5)^t and (4/5)^t will approach 0. NOTE: If one of the fractions was (8/5)^t or 2^t, etc. then as x approaches infinity, that term would approach infinity, so it would be growing over time

5. $a)\lambda^2 - 3\lambda - 4 = 0$
$(\lambda - 4)(\lambda + 1) = 0$

$\lambda = 4,$ $\lambda = -1$
Positive negative
$\therefore$moves away from goes toward (0,0)
(0,0) $\therefore \overrightarrow{v_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\therefore \overrightarrow{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

$4 = 2c_1 - c_2$
$6 = 3c_1 + c_2$
$Add \quad 10 = 5c_1$
$c_1 = 2$

$4 = 2(2) - c_2$
$4 = 4 - c_2$
$-c_2 = 0$
$c_2 = 0$

$$\therefore \begin{bmatrix} u_1(t) \\ u_1(t) \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t}$$

$u_1(t) = 4e^{4t}$
$u_2(t) = 6e^{4t}$

b) $\begin{bmatrix} -4 \\ 16 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$

$-4 = 2c_1 - c_2$
$16 = 3c_1 + c_2$
$Add \quad 12 = 5c_1$
$c_1 = \frac{12}{5}$
$-4 = 2\left(\frac{12}{5}\right) - c_2$
$-4 = \frac{24}{5} - c_2$
$\frac{-20}{5} - \frac{24}{5} = -c_2$
$-\frac{44}{5} = -c_2$
$c_2 = \frac{44}{5}$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \frac{12}{5} \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t} + \frac{44}{5} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

$u_1(t) = \frac{24}{5} e^{4t} - \frac{44}{5} e^{-t}$
$u_2(t) = \frac{36}{5} e^{4t} + \frac{44}{5} e^{-t}$

6. $A = \begin{bmatrix} a & b \\ -3 & -6 \\ 5 & 8 \end{bmatrix}$

$\quad c \quad d$

$$tr(A) = -3 + 8 = 5$$

$$\begin{aligned} \det(A) &= ad - bc \\ &= -3(8) + 6(5) \\ &= -24 + 30 \\ &= 6 \end{aligned}$$

$$\lambda^2 - tr(A)\lambda + \det(A) = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2, \quad \lambda = 3$$

$$\overrightarrow{v_1} = \begin{bmatrix} -6 \\ 2 - (-1) \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \end{bmatrix} \qquad \overrightarrow{v_2} = \begin{bmatrix} -6 \\ 3 + 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = c_1 \begin{bmatrix} -6 \\ 5 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} -6 \\ 5 \end{bmatrix} e^{0} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{0}$$

$$3 = -6c_1 - c_2$$

$$\underline{2 = 5c_1 + c_2}$$

$$Add \quad 5 = -c_1$$

$$c_1 = -5$$

$$2 = 5(-5) + c_2$$

$$2 + 25 = c_2$$

$$c_2 = 27$$

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = -5 \begin{bmatrix} -6 \\ 5 \end{bmatrix} e^{2t} + 27 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t}$$

$$u(t) = 30e^{2t} - 27e^{3t}$$

$$v(t) = -25e^{2t} + 27e^{3t}$$

N. Practice Final Exam

N1. $N = No(\frac{1}{2})^{\frac{t}{h}}$

$$\therefore N = 100(\frac{1}{2})^{\frac{t}{20}}$$

N2. $v(t) = e^{\int P(t)dt}$

$$= e^{\int \frac{1}{t-2}dt} = e^{\ln(t-2)} = t-2$$

$$v(t)u(t) = \int v(t)\, q(t)dt$$

$$(t-2)u(t) = \int(t-2)(4)dt = \int(4t-8)dt$$

$$(t-2)u(t) = \frac{4t^2}{2} - 8t + c$$

$$u(1) = 3 \quad (1-2)(3) = 2(1)^2 - 8(1) + c$$

$$c = -3-2+8$$

$$\mathrm{c} = 3$$

$$(t-2)u(t) = 2t^2 - 8t + 3$$

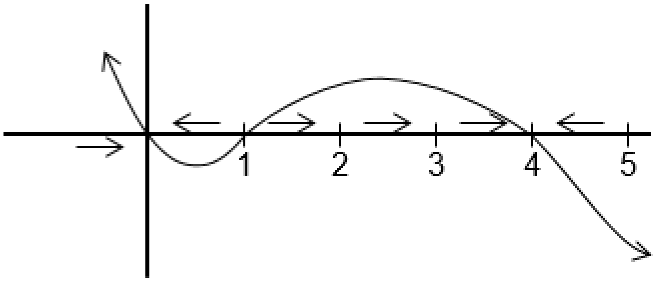
$$u(4) = ?$$

$$(4-2)u = 2(4)^2 - 8(4) + 3$$

$$2u = 32 - 32 + 3$$

$$u = \frac{3}{2}$$

N3.



$N = 0, 1, 4$ equilibrium

$$N = \frac{1}{2} \quad \frac{dN}{dt} = \frac{1}{2}\left(\frac{1}{2} - 1\right)(4 - \frac{1}{2}) < 0$$

$$N = 2 \quad \frac{dN}{dt} = 2(2-1)(4-2) > 0$$

$$N = 5 \quad \frac{dN}{dt} = 5(5-1)(4-5) < 0$$

$$\therefore It\ will\ go\ to\ 4$$

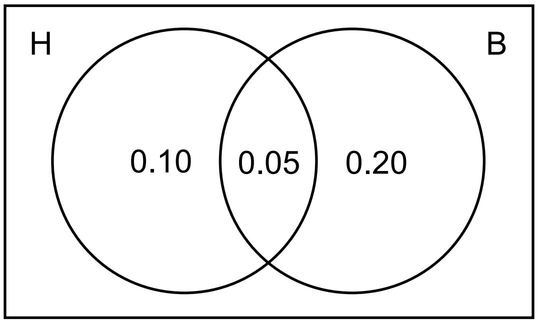
N4. $\boxed{E}$ is false. It is only true if E, F are independent.

N5. a) Let $H = heart\ disease \quad B = high\ blood\ pressure$

$$\Pr(H) = 0.15 \quad \Pr(B) = 0.25 \quad \Pr(H \cap B) = 0.05$$

$$\Pr(H\ or\ B\ but\ not\ both) = 0.10 + 0.20$$

$$= 0.30 \quad or \quad \frac{30}{100}$$



b) $\Pr(H \cap B^c) = \Pr(H) - \Pr(H \cap B)$

$$= 0.15 - 0.05$$

$$= 0.10 \quad \text{SEE VENN DIAGRAM}$$

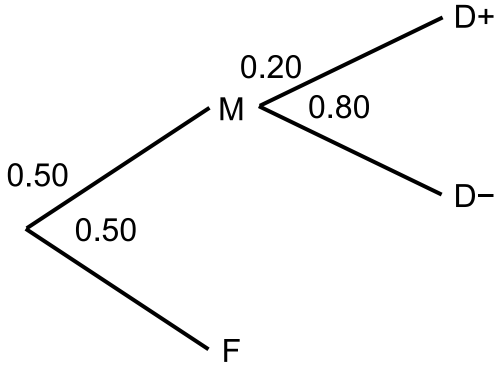
c) $\Pr(B^c/H) = \frac{\Pr(B^c \cap H)}{\Pr(H)} = \frac{0.10}{0.15}$

$$= \frac{10}{15} = \boxed{\frac{2}{3}}$$

d) $\Pr(H/B) = \frac{\Pr(H \cap B)}{\Pr(B)} = \frac{0.05}{0.25}$

$$= \frac{5}{25} = \boxed{\frac{1}{5}}$$

N6.



$\Pr(M \cap D^+) = \Pr(M) \times \Pr(D^+/M)$

$$= 0.5 \times 0.20$$

$$= \frac{1}{2}\left(\frac{2}{10}\right) = \boxed{\frac{1}{10}}$$

N7. a)

x	Pr (x)
1	0.4
2	0.3
3	0.3
	$\overline{1}$

$$
\begin{aligned}
E(x) &= \sum x \text{ Pr } (x) \\
&= 1(0.4) + 2(0.3) + 3(0.3) \\
&= 0.4 + 0.6 + 0.9 \\
&= \boxed{1.9}
\end{aligned}
$$

b)
$$
\begin{aligned}
Var(x) &= \sum x^2 \text{ Pr}(x) - [E(x)]^2 \\
&= 1^2(0.4) + 2^2(0.3) + 3^2(0.3) - 1.9^2 \\
&= 0.4 + 1.2 + 2.7 - 3.61 \\
&= 0.69
\end{aligned}
$$

$$SD(x) = \sqrt{0.69} = 0.83$$

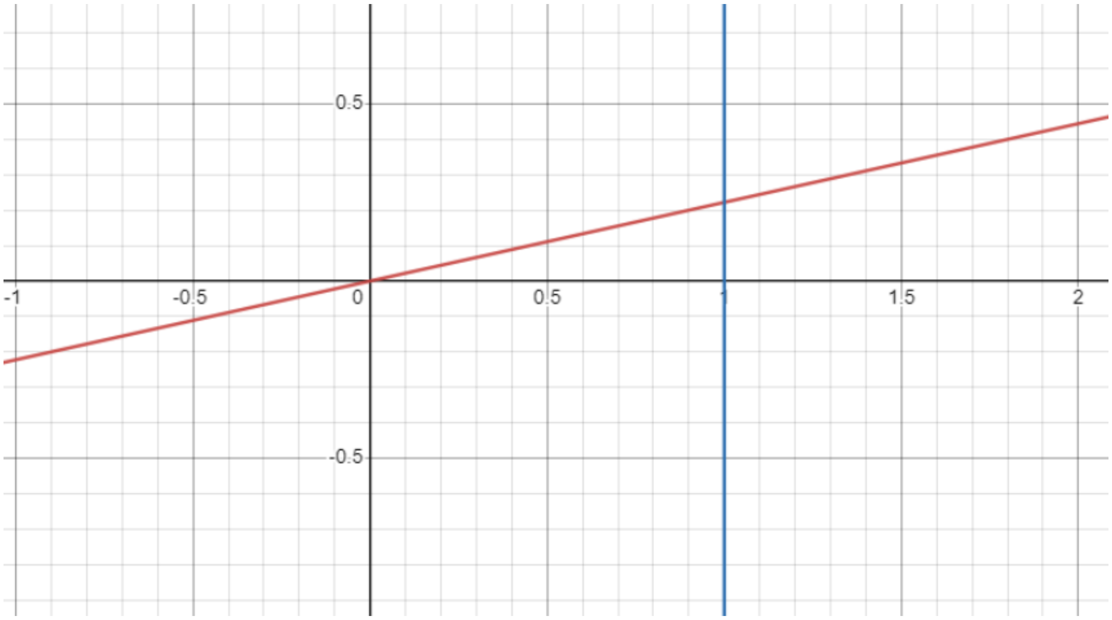
N8.a) $f(x) = F'(x) = \frac{2x}{9} = \frac{2}{9}x$

$$f(x) = \frac{2}{9}x$$

METHOD 1:

$\text{Pr}(0 \leq x \leq 1) = \frac{b \times h}{2} = \frac{1(\frac{2}{9})}{2} = \frac{2}{18} = \frac{1}{9}$ (draw the graph)

Find the area from 0 to 1.



METHOD 2: Use the integral

$\Pr(0 \le x \le 1) = \int_0^1 \frac{2}{9}x dx = \frac{2}{9}\left[\frac{x^2}{2}\right]\frac{1}{0} = \frac{2}{9}\left(\frac{1}{2} - 0\right) = \frac{1}{9}$

b) $E(x) = \int_{-\infty}^{\infty} x\, f(x) dx$

$= \int_0^3 x\left(\frac{2}{9}x\right) dx$

$= \int_0^3 \frac{2}{9}x^2 dx$

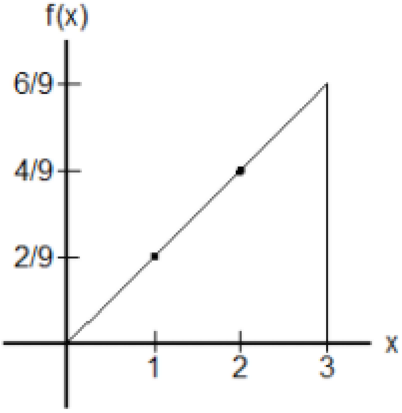
$= \frac{2}{9}\left[\frac{x^3}{3}\right]_0^3$

$= \frac{2}{9}\left[\frac{3^3}{3} - 0\right]$

$= \frac{2}{9}\left[\frac{27}{3}\right]$

$= \frac{2}{9}[9]$

$= 2$



N9. $n = 25 \quad \bar{x} = 5 \quad s^2 = 20$

$t = \frac{\bar{x} - E(x)}{\sqrt{\frac{s^2}{n}}}$

$= \frac{5-0}{\sqrt{\frac{20}{25}}} = \frac{5}{\frac{\sqrt{20}}{5}} = \frac{5\times 5}{\sqrt{20}} = \frac{25}{\sqrt{20}}$

N10. $\bar{x} = 30 \qquad s^2 = 25\, Z = \frac{x - \bar{x}}{s} = \frac{36-30}{5} = \frac{6}{5}$

N11. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+2+3 \\ 8+5+6 \end{bmatrix} = \begin{bmatrix} 7 \\ 19 \end{bmatrix}$

$2\times 3 \;\; 3\times 1 \qquad Ans\ 2\times 1$

N12. invertible if $ad - bc \neq 0$

$$\begin{aligned} ad - bc &= 2(-6) - (3)(-4) \\ &= -12 + 12 \\ &= 0 \end{aligned}$$

$\therefore$ *not invertible since* $ad - bc = 0$

N13. no inverse if $ad - bc = detA = 0$

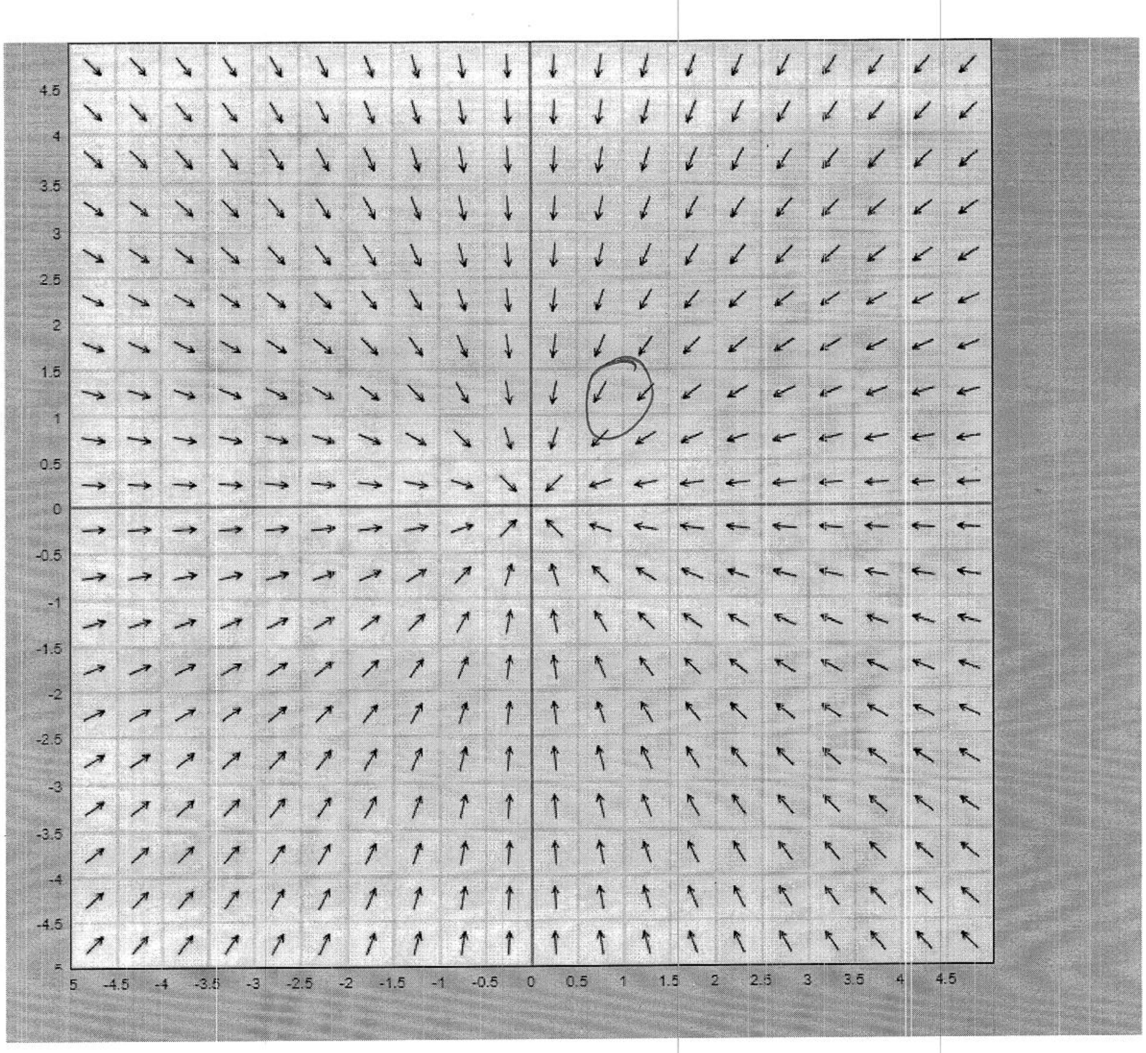
$$\begin{aligned} detA = 3k - (7k + 6) &= 0 \\ 3k - 7k - 6 &= 0 \\ -4k &= 6 \\ k &= -\tfrac{6}{4} = -\tfrac{3}{2} \end{aligned}$$

N14.

$$\begin{array}{c} \\ J \\ A \\ adult \end{array} \begin{array}{c} \begin{array}{ccc} \text{J} & \text{A} & \text{adult} \end{array} \\ \begin{bmatrix} 0 & 0.7 & 2.5 \\ 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \end{bmatrix} \end{array}$$

N15. a) $= 1$ b) $= 3$
$c) = 2$ d) $= 4$

15.



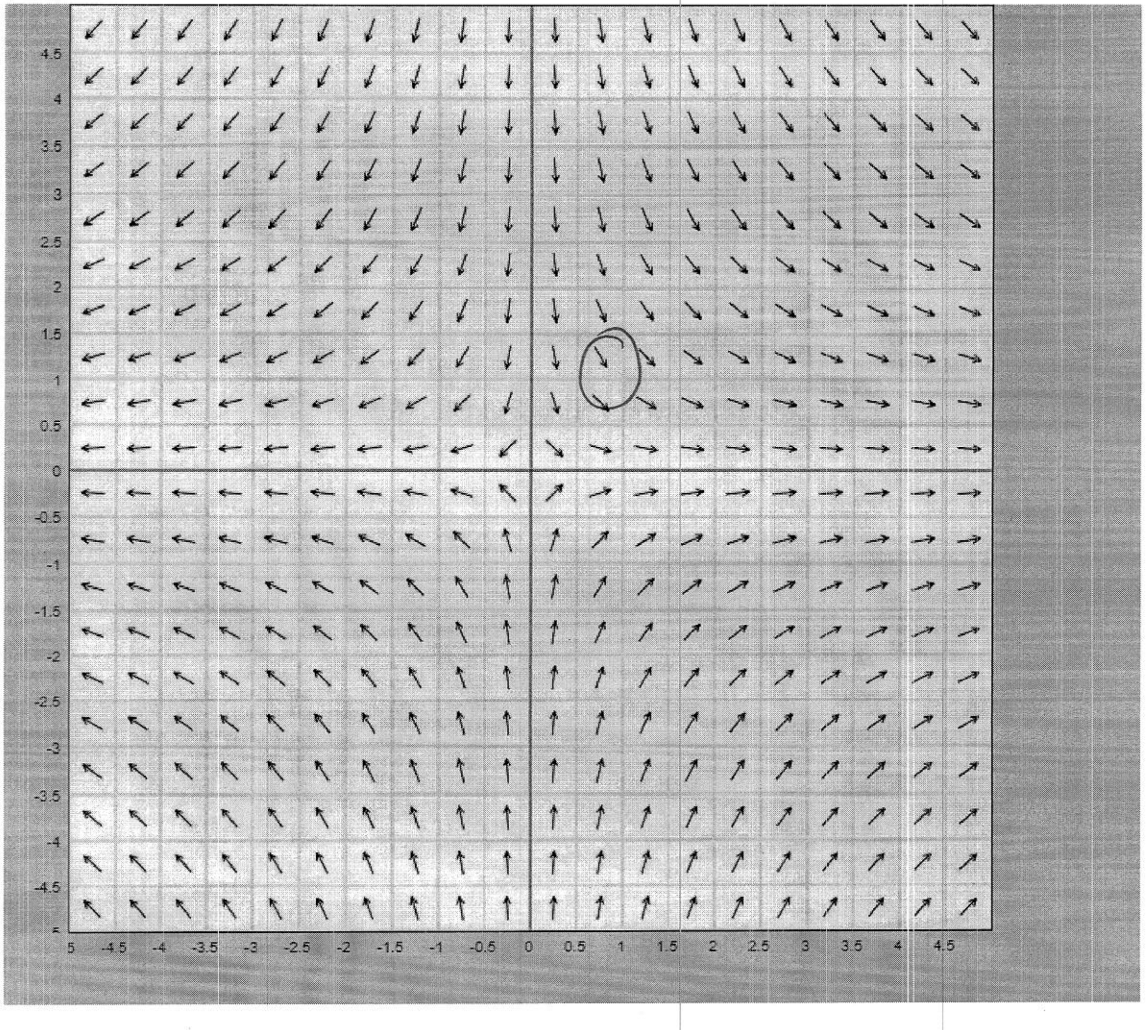
$x' = -x$

$y' = -y$

$$F\begin{bmatrix}1\\1\end{bmatrix} = \left[\begin{array}{c}-1\\-1\end{array}\right]\begin{array}{l}\text{run}\\\text{rise}\end{array}$$ ↙

left + down

15.

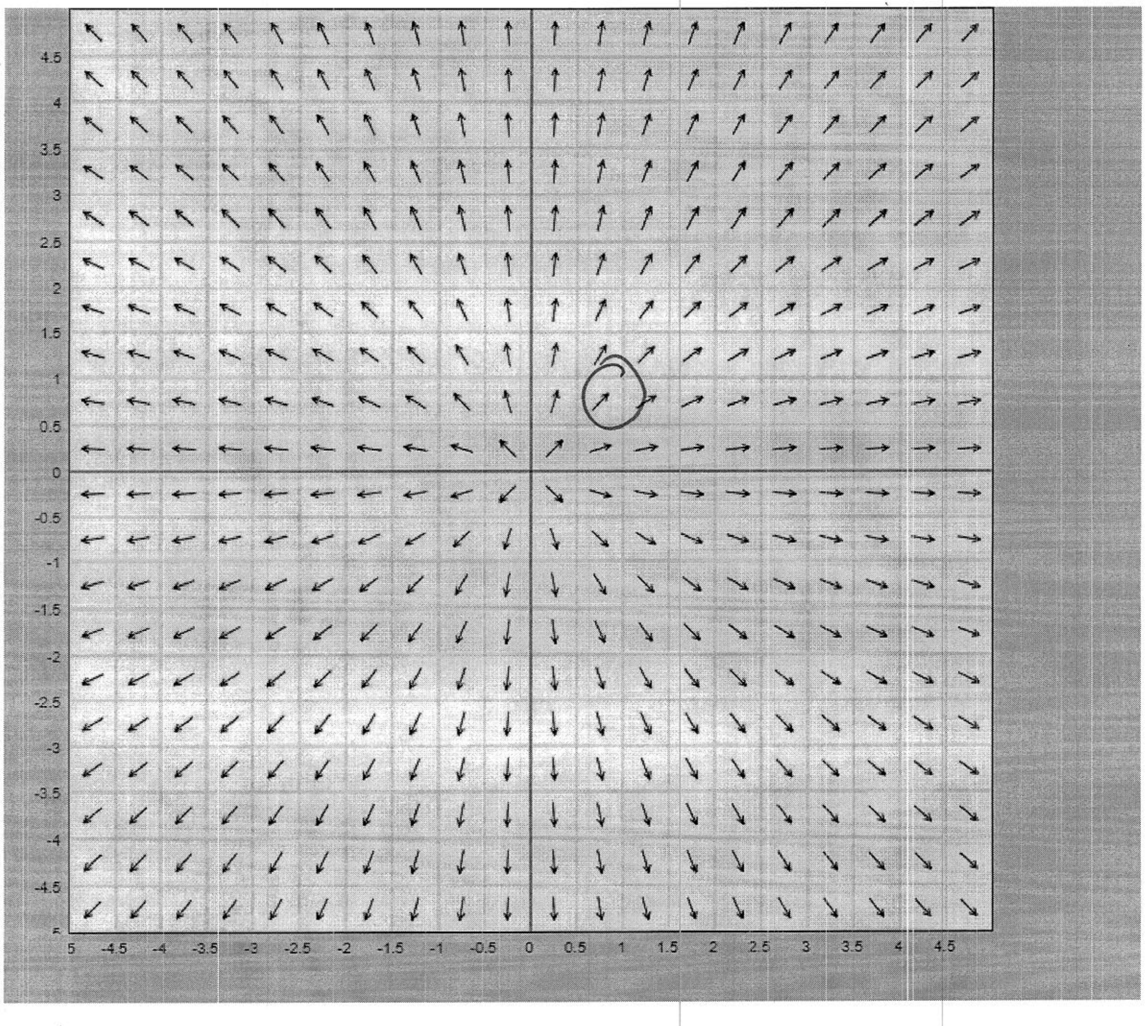


$x' = x$

$y = -y$

$F\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}\begin{matrix}\text{run}\\\text{rise}\end{matrix}$ right + down

15.

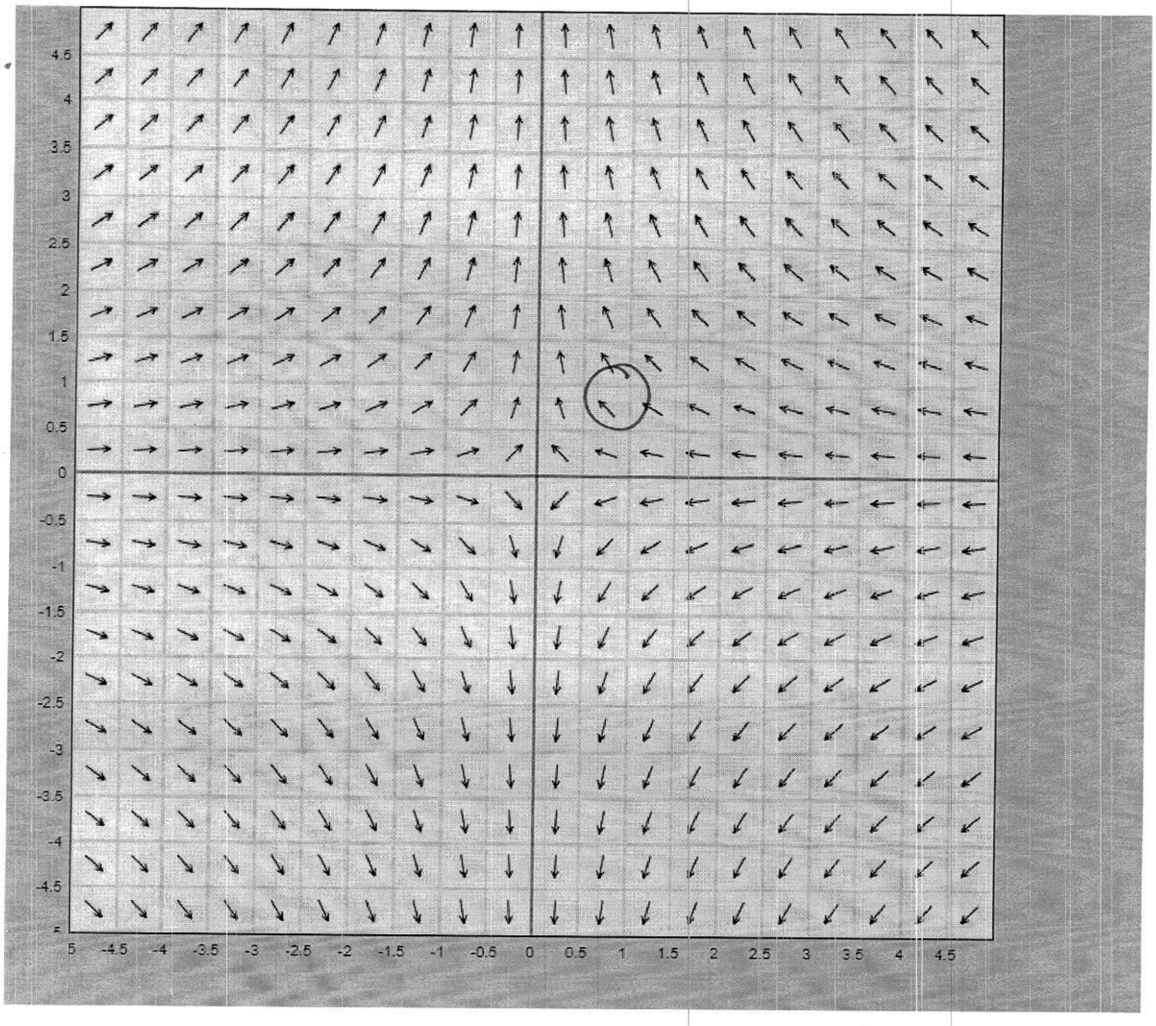


$x' = x$

$y' = y$

$F\begin{bmatrix}1\\1\end{bmatrix} = \begin{array}{c}\text{run}\\\text{rise}\end{array}\begin{bmatrix}1\\1\end{bmatrix} \nearrow$ right + up

15.



$x' = -x$
$y = y$

$$F\begin{bmatrix}1\\1\end{bmatrix} = \begin{array}{l}\begin{bmatrix}-1\\1\end{bmatrix}\end{array}\begin{array}{l}\text{run}\\\text{rise}\end{array}$$

left + up

↖

N16. $tr(A) = -6 + 5 = -1$

$detA = -6(5) - 3(4) = -30 - 12 = -42$

$\lambda^2 - tr(A)\lambda + detA = 0$

$\lambda^2 + \lambda - 42 = 0$

$(\lambda + 7)(\lambda - 6) = 0$

$\lambda = -7,6$

$\lambda = -7$ $\lambda = 6$

$$v_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} \qquad v_2 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -7 - (-6) \end{bmatrix} \qquad = \begin{bmatrix} 3 \\ 6 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -1 \end{bmatrix} \qquad = \begin{bmatrix} 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

N17. $tr(A) = 2 + 4 = 6$

$detA = 2(4) - (5)(-2) = 8 + 10 = 18$

$\lambda^2 - tr(A)\lambda + detA = 0$

$\lambda^2 - 6\lambda + 18 = 0$

$a = 1 \quad b = -6 \quad c = 18$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{36 - 4(1)(18)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{-36}}{2}$$

$$= \frac{6 \pm 6i}{2}$$

$$= 3 \pm 3i$$

$$v_1 = \begin{bmatrix} \lambda - d \\ c \end{bmatrix} = \begin{bmatrix} 3 + 3i - 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 3i - 1 \\ -2 \end{bmatrix}$$

N18. Here the eigenvalues are: $\lambda = -7, 6$ (from a previous question)

$$\vec{v}_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 3 \\ -7-(-6) \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 6-(-6) \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\overrightarrow{x_t} = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t$$

$$\overrightarrow{x_t} = c_1\begin{bmatrix} 3 \\ -1 \end{bmatrix}(-7)^t + c_2\begin{bmatrix} 1 \\ 4 \end{bmatrix}(6)^t$$

At $t = 0$ $\begin{bmatrix} 9 \\ 13 \end{bmatrix} = c_1\begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$9 = 3c_1 + c_2$ (1)

$13 = -c_1 + 4c_2$ (2) multiply by 3

$9 = 3c_1 + c_2$

$39 = -3c_1 + 12c_2$ ADD

$48 = 13c_2$

$c_2 = \frac{48}{13}$ substitute into equation 1

$9 = 3c_1 + c_2$

$9 = 3c_1 + \frac{48}{13}$

$9 - \left(\frac{48}{13}\right) = 3c_2$

$\frac{117}{13} - \frac{48}{13} = 3c_2$

$3c_2 = \frac{69}{13}$ $c_2 = \frac{23}{13}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{23}{13}\begin{bmatrix} 3 \\ -1 \end{bmatrix}(-7)^t + \frac{48}{13}\begin{bmatrix} 1 \\ 4 \end{bmatrix}(6)^t$$

N19.

$$\begin{array}{c|ccc} & P & B & H \\ \hline P & 3/5 & 1/2 & 0 \\ B & 1/5 & 0 & 1/2 \\ H & 1/5 & 1/2 & 1/2 \end{array}$$

$$\begin{bmatrix} 3/5 & 1/2 & 0 \\ 1/5 & 0 & 1/2 \\ 1/5 & 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 3/5\,x + 1/2\,y \\ 1/5\,x + 1/2\,z \\ 1/5\,x + 1/2\,y + 1/2\,z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

From $\boxed{1}$ $3/5\,x + 1/2\,y = x$

$$1/2\,y = 2/5\,x$$

$$10(1/2\,y) = 10(2/5\,x)$$

$$5y = 4x$$

$$y = 4/5\,x$$

Let $x = 5$ *then* $y = 4/5\,(5)$ $\therefore y = 4$

From $\boxed{2}$ $1/5\,x + 1/2\,z = y$

$$1/5\,(5) + 1/2\,z = 4$$

$$1 + 1/2\,z = 4$$

$$1/2\,z = 3 \qquad \therefore z = 6$$

$$\begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix} \begin{matrix} \leftarrow P \\ \leftarrow B \\ \leftarrow H \end{matrix}$$

$$\therefore \Pr(pasta) = \frac{5}{5+4+6} = \frac{5}{15} = \frac{1}{3}$$

N20.

$$A=\begin{bmatrix}\overset{a}{1} & \overset{b}{2}\\ \underset{c}{6} & \underset{d}{-3}\end{bmatrix}$$

$$tr(A)=1+(-3)=-2$$

$$detA=1(-3)-2(6)=-15$$

$$\lambda^2+2\lambda-15=0$$

$$(\lambda+5)(\lambda-3)=0$$

$$\lambda=-5,\ 3$$

$$\lambda_1=-5$$

$$v_1=\begin{bmatrix}b\\ \lambda-a\end{bmatrix}=\begin{bmatrix}2\\ -5-1\end{bmatrix}=\begin{bmatrix}2\\ -6\end{bmatrix}=\begin{bmatrix}1\\ -3\end{bmatrix}$$

$$\lambda_2=3$$

$$v_2=\begin{bmatrix}2\\ 3-1\end{bmatrix}=\begin{bmatrix}2\\ 2\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\vec{u}(t)=c_1v_1e^{\lambda_1 t}+c_2v_2e^{\lambda_2 t}$$

$$\vec{u}(t)=c_1\begin{bmatrix}1\\ -3\end{bmatrix}e^{-5t}+c_2\begin{bmatrix}1\\ 1\end{bmatrix}e^{3t}$$

At $t=0$,

$$\begin{bmatrix}2\\ -1\end{bmatrix}=c_1\begin{bmatrix}1\\ -3\end{bmatrix}e^{0}+c_2\begin{bmatrix}1\\ 1\end{bmatrix}e^{0}$$

$$\begin{array}{r}2=c_1+c_2\\ -1=-3c_1+c_2\\ \hline subtract\ \ 3=4c_1\\ c_1=\frac{3}{4}\end{array}$$

$$2=c_1+c_2$$

$$2=\frac{3}{4}+c_2$$

$$c_2=\frac{8}{4}-\frac{3}{4}=\frac{5}{4}$$

$$\therefore \vec{u}(t)=\frac{3}{4}\begin{bmatrix}1\\ -3\end{bmatrix}e^{-5t}+\frac{5}{4}\begin{bmatrix}1\\ 1\end{bmatrix}e^{3t}$$

$$u_1(t)=\underset{\text{a}}{\frac{3}{4}}e^{-5t}+\underset{\text{b}}{\frac{5}{4}}e^{3t} \qquad u_2(t)=\frac{-9}{4}e^{-5t}+\frac{5}{4}e^{3t}$$

N21.

$$A = \begin{bmatrix} \overset{a}{3} & \overset{b}{-13} \\ \underset{c}{5} & \underset{d}{1} \end{bmatrix}$$

$tr(A) = 3 + 1 = 4$

$detA = 3 + 13(5) = 68$

$\lambda^2 - 4\lambda + 68 = 0$

$\lambda = \frac{4 \pm \sqrt{16-4(1)(68)}}{2(1)}$

$= \frac{4 \pm \sqrt{-256}}{2} = \frac{4 \pm 16i}{2} = 2 \pm 8i \leftarrow \quad a = 2 \quad b = 8$

$\overrightarrow{v_1} = \begin{bmatrix} \lambda - d \\ c \end{bmatrix} = \begin{bmatrix} 2 + 8i - 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 + 8i \\ 5 \end{bmatrix}$

$\vec{v} = \underset{\vec{x}}{\begin{bmatrix} 1 \\ 5 \end{bmatrix}} + i \underset{\vec{y}}{\begin{bmatrix} 8 \\ 0 \end{bmatrix}}$

$a = 2, b = 8 \quad \vec{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$

$u(t) = e^{at}k_1(0)(\cos(bt)\vec{x} - \sin(bt)\vec{y}) + e^{at}k_2(0)(\sin(bt)\vec{x} + \cos(bt)\vec{y}))$

call them k_1 and k_2 for simplicity:

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = e^{2t}k_1(\cos(8t)\begin{bmatrix} 1 \\ 5 \end{bmatrix} - \sin(8t)\begin{bmatrix} 8 \\ 0 \end{bmatrix} + e^{2t}k_2(\sin(8t)\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \cos(8t)\begin{bmatrix} 8 \\ 0 \end{bmatrix})$$

Sub $t = 0$,

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + k_2 \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ call them k_1 and k_2

$2 = k_1 + 8k_2$

$1 = 5k_1 + 0$

$k_1 = \frac{1}{5}$

$2 = \frac{1}{5} + 8k_2$

$\frac{10}{5} - \frac{1}{5} = 8k_2$

$\frac{9}{5} = 8k_2$

$k_2 = \frac{9}{40}$

$k_1(0) = \frac{1}{5} \quad \& \quad k_2(0) = \frac{9}{40}$

$u_1(t) = e^{2t}k_1(\cos(8t) - 8\sin(8t)) + e^{2t}k_2(\sin(8t) + 8\cos(8t))$

$u_1(t) = \frac{1}{5}e^{2t}(\cos(8t) - 8\sin(8t)) + 9e^{2t}(\sin(8t) + 8\cos(8t))$

$= \frac{1}{5}e^{2t}\cos(8t) + 72e^{2t}\cos(8t) - \frac{8}{5}e^{2t}\sin(8t) + 9e^{2t}\sin(8t))$

$= \frac{361}{5}e^{2t}\cos(8t) + \frac{37}{5}e^{2t}\sin(8t)$

$= e^{2t}(\frac{361}{5}\cos(8t) + \frac{37}{5}\sin(8t)$

Best of Luck on The Exam!!!