AMATH 1201 ACE Booklet Solutions (2025)

Material Covered on the Midterm

A. Introduction to Single Equation Models

Example 1. a) 3
b) 2

Example 2. Match each of the following with the most appropriate differential equation listed.
I Second order, non-linear b)
II Second order, linear, homogeneous c)
III First order, linear, inhomogeneous d)
IV Bernoulli differential equation a)

a) $u'(t) = 3u(t) + (u(t))^5$
b) $u''(t) - 3t^2u(t) = e^{u(t)}$
c) $u''(t) = e^t u(t)$
d) $tu'(t) - \frac{1}{t}u(t) = 2\sin(t)$

Example 3.
a) Non-linear so we don't talk about homogeneous (Bernoulli)
b) Linear and inhomogeneous
c) Non-linear so we don't talk about homogeneous
d) Linear and homogeneous (everything on the left in front of u(t) and derivatives is only a function of t).

Example 4.
a)Non-linear
b)Linear
c)Non-linear

Example 5.

LS= $\frac{dx}{dt}$

$x(t) = \frac{-1}{2} + \frac{3}{2}e^{2t}$

$\therefore \frac{dx}{dt} = 0 + \frac{3}{2}e^{2t}(2)$

$= 3e^{2t}$

$RS = 1 + 2x$

$= 1 + 2(\frac{-1}{2} + \frac{3}{2}e^{2t})$

$= 1 - 1 + \frac{6}{2}e^{2t}$

$= 3e^{2t}$

$\therefore LS = RS \quad \therefore it\ is\ a\ solution$

To find the initial size, substitute t=0 into the x(t)

$$x(t) = -\frac{1}{2} + \frac{3}{2}e^{2t}$$

$$x(0) = -\frac{1}{2} + \frac{3}{2}e^{0} = -\frac{1}{2} + \frac{3}{2} = 1$$

Example 6. The answer is D. Per capita means it is a rate that is proportion to the # of individuals in a population

Practice Exam Questions on Introduction to Single Equation Models

A1.

$LS = \frac{db}{dt}$ $\qquad RS = 3b = 3(10e^{3t})$

$b(t) = 10e^{3t}$ $\qquad = 30e^{3t}$

$\therefore \frac{db}{dt} = 10e^{3t}(3) = 30e^{3t}$

$LS = RS \quad \therefore it\ is\ a\ solution$

A2.

$\frac{dG}{dt} = G - 1$

$LS = \frac{dG}{dt} = 0 + e^t = e^t$

$RS = G - 1 = (1 + e^t) - 1 = e^t$

$\therefore LS = RS \quad \therefore it\ is\ a\ solution$

A3. Match each of the following with the most appropriate differential equation listed.
I Second order, non-linear b)
II Third order, linear, homogeneous c)
III First order, linear, inhomogeneous d)
IV Bernoulli differential equation a)

a)$u'(t) = 4u(t) + (u(t))^6$
b)$u''(t) - 6t^3u(t) = \sin u(t)$
$c)u'''(t) = e^t u(t)$
$d)\ t^2u'(t) - \frac{1}{t}u(t) = \cos(t)$

A4.
a) Inhomogeneous
b) Homogeneous
c) inhomogeneous

B. Isometry, Allometry, Log-log Plots, Dimensional Homogeneity

Example 1.

$A = 6x^2 \quad x^2 = \frac{A}{6} \quad \therefore x = (\frac{A}{6})^{\frac{1}{2}}$

$v = x^3$

$V = x^3 = \left[(\frac{A}{6})^{\frac{1}{2}}\right]^3 = (\frac{1}{6})^{\frac{3}{2}}A^{\frac{3}{2}}$

Let the new area be 2A. Let V* be the new volume

$$V^* = (\frac{1}{6})^{\frac{3}{2}}(2A)^{\frac{3}{2}}$$
$$= (\frac{1}{6})^{\frac{3}{2}} \cdot 2^{\frac{3}{2}}A^{\frac{3}{2}}$$
$$= \underbrace{(\frac{1}{6})^{\frac{3}{2}}A^{\frac{3}{2}}}_{V}\ 2^{\frac{3}{2}}$$

$\therefore$ *the volume increases by a factor of* $2^{\frac{3}{2}}$ *or* $\sqrt{8}$

$$V = \left(\frac{1}{6}\right)^{\frac{3}{2}} A^{\frac{3}{2}}$$

Example 2. $\frac{dI}{dt} = \frac{individual}{year} \qquad aI^2 = a\ \ individual^2$

$$\therefore a = \frac{1}{year\ (individual)}$$

Example 3. $P = \rho g h$

LS=$\frac{F}{L^2}$ RS= $\rho g h$

$$=\frac{FT^2}{L^4} \times (\quad) \times L$$

Therefore, the units of g must be $(\quad) = \frac{L}{T^2}$

Example 4. From the graph $m = \frac{2}{3} \qquad y - int = -2$

$m \neq 1 \quad \therefore allometric$

$\ln y = m \ln x + \ln k$

$\therefore \ln k = -2 \quad since\ y - int = -2$

$k = e^{-2}$

Practice Exam Questions on Isometry, Allometry, Log-log Plots, Dimensional Homogeneity

$B1.\qquad aI^3 = a\,(individual)^3$

$$\frac{dT}{dt} = \frac{individual}{year}$$

$$\therefore\ a = \frac{1}{year(individual)^2}$$

B2.

Let V* be new volume

$$V^* = \left(\frac{1}{6}\right)^{\frac{3}{2}}(3A)^{\frac{3}{2}}$$

$$= \left(\frac{1}{6}\right)^{\frac{3}{2}} 3^{\frac{3}{2}} \cdot A^{\frac{3}{2}}$$

$$V^* = \underbrace{\left(\frac{1}{6}\right)^{\frac{3}{2}} A^{\frac{3}{2}}}_{V} \cdot 3^{\frac{3}{2}}$$

$$\therefore\ volume\ increases\ by\ a\ factor\ of\ 3^{\frac{3}{2}}\ or \sqrt{27}$$

B3.What is the equation of in log $-log$ space? Find the slope and y-intercept.

$$In\ V = \ln\left[\left(\frac{1}{6}\right)^{\frac{3}{2}} A^{\frac{3}{2}}\right]$$

$$= \ln\left(A^{\frac{3}{2}}\right) + \ln(\frac{1}{6})^{\frac{3}{2}}$$

$$InV = \frac{3}{2}\ln A + \ln(\frac{1}{6})^{\frac{3}{2}}$$

Slope is $\frac{3}{2}$ $\ and\ y - intercept\ is\ \ \ln(\frac{1}{6})^{\frac{3}{2}}$

B4. $lny = mlnx + \ln(k)$ where the slope is m and the y-intercept is ln(k)

$$lny = \frac{3}{4}lnx + ln4$$

$$lny - lnx^{\frac{3}{4}} + ln4$$

$$lny = \ln(\ 4x^{\frac{3}{4}})$$

$y = 4x^{\frac{3}{4}}$ $\quad m \neq 1$ so it is allometric

B5. LS=v

=m/s

RS=u+at

$$= \frac{m}{s} + \frac{m}{s^2}(s)$$

$$= \frac{m}{s} + \frac{m}{s}$$

$$= \frac{m}{s}$$

Therefore, a must be in $\frac{m}{s^2}$.

C. Recursion Models

Example 1.

a) $P_t = P_{t-1} + 0.15P_{t-1} = 1.15P_{t-1}$ $\quad b = 1.15$

b) General solution is $P_t = b^t \times P_o$

$$P_t = (1.15)^t(200) \quad or \quad 200(1.15)^t$$

c) $P_5 = 200(1.15)^5 = 402.2$ $\quad \therefore 402\ deer$

Example 2. Short Cut

a) $a = 1 + \beta - \gamma = 1 + 1.3 - 0.8 = 1.5$

$P_t = 1.5P_{t-1} - 1500 \quad harvested \quad \therefore negative$

$$P_t = \left(P_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a},\ a \neq 1$$

$$= \left(12\ 500 - \frac{-1500}{1-1.5}\right)(1.5)^t + \frac{-1500}{1-1.5}$$

$$P_t = (12\ 500 - 3000)(1.5)^t + 3000$$

$$P_t = 9500(1.5)^t + 3000$$

Long Method

$P_o = 12\ 500$

$P_t = 1.5P_{t-1} - 1500\ \boxed{1}$ $\quad b = -1500$

$a = 1.5$

$P_o = 12\ 500$

$\Delta P_t = P_t - P_{t-1}$

$= 1.5P_{t-1} - 1500 - P_{t-1} \quad sub\ \boxed{1}$

$= P_{t-1} + 0.5P_{t-1} - 1500 - P_{t-1}$

$= \underbrace{P_{t-1} + 0.5P_{t-1} - 1500}_{\boxed{3}} - P_{t-1}$

$\Delta P_t = 0.5P_{t-1} - 1500$

Consider when $\Delta P_t = 0$

$0.5P_{t-1} - 1500 = 0$

$0.5P_{t-1} = 1500$

$P_{t-1} = 3000$

Define $u_t = P_t - 3000\ \boxed{2}$

$u_{t-1} = P_{t-1} - 3000 \quad sub\ \boxed{3}\ into\ \boxed{2}$

$u_t = P_{t-1} + 0.5P_{t-1} - 1500 - 3000$

$u_t = 1(P_{t-1} - 3000) + 0.5(P_{t-1} - 3000)$

$u_t = 1.05(P_{t-1} - 3000) \leftarrow\ u_{t-1}$

$u_t = 1.05u_{t-1} \quad b = 1.05$

From $\boxed{2}$ $u_t = P_t - 3000$ $\quad sub\ t = 0 \quad P_o = 12\,500$

$U_o = P_o - 3000$

$U_o = 12\,500 - 3000 \ = 9500$

$\therefore U_t = U_o(b)^t$

$U_t = 9500(1.05)^t$

$\therefore from\ \boxed{2} \quad U_t = P_t - 3000$

$P_t = U_t + 3000$

$\therefore P_t = 9500(1.05)^t + 3000$

b)$sub\ n = 5 \ \ P_5 = 9500(1.5)^5 + 3000 = 75\,140.6$

c) $P_t = 1.5P_{t-1} - 1500$ We want $P_t = P_{t-1}$ $;find\ b$

$P_t = 1.5P_{t-1} - b$

$12\,500 = 1.5(12\,500) - b$ $\quad since$ (12,500 is not changing)

$-6250 = -b$

$b = 6250$

Example 3.

i) a)$P_t = bP_{t-1}$ $\quad \therefore b = 0.4$

let $P_o = 10$ $\quad P_1 = 0.4P_o = 0.4(10) = 4$

$P_2 = 0.4P_1 = 0.4(4) = 1.6$

$P_3 = 0.4P_2 = 0.4(1.6) = 0.64$

b) $P_t = b^tP_o$

$P_t = 0.4^t(10)\ or\ \ 10(0.4)^t$

c)$10(0.4)^\infty = 0$ $\quad as\ t \to \infty, \quad P_t \to 0$

In other words, $\lim_{t\to\infty} 10(0.4)^t = \infty,$

ii) a)$P_t = 3P_{t-1} + 4$ $\quad a = 3 \quad b = 4$

$P_o = 10$

$P_1 = 3P_o + 4 = 3(10) + 4 = 34$

$P_2 = 3P_1 + 4 = 3(34) + 4 = 106$

b)$P_t = \left(P_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}, a \neq 1$ $\qquad = \left(10 - \frac{4}{1-3}\right)(3)^t + \frac{4}{1-3}$

$= \left(10 - \frac{4}{-2}\right)(3)^t - 2$

$= (12)(3)^t - 2$

c) $as\ t \to \infty,\ \ P_t \to \infty$

In other words, $\lim_{t\to\infty}[\,(12)(3)^t - 2\,] = \infty,$

Example 4.

a) $P_t = 0.80P_{t-1} \qquad P_0 = 50$

b) $P_t = b^t P_0 = (0.8)^t(50)$

c) $P_5 = (0.8)^5(50) = 16.4$

Example 5. $u(t) = 20(\frac{1}{2})^{\frac{t}{1600}}$

$u(2000) = 20(\frac{1}{2})^{\frac{2000}{1600}} = 20\sqrt[4]{(\frac{1}{2})^5} = 20\sqrt[4]{\frac{1}{32}}8$ g

Example 6. **Short Cut Method for Multiple Choice**

$P_t = (P_{t-1} + 0.04P_{t-1}) - 50$

$P_t = 1.04P_{t-1} - 50 \qquad a = 1.04 \qquad b = -50$

$$Pt = \left(1000 - \frac{-50}{1-1.04}\right)(1.04)^t + (\frac{-50}{1-1.04})$$
$$= \left(1000 + \frac{50}{-0.04}\right)1.04^t + (\frac{-50}{-0.04})$$
$$= (1000 - 1250)1.04^t + 1250$$
$$= -250(1.04)^t + 1250$$

How long until it is worth \$250?

$$250 = -250(1.04^t) + 1250$$
$$-1000 = -250(1.04)^t$$
$$4 = 1.04^t$$
$$t = \frac{ln4}{ln1.04}\ years$$

Long Method

$P_t = 1.04P_{t-1} - 50$ $\boxed{1}$ $\quad b = -50 \quad a = 1.04 \quad P_o = 1000$

$P_t = P_{t-1} + 0.04P_{t-1} - 50$

$\Delta P_t = P_t - P_{t-1}$

$= P_{t-1} + 0.04P_{t-1} - 50 - P_{t-1}$

$\Delta P_t = 0.04P_{t-1} - 50$

Consider when there is no change ie. equilibrium

$\Delta P_t = 0 \quad 0 = 0.04P_{t-1} - 50$

$50 = 0.04P_{t-1}$

$P_{t-1} = 1250$

Define $u_t = P_t - 1250$ which means $P_t = u_t + 1250$ $\boxed{2}$

$u_{t-1} = P_{t-1} - 1250$ which means $P_{t-1} = u_{t-1} + 1250$ $\boxed{2}$

From $\boxed{1}$ $P_t = 1.04P_{t-1} - 50$ *substitute* $\boxed{2}$

$u_t + 1250 = 1.04(u_{t-1} + 1250)$ - 50

$u_t = 1.04u_{t-1} + 1300 - 50 - 1250$

$u_t = 1.04u_{t-1}$, so b=1.04

$\boxed{2}$ $u_t = P_t - 1250$

Substitute $t = 0$ $\quad u_o = P_o - 1250 \quad P_o = 1000$ substitute

$u_o = 1000 - 1250$

$u_o = -250$

$u_t = u_o(b)^t$

$u_t = -250(1.04)^t$ $\boxed{3}$

From $\boxed{2}$ $\overbrace{u_t = P_t - 1250}$

Substitute $\boxed{3}$ *into here* $P_t = u_t + 1250$

$P_t = -250(1.04)^t + 1250$

Example 7. $6U_n = 3U_{n-1} + 2$

$U_n = \frac{3}{6}\ U_{n-1} + \frac{2}{6}$

$U_n = \frac{1}{2}U_{n-1} + \frac{1}{3}$

$U_t = \frac{1}{2}U_{t-1} + \frac{1}{3}$

Same as $U_{t+1} = a\,U_t + b$

$a = \frac{1}{2} \quad b = \frac{1}{3}$

$U_t = \left(U_0 - \frac{b}{1-a}\right)(a)^t + \frac{b}{1-a}$

$U_t = \left(2 - \frac{\frac{1}{3}}{1-\frac{1}{2}}\right)(\frac{1}{2})^t + \frac{\frac{1}{3}}{1-\frac{1}{2}}$

$= \left(2 - \frac{\frac{1}{3}}{\frac{1}{2}}\right)\left(\frac{1}{2}\right)^t + \frac{\frac{1}{3}}{\frac{1}{2}}$

$= \left(2 - \frac{2}{3}\right)\left(\frac{1}{2}\right)^t + \frac{2}{3}$

$U_t = \frac{4}{3}(\frac{1}{2})^t + \frac{2}{3}$ *explicit solution*

$\lim_{t\to\infty}(\frac{4}{3}\left(\frac{1}{2}\right)^t + \frac{2}{3}\right) = \frac{2}{3}$

An extra one to try!

Example 8. Short Cut Method for Multiple Choice

$P_t = 3P_{t-1} + 1 \quad \leftarrow a = 3 \quad b = 1 \ and\ P_o = 1$

$P_t = \left(P_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}$

$Pt = \left(1 - \frac{1}{1-3}\right)(3)^t + (\frac{1}{1-3})$

$Pt = \frac{3}{2}(3^t) - \frac{1}{2})$

Example 8. Long Method!!

$P_t = 3P_{t-1} + 1 \quad \leftarrow a = 3 \quad b = 1 \ and\ P_o = 1$

Find t *so that* $P_t = 2000$

$P_t = 1P_{t-1} + 2P_{t-1} + 1 \ \boxed{1}$

$\Delta P_t = P_t - P_{t-1}$

$= 1P_{t-1} + 2P_{t-1} + 1 - P_{t-1}$

$\Delta P_t = 2P_{t-1} + 1 \quad Consider\ \Delta P_t = 0$

$0 = 2P_{t-1} + 1$

$-1 = 2P_{t-1}$

$P_{t-1} = -\frac{1}{2}$

Define $u_t = P_t - \left(-\frac{1}{2}\right) = P_t + \frac{1}{2} \quad which\ means\ P_t = u_t - \frac{1}{2}$ $\boxed{2}$

And $u_{t-1} = P_{t-1} + \frac{1}{2}$ $which\ means\ P_{t-1} = u_{t-1} - \frac{1}{2}$ $\boxed{2}$

From $\boxed{1}$ $\quad P_t = 3P_{t-1} + 1 \quad substitute\ from$ $\boxed{2}$

$$u_t - \frac{1}{2} = 3(u_{t-1} - \frac{1}{2}) + 1$$

$$u_t = 3u_{t-1} - \frac{3}{2} + 1 + \frac{1}{2}$$

$$u_t = 3u_{t-1} \quad \leftarrow b = 3$$

From $\boxed{2}$ $\quad u_t = P_t + \frac{1}{2}$

Substitute $t = 0 \quad P_o = 1$

$$u_o = P_o + \frac{1}{2}$$

$$u_o = 1 + \frac{1}{2} = \frac{3}{2}$$

$u_t = u_o(b)^t$

$\therefore u_t = \frac{3}{2}(b)^t$

$u_t = \frac{3}{2}(3)^t$ $\boxed{3}$ substitute

From $\boxed{2}$ $\quad u_t = P_t + \frac{1}{2}$

$\frac{3}{2}(3)^t = P_t + \frac{1}{2}$

$\therefore P_t = \frac{3}{2}(3)^t - \frac{1}{2}$

Let $P_t = 12$

$$12 = \frac{3}{2}(3)^t - \frac{1}{2}$$

$\frac{25}{2} = \frac{3}{2}(3)^t$ $multiply\ by\ 2\ and\ \ divide\ by\ 3\ then$ take the ln of both sides

$25 = 3\ (3)^t$

$25/3 = (3)^t$

$\ln\left(\frac{25}{3}\right) = \ln\ (3)^t$

$$t = \frac{\ln\left(\frac{25}{3}\right)}{ln3}$$

Practice Exam Questions on Recursion Models

C1. i) $let\ P_0 = 10 \qquad P_0 = 10$

a) $P_1 - P_0 = 6$

$P_1 - 10 = 6$

$P_1 = 16 \qquad P_2 - P_1 = 6$

$P_2 - 16 = 6 \qquad P_2 = 22$

b) $same\ as\ P_t - P_{t-1} = 6$

$P_t = P_{t-1} + 6 \qquad a = 1 \quad \therefore$

$no\ general\ solution\ using\ the\ formula\ as\ we\ can't$ divide by 1-a if a=1

Since the first term is $\mathrm{P}_0 = 10,$ and we are adding $6(1)$ to it to get 16 and then the next term is adding $6(2)$ to 10 to get 22, etc.

The solution could be written as $P_t = 6t + 10\ by\ inspection.$

c) $as\ \ t \to \infty, \quad P_t \to \infty$

ii) $let\ P_0 = 10$

a) $P_t = 2P_{t-1} + 5 \qquad P_0 = 10$

$P_1 = 2P_0 + 5 = 2(10) + 5 = 25$

$P_2 = 2P_1 + 5 = 2(25) + 5 = 55$

b) $a = 2, \ \ b = 5$

$$P_t = \left(P_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a} = \left(10 - \frac{5}{1-2}\right)(2)^t + \frac{5}{1-2}$$
$$= 15(2)^t - 5$$

c) $as\ \ t \to \infty,\ P_t \to \infty$

iii) $P_0 = 10$

a) $P_t = -1.2P_{t-1} \qquad P_1 = -1.2(10) = -12$

$P_2 = -1.2P_1 = -1.2(-12) = 14.4$

b)$P_t = b^tP_0 \ = (-1.2)^t(10)$

c)$as\ \ t \to \infty,\ \ P_t \to \infty$

C2.a) $P_t = 0.85P_{t-1} \qquad P_0 = 250$

b) $P_t = b^tP_0 = (0.85)^t(250)$

$c)\ P_1 = (0.85)^1(250) = 212.5\ mg$

C3. Since 25 are removed each year, we have: $b = -25 \quad P_0 = 1000$ So, the equation would be $P_t = aP_{t-1} + b$ and we get: $P_t = 1.10P_{t-1} - 25$ since the population is increasing by 10% per year. $P_t = \left(P_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}$

$$P_t = \left(1000 - \frac{-25}{1-1.1}\right)(1.1)^t + \frac{-25}{1-1.1}$$
$$P_t = (1000 - 250)(1.1)^t + 250$$
$$P_t = 750(1.1)^t + 250$$

C4. $a)P_0 = 10 \qquad P_1 = 1.2(10) = 12$
$$P_2 = 1.2P_1 = 1.2(12) = 14.4$$
$$P_3 = 1.2P_2 = 1.2(14.4) = 17.28$$
$$P_t = b^t P_0$$
$$\therefore P_t = (1.2)^t(10) \ \ or \ \ 10(1.2)^t$$

$b)P_0 = 10 \qquad P_t = aP_{t-1} + b \qquad a = 0.2 \quad b = 4$
$$P_1 = aP_0 + b = 0.2P_0 + 4 = 0.2(10) + 4 = 6$$
$$P_2 = 0.2P_1 + 4 = 0.2(6) + 4 = 5.2$$
$$P_3 = 0.2P_2 + 4 = 0.2(5.2) + 4 = 5.04$$
$$P_t = \left(P_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}$$

$$= \left(10 - \frac{4}{1-0.2}\right)^{(0.2)^t} + \frac{4}{1-0.2}$$
$$= (10 - 5)(0.2)^t + 5$$
$$P_t = 5(0.2)^t + 5$$

C5. a) let $P_o = 10 \qquad P_o = 10$
$$P_1 = P_o + 5 = 10 + 5 = 15$$
$$P_2 = P_o + 5 = 15 + 5 = 20$$
$$P_t = aP_{t-1} + b \qquad a = 1 \quad b = 5$$

b)$P_t = \left(P_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}, a \neq 1$
$$= 10 - \frac{5}{1-1} \quad - divide\ by\ 0$$
$no\ general\ solution\ using\ the\ formula\ as\ we\ can't$ divide by 1-a if a=1

Since the first term is $\mathrm{P}_0 = 10$, and we are adding $5(1)$ to it to get 15 and then the next term is adding $5(2)$ to 10 to get 20, etc.
The solution could be written as $P_t = 5t + 10\ by\ inspection.$

c) $as \quad t \to \infty,\ P_t \to \infty$ (numbers keep increasing in part a)

C6. **Short-Cut Method**

$$P_t = (P_{t-1} + 0.04P_{t-1}) - 100$$
$$P_t = 1.04P_{t-1} - 100 \qquad a = 1.04 \quad b = -100$$

$$Pt = \left(Po - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}$$
$$= \left(1000 - \frac{(-100)}{1-1.04}\right)(1.04)^t + \left(\frac{-100}{1-1.04}\right)$$
$$= \left(1000 + \frac{100}{-0.04}\right)(1.04)^t + \left(\frac{-100}{-0.04}\right)$$
$$= (1000 - 2500)1.04^t + 2500$$
$$= -1500(1.04)^t + 2500$$
$$worthless \quad 0 = -1500(1.04)^t + 2500$$
$$1500(1.04)^t = 2500$$
$$1.04^t = 1.\overline{6}$$
$$\ln 1.04^t = \ln 1.\overline{6}$$
$$t = 13\ years$$

Long Method:

C6. $P_t = P_{t-1} + 0.04P_{t-1} - 100 \qquad a = 1.04 \quad b = -100$

$P_o = 1000$

$P_t = 1.04P_{t-1} - 100$ $\boxed{1}$

$\Delta P_t = P_t - P_{t-1} \quad sub\,\boxed{1}$

$= 1.04P_{t-1} - 100 - P_{t-1}$

$\Delta P_t = \underbrace{P_{t-1} + 0.04P_{t-1} - 100}_{\boxed{3}} - P_{t-1}$

$\Delta P_t = 0.04P_{t-1} - 100$

Consider $\Delta P_t = 0$

$0.04P_{t-1} - 100 = 0$

$0.04P_{t-1} = 100$

$P_{t-1} = 2500$

Define $U_t = P_t - 2500$ $\boxed{2}$ which means $P_t = U_t + 2500$

$U_{t-1} = P_{t-1} - 2500$ $\boxed{2}$ which means $P_{t-1} = U_{t-1} + 2500$

From $\boxed{1}$ $P_t = 1.04P_{t-1} - 100$ substitute $\boxed{2}$

$U_t + 2500 = 1.04(U_{t-1} + 2500) - 100$

$U_t = 1.04U_{t-1} + 2600 - 100 - 2500$

$\therefore U_t = 1.04U_{t-1} \quad \leftarrow b = 1.04 \quad U_t = P_t - 2500$

$subst \quad t = 0$

$P_o = 1000$

$U_o = P_o - 2500$

$U_o = 1000 - 2500 \ = \ -1500$

$U_t = U_o(b)^t$

$U_t = -1500(1.04)^t \quad \boxed{4}$

$\therefore from \ \boxed{2} \quad U_t = P_t - 2500$

$P_t = U_t + 2500$

$\therefore P_t = -1500(1.04)^t + 2500 \quad from \boxed{4}$

C7. $P_t = b^t P_o$ and $P_o = 50$

a) $P_t = bP_{t-1} = 1.2P_{t-1}$

b) $P_t = (1.2)^t(50) = 50(1.2)^t$

D. Review of Substitution

Example 2. Integrate $\int \frac{cosx}{(1+sinx)^3}dx$

Substitution

$u = 1 + \sin x \qquad du = \cos x dx$

$\int \frac{1}{u^3}du = \int u^{-3}\,du = \frac{u^{-2}}{-2} + c = \frac{-1}{2u^2} + c = \frac{-1}{2(1+\sin x)^2} + c$

Example 3. Integrate $\int \frac{4sec^2x}{(1+tanx)^2}dx$ Substitution

$u = 1 + \tan x \qquad du = \sec^2 x dx$

$= \int \frac{4du}{u^2} = 4\int u^{-2}du = \frac{4u^{-1}}{-1} + c \quad or \quad \frac{-4}{u^1} + c \quad or \quad \frac{-4}{(1+\tan x)} + c$

Example 4. Integrate $\int \frac{5}{x(1+lnx)^3}dx$ Substitution

$u = 1 + \ln x \qquad du = \frac{1}{x}dx$

$= 5\int \frac{du}{u^3} = 5\int u^{-3}du = 5\frac{u^{-2}}{-2} + c = \frac{-5}{2u^2} + c = \frac{-5}{2(1+\ln x)^2} + c$

Example 5. Integrate $\int_0^1 \frac{2x}{x^2+1}dx$

$u = x^2 + 1 \qquad du = 2xdx$

$x = 0 \quad u = o^2 + 1 = 1$

$x = 1 \quad u = 1^2 + 1 = 2$

$= \int_1^2 \frac{du}{u} = \int_1^2 \frac{1}{u}du = [\ln u]_1^2 = (\ln 2 - \ln 1) = (ln2 - 0)$

$= ln2$

Example 6. Integrate $\int_1^3 \frac{1}{x\sqrt{1-lnx}}dx$

$u = 1 - \ln x \qquad du = \frac{-1}{x}dx \qquad -du = \frac{1}{x}dx$

$x = 1 \quad u = 1 - \ln 1 = 1$

$x = 3 \quad u = 1 - \ln 3$

$= -\int_1^{1-\ln 3} \frac{1}{\sqrt{u}}du = -\int_1^{1-\ln 3} u^{\frac{-1}{2}} = -\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^{1-\ln 3}$

$= -\left[2\sqrt{u}\right]_1^{1-\ln 3} = -\left[2\sqrt{1-\ln 3} - 2\sqrt{1}\right] = -2(\sqrt{1-\ln 3} - 1)$

Practice Exam Questions on Substitution

D1. Substitution $u = e^x + 1 \quad du = e^x dx$

$$\int u^{-3} du = \frac{u^{-2}}{-2} + c = -\frac{1}{2}(e^x + 1)^{-2} + c$$

D2. Substitution Integrate: $\int \frac{\sec^2 x}{(1+\tan x)^3} dx$

Substitution

$u = 1 + \tan x \qquad du = \sec^2 x dx$

$= \int \frac{1}{u^3} du = \int u^{-3} du = \frac{u^{-2}}{-2} + c \quad or \quad \frac{-1}{2u^2} + c$

$= \frac{-1}{2(1+\tan x)^2} + c$

D3. Substitution

Integrate: $\int \frac{1}{x(ln\, x)^2} dx$

Substitution

$u = \ln x \qquad du = \frac{1}{x} dx$

$= \int \frac{1}{u^2} du = \int u^{-2} du = \frac{u^{-1}}{-1} + c = \frac{-1}{u} + c = \frac{-1}{\ln x} + c$

E. Solving Differential Equations

Example 4.

(a) $y = \int 5x^4 dx = x^5 + c$

(b) $y = \int (\frac{4}{x} - x^2 + 4\cos x)dx$

$= 4\ln|x| - \frac{x^3}{3} + 4\sin x + c$

(c) $y = \int(-x^3 + 2\sin x + 6\sec^2 x)dx$

$= \frac{-x^4}{4} - 2\cos x + 6\tan x + c$

Separable Differential Equations

Example 2. a) $\frac{dy}{\sin y} = e^x dx$ $\quad Yes, it\ is\ separable$

*this one we can't solve

b) No, if we multiply by dx we get $dy = (x + 2y)dx$

↑

Has both $x\ and\ y$ ∴ not separable

c) Yes

$\frac{dy}{dx} = e^x \cdot e^y$ $\quad using\ exponent\ rules$

$\frac{dy}{e^y} = e^x dx$

$\int e^{-y}dy = \int e^x dx$

$-e^{-y} = e^x + c$

$e^{-y} = -e^x - c$

$\ln e^{-y} = \ln|-e^x - c|$

$-y = \ln|-e^x - c|$

$y = -\ln|-e^x + c|$

Example 3.

Cross-multiply and get $\int 3u^2 du = \int 2t dt$ and integrate

$\frac{3u^3}{3} = t^2 + c$

$u^3 = t^2 + c$

and solve for u(t)

$u(t) = \sqrt[3]{t^2 + c}$

Example 4.

$\int (u + \cos u) du = \int (3t^4 - 2t) dx$

$\frac{u^2}{2} + \sin u = \frac{3t^5}{5} - t^2 + c$. We can't solve this for u(t) on the left!

Example 5.

$$\int \frac{1}{y+2} dy = \int \frac{1}{x-1} dx$$

$\ln|y+2| = \ln|x-1| + c$

$e^{\ln|y+2|} = e^{\ln|x-1|+c} = e^{\ln|x-1|} e^{c}$

$y + 2 = \pm|x-1|(e^c)$

$y(x) = \pm e^c |x-1| - 2 \qquad \pm\ e^c = constant$= call it k or C

$y(x) = C|x-1| - 2$

Example 6. $\frac{du}{dx} = e^{-u(t)}(2t-4) \qquad y(5) = 0$

$\frac{du}{e^{-u(t)}} = (2t-4)dx$

$\int e^{u(t)} du = \int (2t-4) dx$

$e^{u(t)} = t^2 - 4t + c \qquad sub \quad t = 5,\ u(t) = 0$

$e^0 = 5^2 - 4(5) + c \qquad c = -4$

$\ln e^{u(t)} = \ln|t^2 - 4t + c|$

$u(t) = \ln|t^2 - 4t - 4|$

Example 7.

$\frac{dy}{dt} = \frac{4te^{t^2}}{2y+1} \qquad y(0) = 2$

$\int (2y+1)dy = \int 4te^{t^2}dt$

Substitution:

$let\ u = t^2$
$du = 2tdt$
$2du = 4tdt$

$$\int (2y+1)\,dy = 2\int e^4 du$$

$y^2 + y = 2e^u + c$
$y^2 + y = 2e^{t^2} + c \qquad sub\ (0{,}2)$
$2^2 + 2 = 2e^0 + c$
$4 + 2 = 2 + c \qquad \therefore c = 4$
$\therefore y^2 + y = 2e^{t^2} + 4$ We can't solve for y on the left!

Example 8.

$\int \frac{du}{u(t)} = \int 5dt$

$\ln|u(t)| = 5t + c$

$u(t) = \pm e^{5t} \cdot e^c \qquad \pm e^c = constant$

$u(t) = ce^{5t}$

Linear Differential Equations

Example 1.

$x^2y' + 3xy = 2\sin x$

$y' + \frac{3}{x}y = \frac{2\sin x}{x^2}$

$V(x) = I = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$ (since x>0 $\ln|x| = lnx$)

The answer is $\boxed{c}$

Example 2.

$xy' - (2x+3)y = 2\sec x$

$y' - \frac{(2x+3)}{x}y = \frac{2\sec x}{x}$

$V(x) = I = e^{\int \frac{-2x-3}{x}dx} = e^{\int\left(-2-\frac{3}{x}\right)dx} = e^{-2x-3\ln x} = e^{-2x}e^{\ln x^{-3}} = e^{-2x} \cdot x^{-3} = x^{-3}e^{-2x}$

Example 3.

$xy' + 2y = x^{-3}$

$y' + \frac{2}{x}y = \frac{x^{-3}}{x}$

$y' + \frac{2}{x}y = x^{-4}$ $\quad V(x) = I = e^{\int \frac{2}{x}dx} = e^{2|lnx|} = e^{2\ln x} = e^{\ln x^2} = x^2$ **since x>0, we can take off the absolute value of lnx and just say it is lnx

$V(x)y = \int V(x)Q(x)dx$

$x^2y = \int x^{-4}\, x^2 dx$

$x^2y = \int x^{-2}\, dx$

$x^2y = \frac{x^{-1}}{-1} + c$

$x^2y = \frac{-1}{x} + c$

$y = \frac{-1}{x^3} + cx^{-2}$

Example 4.

$u'(t) + 3u(t) = e^{-t}$

$V(x) = I = e^{\int 3dx} = e^{3t}$

$$e^{3t}u(t) = \int e^{3t}e^{-t}\,dt$$
$$e^{3t}u(t) = \int e^{2t}dt$$
$$e^{3t}u(t) = \frac{e^{2t}}{2} + C$$
$$\therefore u(t) = \frac{e^{2t}}{2e^{3t}} + \frac{C}{e^{3t}}$$
$$u(t) = \frac{1}{2}e^{-t} + Ce^{-3t}$$

What if you have the initial condition (0,3)?

$e^{3t}u(t) = \frac{e^{2t}}{2} + C$ substitue t=0 and u(t) =3 to find c

$$e^{0}(3) = \frac{e^{0}}{2} + C$$

$$c = 3 - \frac{1}{2} = \frac{6}{2} - \frac{1}{2} = \frac{5}{2}$$

$e^{3t}u(t) = \frac{e^{2t}}{2} + \frac{5}{2}$ Divide by e^{3t}

$$u(t) = \frac{1}{2}e^{-t} + \frac{5}{2}e^{-3t}$$

Example 5. $I = V(x) = e^{\int P(x)dx} = e^{\int 3x^2dx} = e^{x^3}$

$$V(x)y = \int V(x)Q(x)dx$$

$Subst \ldots u = x^3 \quad du = 3x^2dx \quad 2du = 6x^2dx$

$$e^{x^3}y = \int e^{x^3}6x^2dx$$
$$e^{x^3}y = \int e^{u}(2du)$$
$$e^{x^3}y = 2e^{u} + c$$
$$e^{x^3}y = 2e^{x^3} + c$$
$$y = \frac{2e^{x^3}}{e^{x^3}} + \frac{c}{e^{x^3}}$$
$$y = 2 + ce^{-x^3}$$

Example 6.Which are separable?

A. $\frac{dy}{dx} = \cos y(x) + xy(x)$ $\quad NO$

B. $\frac{dy}{dx} = \cos y(x) - x^3$ $\quad NO$

C. $\frac{dy}{dx} = (\cos x)y(x) + y(x)$ $\quad YES$

Factor $\frac{dy}{dx} = y(x)[\cos x + 1]$

D. $\frac{dy}{dx} = (\sin y)(x) - y(x)$ $\quad NO$

Example 7. If $y_1(x) and\ y_2(x)$ both solve the inhomogeneous D.E.

$y'(x) = 6p(x)y(x) + 5q(x)$

Find another solution:

$a = 6 \qquad 1 - a = 1 - 6 = -5$

$y(x) = ay_1(x) + (1-a)y_2(x)$

$\therefore y(x) = 6y_1(x) - 5y_2(x)$ is also a solution

Example 8.

Solve: $u'(t) + \frac{1}{t+1}u(t) = t^2$

$$P(t) = \frac{1}{t+1} \qquad Q(t) = t^2$$

Since P(t) is discontinuous at $t = -1$

$D = (-\infty, -1) \cup ((-1, \infty)$

a) $u(o) = 1 \qquad \therefore D = (-1, \infty)$

$v(t) = e^{\int P(t)dt} = e^{\int \frac{1}{t+1}dt} = e^{\ln|t+1|} \quad *$

$= t + 1 \quad since\ t > -1$

$v(t)u(t) = \int v(t)Q(t)$

$(t+1)u(t) = \int (t+1)(t^2)dt$

$(t+1)u(t) = \int (t^3 + t^2)dt$

$(t+1)u(t) = \frac{t^4}{4} + \frac{t^3}{3} + c \quad sub\ (0,1) \quad t = 0, u(t) = 1$

$(0+1)(1) = 0 + 0 + c$

$c = 1$

$\therefore (t+1)u(t) = \frac{t^4}{4} + \frac{t^3}{3} + 1$

$(t+1)u(t) = \frac{3t^4}{12} + \frac{4t^3}{12} + \frac{12}{12}$

$(t+1)u(t) = \frac{3t^4 + 4t^3 + 12}{2}$

$u(t) = \frac{3t^4 + 4t^3 + 12}{2(t+1)}$

b) Same equation as a)

$u(-2) = 1 \;\leftarrow\; D = (-\infty, -1)\ since\ t < -1$

from * in part a)

$v(t) = e^{\ln|t+1|} = -(t+1)\ \ since\ t < -1$

$\therefore\ -(t+1)u(t) = \int -(t+1)(t^2)dt$

$-(t+1)u(t) = \int(-t^3 - t^2)dt$

$-(t+1)u(t) = \frac{-t^4}{4} - \frac{t^3}{3} + c$

$sub\ \binom{-2{,}1}{t,u}\quad -(-2+1)(1) = \frac{-(-2)^4}{4} - \frac{(-2)^3}{3} + c$

$-(-1) = \frac{-16}{4} - \left(\frac{-8}{3}\right) + c$

$1 + \frac{16}{4} - \frac{8}{3} = c$

$c = \frac{12}{12} + \frac{48}{12} - \frac{32}{12}$

$= \frac{28}{12}$

$= \frac{7}{3}$

$\therefore -(t+1)u(t) = \frac{-t^4}{4} - \frac{t^3}{3} + \frac{7}{3}$

$-(t+1)u(t) = \frac{-3t^4 - 4t^3 + 28}{12}$

$u(t) = \frac{-(-3t^4 - 4t^3 + 28}{12(t+1)}\quad or\quad u(t) = \frac{3t^4 + 4t^3 - 28}{12t + 12}$

Bernoulli Differential Equations

Example 1.

$y' + \frac{4}{x}y = x^3y^2, y(2) = -1, x > 0$

$\uparrow n = 2 \ \frac{dy}{dx} + P(x)y = y^nQ(x) \ n = 2$

$Q(x) = x^3 \quad P(x) = \frac{4}{x}$

Use $\frac{du}{dx} + (1-n)up(x) = (1-n)Q(x) \quad u = y^{1-n} = y^{1-2} = y^{-1}$

$u' - \frac{4}{x}u = -x^3 \qquad V(x) = I = e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = e^{\ln x^{-4}} = \frac{1}{x^4}$ **since x>0, we can just take off the absolute value (since x>0 $\ln|x| = lnx$)

V(x) u= $\int V(x)Q(x)dx$

$\frac{1}{x^4}u = \int \frac{1}{x^4}(-x^3)\,dx$

$$\frac{1}{x^4}u = \int \frac{-1}{x}dx$$

$\frac{1}{x^4}u = -\ln|x| + c$

$u = x^4(-\ln|x| + c) \ from \ substitution \ \ u = y^{-1}$

$y^{-1} = x^4(-\ln|x| + c)$

$y = \frac{1}{x^4(c - \ln|x|)} \ * \quad y(2) = -1$

$-1 = \frac{1}{2^4(c - \ln 2)}$

$-16(c - \ln 2) = 1$

$-16c + 16\ln 2 = 1$

$c = \frac{1 - 16\ln 2}{-16} \quad = \frac{-1}{16} + \ln 2$

$\therefore y = \frac{1}{x^4\left[\ln 2 - \frac{1}{16} - ln|x|\right]} \qquad from \ *$

Example 2. $x^3y'(x) = y(x)\big(1 + y(x)\big) \qquad y(1) = 1$

$x^3y'(x) = y(x) + (y(x))^2$
$y'(x) = \frac{1}{x^3}y(x) + \frac{1}{x^3}(y(x))^2$
$y'(x) - \frac{1}{x^3}y(x) = \frac{1}{x^3}(y(x))^2$
$P(x) = -\frac{1}{x^3} \qquad Q(x) = \frac{1}{x^3}$
$y^n = y^2 \quad \therefore n = 2$
$u = y^{1-n} = y^{1-2} = y^{-1}$

$\frac{du}{dx} + (1-n)uP(x) = (1-n)Q(x)$

$\frac{du}{dx} + (-1)u\left(\frac{-1}{x^3}\right) = -1(\frac{1}{x^3})$
$\frac{du}{dx} + x^{-3}u = -x^{-3} \quad linear$

$V(x) = e^{\int x^{-3}dx} = e^{x^{-2}/-2} = e^{-1/2x^2}$
$V(x)u = \int V(x)Q(x)dx$
$e^{-1/2x^2}u = \int e^{-1/2x^2}(-x^{-3})dx$

Substitution $w = \frac{-1}{2x^2} = \frac{-1}{2}x^{-2}$
$dw = -\frac{1}{2}(-2x^{-3})dx$
$dw = x^{-3}dx$
$e^{-1/2x^2}u = -\int e^w dw$
$e^{-1/2x^2}u = -e^w + c$
$e^{-1/2x^2}u = -e^{-1/2x^2} + c$
$u = -1 + ce^{1/2x^2}$
Substitution $u = y^{-1}$
$y^{-1} = -1 + ce^{1/2x^2}$
$y = \frac{1}{-1+ce^{1/2x^2}}$

$y(1) = 1$
$1 = \frac{1}{-1+ce^{1/2}}$
$-1 + ce^{1/2} = 1$
$ce^{1/2} = 2$
$c = 2\big/e^{\frac{1}{2}} \qquad \therefore y(x) = \frac{1}{-1+\frac{2}{e^{\frac{1}{2}}}e^{1/2x^2}}$

Applications

Mixing Problems

Example 1. (0,0) or y(0)=0 since it is pure water at the start.

$\frac{dy}{dt} = 20(2) - \frac{1y}{100+t}$ so that means $\frac{dy}{dt} + \frac{1y}{100+t} = 40$

Example 2. y(0)=0 $V(t) = I = e^{\int \frac{1}{400}} = e^{\frac{1}{400}t}$

$$\frac{dy}{dt} = (in)(in) - \frac{y(out)}{volume}$$
$$\frac{dy}{dt} = (0.1)(5) - \frac{5y}{2000}$$
$$\frac{dy}{dt} = 0.5 - \frac{1}{400}y$$
$$\frac{dy}{dt} + \frac{1}{400}y = 0.5$$
$$V(t)y = \int V(t)Q(t)dt$$
$$e^{\frac{1}{400}t}y = \int e^{\frac{1}{400}t}0.5$$
$$e^{\frac{1}{400}t}y = \frac{0.5e^{\frac{1}{400}t}}{\frac{1}{400}} + c$$
$$e^{\frac{1}{400}t}y = 200e^{\frac{1}{400}t} + c$$
$$y = 200 + \frac{c}{e^{\frac{1}{400}t}}$$
$$solve\ for\ c\ \rightarrow sub(0,0) \qquad 0 = 200 + \frac{c}{e^0}$$

$\therefore c = -200$ $\qquad y = 200 - 200e^{\frac{-1}{400}t}$ and y(0)=100

NOTE: If they said you had 7kg/L at the start and 2000 L in the tank, then y(0) would be y(0)= 7 kg/L (2000L) = 14000 kg, or y(0)=14000

IV Drug Problems

<u>Example 3.</u> $Rate\ in\left(\frac{mg}{hr}\right) = concentration \times flow\ of\ drug$

$$= 150\,\frac{mg}{L} \times \frac{40mL}{h} \leftarrow units\ need\ to\ match$$
$$= 150\,\frac{mg}{L} \times 0.04\ L/hr$$
$$= 6\ mg/hr$$

Rate out = rate metabolized + rate exceeded

Rate metabolized= $k \cdot u(t)$

$$k = \ln\left(\frac{1}{1-p}\right) \leftarrow \%\ of\ drug\ metabolized$$
$$k = \ln\left(\frac{1}{1-0.8}\right)$$
$$k = \ln\left(\frac{1}{0.2}\right) = \ln 5$$
$$\therefore rate\ metabolized = \ln 5 \cdot u(t)$$

Rate excreted=concentration× flow

$$= \frac{u(t)mg}{5\ L} \times \frac{0.04L}{hr}$$
$$= \frac{1}{125}u(t)mg/hr$$

$$rate\ out = \ln 5 \cdot u(t) + \frac{1}{125}u(t)$$
$$= \left(\ln 5 + \frac{1}{125}\right)u(t)$$

$$\frac{du}{dt} = rate\ in - rate\ out$$
$$\frac{du}{dt} = 6 - \left(\ln 5 + \frac{1}{125}\right)u(t)$$

$$\frac{du}{dt} + \overbrace{\left(\ln 5 + \frac{1}{125}\right)}^{\swarrow P(t)} u(t) = 6 \ \leftarrow Q(t)$$
$$\searrow let\ this\ be\ A$$
$$V(t) = I = e^{\int A dt} = e^{At}$$

$$V(t)u = \int V(t)Q(t)dt$$
$$e^{At}u = \int 6e^{At}dt$$
$$e^{At}u = \frac{6e^{At}}{A} + c$$
$$u = \frac{6}{A} + ce^{-At}$$

$$\therefore u(t) = \frac{6}{A} + ce^{-At} \quad substitute\ \ (0,0)$$
$$\uparrow(t,u)$$

$$0 = \frac{6}{A} + ce^{0}$$
$$c = \frac{-6}{A}$$
$$\therefore u = \frac{6}{A} - \frac{6}{A}e^{-At}$$
$$u(t) = \frac{6}{A}(1 - e^{-At})$$
$$A = \left(\ln 5 + \frac{1}{125}\right)$$

$$u(t) = \frac{6}{(ln5 + \frac{1}{125})}(1 - e^{-\left(\ln 5 + \frac{1}{125}\right)t})$$

Population

Example 4. $N(t)\ is\ the\ number\ of\ animals\ at\ time\ \ t$

$$\frac{dN}{dt} = a\left(\frac{1}{time}\right) \times \frac{N}{pigs} + n\left(\frac{pigs}{time}\right) - b(\frac{1}{time}) \times \frac{N}{pigs}$$
$$\frac{dN}{dt} = aN - bN + n$$
$$= (a - b)N + n$$

To find long-term pop find $N(t)\ and\ let\ t \to \infty$

$\boxed{1}$ $N'(t) - (a - b)N = n$

$N'(t) + (b - a)N = n$ Q(t)=n   P(t)= (b-a)N

$\boxed{2}$ $V(t) = I = e^{\int (b-a)dt} = e^{(b-a)t}$

$\boxed{3}$ $(V(t)N) = \int V(t)Q(t)dt$

$\left(e^{(b-a)t}N\right) = \int e^{(b-a)t}n\,dt$

$$e^{(b-a)t}N = \frac{ne^{(b-a)t}}{(b-a)} + c$$
$$N(t) = \frac{n}{b-a} + \frac{c}{e^{(b-a)t}}$$
$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty}\left[\frac{n}{b-a} + \frac{c}{e^{(b-a)t}}\right]$$
$$= \boxed{\frac{n}{b-a}}\ pigs$$

Practice Exam Questions on Differential Equations

E1. $\int y dy = \int x^2 dx$

$\frac{y^2}{2} = \frac{x^3}{3} + c \quad \leftarrow sub\ (1,2)$

$\frac{2^2}{2} = \frac{1^3}{3} + c$

$2 = \frac{1}{3} + c$

$c = 2 - \frac{1}{3} = \frac{5}{3}$

$\therefore \frac{y^2}{2} = \frac{x^3}{3} + \frac{5}{3}$

$y^2 = \frac{2x^3}{3} + \frac{10}{3}$

$y = \pm\sqrt{\frac{2}{3}x^3 + \frac{10}{3}}$

E2. $\int y dy = \int 2x + \sec^2 x$

$\frac{y^2}{2} = x^2 + \tan x + c$

$y^2 = 2x^2 + 2\tan x + 2c$

$y = \pm\sqrt{2x^2 + 2\tan x + k}$

E3. $\int \frac{dy}{y} = \int \frac{\mathbf{1}}{\boldsymbol{x+1}} \boldsymbol{dx}$

$\ln|y| = \ln|x+1| + c$

$e^{\ln y} = \pm e^{\ln|x+1|+c}$

$y = ke^{\ln|x+1|} \quad or \quad y = k|x+1|$

E4. $\frac{dy}{dx} = 3e^x e^y$

$\int \frac{dy}{e^y} = \int 3e^x dx$

$\frac{e^{-y}}{-1} = 3e^x + c$

$e^{-y} = -3e^x - c$ substitute x=0 and y=1

$e^{-1} = -3e^0 - c$

$c = -3 - \frac{1}{e}$

$\therefore e^{-y} = -3e^x + 3 + \frac{1}{e}$

$\ln e^{-y} = \ln\left|-3e^x + 3 + \frac{1}{e}\right|$

$-y = \ln\left|-3e^x + 3 + \frac{1}{e}\right|$

$y = -\ln\left|-3e^x + 3 + \frac{1}{e}\right|$

E5. $\int 2ydy = \int \frac{x}{x^2+3}dx$ *substitution* $u = x^2+3$ $\frac{du}{2} = \frac{2xdx}{2}$

$\int 2ydy = \frac{1}{2}\int \frac{1}{u}du$

$y^2 = \frac{1}{2}\ln|u| + c$

$y^2 = \frac{1}{2}\ln|x^2+3| + c$

$y = \pm\sqrt{\frac{1}{2}\ln|x^2+3| + c}$

E6. $\int ydy = \int \frac{t}{e^{t^2+1}}dt$ $u = t^2+1$ $\frac{du}{2} = \frac{2t}{2}dt$

$\frac{1}{2}du = t\,dt$

$\frac{y^2}{2} = \frac{1}{2}\int \frac{1}{e^u}du$

$\frac{y^2}{2} = \frac{1}{2}\frac{e^{-u}}{-1} + c$

$\frac{y^2}{2} = \frac{-1}{2e^u} + c$

$\frac{y^2}{2} = \frac{-1}{2e^{t^2+1}} + 2c$

$y^2 = \frac{-2}{2e^{t^2+1}} + k$

$y = \pm\sqrt{\frac{-1}{e^{t^2+1}} + k}$

or

$y = \pm\sqrt{\frac{-1}{e^{t^2+1}} + k}$ since k=2c is just a constant

E7. $\int e^y dy = \int \frac{\sin x}{\cos^2 x}dx$ $u = \cos x$ $-du = \sin x dx$

$e^y = -\int u^{-2}du$

$e^y = \frac{-u^{-1}}{-1} + c$

$e^y = \frac{1}{u} + c$

$e^y = \frac{1}{\cos x} + c$

$e^y = \sec x + c$

$y = \ln|\sec x + c|$

E8. $\int y^2 dy = \int \sin x dx$

$\frac{y^3}{3} = -\cos x + c$

$y^3 = -3\cos x + 3c$

$y = \sqrt[3]{-3\cos x + k}$ since k=3c is just a constant. Call it C or k, or whatever!

E9. $\frac{dy}{dt}(2y-2)=\frac{1}{t^2}$

$\int dy(2y-2)=\int t^{-2}dt$

$y^2-2y=\frac{t^{-1}}{-1}+c$

$y^2-2y=\frac{-1}{t}+c \quad sub\ t=1,\ y=0$

$0-0=\frac{-1}{1}+c \quad c=1$

$\therefore y^2-2y=\frac{-1}{t}+1$

E10. $\frac{dN}{dt}=t^3(N+3)$

$\int \frac{dN}{N+3}=\int t^3dt$

$\ln|N+3|=\frac{t^4}{4}+c \quad sub\ t=0, N=5 \ \ to\ find\ c$

$\ln 8=0+c$

$c=\ln 8$

$\therefore \ln|N+3|=\frac{t^4}{4}+\ln 8 \quad sub\ \ t=2$

$\ln|N+3|=\frac{2^4}{4}+\ln 8$

$\ln|N+3|=4+\ln 8$

$N+3=e^{4+\ln 8}$

$N+3=e^4e^{\ln 8}$

$N+3=8e^4$

$\therefore N=8e^4-3$

E11. Linear

$P(x)=-1 \quad Q(x)=e^x \quad V(x)=I=e^{\int P(x)dx}=e^{\int -1dx}=e^{-x}$

$V(x)y=\int V(x)Q(x)dx$

$e^{-x}y=\int e^{-x}e^xdx$

$e^{-x}y=\int 1dx$

$e^{-x}y=x+c$

$y=xe^x+ce^x$

E12. Linear $\frac{dy}{dx}=\frac{x^{\frac{1}{2}}}{x}-\frac{y}{x}$

$\frac{dy}{dx}+\frac{1}{x}y=x^{\frac{-1}{2}}$

P(x)=1/x is not continuous at x=0. Since x>0 so we look at (0,∞) and we get:
$|x|=x$. Remember, the domain can't include any negative numbers, as we have a square root of x.

$$V(x)=I=e^{\int\frac{1}{x}dx}=e^{ln|x|}=x$$

$V(x)y=\int V(x)Q(x)dx$

$xy=\int xx^{\frac{-1}{2}}$ dx

$xy=\int x^{\frac{1}{2}dx}$

$xy=\frac{2}{3}x^{\frac{3}{2}}+c$

$y=\frac{2}{3}x^{\frac{1}{2}}+cx^{-1}$

$y=\frac{2}{3}\sqrt{x}+cx^{-1}$

E13.

$\frac{dy}{dx}+\frac{2}{x}y=\frac{x^{-1}e^{x}}{x}$

$\frac{dy}{dx}+\frac{2}{x}y=x^{-2}e^{x}$ P(x)=2/x and Q(x)= $x^{-2}e^{x}$

Linear V(x)= $I=e^{\int\frac{2}{x}dx}=e^{2ln|x|}$=$e^{ln|x|^{2}}=(-x)^{2}=x^{2}$

P(x) is not continuous at x=0, so we look at (-∞, 0) and we get:

$|x|=-x$

So, V(x)= $e^{2ln|x|}=e^{ln|x|^{2}}=(-x)^{2}=x^{2}$

If we look at (0, ∞)

$|x|=x$

So, V(x)= $e^{2ln|x|}=e^{ln|x|^{2}}=x^{2}$

Both answers are the same!

$V(x)y=\int V(x)Q(x)dx$

$x^{2}y=\int x^{2}x^{-2}e^{x}\,dx$

$x^{2}y=\int e^{x}dx$

$x^{2}y=e^{x}+c$ $y=\frac{e^{x}}{x^{2}}+\frac{c}{x^{2}}$

E14. Linear $V(x) = \ I = e^{\int \frac{1}{x}dx} = e^{ln|x|} = e^{\ln x} = x$ and since the point is (1,2) x>0, so we just take off the absolute value and do nothing

$$\frac{dy}{dx} + \frac{x}{x^2}y = \frac{\ln x}{x^2}$$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\ln x}{x^2}$$

$$V(x)y = \int V(x)Q(x)dx$$

$$(xy) = \int x\,(\frac{\ln x}{x^2})dx$$

$$\int \frac{d}{dx}(xy) = \int \frac{\ln x}{x} \qquad u = \ln x \quad du = \frac{1}{x}dx$$

$$xy = \int udu = \frac{u^2}{2} + c = \frac{(\ln x)^2}{2} + c$$

$$sub\ (1,2) \quad (1)(2) = \frac{(\ln 1)^2}{2} + c \qquad c = 2$$

$$\therefore y = \frac{(\ln x)^2}{2x} + \frac{2}{x}$$

E15. Linear

P(x) is not continuous at x=0, if we look at x>0, we get $|x| = x$

$$V(x) = I = e^{\int \frac{-3}{x}dx} = e^{-3ln|x|} = e^{\ln|x|^{-3}} = |x|^{-3} = x^{-3}$$

$$\frac{dy}{dx} - \frac{3}{x}y = 2x^3$$

$$V(x)y = \int V(x)Q(x)dx$$

$$x^{-3}y = \int x^{-3}(2x^3)dx$$

$$\int \frac{d}{dx}(x^{-3}y) = \int 2dx$$

$$x^{-3}y = 2x + c$$

$$y = 2x^4 + cx^3$$

If $V(x) = I = e^{\int \frac{-3}{x} dx} = e^{-3ln|x|} = e^{\ln|x|^{-3}} = |x|^{-3}$=$(-x)^{-3}$=$-x^{-3}$
x<0, we get $|x| = -x$

$$-x^{-3}y = -\int x^{-3}(2x^3)dx$$

$\int \frac{d}{dx}(x^{-3}y) = \int 2dx$ divide both sides by -1

$$x^{-3}y = 2x + c$$

$$y = 2x^4 + cx^3$$

E16. Linear $\frac{dy}{dx} + \frac{1}{x^2}y = 3e^{\frac{1}{x}}$

$$V(X) = I = e^{\int \frac{1}{x^2}dx} = e^{\int x^{-2}dx} = e^{x^{-1}/-1} = e^{-\frac{1}{x}}$$

$$V(x)y = \int V(x)Q(x)dx$$

$$e^{-\frac{1}{x}}y = \int 3e^{\frac{1}{x}}\, e^{-\frac{1}{x}}dx$$

$$e^{-\frac{1}{x}}y = \int 3dx$$

$e^{-\frac{1}{x}}y = 3x + c$ divide every term by $e^{-\frac{1}{x}}$

$$y = 3xe^{\frac{1}{x}} + ce^{\frac{1}{x}}$$

E17. $V(x) = I = e^{\int P(x)dx} = e^{\int -e^x dx} = e^{-e^x}$

E18.

$a)$ (0,0) $\frac{dy}{dt} = (0.25)(3) - \frac{3y}{400}$

$$\frac{dy}{dt} = 0.75 - \frac{3}{400}y$$

$b)$ $\frac{dy}{dt} = 0.75 - \frac{4y}{400 - 1t}$

E19. $\frac{dy}{dt} = (in)(in) - \frac{y(out)}{volume\ at\ time\ t}$ $(0,25)$ $salt\ at\ start$

$$\frac{dy}{dt} = (0.25)(5) - \frac{y(5)}{1000} \quad OR \quad \frac{dy}{dt} = 1.25 - \frac{1}{200}y$$

E20. Linear a) $P(x) = \frac{-1}{x}$

$V(x) = I = e^{\int f(x)dx} = e^{\frac{-1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$

$b)\ \frac{dy}{dx} - \frac{3x^2y}{x} = \frac{x^3}{x}$

$\frac{dy}{dx} - 3xy = x^2$

$V(x) = I = e^{\int P(x)dx} = e^{\int -3x} = e^{\frac{-3x^2}{2}}$

$c)\ \frac{dy}{dx} + y = x^2$

$P(x) = 1$

$V(x) = I = e^{\int 1dx} = e^x$

E21. Linear $\frac{dy}{dx} - \frac{2y}{x} = \frac{x^2}{x} \quad \rightarrow \quad \frac{dy}{dx} - \frac{2y}{x} = x \qquad P(x) is\ \frac{-2}{x} \quad x\ is\ Q(x)$

$$V(x) = I = e^{\int \frac{-2}{x}dx}$$

$$V(x) = I = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

** note: P(x)j is discontinuous at x=0, but here if you take x>0, you get V(x)= $x^{-2} = \frac{1}{x^2}$

As when you remove the absolute value, if it is a positive quantity under the absolute value, you just take off the absolute value and do nothing

If x<0, you get $V(x) = I = e^{-2ln|x|} = e^{\ln|x|^{-2}} = (-x)^{-2} = \frac{1}{x^2}$ so they are equal!!

$V(x)y = \int V(x)Q(x)dx$

$x^{-2}y = \int x^{-2}xdx$

$x^{-2}y = \int \frac{1}{x}dx$

$x^{-2}y = \ln|x| + c$

$y = x^2 \ln|x| + cx^2$

E22. Linear $\frac{dy}{dx}+y=e^{-x}$ $P(x)=1$ $Q(x)=e^{-x}$ y(0)=2

$V(x)=I=e^{\int P(x)dx}=e^{\int 1\,dx}=e^{x}$

$V(x)y=\int V(x)Q(x)dx$

$e^{x}y=\int e^{-x}\,e^{x}dx$

$e^{x}y=\int 1\text{dx}$

$e^{x}y=x+c$

$y=\frac{x}{e^{x}}+\frac{c}{e^{x}}$ OR $xe^{-x}+ce^{-x}$ sub (0,2) $2=0e^{0}+ce^{0}$ $c=2$

$y=xe^{-x}+2e^{-x}$

E23. Linear $y(\pi)=0$. Since x=pi>0, when you take off the absolute value, you just get x:

$V(x)=I=e^{\int\frac{3}{x}dx}=e^{3ln|x|}=e^{\ln|x|^{3}}=x^{3}$ since $|x|=x$ as x>0

$\frac{dy}{dx}+\frac{3}{x}y=\frac{\cos x}{x^{3}}$

$V(x)y=\int V(x)Q(x)dx$

$x^{3}y=\int\frac{x^{3}\cos x}{x^{3}}dx$

$x^{3}y=\int\cos x\,dx$

$x^{3}y=\sin x+c$ $find\ c$ $sub\ \ x=\pi$ $y=0$

$0=\sin\pi+c$

$c=0$

$y=x^{-3}\sin x+cx^{-3}$

$\therefore y=x^{-3}\sin x$

E24. Linear $3xy' + y = 12x$

$y' + \frac{1}{3x}y = \frac{12x}{3x}$

$y' + \frac{1}{3x}y = 4 \qquad V(x) = I = e^{\int \frac{1}{3x}dx} = e^{\frac{1}{3}\ln x} = e^{\ln x^{\frac{1}{3}}} = x^{\frac{1}{3}}$

$V(x)y = \int V(x)Q(x)dx$

$x^{\frac{1}{3}}y = \int 4x^{\frac{1}{3}}dx$

$x^{\frac{1}{3}}y = \frac{4x^{\frac{4}{3}}}{\frac{4}{3}} + c$

$x^{\frac{1}{3}}y = 3x^{\frac{4}{3}} + c$

$y = \frac{3x^{\frac{4}{3}}}{x^{\frac{1}{3}}} + \frac{c}{x^{\frac{1}{3}}}$

$\therefore y = 3x + cx^{-\frac{1}{3}}$

The answer is $\boxed{A}$

E25. Bernoulli

$\frac{dy}{dx} - \frac{y}{x} = y^9 \qquad n = 9 \quad P(x) = \frac{-1}{x} \qquad Q(x) = 1$

$\frac{du}{dx} + (1-n)u\,P(x) = (1-n)Q(x) \quad \textit{substitution} \quad u = y^{1-n}$

$u = y^{1-9} = y^{-8}$

$\frac{du}{dx} + \frac{8}{x}u = -8 \qquad V(x) = I = e^{\int \frac{8}{x}dx} = e^{8\ln x} = x^8$

$V(x)u = \int V(x)Q(x)dx$

$x^8 u = \int x^8(-8)dx$

$x^8 u = \frac{-8x^9}{9} + c$

$\frac{x^8 u}{x^8} = \frac{-8x^9}{9x^8} + \frac{c}{x^8}$

$u = \frac{-8}{9}x + cx^{-8}$

$y^{-8} = \frac{-8}{9}x + cx^{-8}$

$y = \left(\frac{-8}{9}x + cx^{-8}\right)^{\frac{-1}{8}}$

E26. Bernoulli

$y' - y = y^2 e^x$

$\uparrow \qquad \uparrow$

$P(x) \qquad Q(x) \qquad u = y^{1-n} = y^{1-2} = y^{-1}$

$\frac{du}{dx} + (1-n)uP(x) = (1-n)Q(x)$

$\frac{du}{dx} + (-1)u(-1) = -1e^x$

$\frac{du}{dx} + u = -e^x$

$V(x) = I = e^{\int 1dx} = e^x$

$V(x)u \int V(x)Q(x)dx$

$e^x u = \int e^x(-e^x)dx$

$e^x u = \int -e^{2x} dx$

$e^x u = \frac{-e^{2x}}{2} + c$

$u(x) = \frac{-e^{2x}}{2e^x} + \frac{c}{e^x}$

$u(x) = \frac{-e^x}{2} + ce^{-x}$

$y^{-1} = \frac{-e^x}{2} + ce^{-x}$

$\frac{1}{y} = \frac{-e^x}{2} + ce^{-x}$

$y = \frac{1}{\frac{-e^x}{2} + ce^{-x}}$

E27. Bernoulli

$y' + \frac{1}{x}y = -xy^3$ $\quad\quad u = y^{1-n} = y^{1-3} = y^{-2}$

$\frac{du}{dx} + (1-n)uP(x) = (1-n)Q(x)$

$\frac{du}{dx} + (-2)u\left(\frac{1}{x}\right) = -2(-x)$

$\frac{du}{dx} - \frac{2}{x}u = 2x$

$V(x) = I = e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$

$I = \frac{1}{x^2}$

$$V(x)u = \int V(x)Q(x)dx$$

$$\frac{1}{x^2}u = \int \frac{1}{x^2} \cdot 2xdx$$

$\frac{1}{x^2}u = \int \frac{2}{x}dx$

$\frac{1}{x^2}u = 2\ln|x| + c$

$u(x) = x^2(2\ln|x| + c)$

y^{-2}=$x^2(2\ln|x| + c)$

$\frac{1}{y^2}$=$x^2(2\ln|x| + c)$

$$y^2 = \frac{1}{x^2(2\ln|x| + c)}$$

$$y = \pm\sqrt{\frac{1}{x^2(2\ln|x| + c)}}$$

E28. A)

$(0{,}0)\quad t, salt \qquad V(t) = I = e^{\int \frac{1}{400}} = e^{\frac{1}{400}t}$

$\frac{dy}{dt} = (in)(in) - \frac{y(out)}{volume}$

$\frac{dy}{dt} = (0.1)(5) - \frac{5y}{2000}$

$\frac{dy}{dt} = 0.5 - \frac{1}{400}y$

$\frac{dy}{dt} + \frac{1}{400}y = 0.5$

$V(t)y = \int V(t)Q(t)dt$

$e^{\frac{1}{400}t}y = \int e^{\frac{1}{400}t}0.5$

$e^{\frac{1}{400}t}y = \frac{0.5e^{\frac{1}{400}t}}{\frac{1}{400}} + c$

$e^{\frac{1}{400}t}y = 200e^{\frac{1}{400}t} + c$

$y = 200 + \frac{c}{e^{\frac{1}{400}t}}$

$solve\ for\ c\ \rightarrow sub(0{,}0) \qquad 0 = 200 + \frac{c}{e^0}$

$\therefore c = -200 \qquad y = 200 - 200e^{\frac{-1}{400}t}$

b) $0.05(2000) = 100$

So, $100 = 200 - 200e^{-\frac{1}{400}t}$

$-100 = -200e^{-\frac{1}{400}t}$

$0.5 = e^{-\frac{1}{400}t}$

$ln0.5 = lne^{-\frac{1}{400}t}$

$ln0.5 = -\frac{1}{400}tlne$

t= -400ln0.5

E29. Bernoulli $\int xe^x dx = xe^x - \int e^x dx$

$6y' - 2y = xy^4, y(0) = -2$

$\downarrow n = 4$

$u = y^{1-n} \quad let\ u = y^{-3}$

Divide the original equation by 6 so it has the right form

$y' - \frac{1}{3}y = \frac{1}{6}xy^4$ where P(x)=-1/3 and Q(x)=1/6 x

$$\frac{du}{dx} + (1-n)u\,P(x) = (1-n)Q(x)$$

$$u' + (-3)\left(-\frac{1}{3}\right)u = -3\left(\frac{1}{6}x\right)$$

$u' + u = \frac{-1}{2}x$

$V(x) = I = e^{\int 1\,dx} = e^x$

$$V(x)u = \int V(x)Q(x)dx$$

$$e^x u = \int \frac{-1}{2}e^x \cdot x dx$$

$e^x u = \frac{-1}{2}[xe^x - \int e^x dx]$ *integral given in the question*

$e^x u = \int \frac{-1}{2}xe^x + \frac{1}{2}e^x + c$

$e^x u = \frac{-1}{2}(x-1)e^x + c$

$u = \frac{\frac{-1}{2}(x-1)e^x + c}{e^x}$

$u = \frac{-1}{2}(x-1) + ce^{-x} \quad$ *substitution was* $u = y^{-3}$

$\therefore y^{-3} = \frac{-1}{2}(x-1) + ce^{-x}$

Subst $(0,-2)$ $\quad (-2)^{-3} = \frac{-1}{2}(0-1) + ce^0$

$\frac{-1}{8} = \frac{1}{2} + c$

$c = \frac{-1}{2} - \frac{4}{8} = \frac{-5}{8}$

$c = \frac{-1}{2} - \frac{u}{8} = \frac{-5}{8}$

$$y(x) = \frac{1}{\sqrt[3]{\frac{-1}{2}(x-1)-\frac{5}{8}e^{-x}}}$$

$$y(x) = \frac{1}{\sqrt[3]{\frac{-4(x-1)-5e^{-x}}{8}}}$$

$$y(x) = \frac{1}{\sqrt[3]{\frac{-1}{8}}\sqrt[3]{+4(x-1)+5e^{-x}}}$$

$$= \frac{1}{\frac{-1}{2}\sqrt[3]{4x-4+5e^{-x}}}$$

$$= \frac{-2}{\sqrt[3]{4x-4+5e^{-x}}}$$

E30. Solve: $u'(t) = 10u(t)$

$\int \frac{du}{u(t)} = \int 10tdt$

$\ln|u(t)| = \frac{10t^2}{2} + c$
$\ln|u(t)| = 5t^2 + c$

$e^{ln|u(t)|} = e^{5t^2+c}$ ...take off the absolute value and add +/- and then the +/- combined with e^c is just some constant

$u(t) = \pm e^{5t^2} \cdot e^c$ multiplying with the same base, add the exponents, so this is working backwards

$u(t) = ce^{5t^2}$

E31.

$$\int \frac{1}{y+1}\,dy = \int \frac{1}{x-3}\,dx$$

$$\ln|y+1| = \ln|x-3| + c$$

$e^{\ln|y+1|} = e^{\ln|x-3|+c} = e^{\ln|x-3|}e^{c}$ multiplying with the same base, add the exponents, so this is working backwards

$$y + 1 = \pm|x-3|(e^{c})$$

$$y(x) = \pm e^{c}|x-3| - 1 \qquad \pm e^{c} = \mathit{some\ constant}$$

$$y(x) = k|x-1| - 2 \quad \mathit{final\ constant\ can\ be\ } k, C, \mathit{or\ whatever}!$$

More Practice if you want them!!!

E32.Which are separable?

A. $\frac{dy}{dx} = \cos y(x) + xy(x)$ $\quad NO$

B. $\frac{dy}{dx} = \cos y(x) - x^3$ $\quad NO$

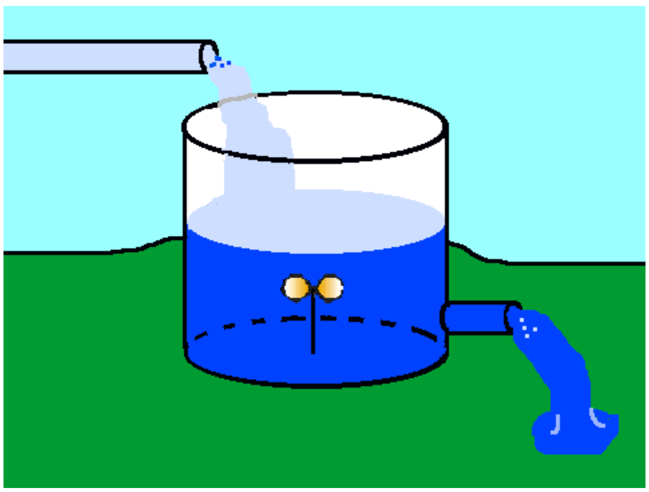
C. $\frac{dy}{dx} = (\cos x)y(x) + y(x)$ $\quad YES$

Factor $\frac{dy}{dx} = y(x)[\cos x + 1]$

D. $\frac{dy}{dx} = (\sin y)(x) - y(x)$ $\quad NO$

E33. Initially a tank contains 2000L of pure water. A valve is opened, allowing a brine solution containing 0.1kg of salt per L to enter the tank at 5L/min. The solution in the tank is stirred constantly at 5L/min. Determine:

a) the amount of salt in the tank t minutes after opening the valve
b) the instant when the concentration reaches 0.05kg per L



E34. A tank contains 400L pure water. Brine with a concentration of 0.25 kg of salt per L begins to flow into the tank at 3L/min and the well stirred solution flows out of the tank at 3L/min.

a) When will the concentration of salt in the tank be 0.2 kg/L?

b) How would the solution be different if the solution flows out at 4L/min?

E35. A tank initially contains 1000L of brine with 25kg of salt in solution. Brine containing 0.25kg of salt per L flows into the tank at 5L/min and the well stirred solution flows out at the same rate. Find the amount of salt in the tank after 45 min.

E36. A tank contains 200L pure water. Brine with a concentration of 0.2 kg of salt per L begins to flow into the tank at 5L/min and the well stirred solution flows out of the tank at 5L/min. What will the concentration of salt in the tank be after 8 minutes?

E37. a) $I = e^{\int \frac{1}{400} dt} = e^{\frac{1}{400}t}$

$(0,0) \quad \leftarrow \textit{pure water}$

$\frac{dy}{dt} = (0.1)(5) - \frac{5y}{2000}$

$\frac{dy}{dt} + \frac{1}{400} y = 0.5$

$\int \frac{d}{dt}\left(e^{\frac{1}{400t}} y\right) = \int 0.5 e^{\frac{1}{400t}}$

$e^{\frac{1}{400t}} y = \frac{0.5 e^{\frac{1}{400t}}}{\frac{1}{400}} + c$

$y = 0.5\left(\frac{400}{1}\right) + c e^{\frac{-1}{400t}}$

$y = 200 + c e^{\frac{-1}{400t}}$ sub (0,0)

$0 = 200 + c e^{0}$

$c = -200$

$\therefore y = 200 - 200 e^{\frac{-1}{400t}}$

b) $(2000)0.05 = 200 - 200 e^{\frac{-1}{400t}}$

$100 - 200 = -200 e^{\frac{-1}{400t}}$

$\frac{-100}{-200} = e^{\frac{-1}{400t}}$

$0.5 = e^{\frac{-1}{400t}}$

$\text{Ln}\, 0.5 = \ln e^{\frac{-1}{400t}}$

$\text{Ln}\, 0.5 = \frac{-1}{400} t$

$t = -400 \ln 0.5$

E38. $\frac{dy}{dt} = (0.25)(3) - \frac{3y}{400}$ $(0,0)$ *pure water*

$\frac{dy}{dt} = 0.75 - \frac{3y}{400}$ $I = e^{\int \frac{3}{400} dt} = e^{\frac{3}{400}t}$

$\frac{dy}{dt} + \frac{3}{400} y = 0.75$

$\int \frac{d}{dt}\left(e^{\frac{3}{400}t} y\right) = \int 0.75 e^{\frac{3}{400}t}$

$e^{\frac{3}{400}t} y = \frac{0.75 e^{\frac{3}{400}t}}{\frac{3}{400}} + c$

$e^{\frac{3}{400}t} y = 100 e^{\frac{3}{400}t} + c$ $\quad sub\ (0,0) \ \rightarrow c = 400$

$y = 100 - \frac{100}{e^{\frac{3}{400}t}}$

b) $sub\ y = 0.2\ and\ solve\ for\ t$

$equation\ would\ be$ $\frac{dy}{dt} = 0.75 - \frac{3y}{400 - 1t}$ $\leftarrow lose\ 1\,\frac{L}{min}$C4. $\frac{dy}{dt} = (0.25)(5) - \frac{y(5)}{1000}$ $(0,25) \leftarrow salt\ at\ start$

$\frac{dy}{dt} = 1.25 - \frac{1}{200} y$

$\frac{dy}{dt} + \frac{1}{200} y = 1.25$ $\quad I = e^{\int \frac{1}{200} dt} = e^{\frac{1}{200}t}$

$\int \frac{d}{dt}\left(e^{\frac{1}{200}t} y\right) = \int e^{\frac{1}{200}t}(1.25)$

$e^{\frac{1}{200}t} y = \frac{1.25 e^{\frac{1}{200}t}}{\frac{1}{200}} + c$

$y = 250 + \frac{c}{e^{\frac{1}{200}t}}$ $\quad sub\ (0,25)\ to\ find\ c$

$25 = 250 + \frac{c}{e^0}$

$c = -225$

$\therefore y = 250 - \frac{225}{e^{\frac{1}{200}t}}$ $at\ 45\ minutes\ sub\ t = 45\ and\ find\ y$

$y = 250 - \frac{225}{e^{\frac{1}{200}(45)}}$

E39. $\frac{dy}{dt} = (0.25)(5) - \frac{y(5)}{1000}$ $\quad (0,25) \leftarrow salt\ at\ start$

$\frac{dy}{dt} = 1.25 - \frac{1}{200}y$

$\frac{dy}{dt} + \frac{1}{200}y = 1.25 \quad I = e^{\int \frac{1}{200}dt} \quad = e^{\frac{1}{200}t}$

$\int \frac{d}{dt}\left(e^{\frac{1}{200}t}y\right) = \int e^{\frac{1}{200}t}(1.25)$

$e^{\frac{1}{200}t}y = \frac{1.25e^{\frac{1}{200}t}}{\frac{1}{200}} + c$

$y = 250 + \frac{c}{e^{\frac{1}{200}t}} \quad sub\ (0,25)\ to\ find\ c$

$25 = 250 + \frac{c}{e^0}$

$c = -225$

$\therefore y = 250 - \frac{225}{e^{\frac{1}{200}t}}$

$at\ 45\ minutes\ sub\ t = 45\ and\ find\ y$

$y = 250 - \frac{225}{e^{\frac{1}{200}(45)}}$

E40. $\frac{dy}{dt} = (0.2)(5) - \frac{y(5)}{200}$

$\frac{dy}{dt} = 1 - \frac{1}{40}y$

$\frac{dy}{dt} + \frac{1}{40}y = 1$

$I = e^{\int \frac{1}{40}dt} \quad = e^{\frac{1}{40}t}$

$\int \frac{d}{dt}(e^{\frac{1}{40}t}\ y) = \int e^{\frac{1}{40}t}$

$e^{\frac{1}{40}t}y = \frac{e^{\frac{1}{40}t}}{\frac{1}{40}} + c$

$y = 40 + \frac{c}{e^{\frac{1}{40}t}} \quad sub\ (0,0)$

$c = -40$

$y = 40 - \frac{40}{e^{\frac{1}{40}t}} = 40 - 40e^{\frac{-1}{40}t}$

$sub\ \ t = 8\ \ to\ find\ y \quad y = 40 - 40e^{\frac{-1}{40}(8)} = 40 - 40e^{-\frac{1}{5}}$kg is the mass and if they want concentration, divide by original volume of 200L

Quiz 1: Practice on Sections A to I

1. $A = 4\pi(3r)^2 = 4\pi(9r^2) = 9(4\pi r^2)$

   $\therefore 9\ times$

   $V = \frac{4}{3}\pi(3r)^3 = \frac{4}{3}\pi(27r^3) = 27(\frac{4}{3}\pi r^3)$

   $\therefore 27\ times$

2. $speed = m/s$

   $\frac{D}{t} = \frac{m}{s}$ $\quad \therefore time\ must\ be\ in\ seconds$

3. $= 2^{\frac{1}{10}}$

   $u(t) = c \times 2^{\frac{t}{10}} = c \times 2^{\frac{1}{10}t}$

4. a) homogeneous
   b) inhomogeneous
   c) inhomogeneous
   d) homogeneous

5. $P_{t+1} = 3000(1.08)^t - n \qquad a = 1.08 \qquad P_0 = 3000 \quad b = -n$

   $Pt = \left(Po - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}$

$$P_t = \left(3000 - \frac{(-n)}{1 - 1.08}\right)(1.08)^t + \frac{-n}{1 - 1.08}$$

= $(3000 - 12.5n)(1.08)^t + 12.5n$

worthless after 10 years, so let t=10

$0 = (3000 - 12.5n)(1.08)^{10} + 12.5n$

$0 = 6476.77 - 26.98656247n + 12.5n$

$-6476.77 = -14.48656247n$

$n = \$447.09$

6. a) linear, 1st order
   b) non-linear, 1st order
   c) non-linear, 2nd order
   d) linear, 2nd order

7. $\frac{du}{dt} = t^2$

$\int du = \int t^2 dt$

$u(t) = \frac{t^3}{3} + c$

8. $\frac{du}{dt} = 2u(t)$

$\frac{1}{u(t)} du = 2dt$

$\int \frac{1}{u(t)} du = \int 2dt$

$\ln|u(t)| = 2t + c$

$e^{\ln|u(t)|} = e^{2t+c}$

$e^{\ln|u(t)|} = e^{2t} \cdot e^{c}$

$u(t) = ae^{2t}$

9. a) Linear $u'(t) - u(t) = \frac{t^3}{3} + t\cos t$

$v(t) = e^{\int -1dt} = \boxed{e^{-t}}$

b) $u'(t) + u(t) = t$

$v(t) = e^{\int 1dt} = \boxed{e^{t}}$

c) $u'(t) - (t+2)u(t) = t$

$v(t) = e^{\int -(t+2)dt} = e^{\int(-t-2)dt} = \boxed{e^{\frac{-t^2}{2} - 2t}}$

$\therefore$
1. $b)$
2. c)
3. a)

10. $Linear \;\; v(t) = e^{\int -\sin t \, dt} = e^{\cos t}$

$$V(t)u = \int V(t)Q(t)dt$$

$e^{\cos t}u(t) = \int e^{\cos t} \sin t \, dt$ substitution $u = \cos t \quad -du = \sin t \, dt$

$e^{\cos t}u(t) = \int -e^{u} du$

$e^{\cos t}u(t) = -e^{u} + c$

$\frac{e^{\cos t}u(t)}{e^{\cos t}} = \frac{-e^{\cos t}}{e^{\cos t}} + \frac{c}{e^{\cos t}}$

$u(t) = -1 + ce^{-\cos t}$

11. Linear $v(t) = e^{\int 1dt} = e^t$

$$V(t)u = \int V(t)Q(t)dt$$

$$e^t u(t) = \int 5e^t dt$$
$$e^t u(t) = 5e^t + c \quad \leftarrow sub \ \ u(t) = 3, \ \ t = 1$$
$$e^1(3) = 5e^1 + c$$
$$c = -2e$$
$$e^t u(t) = 5e^t - 2e$$
$$u(t) = 5 - \frac{2e^1}{e^t}$$
$$u(t) = 5 - 2e^{1-t}$$

12. $Linear\ y' - \frac{1}{x}y = 2\ln x + 1$ Since x=2

$$v(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

$$V(x)y = \int V(x)Q(x)dx$$
$$\frac{1}{x}y = \int \frac{1}{x}(2\ln x + 1)dx$$
$$\frac{1}{x}y = \int \frac{2\ln x\ dx}{x} + \int \frac{1}{x}dx$$

↘substitution $u = \ln x$ $du = \frac{1}{x}dx$

$$\frac{1}{x}y = \int 2udu + \ln|x| + c$$
$$\frac{1}{x}y = \frac{2u^2}{2} + \ln|x| + c$$
$$\frac{1}{x}y = u^2 + \ln|x| + c$$
$$\frac{1}{x}y = (\ln x)^2 + \ln|x| + c \qquad \text{substitute } x = 2 \quad y = 1$$
$$\frac{1}{2}(1) = (\ln 2)^2 + \ln 2 + c$$
$$c = \frac{1}{2} - (\ln 2)^2 - \ln 2$$
$$\therefore \frac{1}{x}y = (\ln x)^2 + \ln|x| + \frac{1}{2} - (\ln 2)^2 - \ln 2$$
$$y = x\left[(\ln x)^2 + \ln|x| + \frac{1}{2} - (\ln 2)^2 - \ln 2\right]$$

F. Stability Analysis of Autonomous Differential Equations

* arrows to right indicate $f(y)$ is positive (above axis)
* arrows to left indicate $f(y)$ is negative (below axis)

Example 1. Find the equilibrium values of the differential equation

$y^2 + 2y - 8 = 0$

$(y+4)(y-2) = 0 \qquad y = 2, -4$

Stable	Unstable
$\rightarrow\leftarrow$	$\leftarrow\rightarrow$
-4	2

	-4	2
$\boxed{-5}$	$\boxed{0}$	$\boxed{3}$
$(-5)^2 + 2(-5) - 8$		$3^2 + 2(3) - 8$
$\boxed{+}$	$\boxed{-}$	$\boxed{+}$
↗	↘	↗

$\therefore$ *equilibrium values* $\hat{y} = 2, -4$
Unstable stable

Example 2.

$y^4 - 16y^2 = 0$

$y^2(y^2 - 16) = 0 \qquad y = 0, 4, -4$

-4 0 4

-5	-1	1	5
$\boxed{+}$	$\boxed{-}$	$\boxed{-}$	$\boxed{+}$
↗	↘	↘	↗

$\rightarrow\leftarrow$	$\leftarrow\leftarrow$	$\leftarrow\rightarrow$
Stable	semi-Stable	unstable

Example 3. $y' = g(y) = y^3 - 4$

$y^3 - 4 = 0$

$y^3 = 4 \qquad y = 1.59 \qquad \therefore \hat{y} = 1.59$ *is an equilibrium point*

$y'' = g'(y) = 3y^2 + 0 = 3y^2$

at $\hat{y} = 1.59 \qquad y'' = 3(1.59)^2 > 0$

$\therefore \hat{y} = 1.59$ *is unstable*

Logistic Growth Model

$\frac{dN}{dt} = r\left(1 - \frac{N}{k}\right)N = 0 \qquad note\!: r, k\ are\ \boxed{+}\ constants$

Example 4.

$r = 0\ \ (not\ possible\ since\ r > 0) \quad or \quad N = 0 \quad or \quad 1 - \frac{N}{k} = 0$

$1 = \frac{N}{k} \qquad k = N \quad or\ N = k$

$\therefore equilibria\ are\ \widehat{N} = 0, \widehat{N} = k$

$$g(N) = r\left(1 - \frac{N}{k}\right)N = r\left(N - \frac{N^2}{k}\right) = rN - \frac{N^2 r}{k} = rN - \frac{r}{k}N^2$$

$$g'(N) = r - \frac{r}{k}(2N) = r - \frac{2rN}{k}$$

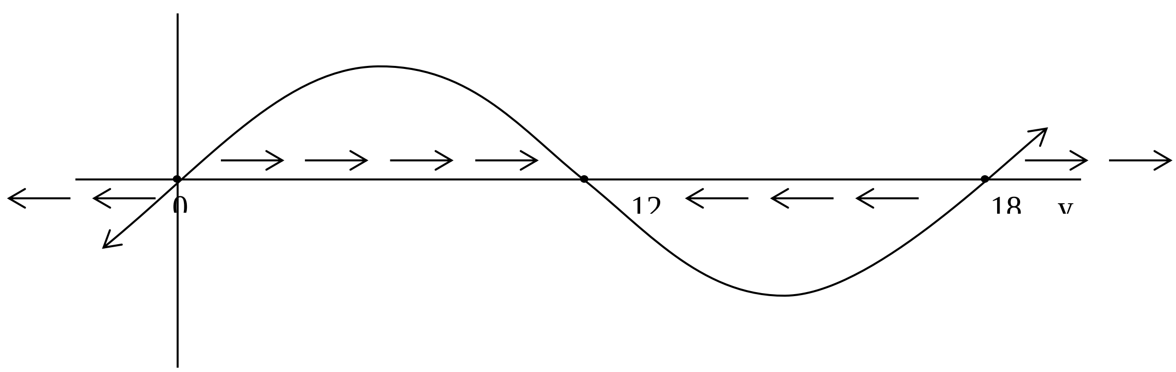
$at\ \widehat{N} = 0 \quad g'(0) = r - 0 = r > 0 \qquad \therefore unstable\ at\ \widehat{N} = 0$

$at\ \widehat{N} = k \quad g'(k) = \frac{r-2r}{k}(k) = r - 2r = -r < 0$

$\therefore stable\ at\ \widehat{N} = k$

Example 5. Stable at b and unstable at a and c.

Example 6. Answer the questions, given the phase plot below:



a) Identify where the graph is increasing and decreasing.
The graph is above the x-axis from 0 to 12, so it is increasing there, as well as from 18 onward
The graph is below the x-axis from 12 to 18, so it is decreasing there as well as from 0 backward
The equation is dy/dt = y (y-12)(y-18). Make sure you test a point in each region to make sure if it is positive, the graph is above the x-axis and negative is below the x-axis. In another scenario it might be dy/dt = -y (y-12)(y-18).

b) Find the equilibria and asses the stability of each of them.
The equilibria are the points where the graph crosses the x-axis, so they are at 0, 12 and 18
At x=0, we have an unstable equilibrium
At x=18, we have an unstable equilibrium (arrows point away from each other)
At x=12, we have a stable equilibrium (arrows point towards each other)

c) Given the following initial conditions, what will the value of y(t) tend towards?
If y(0) =2, y(t) will increase to 12
If y(0)=10, y(t) will increase to 12
If y(0)=15, y(t) will decrease to 12
If y(0)=18, y(t) will remain at 18 (equilibrium)

NOTE: If y(0)=22, y(t) will increase to infinity

Example 7.

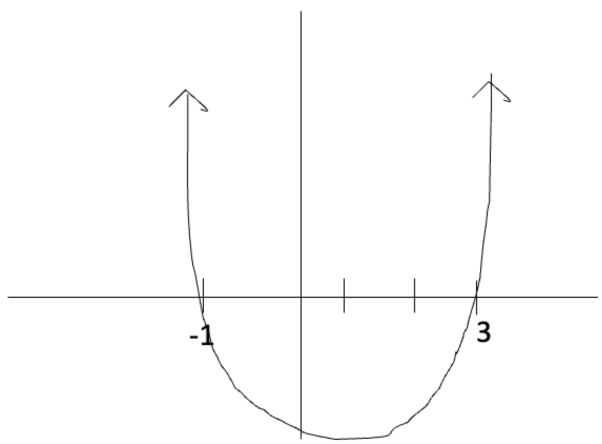
0 test point

-1	1
$(-1)^5$	$(+1)^5$
$\boxed{-}$	$\boxed{+}$

$\leftarrow\leftarrow\leftarrow$ $\rightarrow\rightarrow\rightarrow$

$\therefore$ *unstable*

Example 8. The correct graph is:



It is increasing (above the x-axis from 3 to infinity and from negative infinity to -1. It is below the x-axis, or decreasing from -1 to 3. See the number line below: It is stable at -1 and unstable at 3.

Increasing from 3 to ∞ and from $-\infty$ *to* -1.
Decreasing from -1 *to* 3

$\rightarrow\leftarrow$ $\leftarrow\rightarrow$

-1 3
Stable unstable

Practice Exam Questions on Stability Analysis of Autonomous Differential Equations

F1. (a)

$\rightarrow\leftarrow$

4 $\quad stable$

(b) $y^3 - 3y^2 = 0$

$y^2(y-3) = 0$

y=0, 3

$\leftarrow\leftarrow \qquad \leftarrow\rightarrow$

0 $\qquad$ 3 $\quad unstable\ at\ \ y = 3$

(c)

$2e^{2y} - 4e^y = 0$

$2e^y(e^y - 2) = 0$

$e^y = 0\ means\ no\ solution$ $\quad e^y = 2$ take ln of both sides and y=ln2

$\leftarrow\rightarrow$

$\ln 2$ $\qquad unstable\ at\ \ y = \ln 2$

(d) $0 = (y-3)(y+2) \qquad y = 3\ \ or\ \ y = -2$

-2 $\qquad$ 3

-3	0	4
$(-3)^2 + 3 - 6$	$\boxed{-}$	$4^2 - 4 - 6$
$\boxed{+}$ $\nearrow$	$\searrow$	$\boxed{+}$ $\nearrow$

$\longrightarrow\longleftarrow \qquad \longleftarrow\longrightarrow$

Stable at -2 $\qquad$ unstable at 3

F2.

k is a constant k> 0

$k(A - H) = 0 \qquad H = A$

$\therefore equilibria\ \ \hat{H} = A$

$$g(H) = k(A - H)$$

$$g'(H) = k(-1) = -k < 0 \qquad \therefore stable\ at\ \hat{H} = A$$

F3.

$kP = 0 \qquad P = 0 \quad since\ k\ is\ a\ constant\ \ > 0$

$\therefore equilibria\ \ \hat{P} = 0$

F4.

$0.04(20 - H) = 0 \qquad \therefore H = 20$

Equilibrium $\hat{H} = 20$

$let\ g(H) = 0.04(20 - H) = 0.8 - 0.04H$

$g'(H) = -0.04 < 0 \quad \therefore \quad stable\ at\ \hat{H} = 20$

F5.

$H \geq 0 \quad and \quad H = 10$

$$\frac{dN}{dt} = 2N\left(1 - \frac{N}{100}\right) - H = 2N - \frac{2N^2}{100} - H$$

$$\frac{dN}{dt} = 0 \qquad 2N - \frac{2N^2}{100} - H = 0$$

$$2N - \frac{2N^2}{100} - 10 = 0 \quad (divide\ by - 2)$$

$$-N + \frac{1}{100}N^2 + 5 = 0 \qquad (\times 100)$$

$$N^2 - 100N + 500 = 0$$

$$N = \frac{100 \pm \sqrt{(-100)^2 - 4(1)(500)}}{2(1)} = \frac{100 \pm 89.44}{2}$$

$$N = 94.7 \quad or \ \ N = 5.28$$

$equilibria\ \ \hat{N} = 5.28 \quad and\ 94.7$

F6. $a = unstable \quad b = stable \quad c = unstable$

$$\begin{array}{ccccccc} & \leftarrow\rightarrow & & \rightarrow\leftarrow & & \leftarrow\rightarrow & \\ - & a & + & b & - & c & + \end{array}$$

F7.

	-2	0	2
-3	-1	1	3
$\boxed{-}$	$\boxed{+}$	$\boxed{-}$	$\boxed{+}$

$\leftarrow -2 \rightarrow$ Unstable $\quad \rightarrow 0 \leftarrow$ stable $\quad \leftarrow 2 \rightarrow$ unstable

$y' = 0$

$\therefore y^5 - 16y = 0$

$y(y^4 - 16) = 0$

$y = 0, \ \ y^4 = 16$

$y = 2, -2 \quad \therefore \ \ A\ is\ the\ answer$

F8. $find\ equilibrium$

$$rV\left(1-\frac{V}{k}\right)=0$$

$$rV=0 \quad or \quad 1-\frac{V}{k}=0$$

$$V=0 \qquad\qquad 1=\frac{V}{k}$$

$$V=k$$

$classify\ equilibrium$

$V=0 \qquad V=k$

-1	$k-1$	$k+1$
$\boxed{-}$	$\boxed{+}$	$\boxed{-}$

$$\frac{dV}{dt}=rV\left(1-\frac{V}{k}\right)=rV-\frac{rV^2}{k}$$

$$\frac{dV}{dt}\quad at\ V=-1,\ \frac{dV}{dt}=r(-1)-\frac{r(-1)^2}{k}=-r-\frac{r}{k}$$

$$=-r(1+\frac{1}{k})$$

$$<0 \quad since\ r,k\ \boxed{+}\ constants$$

$\leftarrow 0 \rightarrow \qquad \rightarrow k \leftarrow$

Unstable at $V=0$ stable $at\ V=k$

Method 1:

$$\frac{dv}{dt}=rV-\frac{r}{k}V^2$$

$$g(V)=rV-\frac{r}{k}V^2$$

$$g'(V)=r-\frac{r}{k}(2V)$$

$$g'(0)=r-\frac{r}{k}(2(0))$$

$$=r-0=r>0$$

<u>Unstable at 0</u>

$$g'^{(k)}=r-\frac{r}{\cancel{k}}(2\cancel{k})=r-2r$$

$$=-r<0$$

<u>Stable at k</u>

Method 2:

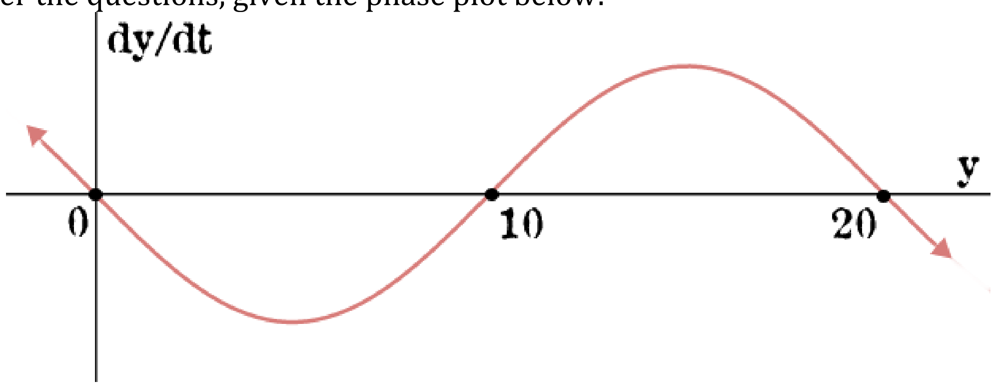
$at\ V = k + 1$

$$\frac{dV}{dt} = r(k+1) - r\frac{(k+1)^2}{k}$$
$$= rk + r - \frac{r(k^2+2k+1)}{k}$$
$$= rk + r - \frac{rk^2}{k} - \frac{2rk}{k} - \frac{r}{k}$$
$$= rk + r - rk - 2r - \frac{r}{k}$$
$$= -r - \frac{r}{k}$$
$$= -r\left(1 + \frac{1}{k}\right) < 0 \quad since\ r, k\ are\ \boxed{+}\ constants$$

$at\ V = k - 1$

$$\frac{dV}{dt} = r(k-1) - r\frac{(k-1)^2}{k}$$
$$= rk - r - \frac{r(k^2-2k+1)}{k}$$
$$= rk - r - \frac{rk^2}{k} + \frac{2rk}{k} - \frac{r}{k}$$
$$= rk - r - rk + 2r - \frac{r}{k}$$
$$= r(1 - \frac{1}{k}) > 0$$

F9. Answer the questions, given the phase plot below:



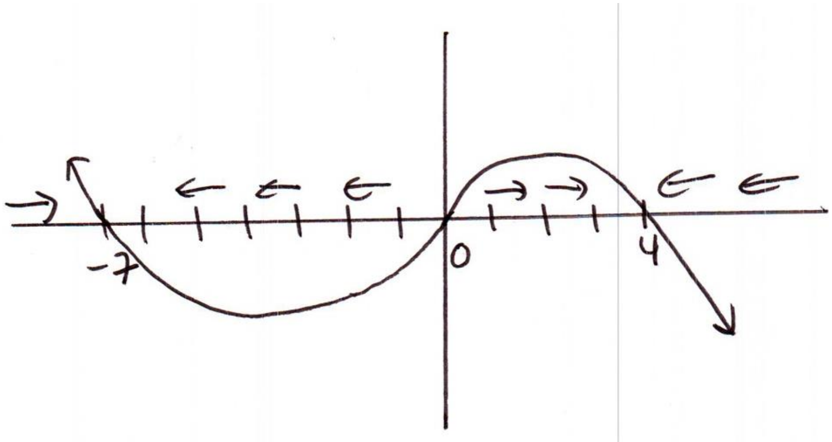
a) Identify where the graph is increasing and decreasing.
The graph is increasing from 10 to 20 and from 0 backwards and it is decreasing from 0 to 10 and from 20 onwards

b) Find the equilibria and asses the stability of each of them.
The equilibria are 0, 10 and 20
At 0, and 20 we have stable equilibria and at 10, we have an unstable equilibria

c) Given the following initial conditions, what will the value of y(t) tend towards?
If y(0) = 2, y(t) will decrease to 0.
If y(0)=10, y(t) will stay at 10 (it is an equilibria)
If y(0)=15, y(t) will increase to 20
If y(0)=18, y(t) will increase to 20

F10. a) The equilibrium are $\underline{0, 4, -7}$

b)



test point $y = 5$ $\quad \frac{dy}{dx} = -y(y-4)(y+7)$

$= -5(5-4)(5+7)$

$< 0 \quad \therefore left\ (below\ axis)$

Test $y = 1$ $\quad \frac{dy}{dx} = -1(1-4)(1+7) > 0 \ \therefore right\ (above\ x\ axis)$

Test $y = -1$ $\quad \frac{dy}{dx} = -(-1)(-1-4)(-1+7) < 0 \quad \therefore left\ (below\ axis)$

$\therefore -7\ is\ stable$ (arrows pointing towards each other)

c)Start at 2, it will go to 4

d) $\lim_{x\to\infty} y(x) = 0 \ \ (stays\ at\ equilibrium)$

Quiz 2: Practice on Sections I and J

1. a=175, b=375, n=8000
   a. $\frac{dN}{at} = (a-b)N + n$
      = (175 – 375)N + 8000
      = -200N + 8000

   b. $\lim_{n\to\infty} \quad \frac{n}{b-a} = \frac{8000}{375-175} = \frac{8000}{200} = 40$

2.
   a. V = 2000 + 5t – 8t
      V = 200 – 3t

   b. $\frac{dA}{dt} = (10)(5) - \frac{8A}{200-3t} = 50 - \frac{8A}{200-3t}$

3. 200 mg/L * 0.06L/hr = 12mg/hr
   K $= \text{In}\left(\frac{1}{1-0.6}\right) = In\left(\frac{1}{0.4}\right) = ln2.5$

   Rate out = In2.5 u(t) $+ \frac{u(t)}{5}$*0.06 L/hr

   = In2.5 u(f) + 0.012u(t)

   $\frac{du}{dt} = 12 - [In2.5 + 0.012\,u(t)]$

   $$\frac{du}{dt} + [In2.5 + 0.012]u(t) = 12$$

   Let P(t)= $[In2.5 + 0.012] = \text{A}$ $\qquad Q(t) = 12$

   $V(t) = e^{\int Adt} = e^{At}$

   $$\int \frac{d}{dt} V(t)\, u(t) = \int V(t)Q(t)dt$$

   $$\int \frac{d}{dt} e^{At}\, u(t) = \int e^{At} * 12dt$$

   $$e^{At}\, u(t) \frac{12e^{At}}{A} + c$$

$$U(t) = \frac{12}{A} + ce^{-At}$$

(0,0)

$$0 = \frac{12}{A} + ce^{0}$$

$$c = \frac{-12}{A}$$

$$u(t) = \frac{12}{A} - \frac{12}{A}e^{-At}$$

$$u(t) = \frac{12}{A}(1 - e^{-At})$$

$$u(t) = \frac{12}{In2.5 + 0.012}(1 - e^{-(In2.5+0.012)t})$$

Let t = 1

$$u(1) = \frac{12}{In2.5 + 0.012}(1 - e^{-(In2.5+0.012)(1)})$$

$$= 7.82$$

4. $y'(t) - \frac{1}{t}y(t) = 4\big(y(t)\big)^4$

$P(t) = -\frac{1}{t}$ $\qquad Q(t) = 4$

a) $u = y^{1-n} = y^{1-4} = y^{-3}$

b) $\frac{dy}{dt} + (1-n)\,P(t)\,u(t) = (1-n)\,Q(t)$

$$u'(t) - 3\left(-\frac{1}{t}\right)u(t) = -3(4)$$

$$u'(t) + \frac{3}{t}u(t) = -12$$

c) $V(t) = e^{\frac{3}{t}} = e^{3lnt} = e^{Int^3} = t^3$

$\int \frac{d}{dt}\,(t^3 u(t)) = \int t^3(-12)dt$

$t^3 u(t) = -\frac{12t^4}{4} = c$

$t^3 u(t) = -3t^4 + c$

$u(t) = -3t + \frac{c}{t^3}$

$1^{-3} = -3(1) = \frac{c}{1^3}$

$1 = -3 = c$

$c = 4$

Therefore, $y^{-3} = 3t + \frac{4}{t^3}$

$y^3 = \frac{1}{-\frac{3f}{1} + \frac{4}{t^3}}$

$y^3 = \frac{1}{\frac{-3t^4+4}{t^3}}$

$y^3 = \frac{t^3}{-3t^4+4}$

$y(t) = \left(\frac{t^3}{-3t^4+4}\right)^{\frac{1}{3}}$

$y(t) = \frac{t}{(-3t^4+4)^{(\frac{1}{3})}}$

5. a) $y = u^{1-n} = u^{1-4} = u^{-3}$

b) $\frac{dy}{dt} + (1-n)\,P(t)\,y(t) = (1-n)\,Q(t)$

$\frac{dy}{dt} + (-3)(1)y(t) = -3(1)$ Q(t)= -3

$$\frac{dy}{dt} - 3y(t) = -3$$

$v(t) = e^{\int -3dt} = e^{-3t}$

$$\int \frac{d}{dt}\left(e^{-3t}y\right) = \int -3e^{-3t}dt$$

$e^{-3t}y = e^{-3t} + c$

$y = 1 + ce^{3t}$

$u^{-3} = 1 + ce^{3t}$

$$\left(\frac{1}{2}\right)^{-3} = 1 + ce^{0}$$

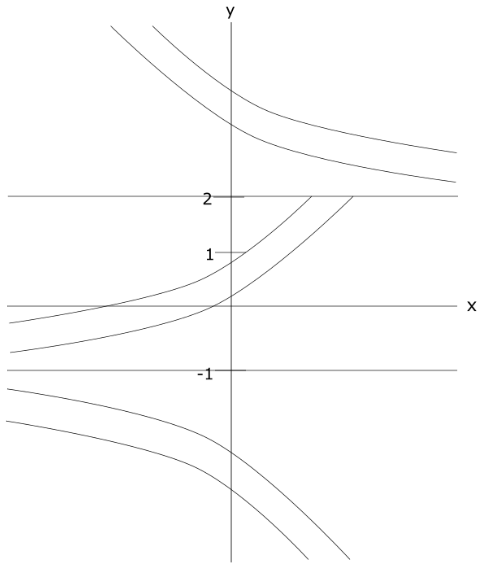
$(2)^3 = 1 + c$

$c = 7$

$u^{-3} = 1 + 7e^{3t}$

$u^3 = \frac{1}{1+7e^{3t}}$ We get: $u = \frac{1}{\sqrt[3]{(1+7e^{3t})}}$

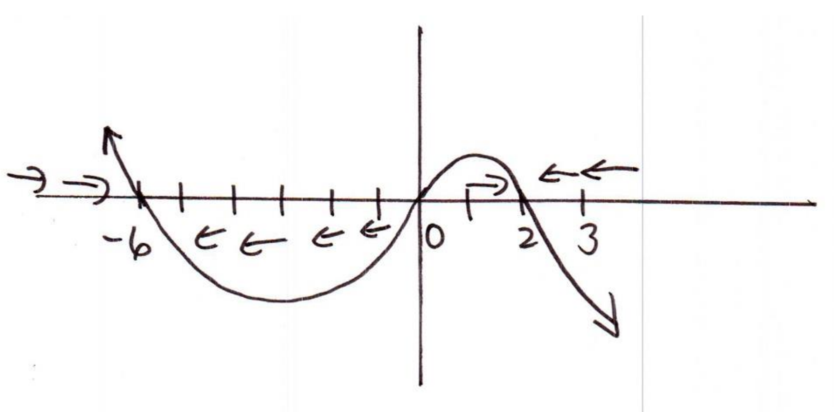
6. From -1 and below, it is below the x-axis, so it is decreasing and the lines go down and right. From -1 to 2, it is above the x-axis, so increasing and lines go up and right. From 2 onward, the graph is below the x-axis so it is decreasing and the lines go down and right.



7.a) The equilibrium are: 0, 2 and -6
b) Test Point y=3
dy/dx= -3(3-2)(3+6) < 0, so arrows point to the left

test point y=1
dy/dx = -1(1-2)(1+6) >0, so arrows point to the right

So, -6 is stable (arrows pointing towards each other)
c) start at 3, it would go down to 2
d) start at 0, it would stay at 0 (an equilibrium)



G. Practice Exam

G1. slope$=\frac{1}{2} \neq 1 \quad \therefore allometry$

G2. $\frac{LM}{T^2} = \frac{M}{L} \cdot v^2 \qquad \therefore v^2 = \frac{L^2}{T^2}$

G3. $P_{t+1} = 1.01P_t - 200$

G4. $I = x^2 \quad B$
$I = x^3 \quad A$
$I = x^{-3}e^{-2x} \quad C$

A) $y' + \frac{3xy}{x^2} = \frac{2\sin x}{x^2}$
$y' + \frac{3y}{x} = \frac{2\sin x}{x^2}$
$I = v(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

B) $y' + \frac{2}{x}y = \frac{x^{-3}}{x}$
$y' + \frac{2}{x}y = x^{-4}$
$I = e^{\int \frac{2}{x}} = e^{2\ln x} = e^{\ln x^2} = x^2$

C) $y' - \left(\frac{2x+3}{x}\right)y = \frac{2\sec x}{x}$
$I = e^{-\int\left(2+\frac{3}{x}\right)dx} = e^{-2x-3\ln x} = e^{-2x}e^{\ln x^{-3}} = e^{-2x}x^{-3}$

G5. $\frac{xy'}{x} + \frac{2y}{x} = \frac{x^{-3}}{x}$

$y' + \frac{2}{x}y = x^{-4}$

$v(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

$\int \frac{d}{dx}(v(x)y) = \int v(x)Q(x)$

$x^2 y = \int x^2\, x^{-4} dx$

$x^2 y = \int x^{-2} dx$

$x^2 y = \frac{x^{-1}}{-1} + c$

$x^2 y = \frac{-1}{x} + c$

Substitute $y(1) = 1$ to find c

$1^2(1) = \frac{-1}{1} + c$

$c = 2$

$x^2 y = \frac{-1}{x} + 2$

$y = \frac{-1}{x^3} + \frac{2}{x^2}$

$y(2) = \frac{-1}{2^3} + \frac{2}{2^2}$

$= \frac{-1}{8} + \frac{2}{4}$

$= \frac{-1}{8} + \frac{4}{8}$

$= \frac{3}{8}$

G6. $v(t) = e^{\int \frac{1}{t-3dt}} = e^{\ln(t-3)} = t - 3$

$\frac{d}{dt}(v(t)u) = v(t)Q(t)$

$\int \frac{d}{dt}(t-3)u = \int (t-3)(1)dt$

$(t-3)u = \frac{t^2}{2} - 3t + c$

Substitute $u(0) = 3$

$\uparrow \quad \uparrow$

$t \quad u$ $\quad (0-3)(3) = \frac{0^2}{2} - 2(0) + c$

$c = -9$

$\therefore (t-3)u = \frac{t^2}{2} - 3t - 9$

$u = \frac{1}{t-3}\left[\frac{t^2}{2} - 3t - 9\right]$

G7. a) $500 + 12t - 8t = 500 + 4t$

b) $\frac{dA}{dt} = (rate\ in)(rate\ in) - \frac{A(rate\ out)}{volume\ at\ time\ t}$

$\frac{dA}{dt} = (12)(12) - \frac{8A}{500+4t}$

$\frac{dA}{dt} = 144 - \frac{8A}{500+4t}$

G8. $\therefore n = 4$

a) $u = y^{1-n} = y^{1-4} = y^{-3}$ $\qquad \therefore p = -3$

b) $y'(t) - \frac{1}{t}y(t) = 4[y(t)]^4$

$P(t) = \frac{-1}{t} \qquad Q(t) = 4$

c) $\frac{du}{dt} + (1-n)u\,p(t) = (1-n)Q(t)$

$\frac{du}{dt} + (-3)u\left(\frac{-1}{t}\right) = -3(4)$

$\frac{du}{dt} + \frac{3}{t}u = -12$

$\frac{d}{dt}(v(t)u) = v(t)Q(t)$

$v(t) = e^{\int \frac{3}{t}dt} = e^{3\ln t} = e^{\ln t^3} = t^3$

$\int \frac{d}{dt}(t^3 u) = \int t^3 \cdot (-12)\mathrm{dt}$

$t^3 u = \frac{-12t^4}{4} + c \quad substitute\ u = y^{-3} \qquad y(1) = 1$

$t^3 y^{-3} = -3t^4 + c$

$1^3(1)^{-3} = -3(1)^4 + c$

$c = 4$

$t^3 y^{-3} = -3t^4 + 4$

$y^{-3} = \frac{-3t^4}{t^3} + \frac{4}{t^3}$

$y^{-3} = \frac{-3t^4+4}{t^3}$

$y^3 = \frac{t^3}{-3t^4+4}$

$y = \sqrt[3]{\frac{t^3}{-3t^4+4}}$

G9. a) No, it will go to 20

b) 60 rats

c) 20 rats

G10. The first graph is correct. It is above the x-axis for greater than 0, so it is increasing there. It is below the x-axis between -1 and 0, so it is decreasing there and it is above the x-axis below -1, so it will be increasing below -1.

G11.

$P_t = 1.04P_{t-1} - 50 \qquad P_o = 850 \qquad a = 1.04 \qquad b = -50$

$P_t = \left(P_o - \frac{b}{1-a}\right)(a^t) + \frac{b}{1-a}$

$P_t = \left(850 - \frac{-50}{1-1.04}\right)(1.04)^t + \frac{-50}{1-1.04}$

$P_t = (850 - 1250)(1.04)^t + 1250$

$P_t = -400(1.04)^t + 1250$

$\mathbf{50 = -400(1.04)^t + 1250}$

$\mathbf{-1200 = -400(1.04)^t}$

$\mathbf{3 = 1.04^t}$

$\boldsymbol{t = \frac{\ln 3}{\ln 1.04}}$

Long Method:

$P_t = 1.04P_{t-1} - 50 \quad \boxed{1}$

$\Delta P_t = P_t - P_{t-1} \quad sub\,\boxed{1}$

$= 1.04P_{t-1} - 50 - P_{t-1}$

$\Delta P_t = \underbrace{P_{t-1} + 0.04P_{t-1} - 50}_{\boxed{3}} - P_{t-1}$

$\Delta P_t = 0.04P_{t-1} - 50$

Consider $\Delta P_t = 0$

$0.04P_{t-1} - 50 = 0$

$0.04P_{t-1} = 50$

$P_{t-1} = 1250$

Define $U_t = P_t - 1250$ $\boxed{2}$ which means $P_t = U_t + 1250$

$U_{t-1} = P_{t-1} - 1250$ $\boxed{2}$ which means $P_{t-1} = U_{t-1} + 1250$

From $\boxed{1}$ $P_t = 1.04P_{t-1} - 50$ substitute $\boxed{2}$

$U_t + 1250 = 1.04(U_{t-1} + 1250) - 50$

$U_t = 1.04U_{t-1} + 1300 - 50 - 1250$

$\therefore U_t = 1.04U_{t-1} \quad \leftarrow b = 1.04 \quad U_t = P_t - 1250$

$subst \quad t = 0$

$P_o = 850$

$U_o = P_o - 1250$

$U_o = 850 - 1250 \;=\; -400$

$U_t = U_o(b)^t$

$U_t = -400(1.04)^t \quad \boxed{4}$

$\therefore from\ \boxed{2} \quad U_t = P_t - 1250$

$P_t = U_t + 1250$

$\therefore P_t = -400(1.04)^t + 1250 \qquad from\,\boxed{4}$...see purple above!!

H. Basic Probability

Example 1. a) $S = \{1,2,3,4,5,6\}$
b)$S = \{(1,1)(1,2)(1,3) \dots (6,1)(6,2)(6,3)(6,4)(6,5)(6,6)\}$
c)$S = \{HHH, HTH, HTT, THH, THT, TTH, TTT, HHT\}$

Mutually Exclusive VS. Independence

Example 2. $E = \{2,4,6\}$ $F = \{1,2,3,4\}$
a) $= \{1,2,3,4,6\}$
b) $= \{2,4\}$
c) $= \{6\}$
d) $no\ since\ b)\ is\ not\ the\ empty\ set$

Example 3. $\Pr(at\ least\ 1\ tail) = 1 - \Pr\ (no\ tails)$
$= 1 - \Pr\ (HHHHHH)$
$= 1 - \left(\frac{1}{2}\right)^6$
$= 1 - \frac{1}{64}$
$= \frac{63}{64}$

Example 4.
Pr(sum seven)=Pr{(1,6)(2,5)(3,4)(4,3)(5,2)(6,1)}=6/36=1/6=0.167

Example 5.
a) $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr\ (A \cap B)$
$0.35 = 0.10 + \Pr(B) - 0.05$
$0.25 = \Pr(B) - 0.05$
$\Pr(B) = 0.30$

b) $\Pr(A \cup B^C) = \Pr(A) + \Pr(B^C) - \Pr\ (A \cap B^C)$
=0.10 + 0.70 – 0.05=0.75

Example 6. a) $\Pr(red) = 0.5$ b) $\Pr(face\ and\ heart) = \frac{3}{52} = 0.058$
c) $\Pr(face\ or\ heart) = \Pr(face) + \Pr(heart) - \Pr\ (face\ and\ heart)$
$= \frac{12}{52} + \frac{13}{52} - \frac{3}{52} = \frac{22}{52} = 0.423$

Example 7.
$\Pr(A \cap B) = \Pr(A) \times \Pr(B) = 0.2(0.5) = 0.10$ since A, B are independent
Pr(A or B)=Pr(A) + Pr(B) - 0.06
=0.2 + 0.5 - 0.10
=0.60

So, a) is true since they are independent...b) is true

To check c)...$\Pr(\bar{A} \cap \bar{B}) = 1 - \Pr(A \cup B) = 1 - 0.6 = 0.4$

The answer is d). only a) and b) are true.

Example 8.
a) They are independent because the first flip being tails won't affect the second flip.

b) They are independent, since "ace" and "spades" don't affect each other. One is the type of suit and one is the denomination...ie. we can get an ace of spades

c) These are disjoint, since one card can't be both a spade and a heart, ie. prob. of both = 0

Example 9. a) $\frac{1}{4}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{64}$ b) $(\frac{13}{52})(\frac{12}{51})(\frac{11}{50})$

c)$\Pr\,(\textit{at least one club}) = 1 - \Pr\,(\textit{no clubs})$
$= 1 - \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = 1 - \frac{27}{64} = \frac{37}{64}$

Example 10. $\Pr(A\ or\ B) = \Pr(A) + PR(B) - PR(A \cap B)$
$0.8 = 0.4 + 0.5 - \Pr\,(A \cap B)$
$0.8 = 0.9 - \Pr\,(A \cap B)$
$\Pr(A \cap B) = 0.1 \neq 0 \quad \therefore\ not\ mutually\ exclusive$
Independent
$check\ \Pr(A \cap B) = \Pr\,(A) \times \Pr\,(B)$
$0.1 \qquad\qquad 0.4 \times 0.5 = 0.2$
$\therefore\ no\quad The\ answer\ is\ (d).$

Example 11. $\Pr(E \cup F) = 0.3 + 0.4 = 0.7$
$\Pr(E^c \cap F^c) = 1 - 0.7 = 0.3$
$\Pr(E \cap F^c) = \Pr\,(E) = 0.30$ since all of E is outside of F (mutually exclusive)

Example 12. $\boldsymbol{a})$ $\Pr(swing \cup slide) = \Pr(swing) + \Pr(slide) - \Pr\,(both)$

$$= \frac{50}{80} + \frac{60}{80} - \frac{40}{80}$$

$$= \frac{110-40}{80} = \frac{70}{80} = \frac{7}{8}$$

b)$\Pr(neither) = 1 - \frac{7}{8} = \frac{1}{8}$

Example 13.

(a) Since $\Pr(A^C \cap B^C) = 0.20$, we know that Pr(AUB)= 1 – 0.20=0.80

From the union formula,
$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
0.80=0.7+0.4-$\Pr(A \cap B)$
$\Pr(A \cap B)$=0.30

b) $Pr(B \cup A^C)^C = \Pr[B^C \cap (A^C)^C] = \Pr(B^C \cap A) = 0.7 - 0.3 = 0.4$

Practice Exam Questions on Probability

H1. $\Pr(A) = 1 - \Pr(O) - \Pr(B) - \Pr(AB) = 1 - 0.50 - 0.20 - 0.05 = 0.25$. The answer is (c).

H2. Pr(both aces)= $\frac{4}{52} \times \frac{3}{51}$=0.00452

H3. Let E denote the event that at least one of the four mosquitoes was a carrier of the virus. Then $\overline{E}$ denotes the event that none of the four mosquitoes was a carrier of the virus. Since each mosquito has a 90% of not being a carrier of the virus,

$$\Pr(\overline{\mathrm{E}}) = (0.90)^4 = 0.6561 .$$

Therefore $\Pr(E) = 1 - \Pr(\overline{E}) = 1 - (0.90)^4 = 0.3439 = 34.39\%$.

H4. The probabilities of drawing 1 red ball, 1 green ball, or 1 yellow ball are

$$\Pr(R) = \frac{5}{10}, \qquad \Pr(G) = \frac{3}{10}, \qquad \Pr(Y) = \frac{2}{10},$$

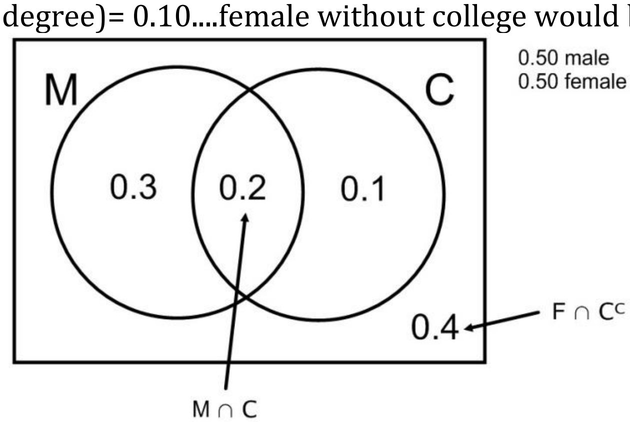
respectively.
The probabilities of drawing 2 red balls, 2 green balls, or 2 yellow balls are

$$\Pr(RR) = \left(\frac{5}{10}\right)^2, \qquad \Pr(GG) = \left(\frac{3}{10}\right)^2, \qquad \Pr(YY) = \left(\frac{2}{10}\right)^2,$$

respectively.
The probability of drawing 2 balls of the same colour is therefore

$$\Pr(RRorGGorYY) = \left(\frac{5}{10}\right)^2 + \left(\frac{3}{10}\right)^2 + \left(\frac{2}{10}\right)^2 = 0.25 + 0.09 + 0.04 = 0.38 .$$

H5. If 30% have a college degree and 20% of men have a college degree, then 10% of the women have a college degree
Pr(female and college degree)= 0.10....female without college would be 0.4, if they asked!



H6.

The probability of *not* catching a fish each time you cast your line is $1-\frac{1}{4}=\frac{3}{4}$.

The probability of *not* catching a fish on the first two attempts is $(\frac{3}{4})^2=\frac{9}{16}$.

The probability of catching at least one fish within the first two attempts is thus $1-\frac{9}{16}=\frac{7}{16}$.

The answer is (b).

H7. a) Pr(F)=0.40 and Pr(N)=0.30, Pr(F and N)=0.20

b) Pr(F or N)= Pr(F) + P(N) – Pr(F and N)=0.40+0.30-0.20=0.50

H8. Using the table, find for a randomly selected individual from this population the probability that he or she:

a) Is in the age interval 40-49

Pr(40-49)=(10+15+50+70)/400
=145/400=0.3625

b) Is in the age interval 40-49 and weighs 170-189 lbs

=50/400

H9. There are 8 possible outcomes for three children and only 3 consist of exactly 2 girls...GGB, GBG and BGG...so, 3/8=0.375

H10. Pr(false)=5/90=0.056

H11. a) Pr(AB)= 5/100=0.05

b) Pr(O or Rh-)= Pr(O) + Pr(RH-) - Pr(both)
=45/100 + 14/100 - 6/100
=53/100=0.53

c) Pr(A and Rh+)= 35/100=0.35

H12. Pr(both A)=40/100 times 39/99

H13. a) Pr(all 4's)=1/6 x 1/6 x 1/6 = 1/216=0.158

b) Pr(no 4)=5/6 x 5/6 x 5/6 = 125/216=0.579

c) Pr(not all 4's) = 1 - Pr(all 4's) = 1 - 1/216 = 215/216=0.995

H14. a) Pr(2)= 3/12 since there are 3 number 2's out of 12 numbers

b) Pr(not a 1)= 1- Pr(get a 1) = 1 - 2/12 = 10/12

c) Pr(not a 2) = 1- 3/12 = 9/12

d) Pr(first 1, second 3) = 2/12 x 6/11 since you don't put them back, you would only have 11 left after removing the first "1"

e) Pr(first 1, second not a 3) = 2/12 x 5/11 = 10/132

H15. Pr(1st die 6 or second 5)= Pr(1st 6) + Pr(2nd 5) - Pr(both)

= 6/36 + 6/36 - 1/36
=11/36

H16. Pr(1st die 6 and 2nd 5 = Pr(getting (6,5) = 1/36 since this is one of 36 possible outcomes

H17. Pr(1st not a 5 or second not a 4)= to be in one set or the other or both...every element in the 36 either has 1st not a 5 OR second not a 4, except the outcome (5,4)...so,
1-1/36=35/36

H18.

a) $\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{64}$

b) $= 1 - \Pr\,(no\ clubs)$

$= 1 - \frac{3}{4}\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = 1 - \frac{27}{64} = \frac{37}{64}$

H19.
a) $\frac{1}{6}$
b) $5 \; or \; 6 \quad \frac{2}{6} = \frac{1}{3}$
c) $4, 2, 6 \quad \frac{3}{6} = \frac{1}{2}$
d) $2, 4, 6 \quad \frac{3}{6} = \frac{1}{2}$

H20. A) 6/12 = ½
b) red or yellow = 8/12 = 2/3
c) (6+4)/12 = 10/12 = 5/ 6

H21.
$A - 4 \quad T - 5 \quad G - 4 \quad C - 3 \quad /16$
$\frac{4+4}{16} = \frac{8}{16} = \frac{1}{2}$

H22. $\frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$ without would be $\frac{3}{10} \times \frac{2}{9} = \frac{6}{90}$

H23. A) Yes, they are disjoint because you can't belong to more than one weight category

b) Pr(D)=1-0.02-0.39-0.35=0.24

H24. Pr(A and B)=Pr(A)Pr(B)= 0.2(0.5)=0.10

H25. $\Pr(heterozygous) = \frac{2}{4} = \frac{1}{2}$
$\Pr(\text{at least one B}) = \frac{3}{4}$

H26. a) $= \Pr(E) \times \Pr(F)$
$= 0.3 \times 0.4 = 0.12$
b) $= \Pr(E) + \Pr(F^c) - \Pr(E) \times \Pr(F^c)$
$= 0.3 + 0.6 - 0.3 \times 0.6$
$= 0.9 - 0.18 = 0.72$
c) $= \Pr(E) \times \Pr(F^c)$
$= 0.3 \times 0.6 = 0.18$

H27. Pr(sum greater than 10)={(5,6)(6,5)(6,6)}=3/36=1/12=0.083

H28. P(A and B)= 1/36 only one outcome since it would be only {(6,1)}
P(AUB)=P(A or B)=P(A)+P(B)- P(A and B)
=6/36 + 6/36 - 1/36
=11/36

H29. $\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$

$0.86 = 0.50 + 0.35 - \Pr(E \cap F)$

$\Pr(E \cap F) = 0.01 \neq 0$

$\therefore E, F \text{ are NOT mutually exclusive}$

Check to see if E and F are independent

$\Pr(E) \times \Pr(F)$

$= 0.5 \times 0.35$

$= 0.175$

$\Pr(E \cap F) = 0.01$

$\therefore \Pr(E \cap F) \neq \Pr(E) \times \Pr(F)$

$\therefore E, F \text{ are not independent}$

H30. BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG are all the possibilities.
Pr(exactly 2 girls)=3/8=0.375

H31. $\Pr(A) = \frac{1}{2}$ (*half the rolls are even*)

$\Pr(B) = \frac{6}{36} = \frac{1}{6}$ [(1,6)(2,6)(3,6)(4,6)(5,6)(6,6)]

$\Pr(A \cap B) = \frac{3}{36} = \frac{1}{12}$ [(2,6)(4,6)(6,6)]

$\Pr(A) \times \Pr(B) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

$\therefore \Pr(A \cap B) = \Pr(A) \times \Pr(B)$ $\therefore$ *they are independent*

I. Conditional Probability

Example 1.

$$\Pr(F/E) = \frac{\Pr(F \cap E)}{\Pr(E)} = \frac{0.30}{0.40} = \frac{3}{4} = 0.75$$

Example 2. $\Pr(N) = \frac{35}{100}$ $\Pr(R) = \frac{50}{100}$ $\Pr(N \cup R) = \frac{80}{100}$

$\Pr(R/\mathrm{N}) = \frac{\Pr(R \cap N)}{\Pr(N)}$

Find $\Pr(R \cap N)$

$\Pr(N \cup R) = \Pr(N) + \Pr(R) - \Pr(N \cap R)$

$\frac{80}{100} = \frac{35}{100} + \frac{50}{100} - \Pr(N \cap R)$

$\Pr(N \cap R) = \frac{5}{100}$

$\therefore \Pr(R/\mathrm{N}) = \frac{5/100}{35/100} = \frac{5}{35} = \frac{1}{7}$

Example 3. $\Pr(E/\mathrm{F}) = \frac{\Pr(E \cap F)}{\Pr(F)}$

$\frac{1}{3} = \frac{\Pr(E \cap F)}{1/4}$

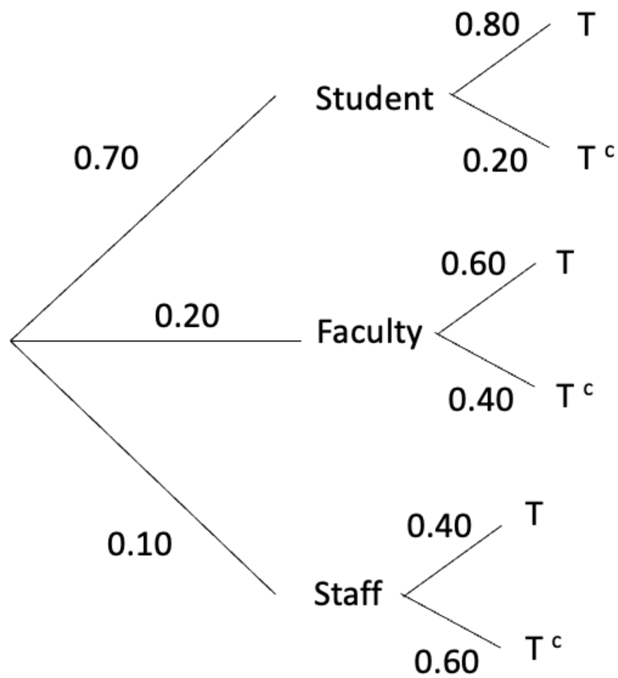
$\Pr(E \cap F) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$

$\Pr(F/\mathrm{E}) = \frac{\Pr(E \cap F)}{\Pr(E)}$

$= \frac{1/12}{2/3}$

$= \frac{1}{12} \times \frac{3}{2} = \frac{3}{24} = \frac{1}{8}$

Example 4.a)



$\Pr(T) = \Pr(student \cap T) + \Pr(faculty \cap T) + \Pr\,(staff \cap T)$
$= 0.70(0.8) + 0.2(0.6) + 0.1(0.4)$
$= 0.72$

b) $\Pr\,(student/T) = \frac{\Pr\,(student \cap T)}{\Pr\,(T)}$

$= \frac{0.70 \times 0.80}{0.72} = \frac{56}{72} = \frac{7}{9}$

Example 5.

a) $S\ Reduced = \{(4,1)(4,2)(4,3)(4,4)(4,5)(4,6)\}$

$\nwarrow \frac{1}{6}$ Since the reduced sample space is what is given, ie. 1st die is a 4 and then we circle how many of these 6 outcomes have a sum greater than 9 and there is only 1 outcome

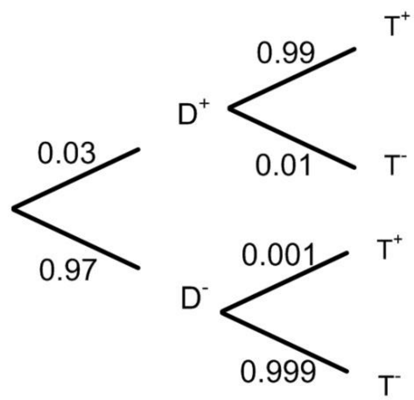
b) $SReduced = \{(1,1)(2,1)(3,1)(4,1)(5,1)(6,1)\} = \frac{0}{6} = 0$

Example 6.

a) $\Pr(5\ boys) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

b) Pr(at least 1 boy)=1-Pr(no boys) = 1 – 1/32 = 31/32

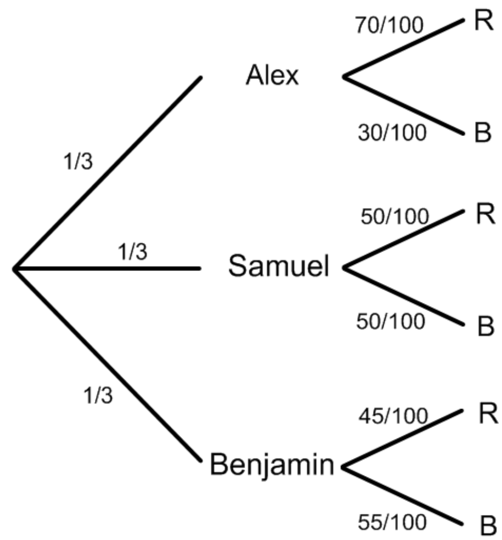
Example 7. $\Pr(T+) = \Pr(D^+ \cap T^+) + \Pr(D^- \cap T^+)$
$= 0.03(0.99) + 0.97(0.001)$
$= 0.0307$



Or (3/100)(99/100)+(97/100)(1/1000)
=(30/1000)(99/100)+(97/100)(1/1000)
=(2970 + 97)/100000
=3067/10000

Example 8.

a) $\Pr(B) = \Pr(Alex \cap B) + \Pr(Samuel \cap B) + \Pr(Ben \cap B)$
$= \Pr(A) \times \Pr(B/A) + \Pr(S) \times \Pr(B/S) + \Pr(B) \times \Pr(B/B)$
$= \frac{1}{3}\left(\frac{30}{100}\right) + \frac{1}{3}\left(\frac{50}{100}\right) + \frac{1}{3}\left(\frac{55}{100}\right)$
$= \frac{30}{300} + \frac{50}{300} + \frac{55}{300}$
$= \frac{135}{300} = \frac{9}{20} = 0.45$



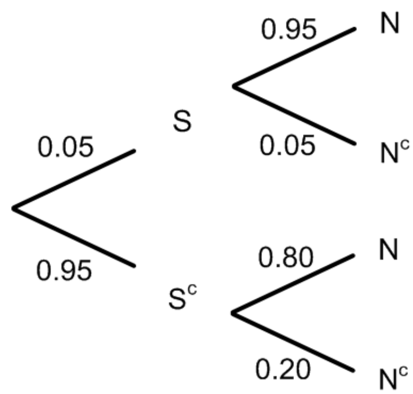
For b) find "what is the probability you got the marble from Samuel if you know it is a blue "

b) $\Pr(Samuel/B) = \frac{\Pr(Samuel \cap B)}{\Pr(B)}$

$$= \frac{(\frac{1}{3})(\frac{50}{100})}{\frac{9}{20}} = \frac{1}{6}\left(\frac{20}{9}\right) = \frac{20}{54} = \frac{10}{27}$$

Example 9.

$\Pr(N^c) = \Pr(S \cap N^c) + \Pr(S^c \cap N^c)$
$= \Pr(S) \times \Pr(N^c/S) + \Pr(S^c) \times \Pr(N^c/S^c)$
$= 0.05(0.05) + 0.95(0.20)$
$= 0.1925$

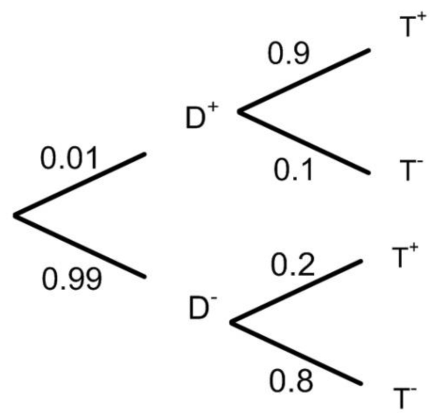


Or (5/100)(5/100)+(95/100)(20/100)
=(25+1900)/10000=1925/10000=0.1925 or 77/400

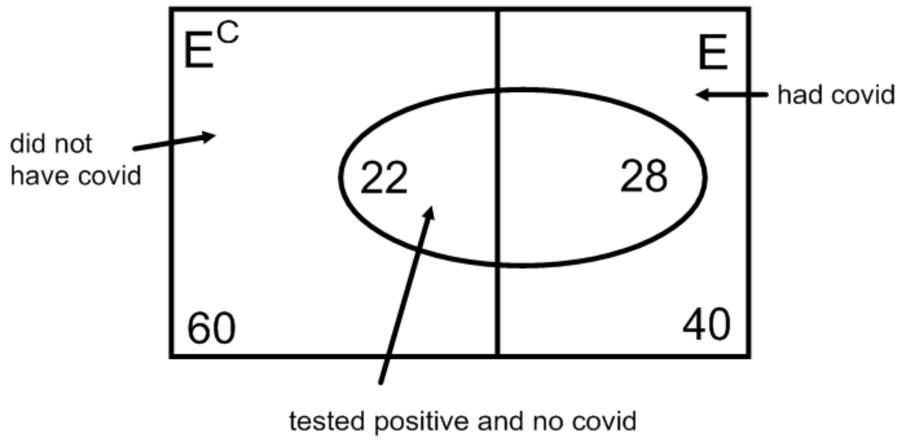
$\Pr(S^c/N) = 1 - \Pr(S/N)$
$= 1 - \frac{\Pr(S \cap N)}{\Pr(N)}$
$= 1 - \frac{0.05(0.95)}{1-0.1925}$
$= 0.941$

Example 10. $\Pr(T^+/D^+) = sensitivity = 0.9$
$\Pr(T^-/D^-) = specificity = 0.8$

$\Pr(D^-/T^-) = \frac{\Pr(D^- and\ T^-)}{\Pr(T^-)} = \frac{0.8(0.99)}{0.01(0.1)+0.99(0.8)}$=0.999
$\uparrow$ *given*



Example 11.



$\therefore 60 - 22 = 38 \quad tested\ negative\ and\ no\ covid$

Sensitivity

$$\Pr(T^+/D^+) = \frac{\Pr\,(T^+ \cap D^+)}{\Pr\,(D^+)}$$

$$= \frac{\frac{28}{100}}{\frac{40}{100}}$$

$$= \frac{28}{40} = 0.70$$

$\Pr(type\ II\ error) = 1 - 0.70 = 0.30$

Specificity

$$\Pr(T^-/D^-) = \frac{\Pr\,(T^- \cap D^-)}{\Pr\,(D^-)}$$

$$= \frac{\frac{38}{100}}{\frac{60}{100}}$$

$$= \frac{38}{60} = 0.63$$

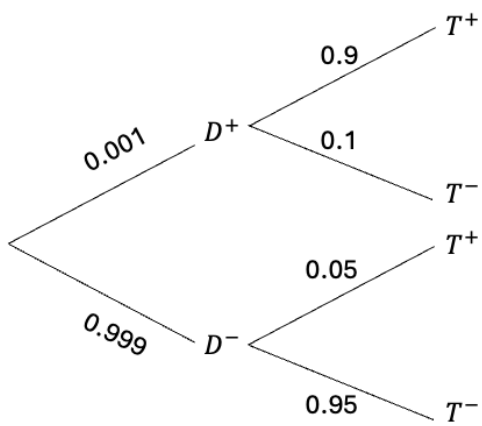
$\Pr(type\ I\ error) = 1 - 0.63 = 0.37$

ADD THIS QUESTION TO YOUR BOOKLET AND DO IT!!!

Example 12. A population has 1000 people, and the prevalence of cancer is 0.001. The sensitivity is 0.90 and the specificity is 0.95. Find each of the following:

a) the number of people who have the disease in the population

$\Pr(T+/D+) = 0.9$ (Sensitivity)
$\Pr(T-/D-) = 0.95$ (Specificity)



b) the probability of a false positive and the number of false positives in this population

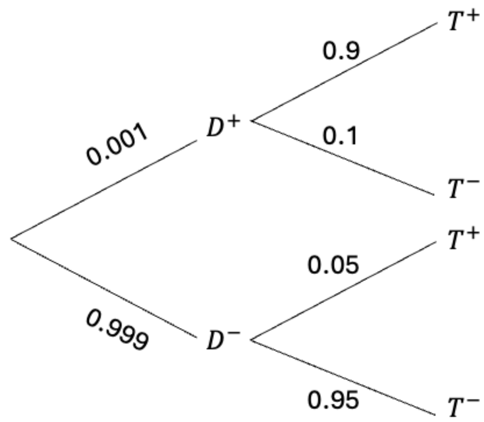
c) the probability of a false negative and the number of false negatives in the population

d) the number of true positives and true negatives?

e) If an individual in the population is tested for the disease and the test comes back positive, is it more likely they actually have the disease or that the test is wrong?

<u>Example 12. Solution:</u>

Pr $(T+/D+) = 0.9$ (Sensitivity)
Pr $(T-/D-) = 0.9$ (Specificity)



a) $E(x) = \text{np} = 1000(0.001) = 10\ people$

b) false $+$
Pr $(T+/D-) = 0.05$
Number false $+= 0.05\ \times$ n
Where n= number <u>without</u> disease

$= 0.05(1000 - 10)$
$= 0.05(990)$
<u>$= 49.5$ people</u>

* false positives only occur if the person doesn't have the disease

c) $\Pr(\text{false} -) = \Pr(T-/D+)$
$= 0.10$

$= 0.10\ \times$ n where n =# number with the disease
<u># false$- = 0.10(10) = 1\ person$</u>

d) <u>true $+$</u>
Pr $(T+/D+) = 0.9$
# of true+ $= 0.9\ \times 10$ =<u>9 people</u>

<u>true $-$</u>
Pr $(T-/D-) = 0.95$
# of true negatives
$= 0.95\ \times 990$
<u>$= 940.5$ people</u>

NOTE: 49.5 + 1 +9 +940.5 = 1000 people!!

e) We see from b), c), and d) that there are 49.5 people with a false + and only 9 people with a true positive, so it is much more likely it was an error in the test than they actually have the disease

Practice Exam Questions on Conditional Probability

I1.

a) $\Pr(T) = Tr(Box\ 1\ and\ T) + \Pr(Box\ 2\ and\ T) + \Pr(Box\ 3\ and\ T)$

$= \frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{2}{4}$

$= \frac{1}{9}+\frac{1}{6}+\frac{1}{6}=\frac{4}{36}+\frac{6}{36}+\frac{6}{36}=\frac{16}{36}=\frac{8}{18}=\frac{4}{9}$

b) $\Pr(2nd|T) = \frac{\Pr(2nd\cap T)}{\Pr(T)} = \frac{1/3\times 1/2}{4/9} = \frac{1/6}{4/9} = \frac{1}{6}\left(\frac{9}{4}\right) = \frac{9}{24} = \frac{3}{8}$

I2.

a) $\Pr(B) = \Pr(W\cap B) + \Pr(W^C\cap B^c)$

$= 0.60\times 0.50 + 0.40\times 0.30$

$= 0.30 + 0.12$

$= 0.42$

$b)\ \Pr(M|B) = \frac{\Pr\ (M\cap B)}{\Pr\ (B)} = \frac{0.40\times 0.30}{0.42} = \frac{0.12}{0.42} = \frac{12}{42} = \frac{2}{7}$

I3. $\Pr(k) + \Pr(k^c k) + \Pr(k^c k^c k) + \Pr(k^c k^c k^c k)\ OR$

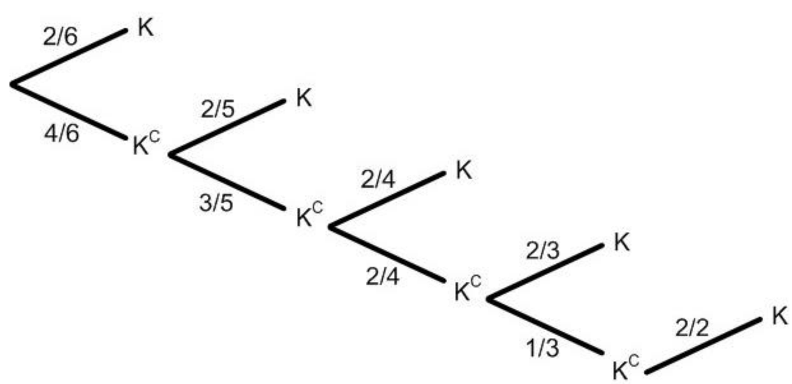
$1 - \Pr(k^c k^c k^c k^c k)$

$= 1 - \frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\times 1$

$= 1 - \frac{4}{5\times 4\times 3}$

$= 1 - \frac{1}{15}$

$= \frac{14}{15}$



I4.

a)

$\Pr(B) = \Pr(Box\ A\ and\ B) + \Pr(Box\ B\ and\ B)$

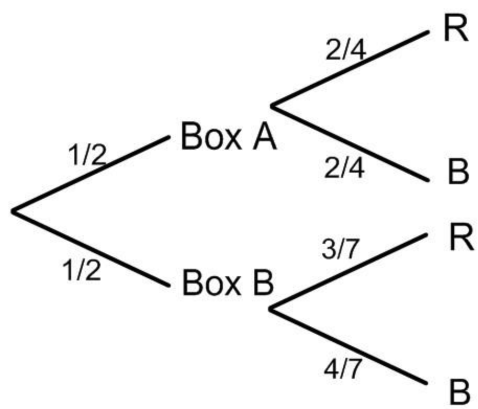
$= \frac{1}{2} \times \frac{2}{4} + \frac{1}{2} \times \frac{4}{7}$

$= \frac{1}{4} + \frac{2}{7}$

$= \frac{7}{28} + \frac{8}{28} = \frac{15}{28}$

b)

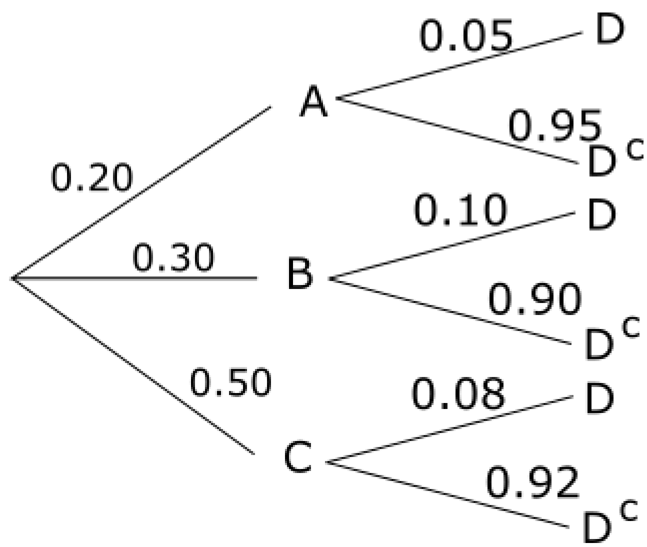
$$\Pr(BoxA|R) = \frac{\Pr\,(BoxA\ and\ R)}{\Pr\,(R)} = \frac{{}^1\!/_2\left(\frac{2}{4}\right)}{{}^1\!/_2\,({}^2\!/_4) + {}^1\!/_2\,({}^3\!/_7)} = \frac{1/4}{{}^1\!/_4 + {}^3\!/_{14}}$$



I5.

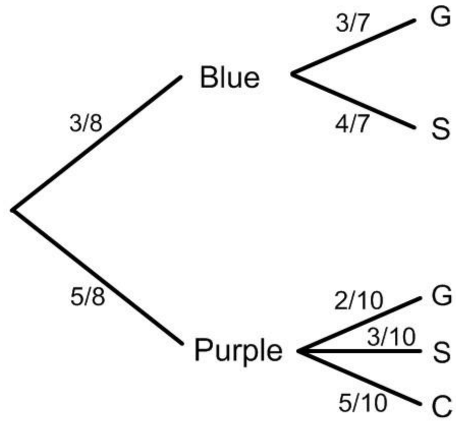
$\Pr(A|D) = \frac{\Pr(\text{ D and A})}{\Pr(D)}$

$= \frac{(0.05)(0.20)}{0.2(0.05) + 0.3(0.10) + 0.5(0.08)}$



I6.

a)
$\Pr(G) = \Pr(B \cap G) + \Pr(P \cap G)$
$= \frac{3}{8} \times \frac{3}{7} + \frac{5}{8} \times \frac{2}{10}$
$= \frac{9}{56} + \frac{10}{80}$



b)
$\Pr(P|S) = \frac{\Pr\text{ (P and S)}}{\Pr(S)}$
$= \frac{{}^{3}/_{10}\left({}^{5}/_{8}\right)}{{}^{3}/_{8}\left({}^{4}/_{7}\right) + {}^{5}/_{8}\left({}^{3}/_{10}\right)} = \frac{{}^{15}/_{80}}{{}^{12}/_{56} + {}^{15}/_{80}}$

I7.
Pr(F/E)=0.20

$\Pr(F/E^C)=0.10$
Pr(E)=0.40

Find $\Pr(E \cap F)$

Pr(F/E)=$\frac{\Pr\ (F \cap E)}{\Pr\ (E)}$

0.20=$\frac{\Pr\ (F \cap E)}{0.40}$

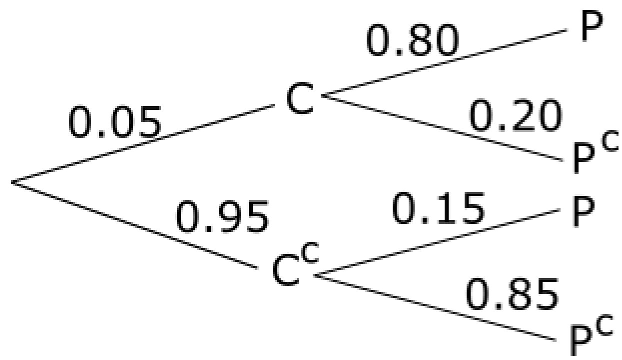
$\Pr\ (F \cap E)$=(0.20)(0.40)=0.08

The answer is C.

I8.

Draw a Tree diagram

$\Pr(C/P) = \frac{\Pr(C \cap P)}{\Pr(P)} = \frac{0.05(0.80)}{0.05(0.80)+0.95(0.15)}$



I9.

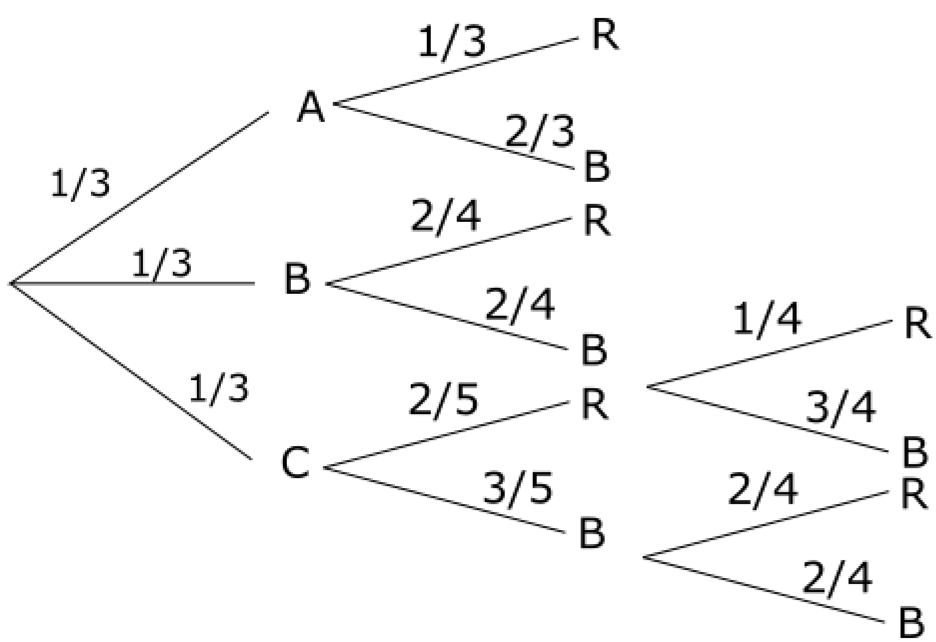
Pr(red) = Pr(red from Box A) + Pr(red from Box B) + Pr(red from Box C)

$= \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{2}{4}\right) + \frac{1}{3}\left(\frac{2}{5}\right) = \frac{1}{9} + \frac{1}{6} + \frac{2}{15}$

I10.

Pr(C and 2nd red) = Pr(C and BR) + Pr(C and RR)

$= \frac{1}{3} \times \frac{3}{5} \times \frac{2}{4} + \frac{1}{3} \times \frac{2}{5} \times \frac{1}{4} = \frac{1}{10} + \frac{1}{30} = \frac{3}{30} + \frac{1}{30} = \frac{4}{30} = \frac{2}{15}$



I11. **(a)** What is the probability that a randomly selected individual is experiencing hypertension?

$$\Pr(\text{hypertension}) = \frac{\#\text{ with hypertension}}{\text{total }\#} = \frac{21+36+30}{180} = \frac{87}{180} \approx 0.48$$

(b) Given that a heavy smoker is selected at random from this group, what is the probability that the person is experiencing hypertension?

$$\Pr(\text{hypertension} \mid \text{heavy smoker}) = \frac{\Pr(\text{hypertension} \cap \text{heavy smoker})}{\Pr(\text{heavy smoker})} = \frac{30}{30+19} \approx 0.61$$

(c) Are the events "hypertension" and "heavy smoker" independent? Give supporting calculations.

Since $\Pr(\text{hypertension} \mid \text{heavy smoker}) = \frac{30}{49} \neq \frac{87}{180} = \Pr(\text{hypertension})$, the two events are *not* independent.

I12. Consider the following events:

$A =$ "adult selected has a college level education";
$B =$ "adult selected is a male with the highest level of education being secondary";
$C =$ "adult selected is a female".

(a) Are the events A and B disjoint? Explain.

Yes. They are disjoint because an adult cannot have a college level education and have his highest level of education be secondary.

(b) Are the events A and C disjoint? Explain.

No. They are not disjoint since females can have a college level education.

(c) What is the probability that an adult selected at random either has a college level education or is female?

Pr(college or female) = Pr(college) + Pr(female) - Pr(college and female)

$$= \frac{22+17}{200} + \frac{45+50+17}{200} - \frac{17}{200} = \frac{39}{200} + \frac{112}{200} - \frac{17}{200} = \frac{134}{200} = 0.67.$$

(d) What is the probability that an adult selected at random has a college level education given that the adult is a female?

$$\Pr(\text{college} \mid \text{female}) = \frac{\Pr(\text{college and female})}{\Pr(\text{female})} = \frac{\frac{17}{200}}{\frac{112}{200}} = \frac{17}{112} \approx 0.15$$

(e) Are the events A and C independent?

$\Pr(A) = \dfrac{22+17}{200} = \dfrac{39}{200}$; $\quad \Pr(C) = \dfrac{45+50+17}{200} = \dfrac{112}{200} = \dfrac{14}{25}$;

$\Pr(A \cap C) = \dfrac{17}{200} = 0.085$; $\quad \Pr(A)\Pr(C) = \dfrac{39}{200} \cdot \dfrac{14}{25} = \dfrac{273}{2500} = 0.1092$;

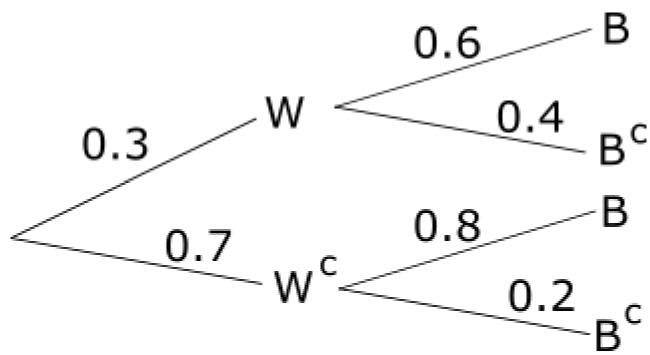
Since $\Pr(A \cap C) \neq \Pr(A)\Pr(C)$, the events A and C are not independent.

I13.

Pr(B)=Pr(W$\cap B$) + Pr $(W^C \cap B) = 0.3(0.6) + (0.70)(0.80) = 0.18 + 0.56 = 0.74$

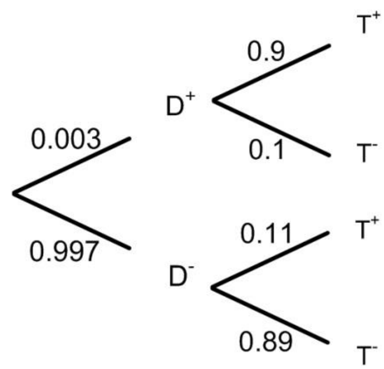
I14.

Pr(W^C/B)=$\frac{\Pr(W^c \cap B)}{\Pr(B)} = \frac{0.7(0.8)}{0.84} = \frac{56}{84}$



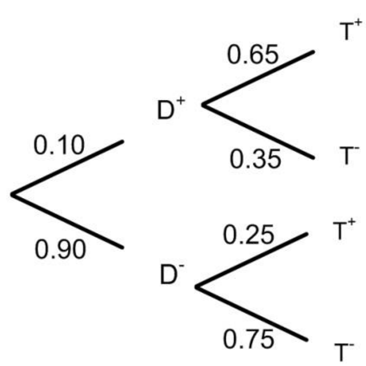
I15.

$$\Pr(D^+/T^+) = \frac{\Pr\,(D^+ \text{ } and \text{ } T^+)}{\Pr\,(T^+)}$$
$$= \frac{(0.90)(0.003)}{0.003(0.9)+0.997(0.11)} = 0.024$$



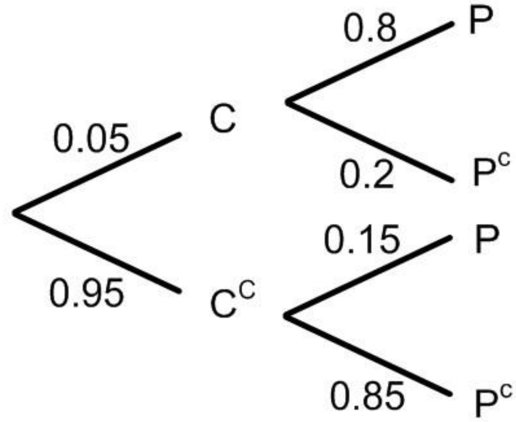
I16.

$\Pr(T^+/D^+) = 0.65$
$\Pr(T^-/D^-) = 0.75$



$$\Pr(D^+/T^+) = \frac{\Pr(T^+ and\ D^+)(}{Pr(T^+)} = \frac{0.65(0.10)}{0.10(0.65)+0.9(0.25)} = 0.224$$

I17.
Draw a Tree diagram

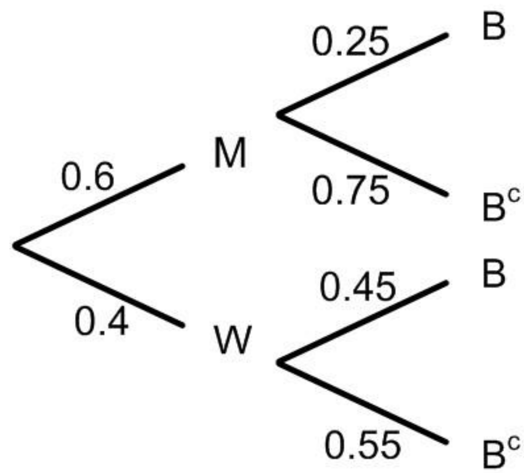


$\Pr(C^C/P) = \frac{\Pr(C^C \; and \; P)}{\Pr(P)} = \frac{0.95(0.15)}{0.95(0.15)+0.05(0.80)} = 0.781$

b) Sensitivity=0.80 (test positive/have disease)
c) Specificity=0.85 (test negative/don't have disease)

d) $\Pr(type\ II\ error) = 1 - 0.80 = 0.20$
$\Pr(type\ I\ error) = 1 - 0.85 = 0.15$

I18.

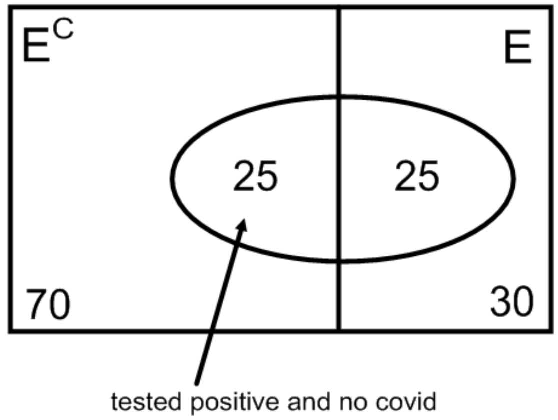


$$\Pr(W|B) = \frac{\Pr(B|W)\Pr(W)}{\Pr(B)}$$
$$= \frac{0.45(0.4)}{0.6(0.25) + 0.4(0.45)}$$

I19. a) $\Pr(C) = 0.5(0.10) + 0.5(0.01) = 0.055 \; or \; 5.5\%$
b) $\Pr(M/C^c) = \frac{\Pr(C^c/M)\Pr(M)}{\Pr(C^c)} = \frac{0.90(0.5)}{1-0.055} = 0.476$

I20.

$\therefore 70 - 25 = 45$ *tested negative and no covid*



Sensitivity $\Pr\left(\frac{T^+}{D^+}\right) = \frac{\Pr(T^+ \cap D^+)}{\Pr(D^+)}$

$= \frac{\frac{25}{100}}{\frac{30}{100}} = \frac{25}{30} = 0.83$

$\Pr(type\ II\ error) = 1 - 0.83 = 0.17$

Specificity $\Pr\left(\frac{T^-}{D^-}\right) = \frac{\Pr(T^- \cap D^-)}{Pr(D^-)}$

$= \frac{\frac{45}{100}}{\frac{70}{100}}$

$= \frac{45}{70} = 0.64$

$\Pr(type\ I\ error) = 1 - 0.64 = 0.36$

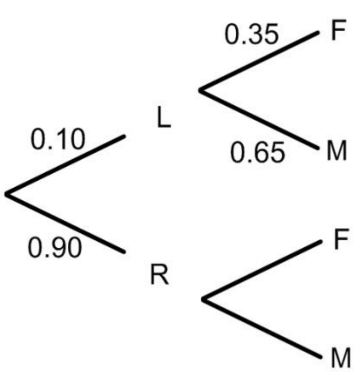
I21. $\Pr(L) = 0.10$ $\Pr(M) = 0.60$

a) $\Pr(L \cap M) = \Pr(L) \times \Pr(M) = 0.10 \times 0.60 = 0.06$ *or* 6%

b) $\Pr(F/L) = 0.35$

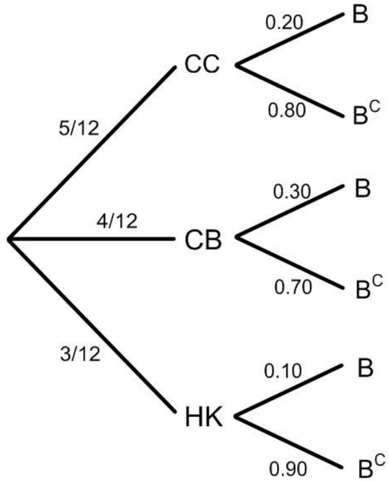
$\Pr(L \cap M) = \Pr(L) \times \Pr(M/L)$

$= 0.10 \times 0.65 = 0.065$



I22.

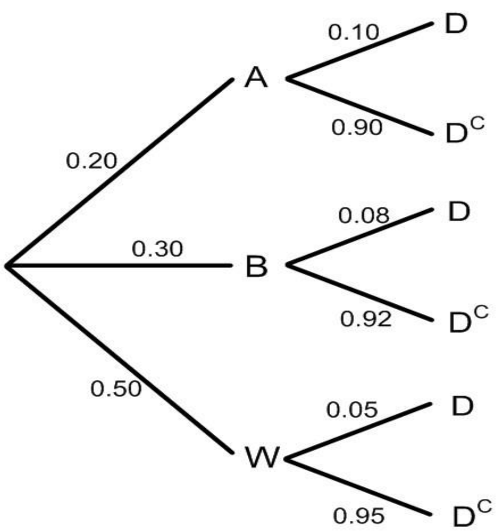
$$\Pr(CC/B^C) = \frac{\Pr\,(CC\ and B^C)}{\Pr(B^C)} = \frac{\frac{5}{12}(0.8)}{\frac{5}{12}(0.8) + \frac{4}{12}(0.7) + \frac{3}{12}(0.9)}$$



I23.

$$\Pr(A|D) = \frac{\Pr\,(\text{A} \cap D)}{\Pr(D)}$$

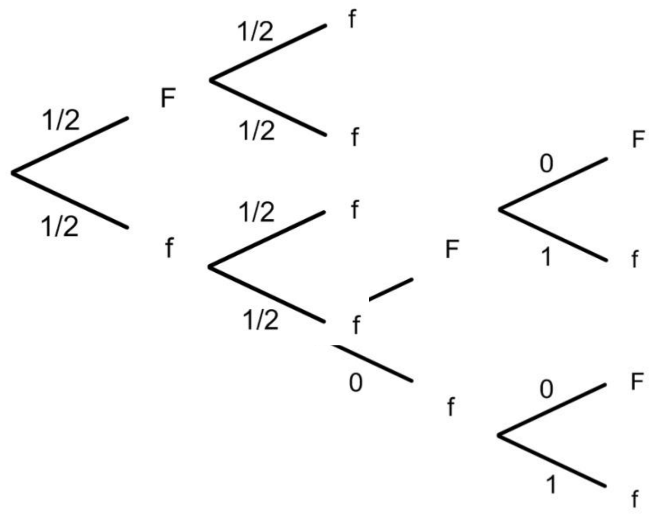
$$= \frac{(0.10)(0.20)}{0.2(0.1) + 0.3(0.08) + 0.5(0.05)} = 0.29$$



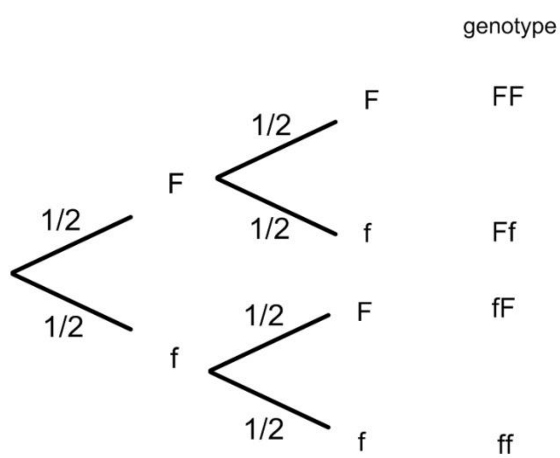
J. Genetics

J1. $a)$ $\Pr(FF) = (1)(0) = 0$

b)$\Pr(ff) = 0(1) = 0$
$\Pr(Ff) = 1(1) + a(a) = 1$
c)$Purple = Ff \quad \therefore 1$

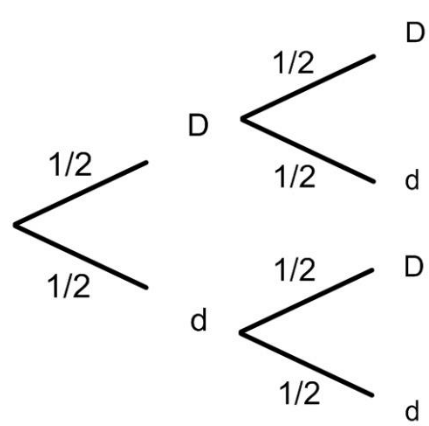


J2. a)
b) Pr(white)=ff = 1/4
c)$\Pr(Ff) = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{2}$



J3.
Yy $yellow$
yy $green$
Yy $yellow$
yy $green$
$\Pr(green) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

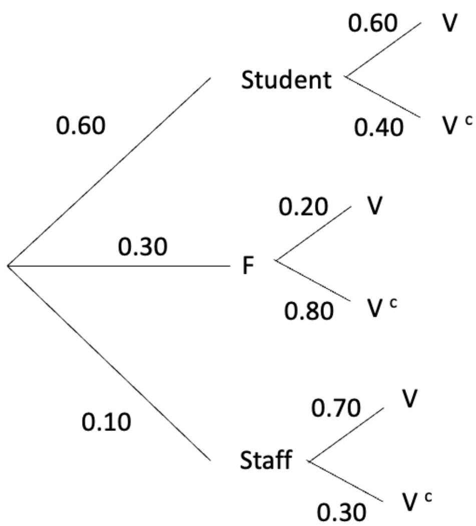
J4.
Ff $- purple$
Ff $- purple$
ff $- white$
ff $- white$
$\Pr(ff) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$



J5. $\Pr(Dd) = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Quiz 3: Practice on Sections L and M

1.a)



$$\Pr(V) = \Pr(student \cap V) + \Pr(F \cap V) + \Pr\,(staff \cap V)$$
$$= 0.6(0.6) + 0.3(0.2) + 0.10(0.70)$$
$$= 0.49$$

b) $\Pr\,(staff/\mathrm{V}) = \frac{\Pr\,(staff \cap V)}{\Pr\,(V)}$

$$= \frac{0.10(0.70)}{0.49}$$
$$= 0.143 \quad or \quad 14.3\%$$

2. $\Pr(at\ least\ 1\ hand) = 1 - \Pr\,(no\ hands)$

$$= 1 - \Pr\,(TTTT)$$
$$= 1 - \left(\frac{1}{2}\right)^4$$
$$= 1 - \frac{1}{16}$$
$$= \frac{15}{16}$$

3. The answer is *independent.*

4. $\Pr(A) = \frac{1}{2} \quad (\frac{1}{2}\ rolls\ are\ even)$

$\Pr(B) = \frac{6}{36} = \frac{1}{6} \quad ((1,1)(1,2)(1,3)(1,4)(1,5)(1,6))$

$\Pr(A \cap B) = \frac{3}{36} = \frac{1}{12} \quad ((1,1)(1,3)(1,5))$

$\Pr(A \cap B) = \Pr\,(A) \times \Pr\,(B)$

$\therefore\ they\ are\ independent$

5. a) $\Pr(S \cup C) = \Pr(S) + \Pr(C) - \Pr(S \cap C)$
$= \frac{40}{75} + \frac{30}{75} - \frac{20}{75}$
$= \frac{50}{75} = 0.667$

b) $\Pr(S^c \cap C^c) = 1 - \Pr(S \cup C) = 1 - 0.667 = 0.333$

6. $\Pr(F/E) = \frac{\Pr(E \cap F)}{\Pr(E)}$
$0.6 = \frac{\Pr(E \cap F)}{0.2}$
$\Pr(E \cap F) = 0.6 \times 0.2 = 0.12$

7. a) $E^c = \textit{there are no queens among the } 5 \textit{ cards drawn}$

b) $E \cap F = \textit{there is at least one queen and all } 5 \textit{ cards are red}$

∴ You must have either the Queen of Hearts or the Queen of Diamonds (or both) among the 5 cards chosen

8. $\Pr(E \cap F) = 0$

9. $\Pr(E \cap F) = \Pr(E) \times \Pr(F)$

K. Practice Exam Questions on Probability

K1. $\frac{8+3+1}{8+3+1+2} = \frac{12}{14} = 0.857$

K2. $1 - \Pr(no\ girls) = 1 - \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= 1 - \frac{1}{8} = \frac{7}{8}$

K3. Pr $(roll\ 4$ /even$)= \frac{1}{3}$ $\nwarrow roll\ 4$

$S_{reduced} = \Omega\ reduced = \{only\ even\} = \{2, 4, 6\}$

↓

Denominator

K4. Pr (roll 5/even)= 0 ← impossible since 5 is not even

K5. Pr(1T, 2H) = Pr(THH, HHT, HTH) = $\frac{3}{8}$ $(draw\ a\ tree)$

K6. a) $\frac{1}{4}\left(\frac{1}{4}\right) = \frac{1}{16}$

b) $\frac{4}{52}$) $\left(\frac{4}{52}\right) = 0.005917$

c) $\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4} = 0.25$

K7. $B^C = \{2, 4, 6\}$

a) $= \{2, 4\}$

b) $= \{1, 2, 3, 4, 5\} = A$

K8. $B^C = \{4, 5, c, d\}$

a) $A \cap B = \{1, 3, a\} \neq C^c = \{1, 5, a\}$

b) $A \cap B^C = \{5\} \neq C^C = \{1, 5, a\}$

c) Always true

d) $A \cap C = \{3\} \neq C^C = \{1, 5, a\}$

e) False

∴ C is the answer.

K9.

a) $= \{1, 2, 3, 4, 5, 6\}$
b) $= \{1, 2, 3, 4\}$
c) $= \{1, 2, 3, 4\}$
d) $= \{\emptyset\}$ $\quad \therefore D\ is\ the\ answer$

K10. Pr(sum 7)

$= \Pr\,((1,6), (6,1), (2,5), (5,2), (3,4), (4,3))$

K11. $\Pr(BBG) = \frac{3}{7} \times \frac{2}{6} \times \frac{4}{5} = \frac{4}{35} = 0.114$

K12. $3x + x = 1$

$4x = 1 \quad x = \frac{1}{4}$

$\Pr(\mathrm{T}) = \Pr(A \cap T) + \Pr\,(B \cap T)$

$= \frac{3}{4}(0.6) + \frac{1}{4}(0.75)$

$= 0.6375$

K13. a) NO should be $(E \cup F)^C = E^c \cap F^c$

b) YES
c)NO
d)YES $\therefore B\ and\ D\ is\ the\ answer$

K14. $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr\,(A \cap B)$

$= 0.20 + 0.50 - 0.15$

$= 0.70 - 0.15 \quad = 0.55$

K15. D is the answer.

K16. D = has disease $\quad D^c =$ doesn't have disease

$P = test\ is +\quad P^c = test\ is -$

$\Pr\,(D/\mathrm{P}) = \frac{\Pr\,(D \cap P)}{\Pr\,(P)} = \frac{0.03 \times 1}{0.03 \times 1 + 0.97 \times 0.15} = 0.17$

Material Since the Midterm

L. Discrete and Continuous Random Variables

Questions emailed to you!

Which of the following are probability density functions?

A) no, the area under the graph isn't 1
B) yes, it is non-negative and the area is 1
C) yes, it is non-negative and the area is 1
D) no, the graph is negative from -1 to 2

Which of the following depicts a cumulative distribution function?

A) Yes, it is non-decreasing, right continuous and $\lim_{x\to\infty} F(x) = 1$ *and* $\lim_{x\to-\infty} F(x) = 0$
B) No, it is decreasing!
C) Yes, it is non-decreasing, right continuous and $\lim_{x\to\infty} F(x) = 1$ *and* $\lim_{x\to-\infty} F(x) = 0$
D) No, it is not right continuous.
E) Yes, it is non-decreasing, right continuous and $\lim_{x\to\infty} F(x) = 1$ *and* $\lim_{x\to-\infty} F(x) = 0$
F) No, $\lim_{x\to\infty} F(x) \neq 1$

Example 1.

$$F(x) = \int_0^x f(s)ds = \int_0^x \frac{3s}{c^4}ds$$
$$= \left[\frac{3s^2}{(c^4)2}\right]_0^x$$
$$= \left[\frac{3s^2}{2c^4}\right]_0^x$$
$$= \frac{3x^2}{2c^4}$$

Example 2.

$$\int_{-a}^{a}(a^4 - x^4)dx = 1$$
$$\left[a^4x - \frac{x^5}{5}\right]_{-a}^{a} = 1$$
$$\left[a^4(a) - \frac{a^5}{5}\right] - \left[a^4(-a) - \frac{(-a)^5}{5}\right] = 1$$
$$a^5 - \frac{a^5}{5} + a^5 - \frac{a^5}{5} = 1$$
$$\frac{2a^5}{1} - \frac{2a^5}{5} = 1$$
$$\frac{10a^5}{5} - \frac{2a^5}{5} = 1$$
$$\frac{8a^5}{5} = 1$$
$$a^5 = \frac{5}{8}$$
$$a = \sqrt[5]{\frac{5}{8}}$$

Example 3.

$$f(x) = \begin{cases} x, 0 \leq x \leq 1 \\ 2 - x, 1 \leq x \leq 2 \\ 0, otherwise \end{cases}$$

$$E(x) = \int_0^1 x \cdot x \, dx + \int_1^2 x(2-x)dx$$

$$= \int_0^1 x^2 dx + \int_1^2 (2x - x^2)dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$= \left(\frac{1}{3} - 0\right) + \left[\left(2^2 - \frac{2^3}{3}\right) - \left(1^2 - \frac{1^3}{3}\right)\right]$$

$$= \frac{1}{3} + \left[4 - \frac{8}{3} - 1 + \frac{1}{3}\right]$$

$$= \frac{1}{3} + \left[3 - \frac{7}{3}\right]$$

$$= \frac{1}{3} + \left[\frac{9}{3} - \frac{7}{3}\right]$$

$$= \frac{1}{3} + \frac{2}{3}$$

$$= 1$$

(b) $E(x^2) = \int_0^1 (x^2)(x)\, dx + \int_1^2 x^2(2-x)dx$

$$= \int_0^1 x^3 + \int_1^2 (2x^2 - x^3)dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \left(\frac{1}{4} - 0\right) + \left[\left(\frac{2(2)^3}{3} - \frac{2^4}{4}\right) - \left(\frac{2}{3}(1)^3 - \frac{1^4}{4}\right)\right]$$

$$= \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}$$

$$= \frac{1}{2} + \frac{14}{3} - \frac{4}{1}$$

$$= \frac{3}{6} + \frac{28}{6} - \frac{24}{6}$$

$$= \frac{7}{6}$$

$$Var(x) = E(x^2) - \mu^2$$

$$= \frac{7}{6} - 1^2$$

$$= \frac{7}{6} - 1$$

$$= \frac{7}{6} - \frac{6}{6} \quad = \frac{1}{6}$$

Example 4. a) $\int_0^1 k(x^2+x) = 1 \quad \textit{since it is a probability function}$

$$k\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1 = 1$$

$$k\left[\frac{1}{3}+\frac{1}{2}\right] = 1$$

$$k\left[\frac{2}{6}+\frac{3}{6}\right] = 1$$

$$k\left(\frac{5}{6}\right) = 1$$

$$k = \frac{6}{5}$$

b) $F(x) = \int_{-\infty}^{x} f(s)ds$

$$F(x) = \int_0^x \frac{6}{5}(s^2+s)ds$$

$$= \frac{6}{5}\left[\frac{s^3}{3}+\frac{s^2}{2}\right]_0^x$$

$$= \frac{6}{5}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]$$

c) $F(x) = \frac{6}{5}(\frac{x^3}{3}+\frac{x^2}{2})$

$$f(x) = F'(x) = \frac{6}{5}(\frac{3x^2}{3}+\frac{2x}{2}) = \frac{6}{5}(x^2+x)$$

d) $E(x) = \int_{-\infty}^{\infty} x\, f(x)dx$

$$= \frac{6}{5}\int_0^1 x(x^2+x)dx$$

$$= \frac{6}{5}\int_0^1 (x^3+x^2)dx$$

$$= \frac{6}{5}\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1$$

$$= \frac{6}{5}\left[\frac{1}{4}+\frac{1}{3}\right]$$

$$= \frac{6}{5}\left[\frac{3}{12}+\frac{4}{12}\right]$$

$$= \frac{6}{5}\left(\frac{7}{12}\right) = \frac{7}{5(2)} = \frac{7}{10}$$

e) $Var(x) = \int_{-\infty}^{\infty} x^2 f(x)dx - [E(x)]^2$

$$= \frac{6}{5}\int_0^1 x^2(x^2+x)dx - \left(\frac{7}{10}\right)^2$$

$$= \frac{6}{5}\int_0^1 (x^4+x^3)dx - \frac{49}{100}$$

$$= \frac{6}{5}\left[\frac{x^5}{5}+\frac{x^4}{4}\right]_0^1 - \frac{49}{100}$$

$$= \frac{6}{5}\left(\frac{1}{5}+\frac{1}{4}\right) - \frac{49}{100}$$

$$= \frac{6}{5}\left(\frac{4+5}{20}\right) - \frac{49}{100}$$

$$= \frac{6}{5}\left(\frac{9}{20}\right) - \frac{49}{100}$$

$$= \frac{54-49}{100} = \frac{5}{100}$$

$$= \frac{1}{20}$$

Example 5.

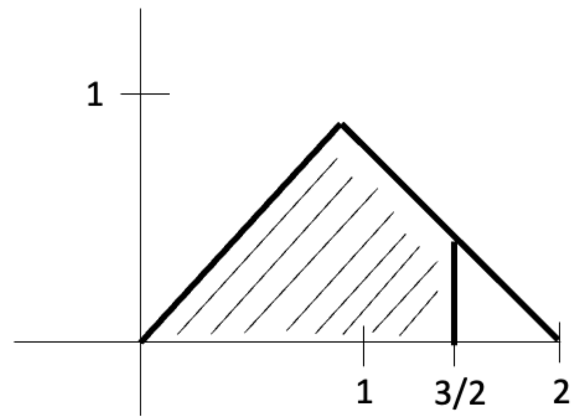
$$F(x) = \begin{cases} 0, x < 0 \\ x^2/4 \quad 0 \le x < 2 \end{cases}$$
$$1 \qquad x \ge 2$$
$$2 \qquad x \ge 2$$

f(x)$= F'(x) = \frac{2x}{4} = \frac{x}{2}$ $\quad derivative$ to find the pdf from cdf

$$E(x) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^2 x\left(\frac{x}{2}\right)dx$$
$$= \int_0^2 \frac{x^2}{2}dx$$
$$= \left[\frac{x^3}{6}\right]_0^2$$
$$= \frac{2^3}{6} - 0 = \frac{8}{6} = \frac{4}{3}$$

Example 6. $\Pr\left(|x| < \frac{3}{2}\right) = \Pr\left(\frac{-3}{2} \le x \le \frac{3}{2}\right)$

$$= \Pr\left(0 \le x \le \frac{3}{2}\right) \quad since\ x > 0$$
$$= 1 - \Pr\left(x > \frac{3}{2}\right)$$
$$= 1 - \frac{bh}{2} = 1 - \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{2}$$
$$= 1 - \frac{\frac{1}{4}}{2}$$
$$= 1 - \frac{1}{8}$$
$$= \frac{7}{8}$$



Method 2: $\Pr\left(|x| < \frac{3}{2}\right)$

$$= \Pr\left(0 < x < \frac{3}{2}\right) = \Pr(0 < x < 1) + \Pr\left(1 < x < \frac{3}{2}\right)$$

$$= \int_0^1 x\,dx + \int_1^{3/2} (2 - x)dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^{3/2}$$

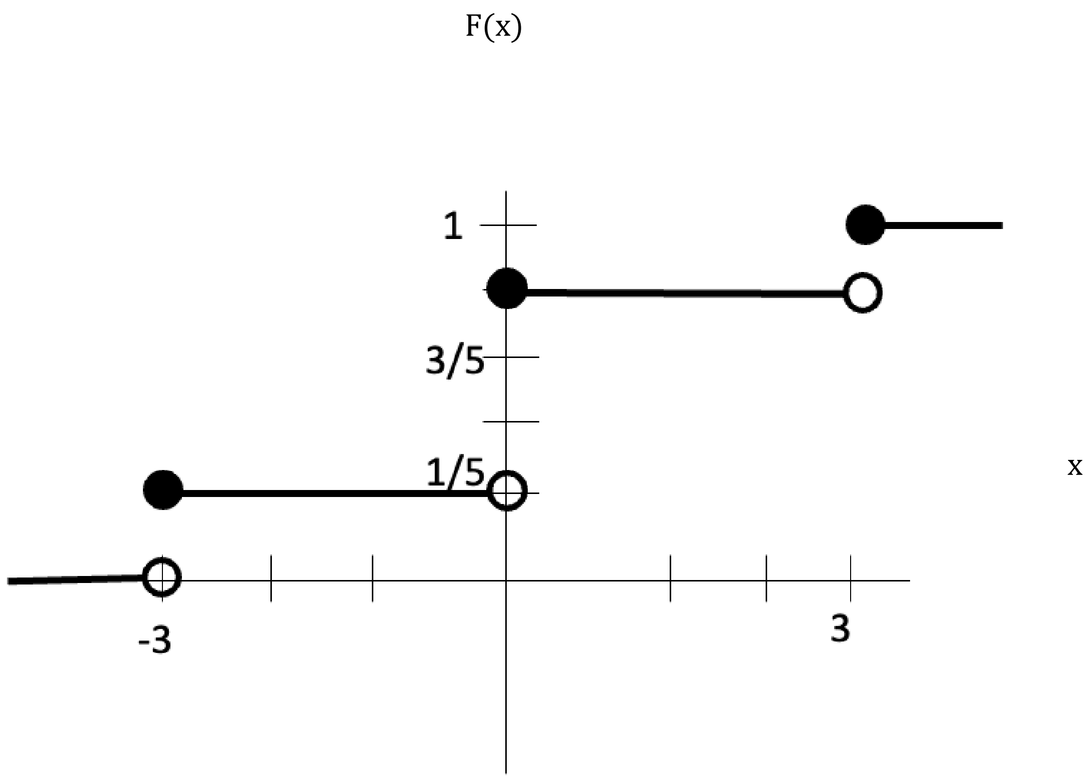
$$= \left(\frac{1}{2} - 0\right) + \left(2\left(\frac{3}{2}\right) - \frac{\left(\frac{3}{2}\right)^2}{2}\right) - \left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{2} + 3 - \frac{9}{8} - 2 + \frac{1}{2}$$

$$= \frac{2}{1} - \frac{9}{8} \quad = \frac{16}{8} - \frac{9}{8} \quad = \frac{7}{8}$$

Example 7. Since this is a discrete graph, not a continuous one with area, we can add up the probabilities going down the chart to get the cumulative values. Ie. F(0)=1/5 +3/5 =4/5

x	$\Pr(x)$	$F(x)$
-3	$\frac{1}{5}$	$\frac{1}{5}$
0	$\frac{3}{5}$	$\frac{4}{5}$
3	$\frac{1}{5}$	1



Example 8.

a) E(2X-3Y)=2E(X) - 3 E(Y)= 2(5) -3(2) = 4

b) V(2X)=2^2V(X)= 4(5)= 20

c) V(-X+2)= $(-1)^2$V(X)=1(5)=5

$\sigma(-X+2)=\sqrt{5}$

Example 9.

$$V(x) = E(x^2) - \left(E(x)\right)^2 = 10 - 2^2 = 10 - 4 = 6$$

$$\sigma(x) = \sqrt{6}$$

Practice Exam Questions on Continuous Random Variables

L1.

A. $\int_0^{\frac{5\pi}{2}} \sin x dx = [-\cos x]_0^{\frac{5\pi}{2}}$

$= -\cos\frac{5\pi}{2} + \cos 0$

$= 0 + 1 = 1$ But, y=sinx is negative from pi to 2pi, so it is not a probability density function.

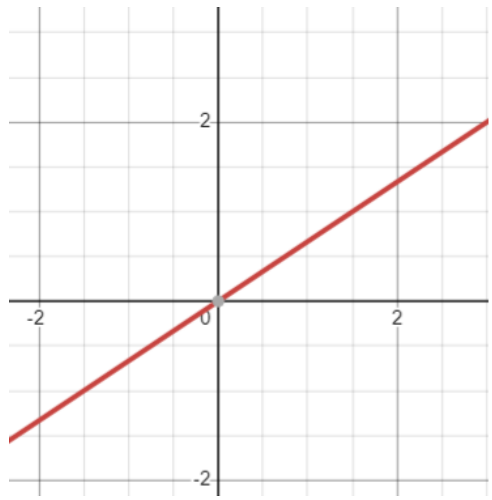
B. $\int_{-1}^{2} \frac{2}{3} x dx = \left[\frac{2}{3}\frac{x^2}{2}\right]_{-1}^{2}$

$= \left[\frac{1}{3}x^2\right]_{-1}^{2}$

$= \frac{1}{3}(2)^2 - \frac{1}{3}(-1)^2$

$= \frac{3}{3} = 1$

But, the graph is below the x-axis from -1 to 0, so it is NOT a probability density function.



L2. $f(x) = 2(x+1)^{-3},\ x > 0$

$F(x) = \int_0^x 2(ts+1)^{-3} dt \quad since\ x > 0$

$= \left[\frac{2(t+1)^{-2}}{-2}\right]_o^x$

$= [-(t+1)^{-2}]_o^x$

$= [-(x+1)^{-2} + (0+1)^{-2}]$

$= \frac{-1}{(x+1)^2} + \frac{1}{1^2}$

$= 1 - \frac{1}{(x+1)^2}$

L3.

$f(x) = 2(x+1)^{-3}; x > 0$

$\Pr(x > 3) = 1 - \Pr(x \leq 3)$

$= 1 - \int_0^3 2(x+1)^{-3} dx$

$= 1 - \left[\frac{2(x+1)^{-2}}{-2}\right]_0^3$

$= 1 - \left[\frac{-1}{(x+1)^2}\right]_0^3$

$= 1 - \left[\frac{-1}{(3+1)^2} + \frac{1}{(0+1)^2}\right]$

$= 1 - \left[-\frac{1}{16} + 1\right]$

$= \frac{1}{16}$

The answer is a).

L4.a) $\int_0^{10} \frac{x^3}{5000}(10-x) = \left[\frac{1}{500}\frac{x^4}{4} - \frac{1}{5000}\frac{x^5}{5}\right]_0^{10}$

$$= \frac{10^4}{4(500)} - \frac{1}{25\,000}10^5 = 5 - 4 = 1$$

b) $= \int_1^4 \frac{x^3}{5000}(10-x)dx = \left[\frac{10x^4}{4(5000)} - \frac{x^5}{5(5000)}\right]_1^4$

$$= \left(\frac{10(4)^4}{20\,000} - \frac{4^5}{25\,000}\right) - \left(\frac{10(1)^4}{20\,000} - \frac{1^5}{25\,000}\right)$$

$$= \left(\frac{2560}{20\,000} - \frac{1024}{25\,000}\right) - \left(\frac{10}{20\,000} - \frac{1}{25\,000}\right)$$

$$= \left(\frac{2550}{20\,000} - \frac{1023}{25\,000}\right) = 0.1275 - 0.04092 = 0.08658$$

c) $\int_6^{10} f(x)dx = \left[\frac{10x^4}{4(5000)} - \frac{x^5}{5(5000)}\right]_6^{10} = \left[\frac{10(10)^4}{4(5000)} - \frac{(10)^5}{5(5000)}\right] - \left[\frac{10(6)^4}{4(5000)} - \frac{(6)^5}{5(5000)}\right]$

$$= \left[\frac{10(10)^4}{4(5000)} - \frac{(10)^5}{5(5000)}\right] - \left[\frac{10(6)^4}{4(5000)} - \frac{(6)^5}{5(5000)}\right]$$

$$= [\,5-4] - [0.648 - 0.31104] = 0.66304$$

L5.

$$CDF = F(x) = \int_0^x 3t^2dt = \left[\frac{3t^3}{3}\right]_0^x = [t^3]_0^x = x^3 - 0 = x^3$$

L6.

$$CDF = F(x) = \int_0^x f(t)dt = \int_0^x \frac{t^3}{4}dt = \left[\frac{t^4}{16}\right]_0^x = \frac{x^4}{16}$$

L7.

$$\Pr\left(0 < x < \frac{1}{4}\right) = \int_0^{0.25} 4x^3dx = \left[\frac{4x^4}{4}\right]_0^{0.25} = [x^4]_0^{0.25}$$

$$= (0.25)^4 - 0^4 = 0.003906$$

L8.

$PDF =$ do the derivative

$$f(x) = F'(x) = \frac{2x(1+x^2) - 2x(x^2)}{(1+x^2)^2}$$

$$= \frac{2x + 2x^3 - 2x^3}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$$

L9.

a) $\mu = \int_{-\infty}^{\infty} xf(x)dx$

$$\mu = \int_0^2 x\left(\frac{1}{2}x\right)dx = \int_0^2 \frac{1}{2}x^2 = \left[\frac{x^3}{6}\right]_0^2 = \frac{2^3}{6} - 0 = \frac{8}{6} = \frac{4}{3}$$

$Var(x) = \int_{-\infty}^{\infty} x^2 f(x) - [E(x)]^2$

$= \int_0^2 x^2\left(\frac{1}{2}x\right)dx - \left(\frac{4}{3}\right)^2 = \int_0^2 \frac{1}{2}x^3 dx - \frac{16}{9}$

$= \left[\frac{1}{2}\frac{x^4}{4}\right]_0^2 - \frac{16}{9} = \left(\frac{1}{8}(2)^4 - 0\right) - \frac{16}{9}$

$= \frac{16}{8} - \frac{16}{9} \quad = \frac{2}{1} - \frac{16}{9} \quad = \frac{18}{9} - \frac{16}{9} \quad = \frac{2}{9}$

b)

a) $E(x) = \int_0^1 x(4x^3)dx = \int_0^1 4x^4 dx = \left[\frac{4x^5}{5}\right]_0^1$

$= \frac{4}{5}(1)^5 - 0 = \frac{4}{5}$

$$b)\ Var(x) = \int_0^1 x^2(4x^3)dx - (\frac{4}{5})^2 = \int_0^1 4x^5 dx - \frac{16}{25}$$

$= \left[\frac{4x^6}{6}\right]_0^1 - \frac{16}{25} = \frac{2}{3}(1)^6 - 0 - \frac{16}{25} \quad = 0.02\overline{6}$

L10. Find E[X]. $f(x) = \frac{1}{80} \quad 0 \le x \le 60$

$f(x) = \frac{100-x}{3200} \quad 60 \le x \le 100$

$E(x) = \int_0^{60} x\left(\frac{1}{80}\right)dx + \int_{60}^{100} \frac{1}{3200}(100-x)xdx$

$= \left[\frac{x^2}{160}\right]_0^{60} + \int_{60}^{100} \frac{1}{3200}(100x - x^2)dx$

$= \left(\frac{3600}{160} - 0\right) + \left[\frac{1}{3200}(\frac{100x^2}{2} - \frac{x^3}{3}\right]_{60}^{100}$

$= 22.5 + \frac{1}{3200}\left[50x^2 - \frac{1}{3}x^3\right]_{60}^{100}$

$= 22.5 + \frac{1}{3200}\left[\left(50(100)^2 - \frac{1}{3}(100)^3\right) - \left(50(60)^2 - \frac{1}{3}(60)^3\right)\right]$

$= 22.5 + \frac{1}{3200}(166\,666.67 - 108\,000)$

$= 22.5 + 18.33$

$= 40.83$

Find F(X).

$$F(x) = \begin{cases} \frac{1}{80}x\,, 0 \le x \le 60 \\ \frac{-1}{6400}x^2 + \frac{1}{32}x - \frac{9}{16}, \qquad 60 \le x \le 100 \end{cases}$$

Step 1 Find $F(x) for\ f(x) = \frac{1}{80}$

$$F(x) = \int_0^x \frac{1}{80}dt = \left[\frac{1}{80}t\right]_0^x = \frac{1}{80}x$$

Step 2 Find $F(x)\ for\ f(x) = \frac{100-x}{3200}$

$$f(x) = \frac{1}{32} - \frac{1}{3200}x$$

$$F(x) = \int_0^{60}\frac{1}{80}dt + \int_{60}^x\left(\frac{1}{32} - \frac{1}{3200}t\right)dt$$

$$= \left[\frac{1}{80}t\right]_0^{60} + \left[\frac{1}{32}t - \frac{1}{3200}\frac{t^2}{2}\right]_{60}^x$$

$$= \left[\frac{1}{80}(60) - 0\right] + \left[\frac{1}{32}x - \frac{x^2}{6400}\right] - \left[\frac{60}{32} - \frac{1}{6400}(60)^2\right]$$

$$= \frac{60}{80} + \frac{1x}{32} - \frac{x^2}{6400} - \frac{60}{32} + \frac{3600}{6400}$$

$$= \frac{-1}{6400}x^2 + \frac{1}{32}x + \frac{4800}{6400} - \frac{12\,000}{6400} + \frac{3600}{6400}$$

$$= \frac{-1}{6400}x^2 + \frac{1}{32}x - \frac{3600}{6400}$$

$$= \frac{-1}{6400}x^2 + \frac{1}{32}x - \frac{9}{16}$$

L11.

$$F(x) = \int_{-\infty}^x f(x)dx = \int_0^x\left(\frac{6t^3}{15} + \frac{6t^2}{10}\right)dx$$

$$= \left[\frac{6x^4}{4(15)} + \frac{6x^3}{3(10)}\right]_0^x$$

$$= \frac{x^4}{10} + \frac{x^3}{5}$$

L12.

Let $f(x) = \begin{cases} 2(1-x) \\ 0 \end{cases}\ if\ 0 \le x \le 1 \quad otherwise$

Find $E(x) + Var(x)$

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)dx \quad = \int_0^1 x(2)(1-x)dx$$

$$= 2\int_0^1(x - x^2)dx \quad = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \quad = 2\left[\left(\frac{1}{2} - \frac{1}{3}\right) - (0-0)\right]$$

$$= 2\left(\frac{3}{6} - \frac{2}{6}\right) \quad = \frac{1}{3}$$

$$E(x^2) = \int_0^1 x^2 2(1-x)dx \quad = \int_0^1(2x^2 - 2x^3)dx$$

$$= \left[\frac{2x^3}{3} - \frac{2x^4}{4}\right]_0^1 \quad = \left(\frac{2}{3} - \frac{2}{4}\right) - (0-0) \quad = \frac{8}{12} - \frac{6}{12}$$

$$= \frac{2}{12} \quad = \frac{1}{6}$$

$$Var(x) = E(x^2) - \mu^2$$

$$= \frac{1}{6} - \left(\frac{1}{3}\right)^2 \quad = \frac{1}{6} - \frac{1}{9} \quad = \frac{3}{18} - \frac{2}{18} \quad = \frac{1}{18}$$

ADD THIS QUESTION AND DO IT!!!

b) The discrete random variable X has the probability distribution function shown below:

X	Pr(X=x)
-1	1/2
0	1/4
1	

Find μ(X) and σ(X).

b) This is a discrete random variable that only takes on values -1, 0 and 1.

X	Pr(X)	X^2
-1	$^1/_2$	1
0	$^1/_4$	0
1	$^1/_4$	1

$$E(X) = -1 \times \frac{1}{2} + 0 \times \frac{1}{4} + 1 \times \frac{1}{4} = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$$

$$E(X^2) = 1\left(\frac{1}{2}\right) + 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{4}\right) = \frac{3}{4}$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{3}{4} - \left(\frac{-1}{4}\right)^2 = \frac{12}{16} - \frac{1}{16} = \frac{11}{16}$$

$$\sigma(X) = \sqrt{\frac{11}{16}}$$

L13. Pr(X</0.8) = 1 – Pr(X>8) = 1 – bh/2 = 1 – (0.2)(0.75)/2 = 1 – 0.075=0.925

L14.

$$a)\ \int_2^3 \frac{x^3}{5000}(10-x)ds = \frac{1}{5000}\int_2^3 (10x^3 - x^4)ds$$

$$= \frac{1}{5000}\left[\frac{10x^4}{4} - \frac{x^5}{5}\right]_2^3$$

$$= \frac{1}{5000}\left[\frac{10(3)^4}{4} - \frac{3^5}{5} - \left(\frac{10(2)^4}{4} - \frac{2^5}{5}\right)\right]$$

$$= \frac{1}{5000}[202.5 - 48.6 - 40 + 6.4]$$

$$= 0.02406$$

b) $E(x) = \int_0^{10} x\, f(x) dx = \int_0^{10} \frac{x \cdot x^3}{5000}(10-x) dx$

$= \int_0^{10} \frac{x^4(10-x)dx}{5000}$

$= \frac{1}{5000} \int_0^{10} (10x^4 - x^5) dx$

$= \frac{1}{5000} \left[\frac{10x^5}{5} - \frac{x^6}{6}\right]_0^{10}$

$= \frac{1}{5000}\left[2(10)^5 - \frac{10^6}{6} - (0-0)\right]$

$=\frac{1}{5000}\left[200\ 000 - \frac{1000\ 000}{6}\right]$

$=\frac{1}{5000}\left[\frac{1\ 200000}{6} - \frac{1000\ 000}{6}\right]$

$= \frac{1}{5000}\left[\frac{200\ 000}{6}\right]$

$=\frac{1}{5000}\left[\frac{200\ 000}{6}\right]$

$= \frac{40}{6} = \frac{20}{3}$

c) $Var(x) = \int_{-\infty}^{\infty} x^2 f(x) - [E(x)]^2$

$= \int_0^{10} x^2 f(x) - (\frac{20}{3})^2$

$= \int_0^{10} \frac{x^2 \cdot x^3(10-x)}{5000} dx - (\frac{20}{3})^2$

$= \frac{1}{5000}\int_0^{10}(10x^5 - x^6) ds - (\frac{20}{3})^2$

$= \frac{1}{5000}\left[\frac{10x^6}{6} - \frac{x^7}{7}\right]_0^{10} - (\frac{20}{3})^2$

$= \frac{1}{5000}\left[\frac{10(10)^6}{6} - \frac{10^7}{7}\right] - (\frac{20}{3})^2$

$= \frac{1}{5000}\left[\frac{10\ 000\ 000}{6} - \frac{10\ 000\ 000}{7}\right] - \left(\frac{20}{3}\right)^2$

$= \left[\frac{2000}{6} - \frac{2000}{7}\right] - \left(\frac{20}{3}\right)^2 = \left[\frac{14000}{42} - \frac{12000}{42}\right] - \left(\frac{20}{3}\right)^2$

$= \left[\frac{2000}{42}\right] - \left(\frac{20}{3}\right)^2 = \frac{1000}{21} - \frac{400}{9}$

=47.619 -44.444=3.18

L15. $f(x) = derivative\ of\ \frac{x+1}{4}$

$f(x) = \frac{1}{4}$

$E(x) = \int_a^b x\, f(x)dx = \int_0^1 \frac{1}{4} x dx$

$= \left[\frac{1}{4}\frac{x^2}{2}\right]_0^1$

$= \left[\frac{x^2}{8}\right]_0^1$

$= \frac{1}{8} - 0 \quad = \frac{1}{8}$

L16. $\mathrm{E}(x) = \int_1^2 x(2x^{-2})dx$

$= \int_1^2 2x^{-1}dx = \int_1^2 \frac{2}{x} dx$

$= 2[\ln x]_1^2$

$= 2[\ln 2 - \ln 1]$

$= 2[\ln 2 - 0]$

$= 2\ln 2 \quad or \quad \ln 2^2 \quad = \ln 4$

$V(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \int_1^2 x^2(2x^{-2})dx$

$= \int_1^2 2dx = [2x]_1^2$

$= 2(2) - 2(1) = \quad \boxed{2}$

$Var(x) = 2 - (\ln 4)^2$

L17. Given $f(x) = \frac{x^3}{4} \quad for \quad 0 < x < 2$

a) Find $F(x)$ $\quad F(x) = \int_o^x \frac{t^3}{4} dt = \left[\frac{t^4}{16}\right]_o^x$

$= \frac{x^4}{16} - 0$

$= \frac{x^4}{16}$

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^4}{16}, & 0 < x < 2 \\ 1 & if \quad x \geq 2 \end{cases}$$

$0 < x < 2$

b) $Find\ E(x)$ $\quad E(x) = \int_0^2 x\left(\frac{x^3}{4}\right)dx$

$= \int_0^2 \frac{x^4}{4}dx$

$= \left[\frac{x^5}{5\cdot 4}\right]_0^2$

$= \left[\frac{x^5}{20}\right]_0^2$

$= \frac{2^5}{20}$

$= \frac{32}{20}$

$= \frac{8}{5}$

L18. Given $f(x) = \{6x - 6x^2, \quad 0 \le x \le 1$

Find $E(x) + Var(x)$

$E(x) = \int_0^1 x(6x - 6x^2)dx$

$= \int_0^1 (6x^2 - 6x^3)dx$

$= \left[\frac{6x^3}{3} - \frac{6x^4}{4}\right]_0^1$

$= \left(\frac{6}{3} - \frac{6}{4}\right) - (0 - 0)$

$= 2 - 1.5$

$= \frac{1}{2}$

$$E(x^2) = \int_0^1 x^2(6x - 6x^2)dx$$

$= \int_0^1 (6x^3 - 6x^4)dx$

$= \left[\frac{6x^4}{4} - \frac{6x^5}{5}\right]_0^1$

$= \left[\frac{3}{2}(1) - \frac{6}{5}(1)\right] - [0 - 0]$

$= \frac{3}{2} - \frac{6}{5}$

$= \frac{15}{10} - \frac{12}{10}$

$= \frac{3}{10}$

$Var(x) = E(x^2) - \mu^2$

$= \frac{3}{10} - (\frac{1}{2})^2$

$= \frac{3}{10} - \frac{1}{4}$

$= \frac{6}{20} - \frac{5}{20}$

$= \frac{1}{20}$

L19. $Let\ f(x) = \frac{3}{8}x^2 \quad 0 \le x \le 2$

Find $E(x)$

$$E(x) = \int_0^2 x\, f(x)dx$$
$$= \int_0^2 \frac{3}{8}x^3 dx$$
$$= \frac{3}{8}\left[\frac{x^4}{4}\right]_0^2$$
$$= \frac{3}{8}\left[\frac{2^4}{4} - 0\right]$$
$$= \frac{3}{8}(4)$$
$$= \frac{12}{8}$$
$$= \frac{3}{2}$$

L20. Given E(X)=2, E(Y)=4 and V(X)=5, find each of the following:

a) E(2X-5Y)=2(2)-5(4)= 4-20=-16

b) V(-2X-5)=$(-2)^2$V(X)= 4(5)=20

c) V($-2X$))=$(-2)^2$V(X)= 4(5)=20

$\sigma(-2X+1)$=$\sqrt{20}$

L21. Given E(X)=5, E(Y)=2 and V(X)=6, find each of the following:

a) E(2X-4Y)=2E(X) - 4 E(Y)= 2(5) -4(2) = 10-8=2

b) V(4X-3)=4^2V(X)=16(6)= 96

c) V(-X+5)= $(-1)^2$V(X)=1(6)=6

$\sigma(-X+2)$=$\sqrt{6}$

L22.

$V(X) = E(X^2) - \left(E(X)\right)^2 = 20 - 4^2 = 20 - 16 = 4$

$\sigma(X) = \sqrt{4} = 2$

L23. $\sigma(X) = 9,\ so\ V(X) = 81$

$V(X) = E(X^2) - \left(E(X)\right)^2$
$81 = E(X^2) - (5)^2$
$E(X^2) = 81 + 25 = 106$

L24.

a) $\int_0^{10} \frac{x^3}{5000}((10-x)dx = \frac{1}{5000}\int_0^{10}(10x^3 - x^4)dx$

$$= \frac{1}{5000}\left[\frac{10x^4}{4} - \frac{x^5}{5}\right]_0^{10}$$

$$= \frac{1}{5000}\left[\frac{10(10)^4}{4} - \frac{10^5}{5} - (0-0)\right] = 1$$

$b)\ CDF = F(x) = \int_{-\infty}^{x} \frac{s^3}{5000}(10-s)ds \quad but\ \ 0 \le x \le 10$

$$\therefore \frac{1}{5000}\int_0^x (10s^3 - s^4)ds = \frac{1}{5000}\left[\frac{10s^4}{4} - \frac{s^5}{5}\right]_0^x$$

$$= \frac{1}{5000}\left[\frac{10x^4}{4} - \frac{x^5}{5} - (0-0)\right]$$

$$= \frac{1}{5000}\left[\frac{5}{2}x^4 - \frac{x^5}{5}\right]$$

$c)\ \int_2^3 \frac{s^3}{5000}(10-s)ds = \frac{1}{5000}\ \int_2^3 (10s^3 - s^4)ds$

$$= \frac{1}{5000}\left[\frac{10s^4}{4} - \frac{s^5}{5}\right]_2^3$$

$$= \frac{1}{5000}\left[\frac{10(3)^4}{4} - \frac{3^5}{5} - \left(\frac{10(2)^4}{4} - \frac{2^5}{5}\right)\right]$$

$$= \frac{1}{5000}\ [202.5 - 48.6 - 40 + 6.4]$$

$$= 0.02406$$

$d)\ E(x) = \int_0^{10} x\, f(x)dx = \int_0^{10} \frac{x \cdot x^3}{5000}(10-x)dx$

$$= \int_0^{10} \frac{x^4(10-x)dx}{5000}$$

$$= \frac{1}{5000}\ \int_0^{10}(10x^4 - x^5)dx$$

$$= \frac{1}{5000}\ \left[\frac{10x^5}{5} - \frac{x^6}{6}\right]_0^{10}$$

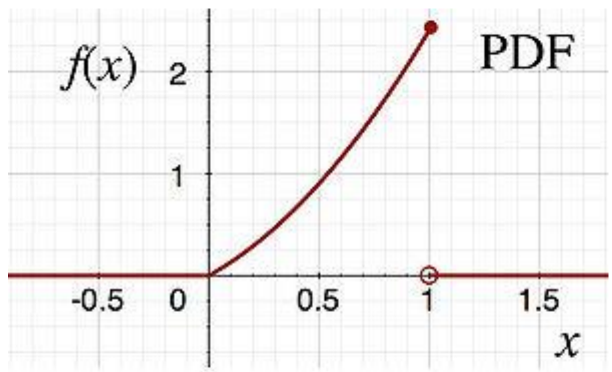
$$= \frac{1}{5000}\left[2(10)^5 - \frac{10^6}{6} - (0-0)\right]$$

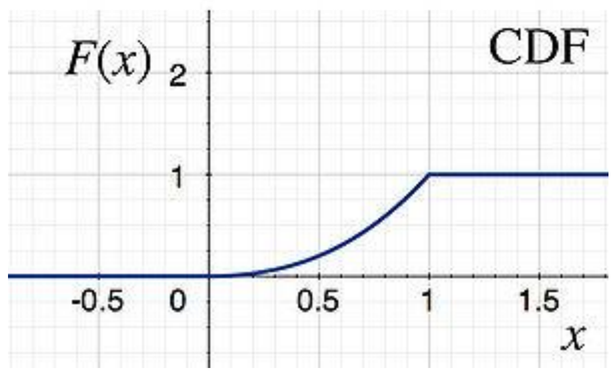
$$= 6.67$$

e) $Var(x) = \int_{-\infty}^{\infty} x^2 f(x) - [E(x)]^2$

$$= \int_0^{10} x^2 f(x) - 6.67^2$$
$$= \int_0^{10} \frac{x^2 \cdot x^3(10-x)}{5000} dx - 6.67^2$$
$$= \frac{1}{5000}\int_0^{10}(10x^5 - x^6)dx - 6.67^2$$
$$= \frac{1}{5000}\left[\frac{10x^6}{6} - \frac{x^7}{7}\right]_0^{10} - 6.67^2$$
$$= \frac{1}{5000}\left[\frac{10(10)^6}{6} - \frac{10^7}{7}\right] - 6.67^2$$
$$= 47.619 - 6.67^2 = 3.1$$

L25. Find the value of k. The integral from 0 to 1 of the function must be =1 since it represents probability.





$$\int_0^1 k(x^2 + x)dx = 1$$
$$k\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = 1$$
$$k\left[\frac{1^3}{3} + \frac{1^2}{2} - (0 - 0\right] = 1$$
$$k\left(\frac{2}{6} + \frac{3}{6}\right) = 1$$
$$k\left(\frac{5}{6}\right) = 1 \; so \; k = 6/5$$

To get the graph of the pdf

Find f(x)

$$f(x) = \frac{6}{5}(t^2 + t)\, dt$$
$$F(x) = \int_0^x \frac{6}{5}(t^2 + t)\, dt$$
$$F(x) = \frac{6}{5}\left[\frac{t^3}{3} + \frac{t^2}{2}\right]_0^x$$
$$F(x) = \frac{6}{5}\left[\frac{x^3}{3} + \frac{x^2}{2} - (0 + 0)\right]$$
$$\therefore f(x) = \frac{6}{5}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)$$

Do a table of values for 0, 0.5, 1 and substitute into f(x)

x	f(x)=6/5(x^2+x)
0	0
0.5	0.9
1	2.4

To get the CDF, type values into F(X)=$\frac{6}{5}(\frac{x^3}{3}+\frac{x^2}{2})$

x	F(X)=$\frac{6}{5}(\frac{x^3}{3}+\frac{x^2}{2})$
0	0
0.5	6/5(0.166)=0.2
1	6/5(1/3+1/2)=6/5(5/6)=1

Last entry has to be 1 for a cumulative graph as probabilities add up to 1

$F(0.5) = \frac{6}{5}(\frac{0.5^3}{3}+\frac{0.5^2}{2})$ (in the next half hour, so less than or equal to 0.5 hr, which is F(0.5))

$$= \frac{6}{5}(\frac{1}{24}+\frac{1}{8})$$
$$= \frac{6}{5}(\frac{1}{24}+\frac{3}{24})$$
$$= \frac{6}{5}(\frac{4}{24})$$
$$= \frac{1}{5}$$

L26.

$$\Pr(50 \le x \le 60) = \int_{50}^{60} \frac{1}{80}\,dx = \left[\frac{1}{80}x\right]_{50}^{60} = \frac{1}{80}(60) - \frac{1}{80}(50)$$
$$= \frac{60}{80} - \frac{50}{80} = \frac{10}{80} = \frac{1}{8}$$

L27. Consider the probability density function

$f(x) = \frac{c}{(x+1)^2} \quad x \ge 0$

Find the CDF

CDF= $F(x) = \int_{-\infty}^{x} c(s+1)^{-2}ds$ since $x \ge 0$

$$= \int_0^x c(s+1)^{-2}ds$$
$$= \left[\frac{c(s+1)^{-1}}{-1}\right]_0^x = \left[\frac{-c}{(s+1)^1}\right]_0^x = \frac{-c}{x+1} + \frac{c}{0+1} = \frac{-c}{x+1} + c$$

M. Pseudorandom Variables and Simulation

Example 1.

a) $9 \div 4 = 2\ R1 \quad \therefore 9\ mod\ 4 = 1$

b) $18 \div 7 = 2\ R4 \quad \therefore 18\ mod\ 7 = 4$

c) $= 4\ since\ 10\ can't \div into\ 4$

d) $= 45 \div 7 = 6\ R3 \quad \therefore 45\ mod\ 7 = 3$

e) $= 5\ since\ 6\ can't \div into\ 5$

Example 2.

a) $U_t = 2U_{t-1} + 1 \quad mod\ 5 \quad U_0 = 1$

$$U_1 = 2U_0 + 1 = 2(1) + 1 = 3\ mod\ 5 = 3$$
$$U_2 = 2U_1 + 1 = 2(3) + 1 = 7\ mod\ 5 = 2$$
$$U_3 = 2U_2 + 1 = 2(2) + 1 = 5\ mod\ 5 = 0$$

b) $U_t = 4U_{t-1} + 4\ \ mod\ 3\ \ U_0 = 2$

$$U_1 = 4U_0 + 4 = 4(2) + 4 = 12\ mod\ 3 = 0$$
$$U_2 = 4U_1 + 4 = 4(0) + 4 = 4\ mod\ 3 = 1$$
$$U_3 = 4U_2 + 4 = 4(1) + 4 = 8\ mod\ 3 = 2$$

Example 3.

Step 1. $F(x) = \int_{-\infty}^{-2} f(s)ds + \int_{-2}^{x} f(s)ds = 0 + \int_{-2}^{x} f(s)ds$

$$= \int_{-2}^{x} \frac{1}{2} s\, ds = \left[\frac{1}{2}\,\frac{s^2}{2}\right]_{-2}^{x} = \left[\frac{s^2}{4}\right]_{-2}^{x}$$

$$= \frac{x^2}{4} - \frac{(-2)^2}{4} = \frac{x^2}{4} - 1$$

Let $y = \frac{x^2}{4} - 1$

Step 2. $x = \frac{y^2}{4} - 1 \quad solve\ for\ y$

$$x + 1 = \frac{y^2}{4}$$
$$4(x+1) = y^2$$
$$y = -\sqrt{4(x+1)} \qquad \therefore F^{-1}(x) = -2\sqrt{x+1}$$

↑

Since the domain is negative

Practice Exam Questions on Pseudorandom Variables and Simulation

M1.

a) $9 \div 2 = 4\ R1 \quad \therefore 9\ mod\ 2 = 1$
b) $18 \div 5 = 3\ R3 \quad \therefore 18\ mod\ 5 = 3$
c) $= 4\ since\ 8\ can't \div into\ 4$
d) $45 \div 8 = 5\ R5 \quad \therefore 45\ mod\ 8 = 5$
e) $= 5\ since\ 7\ can't \div into\ 5$

M2.

$U_t = 2U_{t-1} + 2\ \ mod\ 4 \quad U_0 = 1$

$$U_1 = 2U_0 + 2 = 2(1) + 2 = 4\ mod\ 4 = 0$$
$$U_2 = 2U_1 + 2 = 2(0) + 2 = 2\ mod\ 4 = 2$$
$$U_3 = 2U_2 + 2 = 2(2) + 2 = 6\ mod\ 4 = 2$$

M3.

$$F(x) = \int_{-\infty}^{-10} f(s)ds + \int_{-10}^{x} f(s)ds = 0 + \int_{-10}^{x} f(s)ds$$
$$= \int_{-10}^{x} \frac{1}{50} ds \quad = \left[\frac{1}{50}\frac{s^2}{2}\right]_{-10}^{x} \quad = \left[\frac{s^2}{100}\right]_{-10}^{x} \quad = \frac{x^2}{100} - \frac{(-10)^2}{100} \quad = \frac{x^2}{100} - 1$$

Let $y = \frac{x^2}{100} - 1$

$x \leftrightarrow y \qquad x = \frac{y^2}{100} - 1$

$$x + 1 = \frac{y^2}{100}$$
$$100(x+1) = y^2$$

$y = -\sqrt{100x + 100}$ (negative since the domain is negative)

$\therefore F^{-1}(x) = -\sqrt{100x + 100}$

M4. From the graph $m = -\frac{1}{4}$

$\therefore y = -\frac{1}{4}x\ \ (it\ is\ at\ 1\ at\ x = -4\ \ \therefore start\ integral\ at -4)$

$\boxed{1}\ F(x) = \int_{-4}^{x} -\frac{1}{4}s\,ds$

$= \left[-\frac{1}{4}\frac{s^2}{2}\right]_{-4}^{x}$

$= \left[\frac{-s^2}{8}\right]_{-4}^{x}$

$y = \frac{-x^2}{8} + \frac{(-4)^2}{8}$

$y = \frac{-x^2}{8} + \frac{16}{8}$

$y = \frac{-x^2+16}{8}$

$\boxed{2}$ $Find\ the\ inverse$ (switch x and y first)

$x = \frac{-y^2+16}{8}$

$8x = -y^2 + 16$

$16 - 8x = y^2$

$y^2 = 16 - 8x$

$y = -\sqrt{16 - 8x}$

$\therefore Use\ the\ function\ \ y = -\sqrt{16-8x}$(take the negative square root since x is negative in the original graph.)

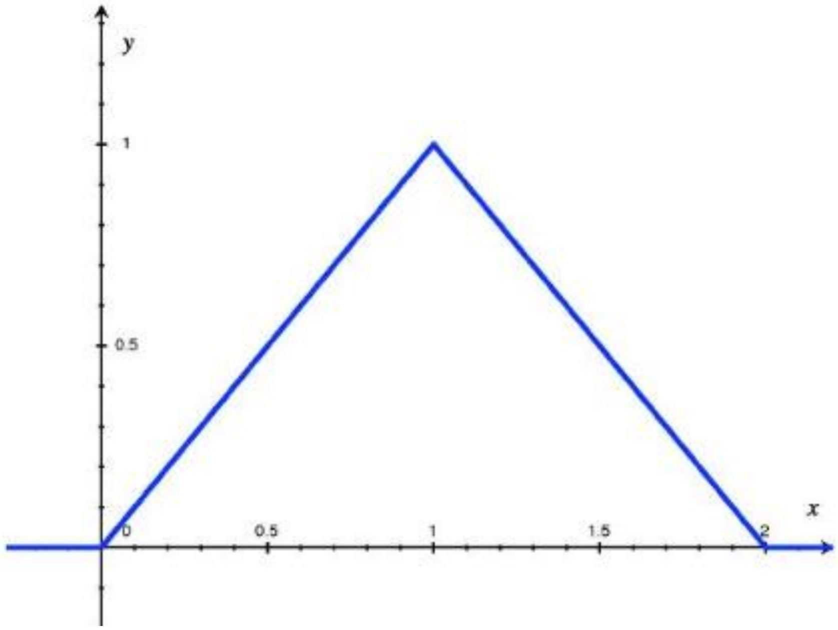
Quiz 4: Practice on Sections A and B

1. $\Pr\left(|x| < \frac{1}{2}\right) = \Pr\left(-\frac{1}{2} \le x \le \frac{1}{2}\right)$

$$= \Pr\left(0 \le x \le \frac{1}{2}\right) \quad since\ x \ge 0$$

$$= \frac{b \times h}{2}$$

$$= \frac{\frac{1}{2}(\frac{1}{2})}{2} = \frac{1}{8} \quad using\ the\ graph$$



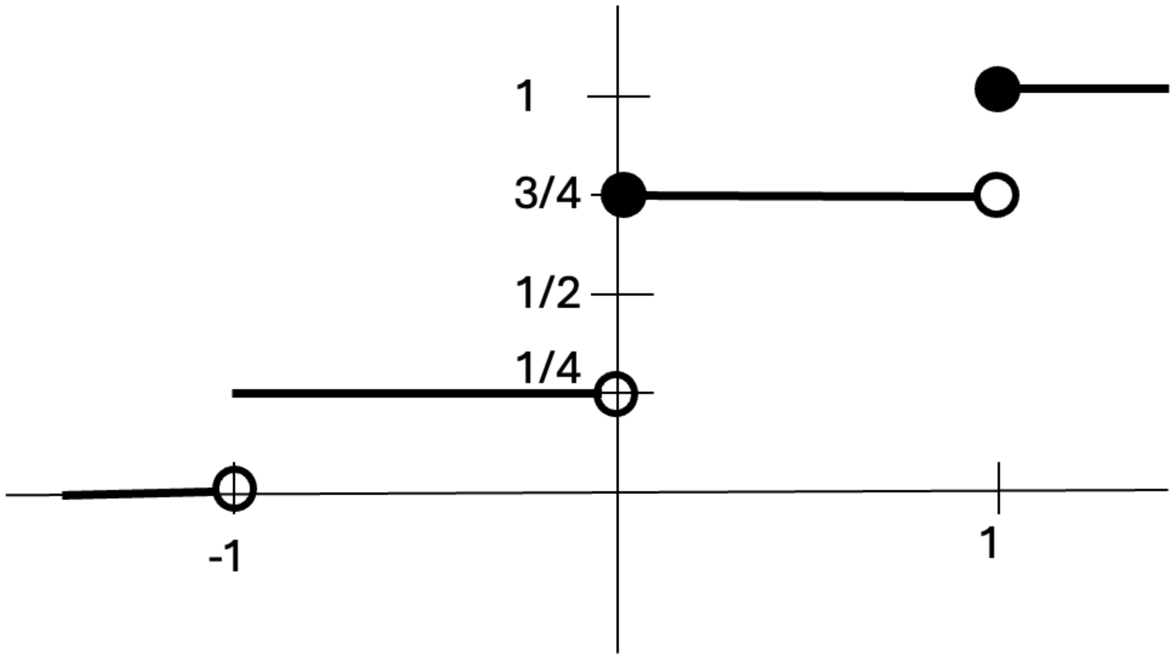
2. $F(x) = \int_0^x f(s)ds = \int_0^x \frac{2s}{c^3} ds$

$$= \left[\frac{2}{c^3}\left(\frac{s^2}{2}\right)\right]_0^x$$

$$= \left[\frac{s^2}{c^3}\right]_0^x$$

$$= \frac{x^2}{c^3}$$

3.



x	$\Pr(x)$	$F(x)$
-1	$\frac{1}{4}$	$\frac{1}{4}$
0	$\frac{2}{4}$	$\frac{3}{4}$
1	$\frac{1}{4}$	1

4. $\int_{-a}^{a}(a^3 - x^3)dx = 1$

$$\left[a^3x - \frac{x^4}{4}\right]_{-a}^{a} = 1$$
$$\left(a^3(a) - \frac{a^4}{4}\right) - \left(a^3(-a) - \frac{(-a)^4}{4}\right) = 1$$
$$a^4 - \frac{a^4}{4} + a^4 + \frac{a^4}{4} = 1$$
$$2a^4 = 1$$
$$a^4 = \frac{1}{2}$$
$$a = \sqrt[4]{\frac{1}{2}}$$

5. a) 3 , 2

   b) 2

6. 0

7. a) $u_1 = (3u_0 + 3) mod\ 5$
$= (3(2) + 3) mod\ 5$
$= 9\ mod\ 5$
$= 4 \quad since\ 9 \div 5 = 1\ \ R4$

b) $u_2 = (3u_1 + 3) mod\ 5$
$= (3(4) + 3) mod\ 5$
$= 15\ \ mod\ 5$
$= 0\ \ since\ 15 \div 5 = 3\ \ R\ 0$

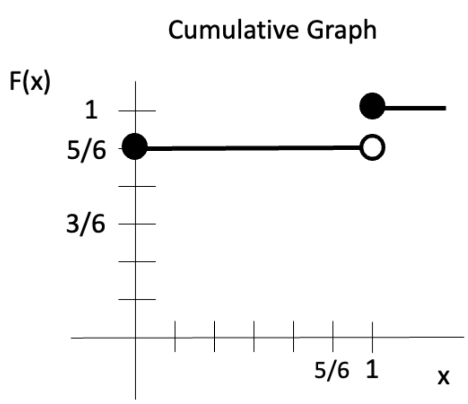
c) $u_3 = (3u_2 + 3) mod\ 5$
$= (3(0) + 3) mod\ 5$
$= 3\ mod\ 5$
$= 3\ since\ 5\ can't\ divide\ into\ the\ 3$

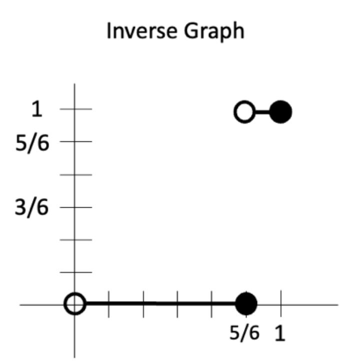
d) $divide\ by\ m \quad \therefore since\ it\ is\ mod\ 5 \quad \div 5$

8.

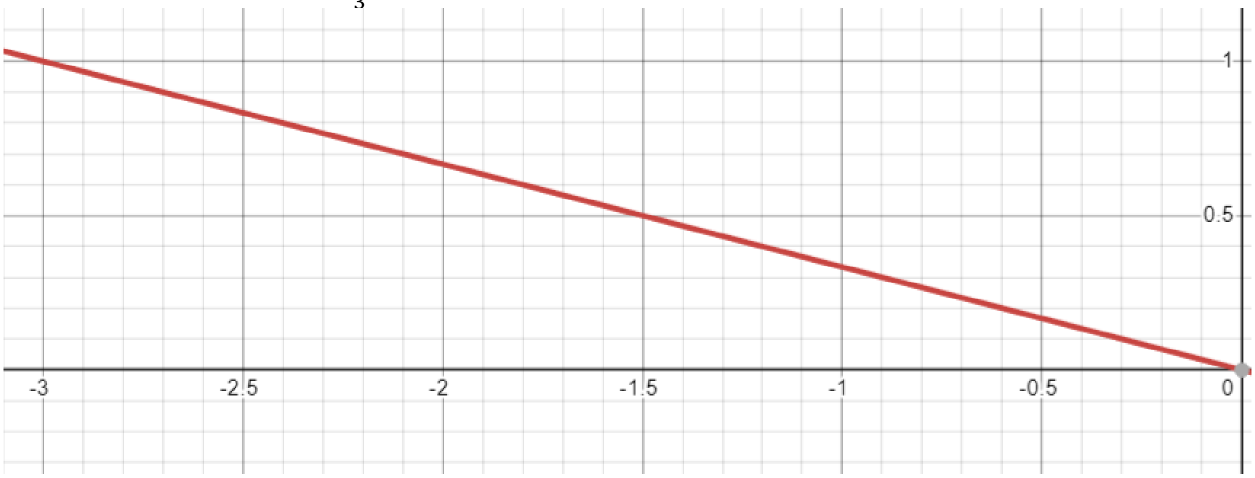
x	$\Pr(x)$	$F(x)$
0	$5x = \frac{5}{6}$	$\frac{5}{6}$
1	$x = \frac{1}{6}$	1
	$\overline{1}$	

$5x + x = 1$
$6x = 1$
$x = \frac{1}{6}$





9.From the graph $m = -\frac{1}{3}$



$\therefore y = -\frac{1}{3}x$ $\quad (it\ is\ 1\ at\ x = -3 \quad \therefore start\ integral\ at - 3$

$\boxed{1}$ $F(x) = \int_{-3}^{x} -\frac{1}{3}s \ \ ds$

$= \left[\frac{-1}{3}\frac{s^2}{2}\right]_{-3}^{x}$

$= \left[\frac{-s^2}{6}\right]_{-3}^{x}$

$y = \left[\frac{-x^2}{6} - \left(-\frac{(-3)^2}{6}\right)\right]$

$y = \left[\frac{-x^2}{6} + \frac{9}{6}\right]$

$y = \frac{-x^9+9}{6}$

$\boxed{2}$ *Find the inverse* (switch x and y)

$x = \frac{-y^2+9}{6}$

$6x = -y^2 + 9$

$6x - 9 = -y^2$

$y^2 = -6x + 9$

$y = -\sqrt{-6x+9}$ $\ (take -$ square root since x is negative in original graph)

$\therefore use\ the\ function \ \ y = -\sqrt{-6x+9}$

N. Working with Data

Example 1.

$\bar{x} = \frac{80+60+70+70+20}{5} = \frac{300}{5} = 60$

$s = \sqrt{\frac{(80-60)^2+(60-60)^2+(70-60)^2+(70-60)^2+(20-60)^2}{5-1}}$

$s = \sqrt{\frac{(20)^2+(0)^2+(10)^2+(10)^2+(-40)^2}{4}}$

$= \sqrt{\frac{400+0+100+100+1600}{4}}$

$= \sqrt{\frac{2200}{4}}$

$= \sqrt{550}$

Example 2.

a) $z_1 = \frac{x-\bar{x}}{s} = \frac{60-67}{10} = -0.7$

b) $z_2 = \frac{x-\bar{x}}{s} = \frac{80-75}{5} = 1$

c) $z_3 = \frac{x-\bar{x}}{s} = \frac{75-80}{3} = -1.67$

Largest to smallest is Z3, Z2, Z1

Example 3. $z = \frac{x-\bar{x}}{s}$

$z = \frac{90-58}{16} = 2$

$2 = \frac{x-62}{8}$

$x = 78$

Example 4. $t = \sqrt{n} \cdot T$

$= \sqrt{n} \cdot \left|\frac{\bar{x}}{s}\right| = \sqrt{10}\left|\frac{5}{3}\right| = \frac{5\sqrt{10}}{3}$

Practice Exam Questions on Working with Data

N1.

a) $\bar{x} = \frac{80+90+72+60}{4} = \frac{302}{4} = 75.5$

$$s = \sqrt{\frac{(80-75.5)^2+(90.75.5)^2+(72-75.5)^2+(60-75.5)^2}{3}}$$

$$s = \sqrt{\frac{4.5^2 + 14.5^2 + (-3.5)^2 + (-15.5)^2}{3}}$$

$$= \sqrt{\frac{483}{3}} = \sqrt{161} = 12.7$$

b) Student 1

$z_1 = \frac{x-\bar{x}}{s} = \frac{80-75.5}{12.7}$

$= 0.35$

Student 4

$z_4 = \frac{60-75.5}{12.7}$

$= -1.22$

Student 4 is more extreme (1.22 standard deviations away from mean)

N2.

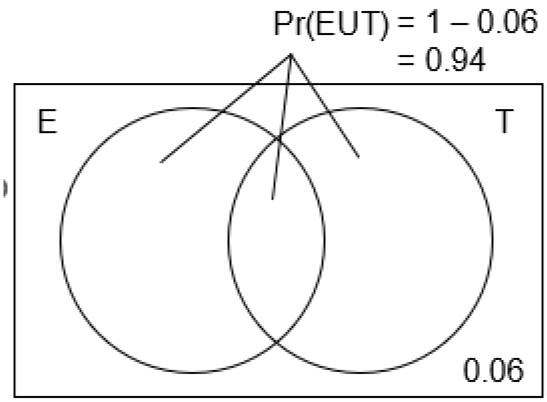
$t = \sqrt{n} \cdot T$

$= \sqrt{n}\left|\frac{\bar{x}}{s}\right| = \sqrt{25}\left|\frac{4}{4}\right| = 5$

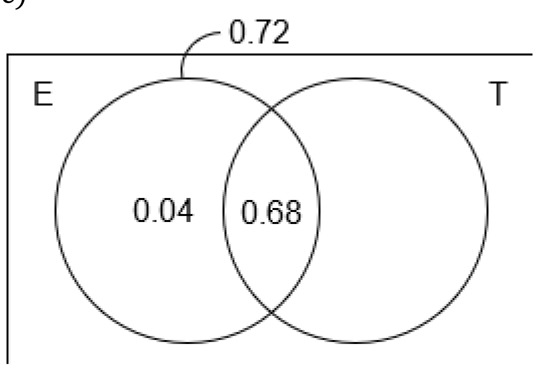
O. Practice Test on Probability

O1.a) $T \cap E^c$ $\quad\quad$ $\Pr(E \cup T) = 1 - 0.06$
$= 0.94$

b) $\Pr(E \cup T) = \Pr(E) + \Pr(T) - \Pr(E \cap T)$
$0.94 = 0.72 + 0.90 - \Pr(E \cap T)$
$\Pr(E \cap T) = 0.68$



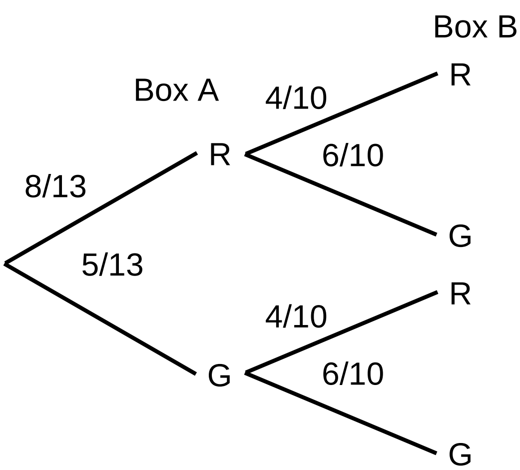
c)



$\Pr(E \cap T^c) = \Pr(E) - \Pr(E \cap T)$
$= 0.72 - 0.68$
$= 0.04$

O2.a)

$\Pr(GG) = \frac{5}{13} \times \frac{6}{10} = \frac{30}{130} = \frac{3}{13}$



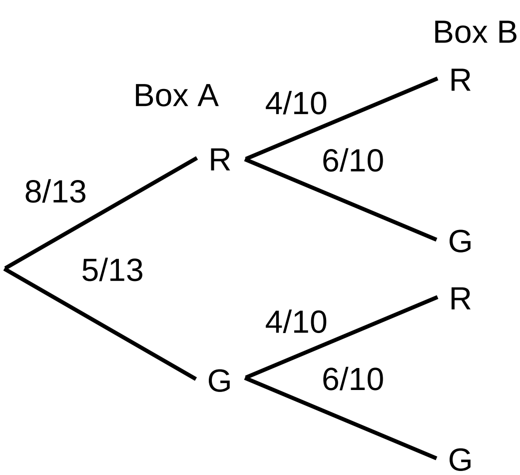
b) $\Pr\,(one\ green\ \cap one\ red)$

$= \Pr(RG) + \Pr\,(GR)$

$= \left(\frac{8}{13}\right)\left(\frac{6}{10}\right) + \left(\frac{5}{13}\right)\left(\frac{4}{10}\right)$

$= \frac{48}{130} + \frac{20}{130}$

$= \frac{68}{130} \quad = \boxed{\frac{34}{65}}$



c) $\Pr\left(box\ {}^{B}/_{R}\right) = \frac{\Pr\,(box\ B \cap R)}{\Pr\,(R)}$

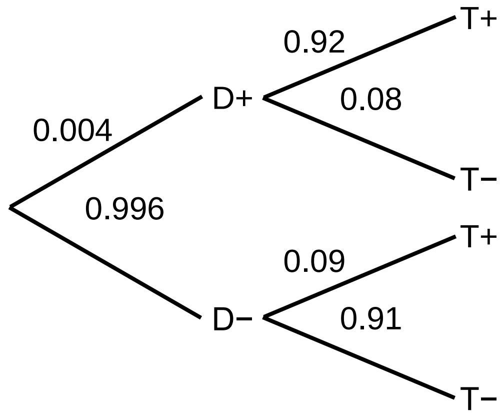
$$= \frac{\frac{1}{3}\left(\frac{4}{10}\right)}{\frac{1}{3}\left(\frac{4}{10}\right)+\frac{2}{3}\left(\frac{8}{13}\right)}$$

$$= \frac{\frac{4}{30}}{\frac{4}{30}+\frac{16}{39}}$$

$$= \frac{0.1333333}{0.543589743}$$

$$= 0.245$$

O3. $\frac{4}{1000} = 0.004$



sensitivity= $0.92 = \Pr\,\left({}^{T\,+}/_{D\,+}\right)$

specificity= $0.91 = \Pr\,\left({}^{T\,-}/_{D\,-}\right)$

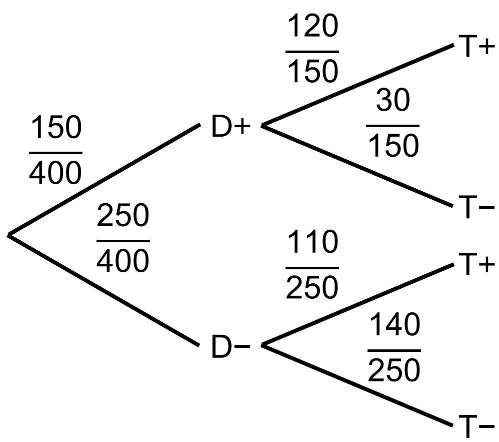
$\Pr(T+) = \Pr(D + \cap\, T+) + \Pr\,(D - \cap\, T+)$

$= 0.004(0.92) + 0.996(0.09)$

$= 0.00368 + 0.08964$

$= 0.09332$

O4.a)

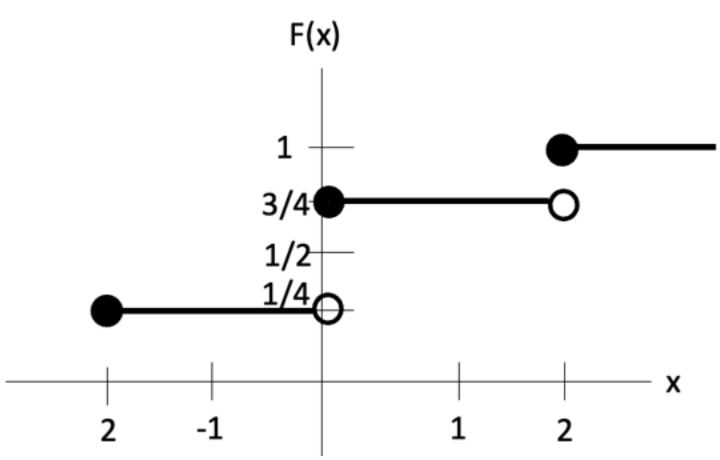


sensitivity= $\Pr\left(T+/D+\right)=\frac{120}{150}=0.8$

b) specificity= $\Pr\left(T-/D-\right)=\frac{140}{250}=\frac{14}{25}=0.56$

O5.

x	$\Pr(x)$	$F(x)$
-2	$\frac{1}{4}$	$\frac{1}{4}$
0	$\frac{1}{2}$	$\frac{3}{4}$
2	$\frac{1}{4}$	1



O6. Probability = Area under graph

$= A_{triangle} + A_{rectangle} + 0.5\ (Area\ from - 1\ to\ 0)$

$= \frac{1}{2}(0.5)(0.5) + (0.5)(0.5) + 0.5$

$= 0.125 + 0.25 + 0.5$

$= 0.875$

O7. $u_{t+1} = 4u_t + 8 \quad mod\ 10$

$u_1 = 4u_0 + 8\ \ mod\ 10$

$u_1 = 4(6) + 8\ \ mod\ 10$

$u_1 = 32\ \ mod\ 10 \qquad 32 \div 10 = 3R2$

$\therefore u_1 = 2$

$u_2 = 4u_1 + 8 \quad mod\ 10$

$u_2 = 4(2) + 8 \quad mod\ 10$

$u_2 = 16 \quad mod\ 10 \quad 16 \div 10 = 1R6$

$\therefore u_2 = 6$

O8. a) $\int_0^k e^{-\frac{x}{4}} dx = 1$

$\left[\frac{e^{\frac{x}{4}}}{-\frac{1}{4}}\right]_0^k = 1$

$\left[-4e^{-\frac{x}{4}}\right]_0^k = 1$

$-4e^{-\frac{k}{4}} + 4e^0 = 1$

$-4e^{-\frac{k}{4}} = 1 - 4$

$-4 = e^{-\frac{k}{4}} = -3$

$e^{-\frac{k}{4}} = \frac{3}{4}$

$\ln e^{-\frac{k}{4}} = \ln(\frac{3}{4})$

$\frac{-k}{4} \ln e = \ln(\frac{3}{4})$

$\frac{-k}{4} = \ln(\frac{3}{4})$

$k = -4\ln(\frac{3}{4})$

b) $F(x) = \int_{-\infty}^{x} f(s) ds$

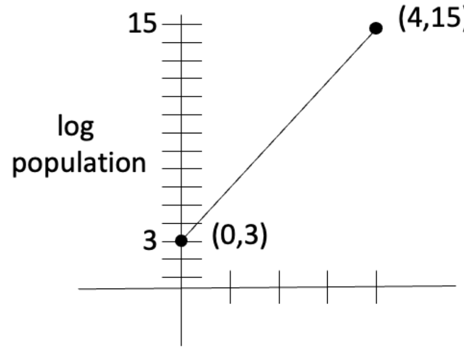
$= \int_0^x e^{-\frac{s}{4}} ds$

$= \left[\frac{e^{-\frac{s}{4}}}{-\frac{1}{4}}\right]_0^x$

$= \frac{e^{-\frac{x}{4}}}{-\frac{1}{4}} - \frac{e^0}{-\frac{1}{4}}$

$= -4e^{-\frac{x}{4}} + 4 \quad or \quad 4 - 4e^{-\frac{x}{4}}$

O9. The value of r is equal to the slope of the line in log-time



slope= $\frac{15-3}{4-0} = \frac{12}{4} = 3$

$$r = slope\ of\ line = \frac{rise}{run} = \frac{12}{4} = 3$$

O10. a) $F(x) = \sqrt{\frac{x}{3}} = \frac{\sqrt{x}}{\sqrt{3}} = \frac{1}{\sqrt{3}}x^{\frac{1}{2}}$

$CDF \rightarrow PDF\ is\ do\ the\ derivative$

$f(x) = F'(x) = \frac{1}{\sqrt{3}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{1}{2\sqrt{3}}x^{-\frac{1}{2}}$

$\therefore E(x) = \int_{-\infty}^{\infty} x\, f(x)dx$

$E(x) = \int_0^3 x(\frac{1}{2\sqrt{3}})x^{-\frac{1}{2}}dx$

$= \int_0^3 \frac{1}{2\sqrt{3}}x^{\frac{1}{2}}dx$

$= \frac{1}{2\sqrt{3}}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$

$= \frac{1}{2\sqrt{3}}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^3$

$= \frac{1}{2\sqrt{3}}\left[\frac{2}{3}(3)^{\frac{3}{2}} - 0\right]$

$= \frac{1}{2\sqrt{3}}\left[\frac{2}{3}\left(\sqrt{3}\right)^3\right]$

$= \frac{1}{2\sqrt{3}}(\frac{2}{3}\left(\sqrt{3}\right)\left(\sqrt{3}\right)\left(\sqrt{3}\right))$

$= \frac{3}{3} = 1$

b) $Var\ (x) = \int_{-\infty}^{\infty} x^2 f(x) - [E(x)]^2$

$= \int_0^3 x^2\left(\frac{1}{2\sqrt{3}}x^{-\frac{1}{2}}\right)dx - 1^2$

$= \int_0^3 \frac{1}{2\sqrt{3}}x^{\frac{3}{2}} - 1$

$= \left[\frac{1}{2\sqrt{3}}\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^3 - 1$

$= \left[\frac{1}{2\sqrt{3}}\left(\frac{2}{5}x^{\frac{5}{2}}\right)\right]_0^3 - 1$

$= \left[\frac{1}{5\sqrt{3}}(3)^{\frac{5}{2}} - 0\right] - 1$

$= \left[\frac{1}{5\sqrt{3}} \times \sqrt{3}\sqrt{3}\sqrt{3}\sqrt{3}\sqrt{3}\right] - 1$

$= \frac{3\times3}{5} - 1$

$= \frac{9}{5} - \frac{5}{5}$

$= \frac{4}{5}$

O11. a) Cheetah

$z = \frac{x-\bar{x}}{s}$

$z = \frac{98-102}{2}$

$= -2$

Lion

$z = \frac{75-80}{3}$

$= -1.67$

∴ Lion runs faster relative to its breed (closer to its mean) If it was $+2$ and $+1.67$, the Cheetah would be faster!

b) $z = \frac{105-102}{2} = 1.5$

∴ $1.5 = \frac{x-80}{3}$

$x = 1.5(3) + 80 = 84.5$

∴ 84.5 km/hr is equivalent to the cheetah's speed.

P. Vectors

Example 1.

b) $= \sqrt{(-2)^2 + (-1)^2 + 1^2}$
$= \sqrt{6}$
c) $= \sqrt{5^2 + (-1)^2 + (-2)^2}$
$= \sqrt{25 + 1 + 4}$
$= \sqrt{30}$

Example 2.

b) $= 5(-2,-1,1) = \ (-10,-5,5)$
$c) = -2(-1,-2,3) = (2,4,-6)$
d) $= \|-3(5,-1,-2)\|$
$= \|(-15,3,6)\| = \sqrt{(-15)^2 + 3^2 + 6^2}$
$= \sqrt{225 + 9 + 36} \quad = \sqrt{270}$

Example 3.
(3,-4)-2(-2,4)=(7, -12)
$\|(7,-12)\| = |(7,-12)| = \sqrt{7^2 + (-12)^2} = \sqrt{49 + 144} = \sqrt{193}$
The answer is C.

Example 4.

b) $= (-2,-1,1) + (5,-1,-2) = (3,-2,-1)$
c) $= (2,-1,-2) + (-1,-2,3) = (1,-3,1)$
d) $= 3(-1,-2,3) - 5(-2,-1,1)$
$= (-3,-6,9) + (10,5,-5) = (7,-1,4)$
e) $= -(-2,-1,1) + 3(5,-1,-2)$
$= (2,1,-1) + (15,-3,-6) = (17,-2,-7)$
f) $= 3(-1,-2,3) - (-2,-1,1) + 2(5,-1,-2)$
$= (-3,-6,9) + (2,1,-1) + (10,-2,-4)$
$= (9,-7,4)$

Practice Exam Questions

P1. $\overrightarrow{AB} = B - A = (5,3) - (2,-3) = (3,6)$

$\overrightarrow{CB} = B - C = (5,3) - (-2,1) = (7,2)$

$\left\|\overrightarrow{AB} - \overrightarrow{CB}\right\| = \left|\overrightarrow{AB} - \overrightarrow{CB}\right| = |(3,6) - (7,2)| = |(-4,4)|$

$= \sqrt{(-4)^2 + 4^2}$

$= \sqrt{32} \quad = \sqrt{16}\sqrt{2} \quad = 4\sqrt{2}$

$\therefore \mathit{The\ answer\ is\ A}$

P2. $3(1,2,1) - (1,3,-1)$

$= (3,6,3) - (1,3,-1) = (2,3,4)$

Therefore, the answer is C).

P3. $2(3,6,1) - 3(0,-1,3)$

$= (6,12,2) - (0,-3,9) = (6,15,-7)$

Therefore, the answer is B).

P4. $\|\vec{u}\| = \sqrt{3^2 + 1^2 + 5^2}$

$= \sqrt{9 + 1 + 25} = \sqrt{35}$

Therefore, the answer is C).

P5. $= \|(1,3,1) + (2,5,1)\|$

$= \|(3,8,2)\|$

$= \sqrt{3^2 + 8^2 + 2^2}$

$= \sqrt{9 + 64 + 4} = \sqrt{77}$

Therefore, the answer is D).

Q. Operations with Vectors

Example 1.

b) $= (-2,-1,1)\cdot(5,-1,-2)$
$= -10 + 1 + (-2) = -11$
c) $= (-1,-2,3)\cdot(2,-1,-2)$
$= -2 + 2 - 6 = -6$

**Dots Product is ALWAYS a Number or Scalar

Example 2. b) $\vec{v}\cdot\vec{w} = (-2,-1,1)\cdot(5,-1,-2)$
$= -10 + 1 - 2 \quad = -11$
$\|\vec{v}\| = \sqrt{(-2)^2 + (-1)^2 + 1^2} = \sqrt{6}$
$\|\vec{w}\| = \sqrt{5^2 + (-1)^2 + (-2)^2} = \sqrt{25 + 1 + 4} = \sqrt{30}$
$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \frac{-11}{\sqrt{6}\sqrt{30}} \quad or \quad \frac{-11}{\sqrt{180}}$

Example 3. $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|} = 0$

$\frac{(1,-4)\cdot(k,3)}{\sqrt{1^2+(-4)^2}\ \sqrt{k^2+3^2}} = 0$

$k - 12 = 0$

$k = 12$

Example 4.
$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$
$\cos 45° = \frac{\vec{u}\cdot\vec{v}}{\sqrt{6}\sqrt{3^2+1^2+4^2}}$
$\cos 45° = \frac{\vec{u}\cdot\vec{v}}{\sqrt{6}\sqrt{26}}$
$\vec{u}\cdot\vec{v} = \frac{1}{\sqrt{2}}\sqrt{6}\sqrt{26} = \sqrt{3}\sqrt{26} = \sqrt{78}$

Example 5.
$\vec{u}\cdot\vec{v} = (k,k)\cdot(-2,5)$
$= -2k + 5k$
$= 3k$
$\|\vec{u}\| = \sqrt{k^2 + k^2} = \sqrt{2k^2} = \sqrt{2}k$
$\|\vec{v}\| = \sqrt{(-2)^2 + 5^2} = \sqrt{29}$
$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{3k}{\sqrt{2}k\sqrt{29}} = \frac{3}{\sqrt{2}\sqrt{29}}$

Example 6. $\begin{matrix} 1 & 2 & 1 & 1 & 2 & 1 \\ -1 & -2 & 4 & -1 & -2 & 4 \end{matrix}$

$(8-(-2),\ -4-4,\ -2+2)$

$\vec{u} \times \vec{v} = (10, -5, 0)$

Example 7. $\begin{matrix} 3 & -3 & -1 & 3 & -3 & -1 \\ -1 & -1 & 2 & -1 & -1 & 2 \end{matrix}$

$\vec{u} \times \vec{v} = (-6-1,\ 1-6, -3-3)$

$= (-7, -5, -6)$

Example 8. $\vec{v} \times \vec{u}$ $\begin{matrix} -1 & 2 & 3 & -1 & 2 & 3 \\ 5 & 2 & -1 & 5 & 2 & -1 \end{matrix}$

$= (-2-6,\ \ 15-1,\ -2-10)$

$= (-8, 14, -12)$

Unit Vectors p.256

It would be $(5, -3, 2)$

Example 9. a) $\overrightarrow{u} \times \overrightarrow{v}$ $\begin{matrix} 1 & 3 & 1 & 1 & 3 & 1 \\ -2 & 4 & -1 & -2 & 4 & -1 \end{matrix}$

$= (-3-4, -2+1,\ 4+6)$

$\overrightarrow{u} \times \overrightarrow{v} = (-7, -1, 10)$

$A = \|(-7, -1, 10)\| = \sqrt{49+1+100} = \sqrt{150}$

b) $\begin{matrix} 1 & 4 & 0 & 1 & 4 & 0 \\ 2 & 9 & 0 & 2 & 9 & 0 \end{matrix}$

$\overrightarrow{u} \times \overrightarrow{v} = (0-0, 0-0, 9-8) = (0,0,1)$

$Area = \|(0,0,1)\| = \sqrt{0+0+1} = 1$

Example 10.

$$\begin{array}{ll} \boxed{1}\ c_1 + 2c_2 = 17 \quad \times(-2) & -2c_1 - 4c_2 = -34 \\ \boxed{2}\ 3c_1 + 4c_2 = 39 & \quad 3c_1 + 4c_2 = 39 \\ \hline & ADD \quad c_1 = 5 \end{array}$$

Substitution $c_1 = 5$ *into* $\boxed{1}$

$$c_1 + 2c_2 = 17$$
$$5 + 2c_2 = 17$$
$$2c_2 = 17 - 5$$
$$2c_2 = 12$$
$$c_2 = 6$$

Practice Exam Questions

Q1. $\vec{u} \cdot \vec{v} = -1 - 6 - 16 = -23$

Therefore, the answer is B).

Q2. $\begin{matrix} 1 & 2 & -3 & 1 & 2 & -3 \\ 1 & 3 & 2 & 1 & 3 & 2 \end{matrix}$

$\vec{u} \times \vec{v} = (4 + 9, -3 - 2, 3 - 2) = (13, -5,\ 1)$

Therefore, the answer is C).

Q3. $\vec{u} \times \vec{v} = \begin{matrix} 5 & 1 & 2 & 5 & 1 & 2 \\ -1 & 1 & 0 & -1 & 1 & 0 \end{matrix}$

$= (0 - 2, -2 - 0,\ 5 + 1) = (-2, -2, 6)$

Therefore, the answer is E).

Q4. C is undefined as once you dot $\vec{v} \cdot \vec{w}$ you get a number and you can't do $\vec{u}\ \cdot$ (some number)

Therefore, the answer is C).

Q5. $\vec{u} \times \vec{v} = \begin{matrix} 1 & 2 & 1 & 1 & 2 & 1 \\ 2 & 1 & 4 & 2 & 1 & 4 \end{matrix}$

$= (8 - 1,\ 2 - 4,\ 1 - 4) = (7, -2, -3)$

$A = \|\vec{u} \times \vec{v}\| = \sqrt{7^2 + (-2)^2 + (-3)^2} = \sqrt{49 + 4 + 9}$

$= \sqrt{62}$

Therefore, the answer is E).

Q6. $\vec{v} \times \vec{w}$ $\begin{matrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 1 & 4 & 1 & 1 & 4 \end{matrix}$

$$= (8-3, 3-4, 1-2) = (5,-1,-1)$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (-2,3,1) \cdot (5,-1,-1)$$

$$= -10 - 3 - 1 = -14$$

Therefore, the answer is A).

Q7. $\vec{u} \times \vec{v} = \begin{matrix} 8 & 1 & 6 & 8 & 1 & 6 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{matrix}$

$$= (2-6, 0-16, 8-0) = (-4,-16,8)$$

Therefore, the answer is E).

Q8. (I) $A = \|\vec{u} \times \vec{v}\| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$ $\quad \therefore false$

(II) $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{6}{5 \cdot 3} = \frac{6}{15} = \frac{2}{5}$ $\quad \therefore false$

Therefore, the answer is D).

Q9. $\vec{u} \cdot \vec{v} = (1,2) \cdot (-1,1) = -1 + 2 = 1$

$$\|\vec{u}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\|\vec{v}\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

$$= \frac{1}{\sqrt{5}\sqrt{2}} = \frac{1}{\sqrt{10}}$$

Q10. $(2, k, 3k) \cdot (2, -5, 1) = 0$

$$4 - 5k + 3k = 0$$

$$-2k = -4$$

$$k = 2$$

Therefore, the answer is C).

Q11.

$c_1 + 2c_2 = 11$ $\boxed{1}$

$2c_1 + 3c_2 = 18$ $\boxed{2}$

$\boxed{1} \times (-2) \quad -2c_1 - 4c_2 = -22$

$$\begin{array}{rr} & 2c_1 + 3c_2 = 18 \\ \hline ADD & -c_2 = -4 \\ & c_2 = 4 \end{array}$$

Substitution $c_2 = 4 \quad into\ \boxed{1}$

$c_1 + 2(4) = 11$

$c_1 + 8 = 11$

$c_1 = 3$

Q12.

$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

$\cos 100° = \frac{\vec{u}\cdot\vec{v}}{\sqrt{3}\sqrt{11}}$

$\vec{u}\cdot\vec{v} = -0.998$

Q13. $Area = det\begin{bmatrix}\vec{u}\\\vec{v}\end{bmatrix}$

$= det\begin{bmatrix}2 & 5\\3 & 10\end{bmatrix} = 20 - 15 = 5$

Or just add a 0 component for z and do cross product and the regular area formula!

R. Vector Fields

Example 1.

a) $F\begin{bmatrix}3\\-3\end{bmatrix}=\begin{bmatrix}3+(-3)\\3(3)-(-3)\end{bmatrix}=\begin{bmatrix}0\\12\end{bmatrix}$

The first element is 0

b) The second element is 12.

c) $F\begin{bmatrix}0\\3\end{bmatrix}=\begin{bmatrix}u+v\\3u-v\end{bmatrix}=\begin{bmatrix}0+3\\3(0)-3\end{bmatrix}=\begin{bmatrix}3\\-3\end{bmatrix}$

The first element is 3.

d) The second element is -3.

Example 2. Let $\vec{z}=-80\vec{x}+70\vec{y}$ Find the first and second elements of $\vec{F}(\vec{z})$

$\vec{F}(\vec{x})=\begin{bmatrix}0\\12\end{bmatrix}$ $\vec{F}(\vec{y})=\begin{bmatrix}3\\-3\end{bmatrix}$

$\mathrm{F}(\vec{z})=-80F(\vec{x})+70F(\vec{y})$

The first element is:

$\vec{F}(\vec{z})=-80(0)+70(3)=210$

The second element is

$\vec{F}(\vec{z})=-80(12)+70(-3)=-1170$

Example 3. *a)* $F(\vec{x})=f\left(\begin{bmatrix}3\\3\end{bmatrix}\right)=-2(3)+5(3)=-6+15=9$

$b)\ F(\vec{y})=f\left(\begin{bmatrix}-1\\4\end{bmatrix}\right)=-2(-1)+5(4)=2+20=22$

$c)\ F(\vec{z})=-4F(\vec{x})+6F(\vec{y})=-4(9)+6(22)=-36+132=96$

Example 4. $F = \begin{bmatrix} 2x + y \\ x \end{bmatrix}$

$(-5,5)$

x, y $\quad F = \begin{bmatrix} 2(-5)+5 \\ -5 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \end{bmatrix} \begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$

$\therefore the\ vector\ goes\ down\ and\ left$

$(5,2)$

x, y $\quad F = \begin{bmatrix} 2(5)+2 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \end{bmatrix} \begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$

$\therefore the\ vector\ goes\ up\ and\ right$

$(-4,-2)$

x, y $\quad F = \begin{bmatrix} 2(-4)+(-2) \\ -4 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix} \begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$

$\therefore the\ vector\ goes\ down\ and\ left$

$(3,-2)$

x, y $\quad F = \begin{bmatrix} 2(3)+(-2) \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$

$\therefore the\ vector\ goes\ up\ and\ right$

$F = \begin{bmatrix} y \\ 2x - y \end{bmatrix}$

$(-5,5)$

x, y $\quad F = \begin{bmatrix} 5 \\ 2(-5)-5 \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \end{bmatrix} \begin{matrix} \leftarrow run \\ \leftarrow rise \end{matrix}$

$\therefore$the vector goes down and right

Example 5.

Show that $F(\vec{x})=3x_1 + 2x_2$ is linear.

$\vec{x} = (x_1, x_2) \quad \vec{y} = (y_1, y_2)$

We want $F(c_1\vec{x} + c_2\vec{y}) = c_1 f(\vec{x}) + c_2 f(\vec{y})$

$$c_1\vec{x} + c_2\vec{y} = c_1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + c_2\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1x_1 + c_2y_1 \\ c_1x_2 + c_2y_2 \end{bmatrix}$$

$$\begin{aligned} F(c_1\vec{x} + c_2\vec{y}) &= 3(c_1x_1 + c_2y_1) + 2(c_1x_2 + c_2y_2) \\ &= c_1(3x_1 + 2x_2) + c_2(3y_1 + 2y_2) \\ &= c_1 f(\vec{x}) + c_2 f(\vec{y}) \end{aligned}$$

$\therefore linear$

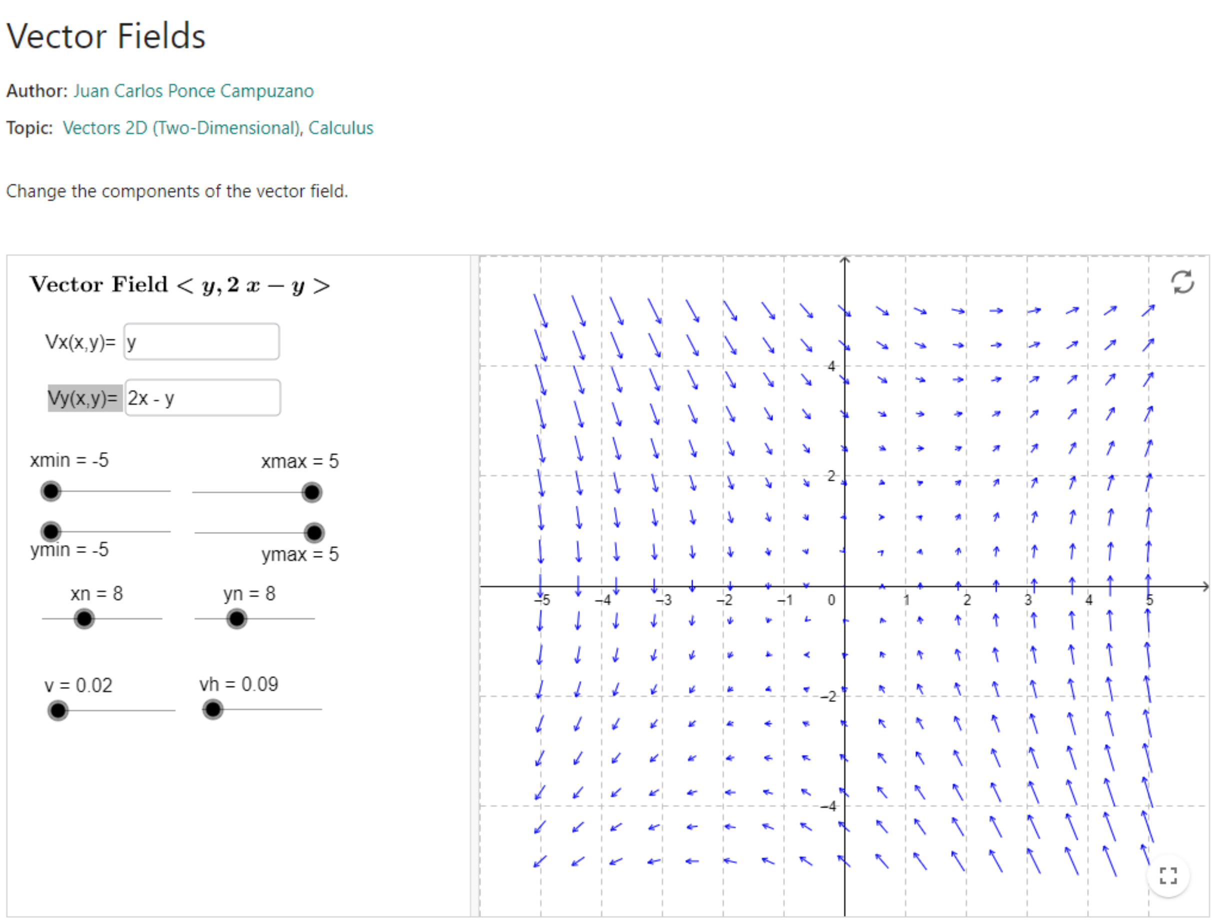
<u>Example 6.</u>

The graphs in order are 2,1,3

$\vec{F}(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} x \\ y \end{bmatrix}$ $Graph\ 2$

$\vec{F}(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} 2x + y \\ x \end{bmatrix}$ $Graph\ 1$

$\vec{F}(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} y \\ 2x - y \end{bmatrix}$ $Graph\ 3$

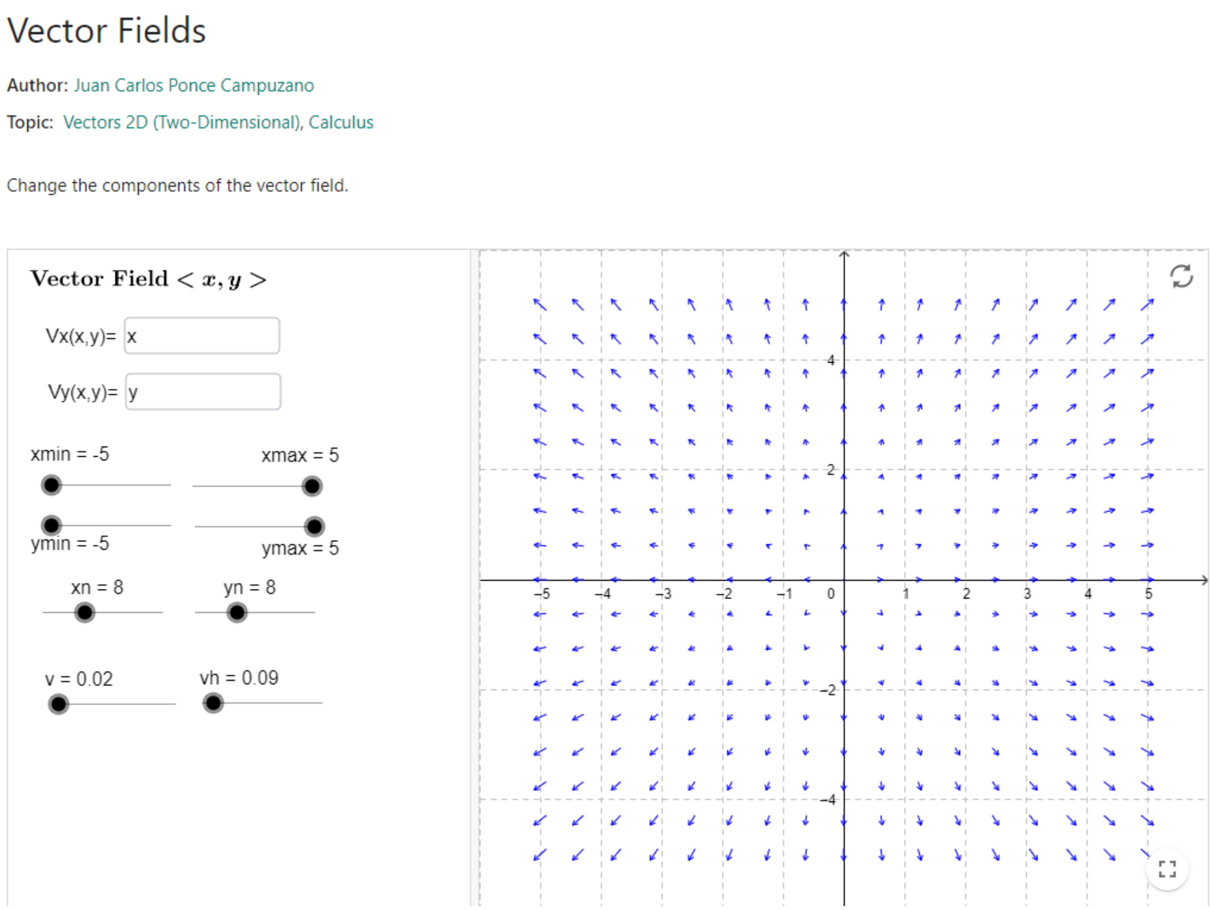


$\vec{F}(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} y \\ 2x - y \end{bmatrix}$ graph 3

Test point (-4,4)

$$\vec{F}(\begin{bmatrix} -4 \\ 4 \end{bmatrix}) = \begin{bmatrix} 4 \\ 2(-4) - 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix}$$

Top number is 4 (run)
Bottom number is -12 (rise)
So, the vector goes down and right

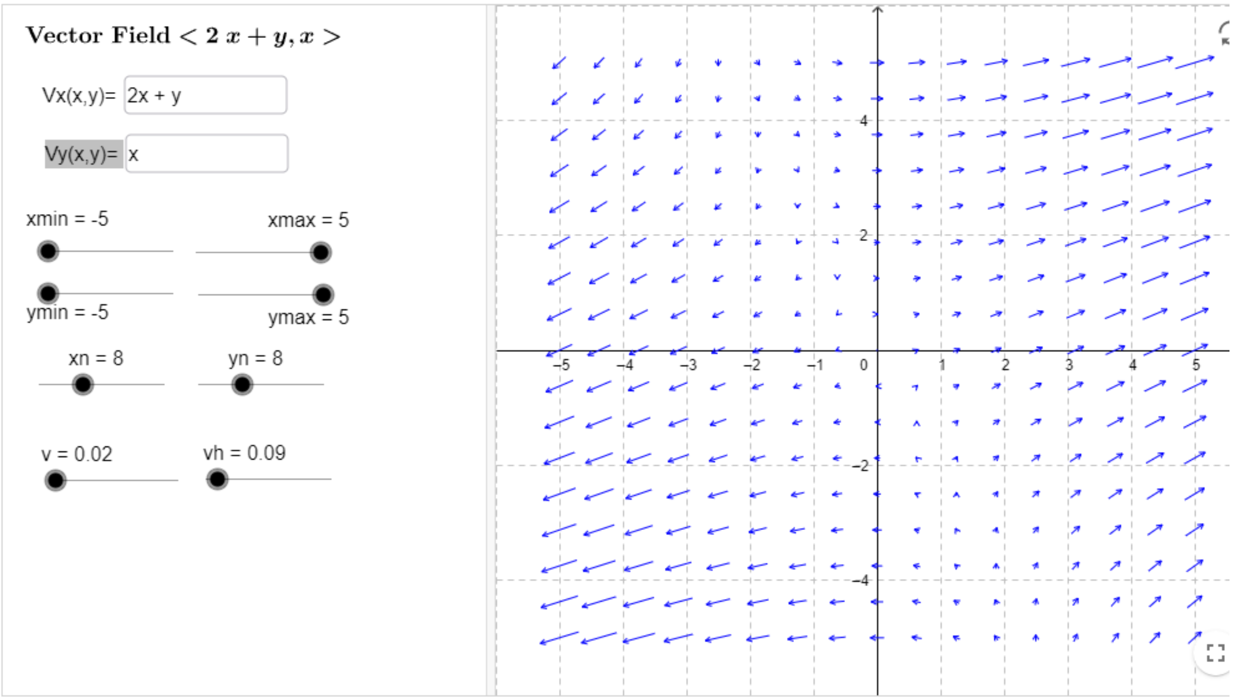


$\vec{F}\left(\begin{bmatrix}u\\v\end{bmatrix}\right) = \begin{bmatrix}x\\y\end{bmatrix}$=graph 2

Test point (-4,4)

$\vec{F}\left(\begin{bmatrix}-4\\4\end{bmatrix}\right) = \begin{bmatrix}-4\\4\end{bmatrix}$

Top number is -4 (run)
Bottom number is 4 (rise)
So, the vector goes up and left



$\vec{F}(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} 2x + y \\ x \end{bmatrix}$ graph 1

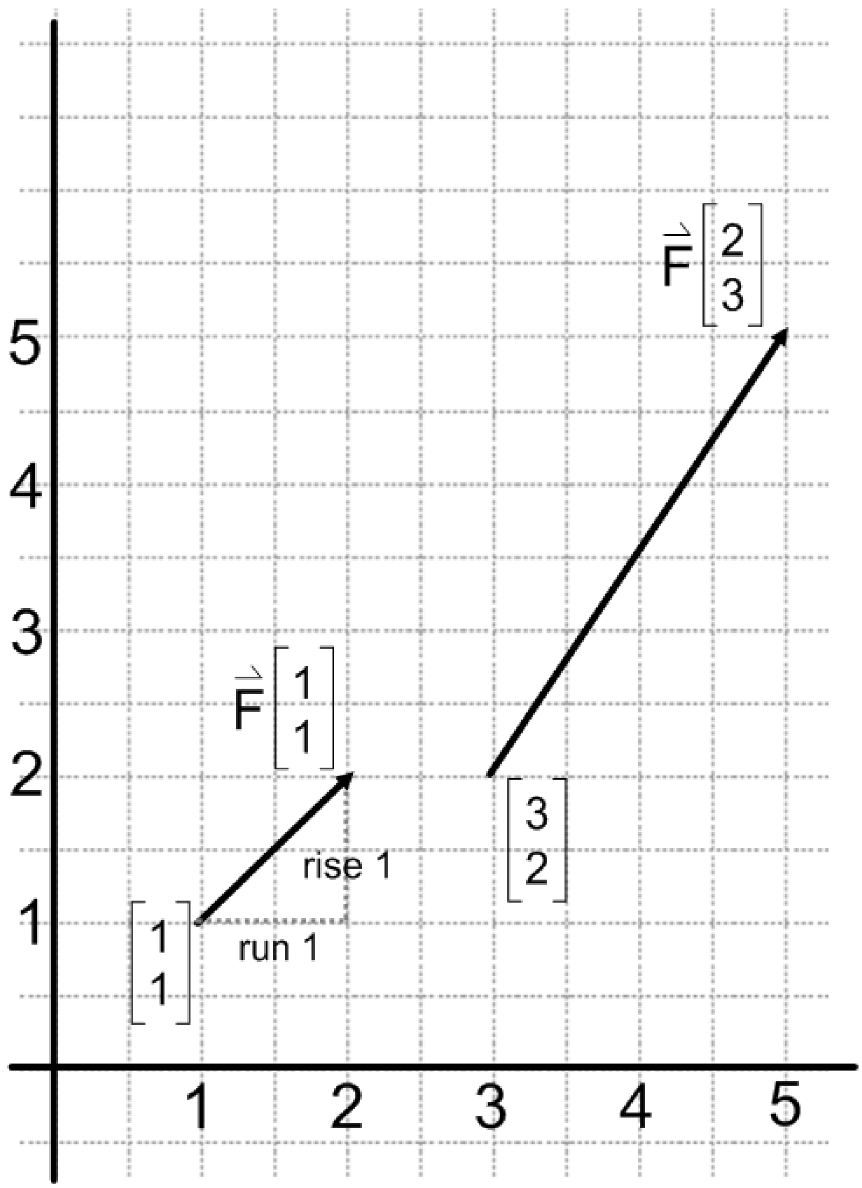
Test point (-4,4)

$$\vec{F}(\begin{bmatrix} -4 \\ 4 \end{bmatrix}) = \begin{bmatrix} 2(-4) + 4 \\ -4 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

Top number is -4 (run)
Bottom number is -4 (rise)
So, the vector goes down and left

Example 7.

$$\vec{F}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}\begin{matrix}\leftarrow run\\\leftarrow rise\end{matrix}$$
$$\vec{F}\left(\begin{bmatrix}3\\2\end{bmatrix}\right)=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}\begin{matrix}\leftarrow run\\\leftarrow rise\end{matrix}$$



Practice Exam Questions

R1.

$$F=\begin{bmatrix}y\\2x+y\end{bmatrix}\quad check\ \left(\begin{matrix}-4,4\\x,y\end{matrix}\right)$$

$$F=\begin{bmatrix}4\\2(-4)+4\end{bmatrix}=\begin{bmatrix}4\\-4\end{bmatrix}\begin{matrix}\leftarrow run\\\leftarrow rise\end{matrix}$$

∴the vector goes down and right

$$\therefore\ the\ 2nd\ graph\ is\ F=\begin{bmatrix}y\\2x+y\end{bmatrix}$$

$$F\begin{bmatrix}x\\2x+y\end{bmatrix}\ check\ \left(\begin{matrix}-4,4\\x,y\end{matrix}\right)$$

$$F=\begin{bmatrix}-4\\2(-4)+4\end{bmatrix}=\begin{bmatrix}-4\\-4\end{bmatrix}\begin{matrix}\leftarrow run\\\leftarrow rise\end{matrix}$$

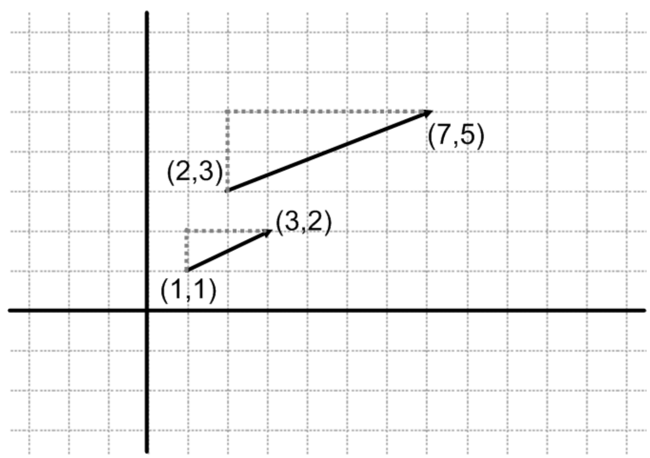
∴vector goes down and left

$$\therefore\ the\ 1st\ graph\ is\ f=\begin{bmatrix}x\\2x+y\end{bmatrix}$$

R2.

$$F\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}\begin{matrix}\leftarrow run\\\leftarrow rise\end{matrix}$$

$$F\left(\begin{bmatrix}2\\3\end{bmatrix}\right)=\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}5\\2\end{bmatrix}\begin{matrix}\leftarrow run\\\leftarrow rise\end{matrix}$$



R3. A) $F\begin{bmatrix}2\\-2\end{bmatrix}=\begin{bmatrix}2+3(-2)\\2-2(-2)\end{bmatrix}=\begin{bmatrix}-4\\6\end{bmatrix}$

$\therefore\ 1st\ element\ is-4\ and\ 2nd\ is\ 6.$

$$B)\ F\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}u+3v\\u-2v\end{bmatrix}=\begin{bmatrix}1+3(2)\\1-2(2)\end{bmatrix}=\begin{bmatrix}7\\-3\end{bmatrix}$$

$\therefore\ 1st\ element\ is\ 7\ and\ 2nd\ is-3.$

$$C)\ F(\vec{z})=-50F(\vec{x})+40F(\vec{y})$$

$$=-50\begin{bmatrix}-4\\6\end{bmatrix}+40\begin{bmatrix}7\\-3\end{bmatrix}$$

$\therefore\ the\ 1st\ element\ is-50(-4)+40(7)=480$ and

$the\ 2nd\ element\ is-50(6)+40(-3)=-300-120=-420$

R4.

a) $F(\vec{x}) = f\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = -3(3) + 7(2) = -9 + 14 = 5$

b) $F(\vec{y}) = f\begin{bmatrix}-1\\3\end{bmatrix} = -3(-1) + 7(3) = 24$

c) $F(\vec{z}) = -5f(\vec{x}) + 7(\vec{y}) = -5(5) + 7(24) = -25 + 168 = 143$

R5. $\vec{F}\left(\begin{bmatrix}u\\v\end{bmatrix}\right) = \begin{bmatrix}3x + y\\x\end{bmatrix}$ $graph$ 3 $\vec{F}\left(\begin{bmatrix}u\\v\end{bmatrix}\right) = \begin{bmatrix}y\\2x + y\end{bmatrix}$ $graph$ 1

$\vec{F}\left(\begin{bmatrix}u\\v\end{bmatrix}\right) = \begin{bmatrix}x\\x + y\end{bmatrix}$ graph 2

Do test points

S. Matrices, Inverses and Determinants

Addition and Subtraction

Example 1.
b) dim is 2x3
c) dim is 1x3
d) dim is 3x3
e) dim is 4x3

Example 4.
$$\begin{bmatrix} 2+3 & -1+2 \\ 1+1 & 3+4 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 7 \end{bmatrix}$$

Example 5.
$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & -3 & 4 \\ 0 & -1 & 0 \end{bmatrix}$$

Multiplication by a Scalar

Example 6.
$$-2A = -2\begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ -10 & -2 \end{bmatrix}$$

Transpose of a Matrix
Example 7.
a) $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

b) $B^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

c) $C^T = \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}$

Matrix Multiplication

8a) $\begin{bmatrix} \mathbf{5} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}$

b) $= \begin{bmatrix} 1(3)+3(2) \\ 2(3)+2(2) \end{bmatrix} = \begin{bmatrix} 3+6 \\ 6+4 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \end{bmatrix}$

c) $= \begin{bmatrix} 1(2)+(-1)(3)+3(1) \\ 2(2)+2(3)+2(1) \\ 1(2)+1(3)+1(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 6 \end{bmatrix}$

Powers of a Matrix

Example 9.

$A^2 = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 4\end{bmatrix}$...the 2,2 entry is 2^2

$A^3 = AA^2 = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 4\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 8\end{bmatrix}$...the 2,2 entry is 2^3=8

$$A^3 = \begin{bmatrix}1 & 0\\0 & 2^{25}\end{bmatrix}$$

Basket-Weave Method p. 280

Example 3.

$$\begin{matrix}1 & -2 & 3 & 1 & -2\\2 & 0 & 4 & 2 & 0\\-3 & 2 & 5 & -3 & 2\end{matrix}$$

Right products $= 0 + 24 + 12 = 36$

Left products $= 0 + 8 - 20 = -12$

detA=right- left $= 36 - (-12) = 48$

Since the determinant is not 0, the vector field is invertible.

Example 4.

$$\begin{matrix}1 & 2 & 3 & 1 & 2\\2 & -1 & 2 & 2 & -1\\3 & 2 & 1 & 3 & 2\end{matrix}$$

$left = -9 + 4 + 4 = -1$ $\qquad$ $right = -1 + 12 + 12 = 23$

$\therefore right - left = 23 - (-1) = 24$

Since the determinant is not 0, the vector field is invertible.

Example 5.

For which value(s) of k is the matrix invertible?

Not invertible $\therefore detA = 0$

$$\begin{matrix}2 & 1 & 0 & 2 & 1\\1 & 2 & 1 & 1 & 2\\0 & -3 & k & 0 & -3\end{matrix}$$

Left $= 0 - 6 + k = k - 6$ $\qquad$ $right = 4k + 0 + 0 = 4k$

$\therefore right - left = 0$

$4k - (k - 6) = 0$

$3k + 6 = 0$

$3k = -6 \qquad k = -2$

So, it is invertible as long as k is not equal to -2.

Example 6.

$$\begin{bmatrix} 2 & 1 & 1 \\ -2 & -2 & 2 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_2 + x_3 = 0 \ \boxed{1}$

$-2x_1 - 2x_2 + 2x_3 = 0 \quad \boxed{2}$

$x_1 + 2x_3 = 0 \quad \rightarrow \boxed{3} \quad from\,\boxed{3}\ x_3 = \frac{-1}{2}x_1$

Substitute into $\boxed{2}$

$-2x_1 - 2x_2 + 2\left(\frac{-1}{2}x_1\right) = 0$

$-2x_1 - 2x_2 - x_1 = 0$

$-3x_1 - 2x_2 = 0$

$\therefore -3x_1 = 2x_2$

$x_2 = \frac{-3}{2}x_1$

$$\therefore vector\ is\ \vec{x} = \begin{bmatrix} x_1 \\ \frac{-3}{2}x_1 \\ \frac{-1}{2}x_1 \end{bmatrix}$$

Example 7.

Let $A = \begin{bmatrix} -3 & 5 \\ 1 & 2 \end{bmatrix}$ $and\ \vec{x} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad Find\ A\vec{x}$

$$\begin{bmatrix} -3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -3-15 \\ 1-6 \end{bmatrix} = \begin{bmatrix} -18 \\ -5 \end{bmatrix}$$

Let $B = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 5 \\ 0 & 4 & 2 \end{bmatrix}$ $and\ \vec{y} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $Find\ the\ 3rd\ entry\ of\ B\vec{y}.$

$= [0, 4, 2] \cdot [1, 0, -1]$

$= 0 + 0 - 2$

$= -2$

Example 8.

F= $\begin{bmatrix} 3 & -1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 4 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

expand along the first column

det F= $(-1)^{1+1}(3)det\begin{bmatrix} 0 & 2 & 2 \\ 4 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

expand along the first column

det F=3$\left[(-1)^{2+1}(4)det\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} + (-1)^{3+1}(1)det\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}\right]$

=3[(-4)(4)+1(-4)]

= 3(-16-4)

= - 60

Practice Exam Questions

S1. Find the determinant of each of the following:

a) det A=7

b) detB=ad - bc = (2)(-7) - (-1)(3) = -14 + 3 = -11

c) C= $\begin{bmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

left products = -8 + 4 + 0 = -4

right products = 0+ 12 – 8 = 4

Det A=right – left = 4 – (-4)

=8

d) D= $\begin{bmatrix} 3 & -1 & 0 \\ 1 & -2 & 2 \\ -1 & 0 & 3 \end{bmatrix}$

left products = 0+0 -3 = -3

right products = -18 +2 + 0 = -16

Det A=right – left = -16+3= -13

e) E= $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & -1 \\ 3 & 4 & 0 \end{bmatrix}$

left products = 0 -4 + 0= -4

right products =0 + 0 + 8 = 8

Det A=right – left = 8 + 4 = 12

S2. Use the basket-weave method to calculate the determinant of B=$\begin{bmatrix} 2 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & 2 & 5 \end{bmatrix}$

left products = 0 -16+(-20)=-4
right products =0+24+12= 36
det A=right – left = 36-(-4)=40

S3. Use the basket-weave method to calculate the determinant of A=$\begin{bmatrix} 2 & 2 & 3 \\ 2 & -1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

left products = -9 + 8 + 4 = 3
right products = -2 + 12 + 12 = 22
Det A=right – left = 22 – 3 = 19

S4.
left products = 0 + 12K + 6K = 18K
right products = 8 + 0 + 0 = 8
Det A=right – left = 8 – 18K

Since it has no inverse, det A = 0,
So, 8 – 18k = 0
8=18k
K=8/18 = 4/9

S5. Determine the value of x so that matrix A=$\begin{bmatrix} 2 & 0 & 4 \\ x & 1 & 3 \\ 2 & 0 & x \end{bmatrix}$ has an inverse.

it has an inverse if detA≠0

If det A=0, 2x-8 = 0 so x=4 and this would be the value for which there is NO inverse
The answer is c). Basket weave and get left = 8+0+0=8 and right=2x+0+0=2x and right – left = 2x – 8 ≠ 0 ,so we get x≠ 4,since you know the determinant can't be 0 since it has an inverse. The answer is c).

S6.

$$\begin{matrix} 2 & 3 & 4 & 2 & 3 \\ 0 & k & 3 & 0 & k \\ 0 & 2k & k & 0 & 2k \end{matrix}$$

$$left = 0 + 12k + 0 = 12k \qquad right = 2k^2 + 0 + 0 = 2k^2$$

$detA = right - left \neq 0 \;\; if\ it\ has\ an\ inverse$

$2k^2 - 12k = 0$ factor
2k(k-6)=0
k=0, 6 no inverse, so it has an inverse if k≠ 0,6

S7. Not invertible, $ie) detA = 0$

$k(k+5) - (-6)(1) = 0$

$k^2 + 5k + 6 = 0$

$(k+2)(k+3) = 0 \quad k = -2, -3$

S8. For which value(s) of k is the matrix invertible?

Not invertible $\therefore detA = 0$

$$\begin{matrix} 2 & 1 & 0 & 2 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 0 & -3 & k & 0 & -3 \end{matrix}$$

Left$= 0 - 6 + k = k - 6 \qquad right = 4k + 0 + 0 = 4k$

$\therefore right - left = 0$

$4k - (k-6) = 0$

$3k + 6 = 0$

$3k = -6 \quad k = -2$

So, it is invertible as long as k is not equal to -2.

S9. a) A=$\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

Expand along the first column

det A= $(-1)^{1+1}(2)det\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

expand along third column

det A= 2[$(-1)^{2+3}(2)det\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$]

=2[(-2)(1)]

= -4

b) $let\ A = \begin{array}{ccc} (1,1) & (1,2) & (1,3) \\ \end{array}$
$$\begin{bmatrix} 1 & 2 & -3 \\ a & b & c \\ d & e & f \end{bmatrix} \text{and given}$$

$\det \begin{bmatrix} a & b \\ d & e \end{bmatrix} = 4$

$\det \begin{bmatrix} a & c \\ d & f \end{bmatrix} = 2$

$\det \begin{bmatrix} b & c \\ e & f \end{bmatrix} = 3$

Find $\det A$

$detA = (-1)^{1+1}(1) \det \begin{bmatrix} b & c \\ e & f \end{bmatrix} + (-1)^{1+2}(2) \det \begin{bmatrix} a & c \\ d & f \end{bmatrix} +$

$(-1)^{1+3}(-3) \det \begin{bmatrix} a & b \\ d & e \end{bmatrix}$

$= (1)(3) + (-2)(2) + (-3)(4)$

$= 3 - 4 - 12$

$= -13$

c) C= $\begin{bmatrix} 2 & -2 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 4 & 5 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ expand along the first column

det C= $(-1)^{1+1}(2)det \begin{bmatrix} 2 & 2 & 2 \\ 4 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$ expand along the bottom row

det C=2$\left[(-1)^{3+3}(1)det \begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix}\right]$ =2$[(2(5) - 2(4)] = 4$

d) D=$\begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ expand along the first column

$$\begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

det D= $(-1)^{1+1}(2)det\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & -1 \end{bmatrix}$ expand along the first column

det D =$2[(-1)^{1+1}(1)det\begin{bmatrix} -3 & -1 \\ 0 & -1 \end{bmatrix}] = 2[3+0] = 6$

Quiz 5: Practice on Sections A to H

1. a) $E(x) = \int_0^4 x\left(\frac{1}{2}x\right)dx$

$= \int_0^4 \frac{1}{2}x^2 dx$

$= \left[\frac{1}{2}\frac{x^3}{3}\right]_0^4$

$= \left[\frac{x^3}{6}\right]_0^4$

$= \frac{4^3}{6}$

$= \frac{64}{6}$

$= \frac{32}{3}$

b) $Var(x) = ?$

$E(x^2) = \int_0^4 x^2\left(\frac{1}{2}x\right)dx$

$= \frac{1}{2}\int_0^4 x^3 dx$

$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^4$

$= \frac{1}{8}[x^4]_0^4$

$= \frac{1}{8}[4^4 - 0]$

$= \frac{4^4}{8}$

$Var(x) = E(x^2) - [E(x)]^2$

$= \frac{4^4}{8} - (\frac{32^2}{3})^2$

$= \frac{256}{8} - \frac{1024}{9}$

$= -81\frac{7}{9}$

2. a) $E(x) = \frac{-2+0-0.8-0.4-0.5}{5} = \frac{-3.7}{5}$ note: there are no probabilities given, so it is just the mean which is to add up all the numbers and divide by how many there are

$= -0.74$

b) $Var(x) = \sum\frac{(x-\bar{x})^2}{n-1}$ (again, no probabilities given) n=5 numbers or 5 data

$$Var(x) = \frac{(-2+0.74)^2 + (0+0.74)^2 + (-0.8+0.74)^2 + (-0.4+0.74)^2 + (-0.5+0.74)^2}{5-1}$$

$= \frac{(-1.26)^2+0.74^2+(-0.06)^2+(0.34)^2+(0.24)^2}{4}$

$\frac{2.31}{4} = 0.58$

3. $Z \ = \dfrac{95-62}{12} = 2.75$

$2.75 = \frac{x-50}{9}$

$x = 74.75$

4. $F\begin{bmatrix}3\\-3\end{bmatrix} = \begin{bmatrix}3+(-3)\\3-2(-3)\end{bmatrix} = \begin{bmatrix}0\\9\end{bmatrix}$

a) The first element is 0.

b) The second element is 9

c) $\vec{F}\begin{bmatrix}0\\4\end{bmatrix} = \begin{bmatrix}u+v\\u-2v\end{bmatrix} = \begin{bmatrix}0+4\\0-2(4)\end{bmatrix} = \begin{bmatrix}4\\-8\end{bmatrix}$

The first element is 4.

d) The second element is -8.

e) $\vec{F}(\vec{x}) = \begin{bmatrix}0\\9\end{bmatrix} \qquad \vec{F}(\vec{y}) = \begin{bmatrix}4\\-8\end{bmatrix}$

$\vec{F}(z) = -90.5(0) + 85.5(4) = 342$

The first element is 342.

f) $\vec{F}(z) = -90.5(9) + 85.5(-8) = -1498.5$

The second element is – 1498.5.

5. a) $f(\vec{x}) = \vec{f}\begin{bmatrix}u\\v\end{bmatrix} = -3u + 4v$

$= f\left(\begin{bmatrix}4\\4\end{bmatrix}\right) = -3(4) + 4(4) = -12 + 16 = 4$

b) $f(\vec{y}) = f(\begin{bmatrix}-2\\2\end{bmatrix} = -3(-2) + 4(2) = 6 + 8 \ = 14$

c) $\vec{z} = -3\vec{x} + 5\vec{y}$

$f(\vec{z}) = -3f(\vec{x}) + 5\vec{f}(y)$

$= -3(4) + 5(14) = -12 + 70 = 58$

6. $A)\ \vec{F}\left(\begin{bmatrix}u\\v\end{bmatrix}\right) = \begin{bmatrix}2x+y\\x\end{bmatrix}$

$\vec{F}\left(\begin{bmatrix}4\\-4\end{bmatrix}\right) = \begin{bmatrix}2(4)-4\\4\end{bmatrix} = \begin{bmatrix}4\\4\end{bmatrix}$ (right, up) So, graph 3.

$B)\ \vec{F}\left(\begin{bmatrix}u\\v\end{bmatrix}\right) = \begin{bmatrix}y\\2x-y\end{bmatrix} \quad \vec{F}\left(\begin{bmatrix}4\\-4\end{bmatrix}\right) = \begin{bmatrix}-4\\2(4)-(-4)\end{bmatrix} = \begin{bmatrix}-4\\12\end{bmatrix}$ (left and up) So, graph 1.

$C)\ \vec{F}\left(\begin{bmatrix}u\\v\end{bmatrix}\right) = \begin{bmatrix}x\\y\end{bmatrix} \quad \vec{F}\left(\begin{bmatrix}4\\-4\end{bmatrix}\right) = \begin{bmatrix}4\\-4\end{bmatrix}$ (right and down) So, graph 2.

7. a) $\begin{bmatrix} 0 & 2 \\ 0.4 & 0.2 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0+6 \\ 2+0.6 \end{bmatrix} = \begin{bmatrix} 6 \\ 2.6 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 \\ 0.4 & 0.2 \end{bmatrix}\begin{bmatrix} 6 \\ 2.6 \end{bmatrix} = \begin{bmatrix} 5.2 \\ 2.92 \end{bmatrix}$$

$\therefore$ 5.2 *will be* in the first category

b) $\therefore$ 2.92 will be in the second category

8. $A\vec{x} = \begin{bmatrix} -2 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2-8 \\ 1-6 \end{bmatrix} = \begin{bmatrix} -10 \\ -5 \end{bmatrix}$

$\therefore the\ first\ element\ is -10\ and\ the\ second\ element\ is -5.$

9. 3rd entry = $\begin{bmatrix} & & \\ & & \\ 5 & -1 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} = [-5+0+4] = -1$

10. $\det A = ad - bc = 1(6) - 2(3) = 0$

$\therefore No$

11. $Area = det\begin{bmatrix} \vec{u} \\ \vec{v} \end{bmatrix} = det\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$

$= 6 - 4 = 2$

T. Matrix Models and Leslie Matrices

Example 2.

$$\vec{x}\cdot\vec{y} = [\beta,\gamma]\cdot[0,7]$$
$$= \beta(0) + 7\gamma$$
$$= 7\gamma$$ The answer is C.

Example 3.

a)

$$G = \begin{array}{c} \\ H \\ A \end{array}\begin{array}{c} \begin{array}{cc} H & A \end{array} \\ \begin{bmatrix} 1.2 & 1.5 \\ 0.5 & 0.3 \end{bmatrix} \end{array}$$

b) $\begin{bmatrix} 1.2 & 1.5 \\ 0.5 & 0.3 \end{bmatrix}\begin{bmatrix} 100 \\ 50 \end{bmatrix} = \begin{bmatrix} 195 \\ 65 \end{bmatrix}$

There would be 195 hatchlings and 65 adults in year 1.

Example 4.

a)

$$G = \begin{array}{c} \\ H \\ J \\ A \end{array}\begin{array}{c} \begin{array}{ccc} H & J & A \end{array} \\ \begin{bmatrix} 0 & 0.2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \end{array}$$

b) $\begin{bmatrix} 0 & 0.2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}\begin{bmatrix} 30 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 0+2+10 \\ 15+0+0 \\ 0+4+0 \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \\ 4 \end{bmatrix}$

$\therefore$ 12 *hatchlings*, 15 *juveniles*, 4 *adults*

c) $\begin{bmatrix} 0 & 0.2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}\begin{bmatrix} 12 \\ 15 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 6 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0.2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 7.2 \\ 3.5 \\ 2.4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0.2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}\begin{bmatrix} 7.2 \\ 3.5 \\ 2.4 \end{bmatrix} = \begin{bmatrix} 3.1 \\ 3.6 \\ 1.4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0.2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}\begin{bmatrix} 3.1 \\ 3.6 \\ 1.4 \end{bmatrix} = \begin{bmatrix} 2.12 \\ 1.55 \\ 1.44 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0.2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}\begin{bmatrix} 2.12 \\ 1.55 \\ 1.44 \end{bmatrix} = \begin{bmatrix} 1.75 \\ 1.06 \\ 0.62 \end{bmatrix}$$

Etc.

$\therefore$ total population decreases to zero.

The answer is A.

Example 5.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_t = \begin{array}{c} \\ x \\ y \\ z \end{array}\begin{bmatrix} x & y & z \\ 0.85 & 0 & 0 \\ 0.15 & 0.80 & 0 \\ 0 & 0.20 & 1 \end{bmatrix}$$

Practice Exam Questions

T1.

$$G = \begin{array}{c} \\ H \\ A \end{array}\begin{array}{c} \begin{array}{cc} \text{H} & \text{A} \end{array} \\ \begin{bmatrix} 0.8 & 1.2 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

b) $\begin{bmatrix} 0.8 & 1.2 \\ 0.6 & 0.4 \end{bmatrix}\begin{bmatrix} 60 \\ 50 \end{bmatrix} = \begin{bmatrix} 108 \\ 56 \end{bmatrix}$

c) $\begin{bmatrix} 0.8 & 1.2 \\ 0.6 & 0.4 \end{bmatrix}\begin{bmatrix} 108 \\ 56 \end{bmatrix} = \begin{bmatrix} 153.6 \\ 87.2 \end{bmatrix} \therefore 87\ adults$

T2.

$$G = \begin{array}{c} \\ H \\ A \end{array}\begin{array}{c} \begin{array}{cc} \text{H} & \text{A} \end{array} \\ \begin{bmatrix} 1.2 & 1.5 \\ 0.5 & 0.3 \end{bmatrix} \end{array} \quad top\ is\ A\ to\ H \quad and\ bottom\ is\ A\ to\ A$$

a)

b) $\begin{bmatrix} 1.2 & 1.5 \\ 0.5 & 0.3 \end{bmatrix}\begin{bmatrix} 40 \\ 20 \end{bmatrix} = \begin{bmatrix} 78 \\ 26 \end{bmatrix}$

c) $\begin{bmatrix} 1.2 & 1.5 \\ 0.5 & 0.3 \end{bmatrix}\begin{bmatrix} 78 \\ 26 \end{bmatrix} = \begin{bmatrix} 132.6 \\ 46.8 \end{bmatrix} \therefore 133\ hatchlings\ and\ 47\ adults\ in\ year\ 3.$

T3.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 0 & 5 \\ 3 & 4 & 0 \\ 0 & 2 & 0 \end{bmatrix} \end{array} \qquad \begin{bmatrix} a_t \\ b_t \\ c_t \end{bmatrix}$$

T4.

a)

$$\begin{array}{c} \\ H \\ A \end{array}\begin{array}{c} \begin{array}{cc} H & A \end{array} \\ \begin{bmatrix} 1.3 & 1.5 \\ 0.6 & 0.5 \end{bmatrix} \end{array}$$

b) $\begin{bmatrix} 1.3 & 1.5 \\ 0.6 & 0.5 \end{bmatrix}\begin{bmatrix} 100 \\ 30 \end{bmatrix} = \begin{bmatrix} 175 \\ 75 \end{bmatrix}$

c)

$$\begin{bmatrix} 1.3 & 1.5 \\ 0.6 & 0.5 \end{bmatrix}\begin{bmatrix} 175 \\ 75 \end{bmatrix} = \begin{bmatrix} 340 \\ 142.5 \end{bmatrix}$$

So, we expect 340 hatchlings and 143 adults in year 3.

T5.

a)

$$G = \begin{array}{c} \\ H \\ J \\ A \end{array}\begin{array}{c} \begin{array}{ccc} H & J & A \end{array} \\ \begin{bmatrix} 0 & 0.3 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \end{array}$$

b) Estimate the population of hatchlings and adults in year 2 if there are 120 birds in the population: 60 hatchlings, 40 juveniles and 20 adults.

$$\begin{bmatrix} 0 & 0.3 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}\begin{bmatrix} 60 \\ 40 \\ 20 \end{bmatrix}\begin{array}{c} H \\ J \\ A \end{array} = \begin{bmatrix} 0+12+20 \\ 36+0+0 \\ 0+20+0 \end{bmatrix} = \begin{bmatrix} 32 \\ 36 \\ 20 \end{bmatrix}$$

$\therefore 32\ hatchlings\ in\ year\ 2$

T6.

a) $C_{t+1} = 0.4C_t$

$a_{t+1} = 0.098C_t + 0.75a_t + 0.77r_t$

$r_{t+1} = 0.27a_t$

$$\begin{array}{c} \\ C_1 \\ a_1 \\ r_1 \end{array}\begin{array}{c} \begin{array}{ccc} C_1 & a_1 & r_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0.4 \\ 0.098 & 0.75 & 0.77 \\ 0 & 0.27 & 0 \end{bmatrix} \end{array}\begin{array}{c} \\ \begin{bmatrix} C_t \\ a_t \\ r_t \end{bmatrix} \end{array}$$

b)

$$\begin{array}{c} \\ h_1 \\ e_1 \\ l_1 \end{array}\begin{array}{c} \begin{array}{ccc} h_1 & e_1 & l_1 \end{array} \\ \begin{bmatrix} 0.996 & 0.55 & 0 \\ 0.004 & 0.2 & 0 \\ 0 & 0.55 & 2 \end{bmatrix} \end{array}\begin{array}{c} \\ \begin{bmatrix} h_t \\ e_t \\ l_t \end{bmatrix} \end{array}$$

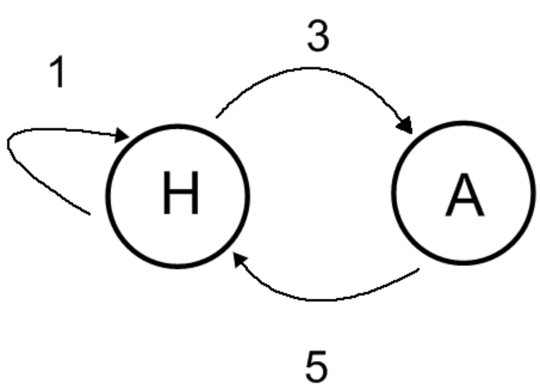
$h_{t+1} = 0.996h_t + 0.55e_t$

$e_{t+1} = 0.004h_t + 0.2e_t$

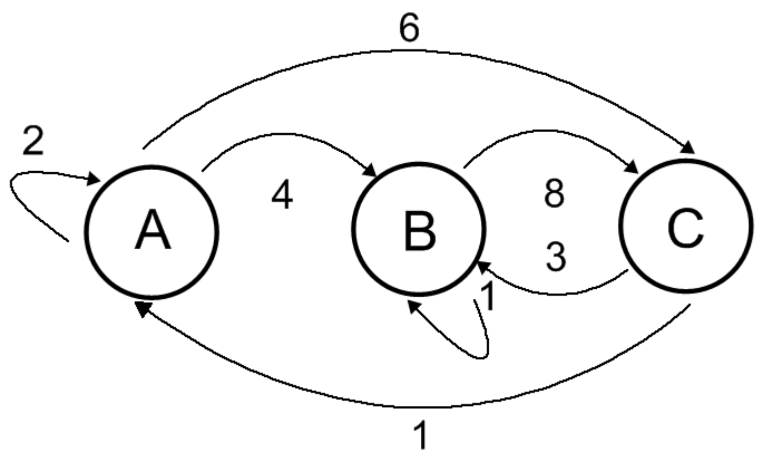
$l_{t+1} = 0.55e_t + 2l_t$

T7.

a)



b)



T8. a)

Construct the corresponding matrix…

$$\begin{array}{c} \\ x \\ y \end{array}\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 4 & 2 \\ 0 & 0.8 \end{bmatrix} \end{array} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$x_i + 1 = 4x_i + 2y_i$$
$$y_i + 1 = 0x_i + 0.8y_i$$

b)
$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 0 & 0 & 2 \\ B & 0 & 0 & 5 & 0 \\ C & 0 & 1 & 0 & 0 \\ D & 3 & 0 & 0 & 0 \end{array}$$

T9. a)

$$G = \begin{array}{c} \\ H \\ A \end{array}\begin{array}{c} \begin{array}{cc} H & A \end{array} \\ \begin{bmatrix} 1.5 & 1.8 \\ 0.5 & 0.4 \end{bmatrix} \end{array}$$

b) $\begin{bmatrix} 1.5 & 1.8 \\ 0.5 & 0.4 \end{bmatrix} \begin{bmatrix} 50 \\ 20 \end{bmatrix}^{H}_{A} = \begin{bmatrix} 75 + 36 \\ 25 + 8 \end{bmatrix} = \begin{bmatrix} 111 \\ 33 \end{bmatrix}$

c) $\begin{bmatrix} 1.5 & 1.8 \\ 0.5 & 0.4 \end{bmatrix} \begin{bmatrix} 111 \\ 33 \end{bmatrix} = \begin{bmatrix} 166.5 + 59.4 \\ 55.5 + 13.2 \end{bmatrix} = \begin{bmatrix} 225.9 \\ 68.70 \end{bmatrix}$

$\therefore$ *approx* 226 *hatchlings* and 69 adults in year 3.

T10.

$$\begin{array}{c|cc} & A & B \\ \hline A & 0.7 & 0.2 \\ B & 0.3 & 0.8 \end{array}$$

T11. a)

$$G = \begin{array}{c|cc} & H & A \\ \hline H & 0.70 & 1.3 \\ A & 0.5 & 0.6 \end{array}$$

b) $\begin{bmatrix} 0.70 & 1.3 \\ 0.5 & 0.6 \end{bmatrix}\begin{bmatrix} 60 \\ 30 \end{bmatrix} = \begin{bmatrix} 42+39 \\ 30+18 \end{bmatrix} = \begin{bmatrix} 81 \\ 48 \end{bmatrix}$

c) $\begin{bmatrix} 0.7 & 1.3 \\ 0.5 & 0.6 \end{bmatrix}\begin{bmatrix} 81 \\ 48 \end{bmatrix} = \begin{bmatrix} 119.1 \\ 69.3 \end{bmatrix}$ $approx$ 69 $adults$ and 119 hatchlings in year 3.

T12.

a) $\begin{bmatrix} x \\ y \end{bmatrix}_1 = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $doesn't\ \ move$

$\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $changes$

The solution is A.

T13. $\begin{bmatrix} 0 & 2 \\ 0.5 & 0.3 \end{bmatrix}\begin{bmatrix} 10 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5.6 \end{bmatrix}$

$\begin{bmatrix} 0 & 2 \\ 0.5 & 0.3 \end{bmatrix}\begin{bmatrix} 4 \\ 5.6 \end{bmatrix} = \begin{bmatrix} 11.2 \\ 3.68 \end{bmatrix}$

Therefore, two years into the future, there will be approximately 11 and 4 individuals.

U. Eigenvalues and Eigenvectors

Example 2. $\det(A - \lambda I) = 0$

$$A - \lambda I$$
$$= \begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ -2 & -1-\lambda \end{bmatrix}$$

$\det(A - \lambda I) = 0 \quad \therefore (2-\lambda)(-1-\lambda) - (3)(-2) = 0$

$-2 - 2\lambda + \lambda + \lambda^2 + 6$

$\lambda^2 - \lambda + 4 = 0$ OR

Short-cut: (Multiple Choice only)

use tr(A) =1 and detA = -2 + 6 = 4

$\lambda^2 - \lambda + 4 = 0$

Example 3.

$a = 1 \quad b = -1 \quad c = 4$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4(1)(4)}}{2(1)} = \frac{1 \pm \sqrt{-15}}{2} = \frac{1 \pm \sqrt{15}i}{2}$$

Example 4.

Find the eigenvalues and eigenvectors for A.

$$\begin{array}{cc} a & b \\ \multicolumn{2}{c}{\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}} \\ c & d \end{array}$$

Short-cut: (Multiple Choice only)

$trA = 2 + 3 = 5$

$detA = 6 - 0 = 6$

$\lambda^2 - tr(A)\lambda + detA = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0 \qquad \lambda = 2, 3$ eigenvalues

$\lambda = 2$

Short-cut: (Multiple Choice only and only for distinct eigenvalues)

$$\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 0 \\ 2-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \; \therefore \overset{eigenvector\ multiple\ of}{must\ use\ the\ other\ form}$$

$$\begin{bmatrix} \lambda - d \\ c \end{bmatrix} = \begin{bmatrix} 2-3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \; \therefore non-zero\ multiples\ of\ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{or} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 3 \qquad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 0 \\ 3-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; \therefore non-zero\ multiples\ of\ \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So, the eigenvalue that corresponds to the eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ *is* 2.

The answer is A.

Long Method:

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{bmatrix} = 0$

$ad - bc = 0$
$(2-\lambda)(3-\lambda) - 0 = 0$
$(2-\lambda)(3-\lambda) = 0$
$\lambda = 2, 3$

$(A - \lambda I)\vec{v} = \vec{0}$
$\begin{bmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 2$

$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$0v_1 + v_2 = 0$ $\boxed{1}$
$v_1 + v_2 = 0$
$v_2 = -v_1$

$\therefore \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 3$

$\begin{bmatrix} 2-3 & 0 \\ 1 & 3-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-v_1 + 0v_2 = 0$
$-v_1 = 0$
$v_1 = 0$

$\therefore$ let $v_2 = t$
Eigenvector is $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ $or\ t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (any multiples). This is called the family of eigenvectors!

Example 5. **Short-cut: (Multiple Choice only)**

$tr(A) = 2 + 2 = 4$

$detA = ad - bc = 2(2) - (-1)(3) = 4 + 3 = 7$

characteristic polynomial $\lambda^2 - tr(A)\lambda + detA = 0$

$$\lambda^2 - 4\lambda + 7 = 0$$

$a = 1, \quad b = -4, \quad c = 7 \qquad \lambda = \frac{4 \pm \sqrt{16 - 4(7)}}{2} \quad \therefore \lambda = \frac{4 \pm \sqrt{-12}}{2}$

$$\lambda = \frac{4 \pm \sqrt{4}\sqrt{-3}}{2} = \frac{4 \pm 2\sqrt{3}i}{2}$$

$$\lambda = 2 + \sqrt{3}i, \; 2 - \sqrt{3}i$$

Short-cut: (Multiple Choice only and only for distinct eigenvalues)

eigenvector $\quad \lambda = 2 + \sqrt{3}i \qquad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -1 \\ 2 + \sqrt{3}i - 2 \end{bmatrix} = \begin{bmatrix} -1 \\ \sqrt{3}i \end{bmatrix}$

$\lambda = 2 - \sqrt{3}i \quad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -1 \\ 2 - \sqrt{3}i - 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -\sqrt{3}i \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3}i \end{bmatrix}$

Long-Method 1:

$\lambda = 2 + \sqrt{3}i$

$A\vec{v} = \lambda\vec{v}$

$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left(2 + \sqrt{3}i\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$2v_1 - v_2 = (2 + \sqrt{3}i)v_1$

$3v_1 - 2v_2 = (2 + \sqrt{3}i)v_2$

$2v_1 - v_2 = 2v_1 + \sqrt{3}i\, v_1$

$2v_1 - 2v_1 - \sqrt{3}iv_1 = v_2$

$v_2 = -\sqrt{3}iv_1$

Let $v_1 = 1$

$v_2 = -\sqrt{3}i$

$\begin{bmatrix} 1 \\ -\sqrt{3}i \end{bmatrix}$

Long-Method 2:

$(A - \lambda I)\vec{v} = \vec{0}$

$\begin{bmatrix} 2-\lambda & -1 \\ 3 & 2-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 2 + \sqrt{3}i$

$\begin{bmatrix} 2-\left(2+\sqrt{3}i\right) & -1 \\ 3 & 2-\left(2+\sqrt{3}i\right) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -\sqrt{3}i & -1 \\ 3 & -\sqrt{3}i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-\sqrt{3}i\, v_1 - v_2 = 0$ $\boxed{1}$

From $\boxed{1}$ $\quad v_2 = -\sqrt{3}i\, v_1$

Let $v_1 = 1$

$v_2 = -\sqrt{3}i$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\sqrt{3}i \end{bmatrix}$

QUESTION: $\begin{bmatrix} i \\ \sqrt{3} \end{bmatrix} \times i = \begin{bmatrix} i^2 \\ \sqrt{3}i \end{bmatrix} = \begin{bmatrix} -1 \\ \sqrt{3}i \end{bmatrix} = \vec{v}$

$\therefore$ $\vec{v}\ and\ \vec{w}$ are equal

$A\overrightarrow{W} = A\vec{v} = \lambda\vec{v} = \left[2+\sqrt{3}i\right]\begin{bmatrix} i \\ \sqrt{3} \end{bmatrix}$

$= \begin{bmatrix} 2i + \sqrt{3}i^2 \\ 2\sqrt{3} + 3i \end{bmatrix} = \begin{bmatrix} 2i - \sqrt{3} \\ 2\sqrt{3} + 3i \end{bmatrix}$

Example 6.

$$A\vec{v} = \lambda\vec{v}$$

$$\begin{bmatrix} 0.8 & 0.1 & 0.3 \\ 0.1 & 0.7 & 0.3 \\ 0.1 & 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 1\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$0.8v_1 + 0.1v_2 + 0.3v_3 = v_1 \quad \boxed{1}$$

$$0.1v_1 + 0.7v_2 + 0.3v_3 = v_2 \quad \boxed{2}$$

$$0.1v_1 + 0.2v_2 + 0.4v_3 = v_3 \quad \boxed{3}$$

From $\boxed{1}$ $\quad 0.1v_2 = v_1 - 0.8v_1 - 0.3v_3$

$$0.1v_2 = 0.2v_1 - 0.3v_3$$

$$v_2 = \frac{0.2v_1 - 0.3v_3}{0.1}$$

$$v_2 = 2v_1 - 3v_3$$

From $\boxed{2}$ $\quad 0.1v_1 + 0.3v_3 = v_2 - 0.7v_2$

$$0.1v_1 + 0.3v_3 = 0.3v_2$$

Substitute $v_2 = 2v_1 - 3v_3$

$$0.1v_1 + 0.3v_3 = 0.3(2v_1 - 3v_3)$$

$$0.1v_1 + 0.3v_3 = 0.6v_1 - 0.9v_3$$

$$0.3v_3 + 0.9v_3 = 0.6v_1 - 0.1v_1$$

$$1.2v_3 = 0.5v_1$$

$$v_3 = \frac{0.5v_1}{1.2}$$

$$v_3 = \frac{5v_1}{12}$$

Let $v_1 = 12$

$$v_3 = 5$$

$$v_2 = 2(12) - 3(5) = 24 - 15$$

$$v_2 = 9$$

$$\therefore \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 5 \end{bmatrix}$$

<u>Example 7.</u> a)

$\det(A-\lambda I)=0$

$$det\left(\begin{bmatrix}2&1&0\\1&2&1\\0&1&2\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}\right)=0$$

$$det\begin{bmatrix}2-\lambda&1&0\\1&2-\lambda&1\\0&1&2-\lambda\end{bmatrix}=0$$

$$\begin{matrix}2-\lambda&1&0&2-\lambda&1\\1&2-\lambda&1&1&2-\lambda\\0&1&2-\lambda&0&1\end{matrix}$$

$left = 2-\lambda+2-\lambda$
$\quad = 4-2\lambda$

$right = (2-\lambda)^3+0+0$
$\quad = (2-\lambda)(2-\lambda)(2-\lambda)$
$\quad = (2-\lambda)(4-4\lambda+\lambda^2)$
$\quad = 8-8\lambda+2\lambda^2-4\lambda+4\lambda^2-\lambda^3$
$\quad = -\lambda^3+6\lambda^2-12\lambda+8$

$detA = right-left = 0$
$-\lambda^3+6\lambda^2-12\lambda+8-(4-2\lambda)=0$
$-\lambda^3+6\lambda^2-10\lambda+4=0$
$\therefore$ *the characteristic polynomial is*

$$-\lambda^3+6\lambda^2-10\lambda+4=0 \text{ or } \lambda^3-6\lambda^2+10\lambda-4=0$$

Use the Factor Theorem to find the eigenvalues (roots).

$-\lambda^3+6\lambda^2-10\lambda+4=0$

$\boxed{\lambda^3-6\lambda^2+10\lambda-4=0}$ $\quad f(\lambda)=\lambda^3-6\lambda^2+10\lambda-4 \leftarrow \pm1, \pm2, \pm4$

$$\begin{array}{c|cccc} & \lambda^3 & \lambda^2 & \lambda & \# \\ 2 & 1 & -6 & 10 & -4 \\ & \downarrow & 2 & -8 & 4 \\ \hline & 1 & -4 & 2 & \lfloor \text{OR} \end{array}$$

$f(2)=0 \ \therefore (\lambda-2)$ is a factor

$1\lambda^2-4\lambda+2$

$$(\lambda-2)\left(\underbrace{\lambda^2-4\lambda+2}_{\text{quadratic function}}\right)=0$$

$\boxed{\lambda=2}$

You can use the quadratic formula to finish!

b) $A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 2\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$2v_1 + v_2 = 2v_1 \ \boxed{1}$

$v_1 + 2v_2 + v_3 = 2v_2 \ \boxed{2}$

$v_2 + 2v_3 = 2v_3 \ \boxed{3}$

From $\boxed{1}$ $v_2 = 0$

From $\boxed{2}$ $v_1 = -v_3$

From $\boxed{3}$ $v_2 = 0$

Let $v_3 = 1$ $\quad \therefore \vec{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

<u>Practice Exam Questions</u>

U1.

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ A = & \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} \\ & \begin{array}{cc} c & d \end{array} \end{array}$$

Short-cut: (Multiple Choice only)

$tr(A) = 6 + (-4) = 2$

$detA = 6(-4) - 16(-1) = -24 + 16 \qquad detA = -8$

$\lambda^2 - 2\lambda - 8 = 0$

$(\lambda - 4)(\lambda + 2) = 0 \qquad \lambda = 4, -2 \ \ eigenvalues$

$\lambda = 4$ eigenvector multiple of $\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 16 \\ 4 - 6 \end{bmatrix} = \begin{bmatrix} 16 \\ -2 \end{bmatrix}$

Short-cut: (Multiple Choice only and only for distinct eigenvalues)

$$\lambda = 4 \ eigenvector = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 16 \\ 4 - 6 \end{bmatrix} = \begin{bmatrix} 16 \\ -2 \end{bmatrix} \ or \ any \ multiple$$

<u>Long method</u>

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 6 - \lambda & 16 \\ -1 & -4 - \lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(6 - \lambda)(-4 - \lambda) - 16(-1) = 0$

$-24 - 6\lambda + 4\lambda + \lambda^2 + 16 = 0$

$\lambda^2 - 2\lambda - 8 = 0$
$(\lambda - 4)(\lambda + 2) = 0$
$\lambda = 4, -2$
One way…
$A\vec{v} = \lambda\vec{v}$

$\lambda = 4$

$$\begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$6v_1 + 16v_2 = 4v_1 \quad \boxed{1}$
$-v_1 - 4v_2 = 4v_2 \quad \boxed{2}$

From $\boxed{1}$ $16v_2 = -2v_1$
$v_1 = -8v_2$ Let $v_2 = 1$

$\vec{v} = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$ same as $\begin{bmatrix} 16 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$

Eigenvector
$\lambda = 4$
Another way…
$(A - \lambda I)(\vec{v}) = \vec{0}$

$$\begin{bmatrix} 6-\lambda & 16 \\ -1 & -4-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6-4 & 16 \\ -1 & -4-4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 16 \\ -1 & -8 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2v_1 + 16v_2 \quad \boxed{1}$
$-v_1 - 8v_2 = 0 \ \boxed{2} \rightarrow \ v_1 = 8v_2$

Let $v_2 = 1$
$v_1 = -8$

$\therefore \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$ or any multiple. The family of eigenvectors is t(1,-8), ie. any multiple of this vector.

U2.

$$A = \begin{array}{cc} a & b \\ \multicolumn{2}{c}{\begin{bmatrix} -4 & 2 \\ 3 & -5 \end{bmatrix}} \\ c & d \end{array}$$

Short-cut: (Multiple Choice only)

$tr(A) = -4+(-5) = -9$

$Det(A) = -4(-5) - 2(3) = 20 - 6 = 14$

$\lambda^2 + 9\lambda + 14 = 0$

$(\lambda + 7)(\lambda + 2) = 0 \qquad \lambda = -7, -2 \; eigenvalues$

$\lambda = -2$

Short-cut: (Multiple Choice only and only for distinct eigenvalues)

$eigenvector\ multiple\ of \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ -2-(-4) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Long Method: $\lambda = -2$

$A\vec{v} = \lambda\vec{v}$

$\begin{bmatrix} -4 & 2 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -2\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$-4v_1 + 2v_2 = -2v_1 \quad \boxed{1}$

$3v_1 - 5v_2 = -2v_2 \quad \boxed{2}$

From $\boxed{1}$ $\quad 2v_2 = -2v_1 + 4v_1$

$2v_2 = 2v_1$

$v_2 = v_1$

$\therefore$ let $v_2 = 1$

$\therefore \; \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Another way…

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} -4 & 2 \\ 3 & -5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} -4-\lambda & 2 \\ 3 & -5-\lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(-4-\lambda)(-5-\lambda) - 2(3) = 0$

$+20 + 4\lambda + 5\lambda + \lambda^2 - 6 = 0$

$\lambda^2 - 9\lambda + 14 = 0$

$(\lambda + 7)(\lambda + 2) = 0$

$\lambda = -7, -2$

Eigenvector
$\lambda = -2$

$$\begin{bmatrix} -4+2 & 2 \\ 3 & -5+2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2v_1 + 2v_2 = 0$
$2v_2 = 2v_1$
$v_2 = v_1$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector. The family of this eigenvector is all multiples of this vector, ie. t(1,1).

U3. $\det(A - \lambda I) = 0$ $\quad \therefore \det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$. In this question, we need the factor theorem.

$$det\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} = 0 \qquad right = -\lambda^3 + 1 + 1$$

$$left = -3\lambda$$

$$detA = right - left$$

$$0 = -\lambda^3 + 2 - (-3\lambda)$$

$$0 = -\lambda^3 + 2 + 3\lambda \quad or \quad \lambda^3 - 3\lambda - 2 = 0$$

$let\ f(\lambda) = \lambda^3 - 3\lambda - 2 \qquad f(-1) = -1 + 3 - 2 = 0$

$(\lambda + 1) is\ a\ factor$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & \downarrow & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & 0 \ \text{R} \end{array}$$

$$\therefore (\lambda + 1)(\lambda^2 - \lambda - 2) = 0$$

$$(\lambda + 1)(\lambda - 2)(\lambda + 1) = 0 \qquad \lambda = -1, -1, 2$$

$\therefore \lambda = -1 \quad find\ eigenvector$

$A\vec{v} = \lambda\vec{v}$

$$\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = -1\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\left.\begin{aligned} v_2 + v_3 &= -v_1 \\ v_1 + v_3 &= -v_2 \\ v_1 + v_2 &= -v_3 \end{aligned}\right\} \textit{solve system} \text{ (all equations are identical)}$$

$From\ equation\ (1) \quad -v_1 = v_2 + v_3$

$v_1 = -v_2 - v_3$

The eigenvector is $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_2 - v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

↘ ↙

eigenvectors

If you want to find the other one:

$\lambda = 2 \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$v_2 + v_3 = 2v_1 \quad \boxed{1}$ which is the same as $2v_1 - v_3 = v_2$

$v_1 + v_3 = 2v_2 \quad \boxed{2}$

$v_1 + v_2 = 2v_3 \quad \boxed{3}$

$\boxed{1} + \boxed{2} \quad 3v_1 = 3v_2$ and we get: $v_1 = v_2$

From $\boxed{3}$ $\quad v_1 + v_2 = 2v_3$ becomes $v_1 + v_1 = 2v_3$ or $2v_1 = 2v_3$ and $v_1 = v_3$

The vector is $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1 \\ v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and let $v_1 = 1$

$\therefore (1,1,1) is\ a\ vector$

U4.

$$A = \begin{array}{cc} a & b \\ \left[\begin{matrix} 7 & -1 \\ 4 & 3 \end{matrix}\right. & \left.\right] \\ c & d \end{array}$$

$$A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$$

Short-cut: (Multiple Choice only)

$tr(A) = 7 + 3 = 10$

$detA = ad - bc$

$= 7(3) - (-1)(4)$

$= 21 + 4 \;\; = 25$

$\lambda^2 - tr(A)\lambda + detA = 0$

$\lambda^2 - 10\lambda + 25 = 0$

$(\lambda - 5)(\lambda - 5) = 0 \qquad \lambda = 5,\; 5$**** Don't use short cut to find your eigenvectors, as you will only obtain one eigenvector, even if there are really 2!!**

<u>Long method</u> $\lambda = 5$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 5\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$7v_1 - v_2 = 5v_1 \quad \boxed{1}$

$4v_1 + 3v_2 = 5v_2 \quad \boxed{2}$

From $\boxed{1}$ $\; 2v_1 = v_2$

$v_2 = 2v_1$

let $\; v_1 = 1$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the eigenvector. The family of this eigenvector is all multiples of it, ie. t(1,2).

U5.a) $\lambda^2 - tr(A)\lambda + \det(A) = 0$

Or $\det(A - \lambda I) = 0$

$$det\left[\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] = 0$$

$$det\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} = 0$$

$$\begin{matrix} 1-\lambda & -1 & 0 & 1-\lambda & -1 \\ -1 & 2-\lambda & -1 & -1 & 2-\lambda \\ 0 & -1 & 1-\lambda & 0 & -1 \end{matrix}$$

$left = 0 + 1(1-\lambda) + 1(1-\lambda)$ $\quad$ $right = (1-\lambda)(2 - 3\lambda + \lambda^2) + 0 + 0$

$= 1 - \lambda + 1 - \lambda$ $\quad$ $= 2 - 3\lambda + \lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3$

$= 2 - 2\lambda$ $\quad$ $= 2 - 5\lambda + 4\lambda^2 - \lambda^3$

$detA = right - left = 0$

$2 - 5\lambda + 4\lambda^2 - \lambda^3 - (2 - 2\lambda) = 0$

$-\lambda^3 + 4\lambda^2 - 3\lambda = 0$

b) $-\lambda(\lambda^2 - 4\lambda + 3) = 0$

$\lambda = 0$ $\quad$ $(\lambda - 3)(\lambda - 1) = 0$

$\lambda = 3{,}1$ $\leftarrow$ *eigenvalues* (3 of them)

c) $A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 3\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$v_1 - v_2 = 3v_1$ $\boxed{1}$ $\rightarrow -v_2 = 2v_1$ *or* $v_2 = -2v_1$

$-v_1 + 2v_2 - v_3 = 3v_3$ $\boxed{2}$

$-v_2 + v_3 = 3v_3$ $\boxed{3}$ $\rightarrow$ $-v_2 = 2v_3$ *or* $v_2 = -2v_3$ *or* $v_3 = -\frac{1}{2}v_2$

let $v_1 = 1$, *then* $v_2 = -2$ *and* $v_3 = 1$

$$\lambda = 3 \text{ has eigenvector } \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$check \quad \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

$$\lambda\vec{v} = 3\vec{v} = 3\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

Find the eigenvector for eigenvalue $\lambda = 0$

$A\vec{v} = \lambda\vec{v}$

$$\lambda = 0 \qquad \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_1 - v_2 = 0 \ \boxed{1} \qquad v_1 = v_2$

$-v_1 + 2v_2 - v_3 = 0 \ \boxed{2}$

$-v_2 + v_3 = 0 \ \boxed{3} \qquad v_3 = v_2$

$let\ v_2 = 1 \quad \to v_1 = 1,\ v_3 = 1$

$$vector \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad for\ \lambda = 0$$

$$check \qquad \underset{A}{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}\underset{\vec{v}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Find the eigenvector for eigenvalue $\lambda = 1$

$$\lambda = 1 \qquad \underset{A}{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}\underset{\vec{v}}{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}} = \underset{\lambda}{1}\underset{\vec{v}}{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}$$

$v_1 - v_2 = v_1 \ \boxed{1} \qquad \to v_2 = 0$

$-v_1 + 2v_2 - v_3 = v_2 \ \boxed{2} \quad \to -v_1 + 0 - v_3 = 0$

$\qquad -v_1 = v_3$

$-v_2 + v_3 = v_3 \ \boxed{3} \quad \to \quad v_2 = 0$

$$let\ v_1 = 1 \qquad vector \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad for\ \lambda = 1$$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \underbrace{1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{\lambda\vec{v}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

U6. a)

$$\det\left[\begin{bmatrix}1 & -2 & -2\\ 4 & -5 & -2\\ 8 & -4 & 5\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right] = 0$$

$$det\begin{bmatrix}1-\lambda & -2 & -2\\ 4 & -5-\lambda & -2\\ 8 & -4 & 5-\lambda\end{bmatrix} = 0$$

$$\begin{matrix}1-\lambda & -2 & -2 & 1-\lambda & -2\\ 4 & -5-\lambda & -2 & 4 & -5-\lambda\\ 8 & -4 & 5-\lambda & 8 & -4\end{matrix}$$

$$left = -16(-5-\lambda) + 8(1-\lambda) - 8(5-\lambda)$$
$$= 80 + 16\lambda + 8 - 8\lambda - 40 + 8\lambda$$
$$= 16\lambda + 48$$
$$right = (1-\lambda)(-5-\lambda)(5-\lambda) + 32 + 32$$
$$= (1-\lambda)(-25 + 5\lambda - 5\lambda + \lambda^2) + 64$$
$$= -25 + 5\lambda - 5\lambda + \lambda^2 + 25\lambda - 5\lambda^2 + 5\lambda^2 - \lambda^3 + 64$$
$$= -\lambda^3 + \lambda^2 + 25\lambda + 39$$
$$detA = -\lambda^3 + 25\lambda + 39 - (16\lambda + 48) = 0$$
$$-\lambda^3 + \lambda^2 + 25\lambda + 39 - 16\lambda - 48 = 0$$
$$-\lambda^3 + \lambda^2 + 9\lambda - 9 = 0$$
$$\lambda^3 - \lambda^2 - 9\lambda + 9 = 0$$
$$\lambda^2(\lambda - 1) - 9(\lambda - 1) = 0$$
$$(\lambda - 1)(\lambda^2 - 9) = 0$$

b) $\lambda = 1, 3, -3 \quad eigenvalues$

c) $\lambda = 1$

$$A\vec{v} = \lambda\vec{v}$$

$$\begin{bmatrix}1 & -2 & -2\\4 & -5 & -2\\8 & -4 & 5\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = 1\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

$v_1 - 2v_2 - 2v_3 = v_1$ $\boxed{1}$ $\quad -2v_2 = 2v_3 \quad v_2 = -v_3$

$4v_1 - 5v_2 - 2v_3 = v_2$ $\boxed{2}$

$8v_1 - 4v_2 + 5v_3 = v_3$ $\boxed{3}$

$8v_1 - 4v_2 + 4v_3 = 0$

$8v_1 - 4(-v_3) + 4v_3 = 0$

$8v_1 + 4v_3 + v_3 = 0$

$8v_1 + 8v_3 = 0$

$8v_1 = -8v_3 \qquad v_1 = -v_3$

$\therefore let\ v_3 = 1 \quad \rightarrow v_2 = -1 \quad v_1 = -1$

$$vector \begin{bmatrix}-1\\-1\\1\end{bmatrix} for\ \lambda = 1$$

$check \quad A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix}1 & -2 & -2\\4 & -5 & -2\\8 & -4 & 5\end{bmatrix}\begin{bmatrix}-1\\-1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\\1\end{bmatrix} \quad and\ \lambda\vec{v} = 1\begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

$\lambda = 3 \qquad A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix}1 & -2 & -2\\4 & -5 & -2\\8 & -4 & 5\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = 3\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

$v_1 - 2v_2 - 2v_3 = 3v_1$ $\boxed{1}$

$4v_1 - 5v_2 - 2v_3 = 3v_2$ $\boxed{2}$

$8v_1 - 4v_2 + 5v_3 = 3v_3$ $\boxed{3}$

From $\boxed{1}$ $\; -2v_2 - 2v_3 = 2v_1 \quad or \quad v_1 = -v_2 - v_3$ $\boxed{4}$

From $\boxed{2}$ $\; 4v_1 - 8v_2 - 2v_3 = 0$

$2v_1 - 4v_2 - v_3 = 0$

$2v_1 = 4v_2 + v_3$ $\boxed{5}$

$v_1 = -v_2 - v_3$ $\boxed{4}$ $\times 2 \qquad 2v_1 = -2v_2 - 2v_3$

$2v_1 = 4v_2 + v_3$ $\boxed{5}$ $\qquad \underline{2v_1 = 4v_2 + v_3}$ $\quad$ subtract

$0 = \; -6v_2 - 3v_3$

$3v_3 = -6v_2 \qquad v_3 = -2v_2$

Sub into $\boxed{5}$ $\quad 2v_1 = 4v_2 + v_3$

$2v_1 = 4v_2 + (-2v_2)$

$2v_1 = 2v_2 \qquad v_1 = v_2$

$let\ v_2 = 1,\ v_3 = -2(1) = -2$

$$vector \begin{bmatrix}1\\1\\-2\end{bmatrix} \quad for\ \lambda = 3$$

$check\ A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 1 & -2 & -2 \\ 4 & -5 & -2 \\ 8 & -4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}$$

$$\lambda\vec{v} = 3\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}$$

$$let\ \lambda = -3$$

$$A\vec{v} = \lambda\vec{v}$$

$$\begin{bmatrix} 1 & -2 & -2 \\ 4 & -5 & -2 \\ 8 & -4 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = -3\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$v_1 - 2v_2 - 2v_3 = -3v_1 \quad \boxed{1}$$

$$4v_1 - 5v_2 - 2v_3 = -3v_2 \quad \boxed{2}$$

$$8v_1 - 4v_2 + 5v_3 = -3v_3 \quad \boxed{3}$$

$$from \boxed{1} \quad -2v_2 - 2v_3 = -4v_1 \qquad v_2 + v_3 = 2v_1$$

$$from \boxed{2} \quad 4v_1 - 2v_3 = 2v_2 \qquad 2v_1 - v_3 = v_2 \quad same\ as \boxed{1}$$

$$from \boxed{3} \quad 8v_1 - 4v_2 = -8v_3 \qquad 2v_1 - v_2 = -2v_3$$

$$2v_1 = -2v_3 + v_2 \quad sub\ into \boxed{1}$$

$$\therefore v_2 + v_3 = -2v_3 + v_2$$

$$3v_3 = 0 \qquad v_3 = 0$$

$$sub\ v_3 = 0 \quad into \boxed{2} \quad 2v_1 - 0 = v_2$$

$$v_2 = 2v_1$$

$$v = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$A\vec{v} = \lambda\vec{v}$$

$$\begin{bmatrix} 1 & -2 & -2 \\ 4 & -5 & -2 \\ 8 & -4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix} \qquad -3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$$

U7. $A\vec{v} = \lambda\vec{v}$ *to find eigenvectors*

$$\lambda = 2 \quad \begin{bmatrix} 2 & 1 & -2 \\ -3 & 0 & 4 \\ -2 & -1 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 2\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$\boxed{1}$ $2v_1 + v_2 - 2v_3 = 2v_1 \quad \therefore v_2 = 2v_3$

$\boxed{2}$ $-3v_1 + 4v_3 = 2v_2$

Subst. from $\boxed{1}$ $\quad -3v_1 + 4v_3 = 2(2v_3)$

$-3v_1 + 4v_3 = 4v_3$

$-3v_1 = 0 \quad v_1 = 0$

$$let\ t = v_3 \quad \therefore vector \begin{bmatrix} 0 \\ 2t \\ t \end{bmatrix} or\ any\ multiple$$

$$let\ t = 1, you\ get \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore B$ *is the solution*

NOTE: [0 0 0] is NEVER an eigenvector

U8. **Short-cut: (Multiple Choice only)**

$tr(A) = 2 + 1 = 3$

$detA = ad - bc = 2(1) - 2(1) = 0$

$\lambda^2 - tr(A) + detA = 0$

$\lambda^2 - 3\lambda = 0$

$\lambda(\lambda - 3) = 0$

$\lambda = 0,3 \quad \therefore\ E$ *is the answer*

Long Method:

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 2-\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix} = 0$$

$ad - bc = 0$

$(2 - \lambda)(1 - \lambda) - 2(1) = 0$

$\lambda^2 - 3\lambda = 0$

$\lambda(\lambda - 3) = 0$

$\lambda = 0, 3$

U9. **Short-cut: (Multiple Choice only)**

$tr(A) = 1 + 3 = 4$

$detA = 1(3) - 0(2) = 3$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 3)(\lambda - 1) = 0$

$\lambda = 1,3$

$\therefore E \text{ is the solution}$

Long method

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{bmatrix} = 0$$

$ad - bc = 0$

$(1 - \lambda)(3 - \lambda) - 0 = 0$

$(1 - \lambda)(3 - \lambda) = 0$

$\lambda = 1, 3$

U10. $A = \lambda^2 - 8\lambda + 12 = 0$

$(\lambda - 2)(\lambda - 6) = 0$

$\lambda = 2,6$

Short-cut: (Multiple Choice only and only for distinct eigenvalues)

$eigenvector \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 3 \\ 2-2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ or any multiple

$\therefore v_1$

$B = \lambda^2 - 8\lambda + 12 = 0$

$\lambda = 2,6\ eigenvector \quad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 3 \\ 2-6 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$\therefore v_2$

$C = \begin{bmatrix} 4 & 2 \\ -2 & 0 \end{bmatrix}$

$\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ 2-4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ *or any multiple*

$\therefore v_3$

Long method

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 2-\lambda & 3 \\ 0 & 6-\lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(2-\lambda)(6-\lambda) - 0 = 0$

$\lambda = 2, 6$

$\det(B - \lambda I) = 0$

$\det\left(\begin{bmatrix} 6 & 3 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 6-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(6-\lambda)(2-\lambda) - 0 = 0$

$(6-\lambda)(2-\lambda) = 0$

$\lambda = 2, 6$

U11. **Short-cut: (Multiple Choice only)**

$$detA = ad - bc = 0(4) - (-1)(0) = 0$$

$$\lambda^2 - 4\lambda = 0$$

$$\lambda(\lambda - 4) = 0$$

$$\lambda = 0{,}4$$

$\therefore A \text{ is the answer}$

U12. a) **Short-cut: (Multiple Choice only)**

$$tr(A) = a + d = 1 + 1 = 2$$

$$detA = ad - bc = 1(1) - 2(-1) = 3$$

$$\lambda^2 - tr(A)\lambda + detA = 0$$

$$\lambda^2 - 2\lambda + 3 = 0 \qquad a = 1 \quad b = -2 \quad c = 3$$

won't factor

$$\lambda = -\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} = \frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm \sqrt{4}\sqrt{2}i}{2}$$

$$\lambda = \frac{2 + 2\sqrt{2}i}{2}, \frac{2 - 2\sqrt{2}i}{2} \quad or \quad 1 + \sqrt{2}\, i \;\; 1 - \sqrt{2}i$$

$$\lambda = 1 + \sqrt{2}i \qquad A = \begin{array}{c} \begin{matrix} \text{a} & \text{b} \end{matrix} \\ \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \\ \begin{matrix} c & d \end{matrix} \end{array}$$

b) **Short-cut: (Multiple Choice only and only for distinct eigenvalues)**

Eigenvector $\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ 1 + \sqrt{2}i - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2}i \end{bmatrix}$

Long method:

If $\lambda = 1 + \sqrt{2}i \qquad \lambda = 1 - \sqrt{2}i$

$$A\vec{v} = \lambda\vec{v}$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left(1 + \sqrt{2}i\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$v_1 + 2v_2 = \left(1 + \sqrt{2}i\right)v_1 \quad \boxed{1}$$

$$-v_1 + v_2 = \left(1 + \sqrt{2}i\right)v_2 \quad \boxed{2}$$

$$\cancel{v_1} + 2v_2 = \cancel{v_1} + \sqrt{2}i\, v_1$$

$$-v_1 + \cancel{v_2} = \cancel{v_2} + \sqrt{2}i\, v_2$$

From $\boxed{1}$ $\quad 2v_2 = \sqrt{2}i\, v_1$

$$v_2 = \frac{\sqrt{2}i}{2}\, v_1$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2}i \end{bmatrix}$$

$\lambda = 1 - \sqrt{2}i$

Eigenvector$=\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ 1 - \sqrt{2}i - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -\sqrt{2}i \end{bmatrix}$

Long Method 1:

$\lambda = 1 - \sqrt{2}i$

$A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = 0$

$$\begin{bmatrix} 1-\lambda & 2 \\ -1 & 1-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-(1-\sqrt{2}i) & 2 \\ -1 & 1-(1-\sqrt{2}i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2i} & 2 \\ -1 & \sqrt{2i} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\sqrt{2i}\, v_1 + 2v_2 = 0$

$2v_2 = -\sqrt{2i}\, v_1$

$2v_2 = -\dfrac{\sqrt{2i}}{2}\, v_1$

Let $v_1 = 2$

$v_2 = -\sqrt{2i}$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -\sqrt{2i} \end{bmatrix}$$

Long Method 2: If $\lambda = 1 + \sqrt{2}i$ $\quad \lambda = 1 - \sqrt{2}i \quad A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (1 - \sqrt{2}i)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$v_1 + 2v_2 = (1 - \sqrt{2}i)v_1 \quad \boxed{1}$

$-v_1 + v_2 = (1 - \sqrt{2}i)v_2 \quad \boxed{2}$

From $\boxed{1}$

$v_1 + 2v_2 = v_1 - \sqrt{2}i\, v_1$

$2v_2 = -\sqrt{2}i\, v_1$

$v_2 = -\frac{\sqrt{2}i}{2}\, v_1$

Let $v_1 = 2$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -\sqrt{2}i \end{bmatrix}$$

U13. a) $\det(A-\lambda I) = det\left(\begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$

$det\begin{bmatrix} 3-\lambda & 2 & 3 \\ 0 & 6-\lambda & 10 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$

In upper Δ form, so you can just multiply along the main diagonal and get the eigenvalues, instead of basket weaving. As a result, the numbers on the main diagonal are always the eigenvalues. (for any matrix in triangular form)

$(3-\lambda)(6-\lambda)(2-\lambda) = 0$

$\therefore \lambda = 3,6,2$

b) Eigenvector for $\lambda = 2$ (smallest eigenvalue)

$A\vec{v} = \lambda\vec{v}$

$\begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 2\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$3v_1 + 2v_2 + 3v_3 = 2v_1 \quad \boxed{1}$

$6v_2 + 10v_3 = 2v_2 \quad \boxed{2}$

$2v_3 = 2v_3 \quad \boxed{3}$

From $\boxed{2}$ $\quad 4v_2 = -10v_3$

$v_2 = \frac{-10}{4}v_3 = \frac{-5}{2}v_3 \quad sub\ \ into \ \boxed{1}$

$3v_1 + 2\left(\frac{-10}{4}v_3\right) + 3v_3 = 2v_1$

$3v_1 - 5v_3 + 3v_3 = 2v_1$

$v_1 = 2v_3 \quad \therefore \left(2, \frac{-5}{2}, 1\right)\ or\ \ (4,-5,2)\ \ is\ an\ eigenvector\ \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$

Here are some more to try!!

U14. Find the eigenvalues for $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}$.
Then, find the eigenvectors for the largest and smallest eigenvalues.

U15. For A= $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$, find the eigenvalues and eigenvectors.

U16. For $A = \begin{bmatrix} 5 & -4 \\ 1 & 3 \end{bmatrix}$, find eigenvalues and the eigenvector for $a - bi$.

U17. For A=$\begin{bmatrix} 2 & -4 \\ 5 & 5 \end{bmatrix}$, find the eigenvalue and the eigenvector for a+bi.

U14. $\det\left(\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}\right) = 0$

$$\det\begin{bmatrix} \overbrace{-2-\lambda}^{1,1} & 0 & 0 & 0 \\ 0 & 4-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 3 & 4-\lambda \end{bmatrix} = 0$$

$$(-1)^{1+1}(-2-\lambda)\det\begin{bmatrix} \overbrace{4-\lambda}^{1,1} & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 3 & 4-\lambda \end{bmatrix} = 0$$

$$(-2-\lambda)((-1)^{1+1}(4-\lambda)\det\begin{bmatrix} 3-\lambda & 0 \\ 3 & 4-\lambda \end{bmatrix} = 0$$

$$(-2-\lambda)(4-\lambda)[(3-\lambda)(4-\lambda)] = 0$$

$$\therefore \lambda = -2, 4, 3, 4$$

Find e-vector for $\lambda = 4$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 4\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$\boxed{1} - 2v_1 = 4v_1 \quad \therefore 6v_1 = 0 \;\; \therefore v_1 = 0$

$\boxed{2}\ 4v_2 = 4v_2 \leftarrow \text{doesn't mean } v_2 = 0$

$\boxed{3}\ 3v_3 = 4v_3 \quad \therefore v_3 = 0$

$\boxed{4}\ 3v_3 + 4v_4 = 4v_4 \quad \therefore v_3 = 0$

∴ eigenvector

$$= \begin{bmatrix} 0 \\ v_2 \\ 0 \\ v_4 \end{bmatrix} = v_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

↑ ↑

2 e-vectors

Find e-vector for $\lambda = -2$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = -2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$\boxed{1}\ -2v_1 = -2v_1 \quad \leftarrow \text{doesn't mean } v_1 = 0$

$\boxed{2}\ 4v_2 = -2v_2 \therefore 6v_2 = 0 \ \therefore v_2 = 0$

$\boxed{3}\ 3v_3 = -2v_3 \quad \therefore 5v_3 = 0 \ \therefore v_3 = 0$

$\boxed{4}\ 3v_3 + 4v_4 = -2v_4 \quad \therefore 3v_3 = -6v_4$

$v_3 = -2v_4$

But $v_3 = 0 \ \therefore v_4 = 0$

$$\therefore \begin{bmatrix} v_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

↑

e-vector for $\lambda = -2$

U15. A= $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$,

Short-cut: (Multiple Choice only)

$tr(A) = 8$

$\det A = ad - bc = 5(3) - (-2)(1)$

$= 17$

$\lambda^2 - tr(A)\lambda + \det A = 0$

$\lambda^2 - 8\lambda + 17 = 0$

Doesn't factor

$\lambda^2 - 8\lambda + 16 = -17 + 16$

$(\lambda - 4)^2 = -1$

$\lambda - 4 = \pm\sqrt{-1}$

$\lambda - 4 = i, \quad \lambda - 4 = -i$

$\lambda = 4 + i, 4 - i$

Find the eigenvector:(Long Method)

$\lambda = 4 + i$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (4+i)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$5v_1 - 2v_2 = 4v_1 + iv_1 \quad \boxed{1}$

$v_1 - iv_1 = 2v_2$

$v_1(1-i) = 2v_2$

$v_2 = \dfrac{v_1(1-i)}{2}$

Let $v_1 = 2 \;\; v_2 = 1 - i$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$$

Note: it could also be

$$\begin{bmatrix} 2 \\ 1-i \end{bmatrix}\begin{matrix} \times 1+i \\ \times 1+i \end{matrix}$$

$$\begin{bmatrix} 2+2i \\ 1+i-i-i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2i \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2+2i \\ 2 \end{bmatrix}$$

$$= 2\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

$$(A - \lambda I)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = 4 + i$

$$\begin{bmatrix} 5-(4+i) & -2 \\ 1 & 3-(4+i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(1-i)v_1 - 2v_2 = 0$

$(1-i)v_1 = 2v_2$

$$v_2 = \frac{(1-i)}{2} v_1$$

Let $v_1 = 2$
$v_2 = 1 - i$
$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$

Note: it could also be

$\begin{bmatrix} 2 \\ 1-i \end{bmatrix} \begin{matrix} \times 1+i \\ \times 1+i \end{matrix}$
$\begin{bmatrix} 2+2i \\ 1+i-i-i^2 \end{bmatrix}$
$= \begin{bmatrix} 2+2i \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2+2i \\ 2 \end{bmatrix}$

$= 2\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$

U16. $A = \begin{bmatrix} 5 & -4 \\ 1 & 3 \end{bmatrix}$ Find eigenvalues and the e-vector for $a - bi$

Long method

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 5 & -4 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det \begin{bmatrix} 5-\lambda & -4 \\ 1 & 3-\lambda \end{bmatrix} = 0$
$ad - bc = 0$
$(5-\lambda)(3-\lambda) + 4 = 0$

$15 - 5\lambda - 3\lambda + \lambda^2 + 4 = 0$

$\lambda^2 - 8\lambda + 19 = 0$
$\lambda^2 - 8\lambda = -19$
$\lambda^2 - 8\lambda + 16 = -19 + 16$
$(\lambda - 4)^2 = -3$
$\lambda - 4 = \pm\sqrt{-3}$
$\lambda = 4 \pm \sqrt{-3}$

$(A - \lambda I)\vec{v} = 0$

$\begin{bmatrix} 5-\lambda & -4 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 4 + \sqrt{-3}$
$\begin{bmatrix} 5-\left(4+\sqrt{-3}\right) & -4 \\ 1 & 3-\left(4+\sqrt{-3}\right) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1-\sqrt{-3}i & -4 \\ 1 & -1-\sqrt{-3}i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_1 + \left(-1-\sqrt{3}i\right)v_2 = 0$
$v_1 = -\left(-1-\sqrt{3}i\right)v_2$
$v_1 = \left(1+\sqrt{3}i\right)v_2$

Let $v_2 = 1$
$\therefore \begin{bmatrix} 1+\sqrt{3}i \\ 1 \end{bmatrix}$ is the eigenvector

U17. $A = \begin{bmatrix} \overbrace{2}^{a} & \overbrace{-4}^{b} \\ \underbrace{5}_{c} & \underbrace{5}_{d} \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 2 & -4 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 2-\lambda & -4 \\ 5 & 5-\lambda \end{bmatrix} = 0$
$ad - bc = 0$
$(2-\lambda)(5-\lambda) + 20 = 0$

$10 - 7\lambda + \lambda^2 + 20 = 0$

$\lambda^2 - 7\lambda + 30 = 0$

$$\lambda = \frac{7 \pm \sqrt{49 - 4(1)(30)}}{2(1)}$$

$$\lambda = \frac{7 \pm \sqrt{71}i}{2}$$

$$\lambda = \frac{7 + \sqrt{71}i}{2}, \lambda = \frac{7 - \sqrt{71}i}{2}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$\left(\begin{bmatrix} 2 & -4 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - \lambda & -4 \\ 5 & 5 - \lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda = \frac{7 + \sqrt{71}i}{2}$$

$$\begin{bmatrix} 2 - \left(\frac{7 + \sqrt{71}i}{2}\right) & -4 \\ 5 & 5 - \left(\frac{7 + \sqrt{71}i}{2}\right) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-3 - \sqrt{71}i}{2} & -4 \\ 5 & \frac{3 - \sqrt{71}i}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$5v_1 + \frac{3 - \sqrt{71}i}{2}v_2 = 0$$

$$5v_1 = -\frac{\left(3 - \sqrt{71}i\right)}{2}v_2$$

$$v_1 = \frac{-3 + \sqrt{71}i}{2}v_2$$

Let $v_2 = 2, v_1 = -3 + \sqrt{71}i$

$\therefore$ the eigenvector is $\begin{bmatrix} -3 + \sqrt{71}i \\ 2 \end{bmatrix}$

Simplifying with Complex Numbers…Try These!

U18. Evaluate each of the following:

1. $(3 + 4i) + (10 - 2i)$

2. $(4 + 7i)(-2 - 3i)$

U19. Evaluate and express in form $a + bi$:

$$\frac{4 - 2i}{1 + 3i}$$

U20. Find the absolute value of $2 + 3\sqrt{2i}$.

U18.
1. $(3+4i)+(10-2i)$

$= 13+2i$

2. $(4+7i)(-2-3i)$

$= -8-12i-14i-21i^2$

$= -8-26i-21(-1)$

$= 13-26i$

U19. Evaluate and express in form $a+bi$

$\frac{4-2i}{1+3i}$

$$\frac{(4-2i)}{(1+3i)}\frac{(1-3i)}{(1-3i)} = \frac{4-12i-2i+6i^2}{1-3i+3i-9i^2}$$

$$= \frac{4-14i+6(-1)}{1-9(-1)}$$

$$= \frac{10-14i}{10} = 1-\frac{7}{2}i$$

U20. Find the absolute value of $2+3\sqrt{2i}$

$|z| = \sqrt{a^2+b^2}$ $\quad a=2 \quad b=3\sqrt{2}$

$$= \sqrt{2^2+\left(3\sqrt{2}\right)^2}$$

$$= \sqrt{4+9(2)} = \sqrt{22}$$

V. Applications of Eigenvalues and Eigenvectors

Example 1

a) $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ $x_t = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t$

$$= c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}(2)^t + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}(3)^t$$

Find the unknowns

At $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $find\ c_1\ and\ c_2 \quad (t = 0)$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}(2)^0 + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}(3)^0$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \end{bmatrix}$$

$1 = -c_1 \quad \therefore c_1 = -1$

$1 = c_1 + c_2$

$1 = -1 + c_2 \quad c_2 = 2$

$$x_t = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t$$

$$\therefore \overrightarrow{x_t} = \left[-1\begin{bmatrix} -1 \\ 1 \end{bmatrix}(2)^t + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}(3)^t\right]$$

b) $let\ t = 3$

$$\overrightarrow{x_3} = \left(-1\begin{bmatrix} -1 \\ 1 \end{bmatrix}(2)^3 + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}(3)^3\right)$$

$$= -8\begin{bmatrix} -1 \\ 1 \end{bmatrix}(2)^3 + 54\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 54 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 8 \\ 46 \end{bmatrix}$$

c) as x approaches infinity, $\overrightarrow{x_t} = \left[-1\begin{bmatrix} -1 \\ 1 \end{bmatrix}(2)^t + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}(3)^t\right]$approaches infinity as both $(2)^t\ and (3)^t$will approach infinity.

Example 2.

$$\begin{array}{c c} & \begin{array}{ccc} W & B & D \end{array} \\ \begin{array}{c} W \\ B \\ D \end{array} & \begin{bmatrix} 3/5 & 1/4 & 1/2 \\ 1/5 & 0 & 0 \\ 1/5 & 3/4 & 1/2 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 3/5 & 1/4 & 1/2 \\ 1/5 & 0 & 0 \\ 1/5 & 3/4 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{5}x + \frac{1}{4}y + \frac{1}{2}z \\ \frac{1}{5}x \\ \frac{1}{5}x + \frac{3}{4}y + \frac{1}{2}z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

From $\boxed{2}$ $\quad y = \frac{1}{5}x \quad let\ x = 5$

$$y = 1$$

From $\boxed{1}$ $\quad \frac{3}{5}x + \frac{1}{4}y + \frac{1}{2}z = x$

$$\frac{3}{5}(5) + \frac{1}{4}(1) + \frac{1}{2}z = 5$$

$$3 + \frac{1}{4} + \frac{1}{2}z = 5$$

$$\frac{1}{2}z = 5 - 3 - \frac{1}{4}$$

$$\frac{1}{2}z = 2 - \frac{1}{4}$$

$$\frac{1}{2}z = \frac{8}{4} - \frac{1}{4}$$

$$\frac{1}{2}z = \frac{7}{4}$$

$$4z = 14$$

$$z = \frac{14}{4} = \frac{7}{2}$$

$$\begin{bmatrix} 5 \\ 1 \\ 7/2 \end{bmatrix} \times 2 = \begin{bmatrix} 10 \\ 2 \\ 7 \end{bmatrix}$$

$$\therefore \frac{10}{10+2+7} = \frac{10}{19} \ \ walk$$

$$bike = \frac{2}{19}$$

$$drive = \frac{7}{19}$$

Practice Exam Questions

V1. $find\ eigenvalues\ \ trA = -9$

$$detA = 20 - 6 = 14$$
$$\lambda^2 + 9\lambda + 14 = 0$$
$$(\lambda + 2)(\lambda + 7) = 0$$
$$\lambda = -2, -7$$

$$\lambda = -2\ eigenvector\quad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ -2 + 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}\ or\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\lambda = -7\ eigenvector\quad \begin{bmatrix} 2 \\ -7 + 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

a)
$$\overrightarrow{x_t} = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t \qquad \therefore t = 0$$
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}(-2)^0 + c_2\begin{bmatrix} 2 \\ -3 \end{bmatrix}(-7)^0]$$
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ -3c_2 \end{bmatrix}$$
$$1 = c_1 + 2c_2$$
$$-2 = c_1 - 3c_2$$
$$-1 = 5c_2 \qquad c_2 = -\tfrac{1}{5}$$
$$1 = c_1 + 2(-\tfrac{1}{5})$$
$$1 = c_1 - \tfrac{2}{5} \qquad c_1 = \tfrac{7}{5}$$
$$\overrightarrow{x_t} = \left(\frac{7}{5}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}(-2)^t + \left(\frac{-1}{5}\right)\begin{bmatrix} 2 \\ -3 \end{bmatrix}(-7)^t$$

b) $t = 3$
$$\overrightarrow{x_3} = \left(\tfrac{7}{5}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}(-2)^3 + \left(\tfrac{-1}{5}\right)\begin{bmatrix} 2 \\ -3 \end{bmatrix}(-7)^3$$

$$= -\tfrac{56}{5}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \tfrac{343}{5}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$
$$\begin{bmatrix} -\frac{56}{5} \\ -\frac{56}{5} \end{bmatrix} + \begin{bmatrix} \frac{686}{5} \\ -\frac{1029}{5} \end{bmatrix}.\ So, \overrightarrow{x_3} = \begin{bmatrix} 126 \\ -217 \end{bmatrix}$$

V2.

$$A = \begin{array}{cc} a & b \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ c & d \end{bmatrix} \qquad trA = 2 \qquad detA = 1 - 4 = -3$$

$$\lambda^2 - 2\lambda - 3 = 0$$
$$(\lambda - 3)(\lambda + 1) = 0 \quad \lambda = 3, -1$$

$$\lambda = 3\ eigenvector \quad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ 3 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -1\ eigenvector \quad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ -1 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\overrightarrow{x_t} = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t \quad \therefore \overrightarrow{x_t} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}(3)^t + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}(-1)^t$$

$$x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad at\ t = 0 \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}(3)^0 + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}(-1)^0$$

$$1 = c_1 + c_2 \quad \boxed{1}$$
$$2 = c_1 - c_2 \quad \boxed{2}$$
$$3 = 2c_1 \qquad c_1 = \tfrac{3}{2} \qquad 1 = \tfrac{3}{2} + c_2$$
$$c_2 = 1 - \tfrac{3}{2} = -\tfrac{1}{2}$$
$$\overrightarrow{x_t} = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(3)^t + \left(\frac{-1}{2}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}(-1)^t$$

$$\overrightarrow{x_2} = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(3)^2 + \left(\frac{-1}{2}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}(-1)^2$$

$$= \frac{3}{2}(9)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \pm \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{27}{2} \\ \frac{27}{2} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 13 \\ 14 \end{bmatrix}$$

V3. From U3. $x_0 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

$$\overrightarrow{x_t} = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t$$

$\overrightarrow{x_t} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} (-1)^t + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} (-1)^t + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (2)^t$ sub $t = 0$

$x_0 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} (-1)^0 + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} (-1)^0 + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (2)^0$

$1 = -c_1 - c_2 + c_3$ $\boxed{1}$

$1 = c_1 + c_3$ $\boxed{2}$ $c_1 = 1 - c_3$

$3 = c_2 + c_3$ $\boxed{3}$ $c_2 = 3 - c_3$ sub into $\boxed{1}$

$1 = -(1 - c_3) - (3 - c_3) + c_3$

$1 = -1 + c_3 - 3 + c_3 + c_3$

$3c_3 = 1 + 4$

$c_3 = \frac{5}{3}$

$c_1 = 1 - \frac{5}{3} = -\frac{2}{3}$

$c_2 = 3 - \frac{5}{3} = \frac{9}{3} - \frac{5}{3} = \frac{4}{3}$

$\therefore \overrightarrow{x_t} = -\frac{2}{3} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} (-1)^t + \frac{4}{3} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} (-1)^t + \frac{5}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (2)^t$

W. Systems of Recursion Models

Example 1. Consider the system of recursion equations:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix}$$

If $x_1(0) = 3, x_2(0) = -5$, then find the solution for $x(t)$.

$tr(A) = 4 + (-7) = -3$

$detA = ad - bc = 4(-7) - (-3)(6) = -10$

$\lambda^2 + 3\lambda - 10 = 0$

$(\lambda + 5)(\lambda - 2) = 0$

$\lambda = -5,\ \ 2$

$$\begin{array}{c} a \quad\ \ b \\ \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \\ c \quad\ \ d \end{array}$$

$\vec{v}_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -3 \\ -5-4 \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} -3 \\ 2-4 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\overrightarrow{x_t} = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t$

$\overrightarrow{x_t} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} (-5)^t + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} (2)^t$

At $t = 0$ $\quad c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$1c_1 + 3c_2 = 3 \ \boxed{1} \quad (x - 3)$

$3c_1 + 2c_2 = -5 \ \boxed{2}$

$-3c_1 - 9c_2 = -9$

$3c_1 + 2c_2 = -5$

$Add \quad -7c_2 = -14$

$c_2 = 2$

$c_1 + 3(2) = 3$

$c_1 = 3 - 6$

$c_1 = -3$

$c_1 = -3, \quad c_2 = 2$

$\therefore \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t$

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} (-5)^t + 2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} (2)^t$

$x_1(t) = -3(1)(-5)^t + 2(3)(2)^t = -3(-5)^t + 6(2)^t$

$x_2(t) = -3(3)(-5)^t + 2(2)(2)^t = -9(-5)^t + 4(2)^t$

b) Find the long-term behaviour:
As $t \to \infty$,
$\vec{x}(t) \to -3\begin{bmatrix}1\\3\end{bmatrix}(-5)^{\infty} + 2\begin{bmatrix}3\\2\end{bmatrix}2^{\infty}$
$\therefore \vec{x}(t) \to \infty$

c) $\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = -3\begin{bmatrix}1\\3\end{bmatrix}(-5)^t + 2\begin{bmatrix}3\\2\end{bmatrix}(2)^t$

The geometric rate of increase of the population is 5 since that is the largest eigenvalue in absolute value. Since 5 >1, it is an increasing population, rather than a shrinking one.

d) The eigenvector associated with the eigenvalue of 5 is $\begin{bmatrix}1\\3\end{bmatrix}$, in the long-run we would expect 1 juvenile for every 3 adults.

Practice Exam Questions

W1. Consider the system of recursion equations:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix}$$

If $x_1(0) = 6$, $x_2(0) = -8$, Find the solution for x(t).

$$tr(A) = 5 + (-2) = 3$$
$$detA = 5(-2) - (-3)(2) = -10 + 6 = -4$$
$$\lambda^2 - 3\lambda - 4 = 0$$
$$(\lambda - 4)(\lambda + 1) = 0$$
$$\lambda = 4, \quad -1$$

$$\begin{array}{c} a \quad b \\ \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix} \\ c \quad d \end{array}$$

$$\vec{v}_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -3 \\ 4 - 5 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -3 \\ -1 - 5 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\overrightarrow{x_t} = c_1 \overrightarrow{v_1} (\lambda_1)^t + c_2 \overrightarrow{v_2} (\lambda_2)^t$$
$$\overrightarrow{x_t} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} (4)^t + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} (-1)^t$$

At $t = 0$ $\quad \begin{bmatrix} 6 \\ -8 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$6 = 3c_1 + c_2 \ (x - 2)$$
$$-8 = c_1 + 2c_2$$

$$\begin{array}{r} -12 = -6c_1 - 2c_2 \\ -8 = c_1 + 2c_2 \\ \hline Add \quad -20 = -5c_1 \end{array}$$
$$c_1 = 4$$
$$-8 = 4 + 2c_2$$
$$-12 = 2c_2$$
$$c_2 = -6$$
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 4 \begin{bmatrix} 3 \\ 1 \end{bmatrix} (4)^t - 6 \begin{bmatrix} 1 \\ 2 \end{bmatrix} (-1)^t$$

b) The dominant eigenvalue in absolute value is 4, so the geometric rate of increase of the population is 4. Since 4>1, the population is growing.

c)The eigenvector associate with eigenvalue 4 is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ so we would expect to count 3 juveniles for every 1 adult in the long-run.

W2. Consider the system of recursion equations:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix}$$

$If\ x_1(0) = 4,\ \ x_2(0) = 7,$ then find the solution for $x(t)$.

$$trA = 1 + 4 = 5$$

$$detA = 1(4) - 1(-2) = 4 + 2 = 6$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2, \quad 3$$

$$\begin{array}{cc} a & b \\ \multicolumn{2}{c}{\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}} \\ c & d \end{array}$$

$$\vec{v}_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 1 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\overrightarrow{x_t} = c_1 \overrightarrow{v_1} (\lambda_1)^t + c_2 \overrightarrow{v_2} (\lambda_2)^t$$

$$\overrightarrow{x_t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (2)^t + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} (3)^t$$

At $t = 0$ $\quad \begin{bmatrix} 4 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$4 = c_1 + c_2$$

$$7 = c_1 + 2c_2$$

$$Subtract \quad -3 = -c_2$$

$$c_2 = 3$$

$$4 = c_1 + 3$$

$$c_1 = 1$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (2)^t + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} (3)^t$$

Another one to try!!

Example. Find the long-term behaviour if $\vec{x}[t] = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{1}{2}\right)^t + 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} (4)^t$ as $\vec{t} \to \infty$

$$\vec{x}[t] = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{1}{2}\right)^\infty + 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} (4)^\infty$$

$$\therefore \vec{x}[t] = 0 + \infty \to \infty$$

X. Systems of Differential Equations

Example 1.
Rewrite the second equation:

$u_1{}'(t) = -5u_1(t) + u_2(t)$
$u_2{}'(t) = u_1(t) + -2u_2(t) + 3$

$\vec{\boldsymbol{u}}(\boldsymbol{t}) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ $\vec{\boldsymbol{\alpha}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ **and** $A = \begin{bmatrix} -5 & 1 \\ 1 & -2 \end{bmatrix}$

$\overrightarrow{\boldsymbol{u}'}(\boldsymbol{t}) = \boldsymbol{A}\vec{\boldsymbol{u}}(\boldsymbol{t}) + \vec{\boldsymbol{\alpha}}$

$\overrightarrow{\boldsymbol{u}'}(\boldsymbol{t}) = \begin{bmatrix} -5 & 1 \\ 1 & -2 \end{bmatrix} \vec{\boldsymbol{u}}(\boldsymbol{t}) + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

Case 1. Real and Distinct Eigenvalues

Example 2. Determine the solution to $\overrightarrow{u'}(t) = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ with initial condition $\vec{u}(0) = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$.

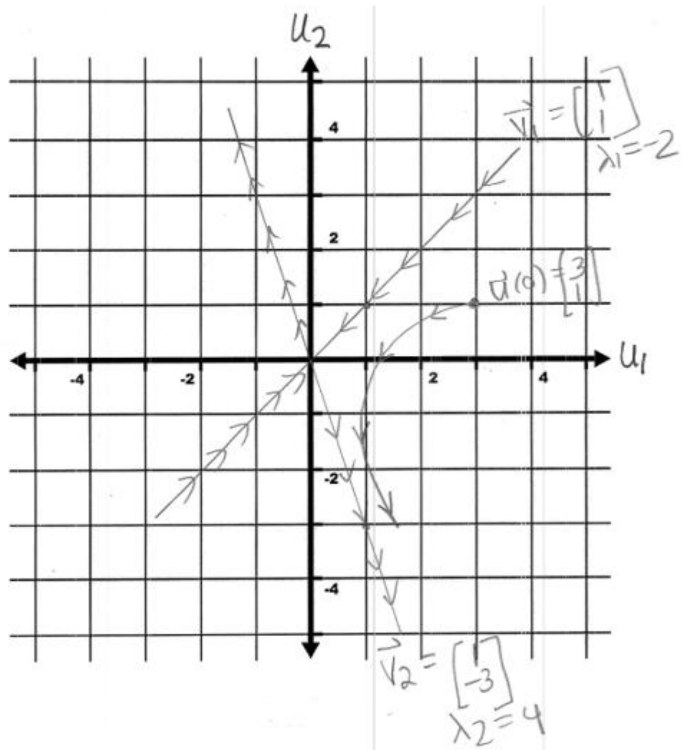
The solution is:
$u(t) = c_1 e^{\lambda_1 t} \overrightarrow{v_1} + c_2 e^{\lambda_2 t} \overrightarrow{v_2}$
Find $c_1 \& c_2$

$$\vec{u}(0) = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

Let $t = 0$ $\begin{bmatrix} 1 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^0 + c_2 e^0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$1 = -c_1 + c_2$ $\boxed{1}$
$8 = c_1 + 2c_2$ $\boxed{2}$
ADD $9 = 3c_2$
$c_2 = 3$ $\quad subst\ into$ $\boxed{2}$
$8 = c_1 + 2(3)$
$8 = c_1 + 6$
$c_1 = 2$
$\therefore \vec{u}(t) = 2e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Example 3.



Case 2. Complex Eigenvalues

Example 4.

Solve $\overrightarrow{u'}(t) = \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 9 & -1 \\ 5 & 7 \end{bmatrix} \\ \begin{array}{cc} c & d \end{array} \end{array} \vec{u}(t)\ with\ \vec{u}(0) = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

1. Find eigenvalues

$$tr(A) = 9 + 7 = 16$$
$$detA = 63 + 5 = 68$$
$$\lambda^2 - 16\lambda + 68 = 0$$
$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\lambda = \frac{16 \pm \sqrt{256 - 4(1)(68)}}{2(1)}$$
$$\lambda = \frac{16 \pm \sqrt{-16}}{2}$$
$$= \frac{16 \pm 4i}{2}$$
$$= 8 \pm 2i$$

2. Find the corresponding eigenvectors

$\begin{bmatrix} \lambda - d \\ c \end{bmatrix}$ *gives a vector with* i *on top*

$\lambda_1 = 8 + 2i$ (*positive imaginary part*) $\lambda_2 = 8 - 2i$

$\begin{bmatrix} 8 + 2i - 7 \\ 5 \end{bmatrix}$ $\begin{bmatrix} 8 - 2i - 7 \\ 5 \end{bmatrix}$

$\therefore \overrightarrow{v_1} = \begin{bmatrix} 1 + 2i \\ 5 \end{bmatrix}$ $\overrightarrow{v_2} = \begin{bmatrix} 1 - 2i \\ 5 \end{bmatrix}$

($\overrightarrow{v_1}$ has the positive imaginary part) (the conjugate of $\overrightarrow{v_1}$)

$$\overrightarrow{v_1} = \begin{bmatrix} 1 + 2i \\ 5 \end{bmatrix} = \underset{\substack{\uparrow \\ \vec{x}}}{\begin{bmatrix} 1 \\ 5 \end{bmatrix}} + i \underset{\substack{\uparrow \\ \vec{y}}}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

3. $\vec{u}(t) = c_1 e^{at}(\cos(bt)\,\vec{x} - \sin(bt)\,\vec{y}) + c_2 e^{at}(\sin(bt)\,\vec{x} + \cos(bt)\,\vec{y})$

$\lambda_1 = 8 + 2i \qquad a = 8,\ b = 2$

$\vec{u}(t) = c_1 e^{8t}\left(\cos(2t)\begin{bmatrix}1\\5\end{bmatrix} - \sin(2t)\begin{bmatrix}2\\0\end{bmatrix}\right) + c_2 e^{8t}(\sin(2t)\begin{bmatrix}1\\5\end{bmatrix} + \cos(2t)\begin{bmatrix}2\\0\end{bmatrix})$

Substitute in initial condition:

$\vec{u}(0)=\begin{bmatrix}5\\5\end{bmatrix}$

$\begin{bmatrix}5\\5\end{bmatrix} = c_1 e^0\left(1\begin{bmatrix}1\\5\end{bmatrix} - 0\right) + c_2 e^0(0 + 1\begin{bmatrix}2\\0\end{bmatrix})$

$\begin{bmatrix}5\\5\end{bmatrix} = c_1\begin{bmatrix}1\\5\end{bmatrix} + c_2\begin{bmatrix}2\\0\end{bmatrix}$

$\begin{bmatrix}5\\5\end{bmatrix} = c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2}$

$\begin{bmatrix}5\\5\end{bmatrix} = c_1\begin{bmatrix}1\\5\end{bmatrix} + c_2\begin{bmatrix}8\\0\end{bmatrix}$

$5 = c_1 + 2c_2 \quad \boxed{1}$

$5 = 5c_1 + 0 \qquad c_1 = 1 \quad \boxed{2}$

Substitute $c = 1$ *into* $\boxed{1}$

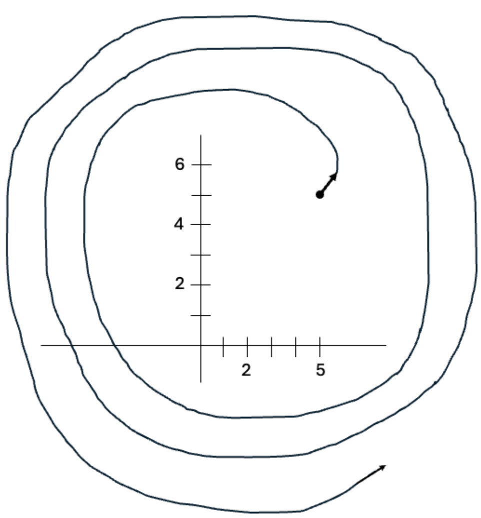
$5 = 1 + 2c_2$

$4 = 2c_2$

$c_2 = 2$

Final Answer:

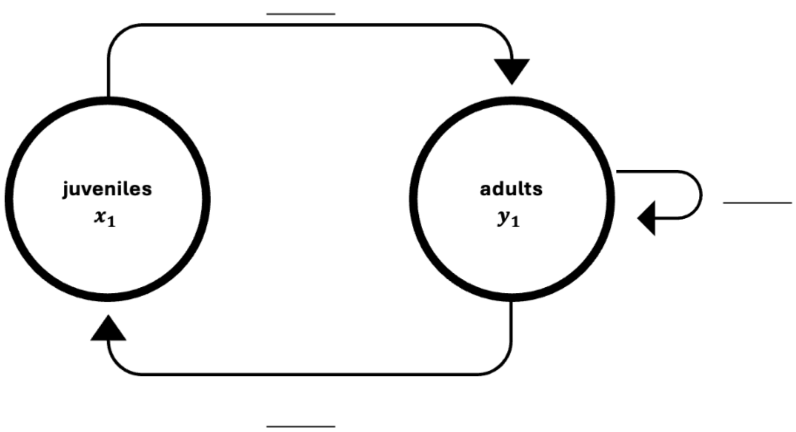
$\boldsymbol{\vec{u}(t) = e^{8t}\left(\cos(2t)\begin{bmatrix}1\\5\end{bmatrix} - \sin(2t)\begin{bmatrix}2\\0\end{bmatrix}\right) + 2e^{8t}(\sin(2t)\begin{bmatrix}1\\5\end{bmatrix} + \cos(2t)\begin{bmatrix}2\\0\end{bmatrix})}$



Here are some more to try!!

1. Consider an age-structured population made up of juveniles and adults. Each adult in year one produces on average 27/4 juveniles counted in the next year. Each juvenile we count in one year survives to be an adult in the next year with probability 25%. Also, each adult in one year survives to be counted as an adult again in the following year with probability 9/4.

a) Fill in the blanks in the diagram below



b) If x_t is the number of juveniles in year t and y_t is the number of adults in year t and $x_0=24$ and $y_0=8$, find an explicit expression for x_t and y_t in terms of t.

c) What is the long-term geometric rate of increase of the population? Is it growing or shrinking?

d) In the long-run, how many juveniles will we count for every adult? Explain.

2. a) Find $u_1(t)$ and $u_2(t)$ such that:

$$u_1'(t) = 4u_1(t) - u_2(t)$$

$$u_2'(t) = 2u_1(t) + u_2(t)$$

with $u_1(0) = 0$ *and* $u_2(0) = 3$.

b) Sketch the solution to the initial value problem in part a) below using a solid curve for $t \in [0, \infty)$. Label your initial condition and show the direction of travel using arrows.

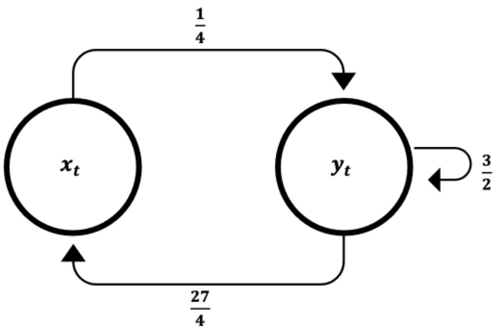
c) Sketch the solution to the differential equations from part a) with initial condition $u_1(0) = -1$ *and* $u_2(0) = -2$.

d) Repeat part c) for $u_1(0) = 2$ *and* $u_2(0) = 2$.

Solution:

1.

a)



b)

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{array}{c c} & \begin{array}{cc} x_t & y_t \end{array} \\ \begin{array}{c} x_t \\ y_t \end{array} & \begin{bmatrix} 0 & 27/4 \\ 1/4 & 3/2 \end{bmatrix} \end{array} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix}$$

And $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 24 \\ 8 \end{bmatrix}$

<u>Find eigenvalues</u> A=$\begin{bmatrix} 0 & 27/4 \\ 1/4 & 3/2 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 0 & \frac{27}{4} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} -\lambda & \frac{27}{4} \\ \frac{1}{4} & \frac{3}{2} - \lambda \end{bmatrix} = 0$$

ad-bc=0

$-\lambda\left(\frac{3}{2} - \lambda\right) - \frac{27}{4}\left(\frac{1}{4}\right) = 0$

$\lambda^2 - \frac{3}{2}\lambda - \frac{27}{16} = 0$ x16

$16\lambda^2 - 24\lambda - 27 = 0$

$(4\lambda - 9)(4\lambda + 3) = 0$

$4\lambda = 9$ $\quad\quad$ $4\lambda = -3$

$\lambda_1 = \frac{9}{4}$ $\quad\quad$ $\lambda_2 = -\frac{3}{4}$

Eigenvectors

$$\begin{bmatrix} -\lambda & \frac{27}{4} \\ \frac{1}{4} & \frac{3}{2}-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda_1 = \frac{9}{4}$

$\frac{3}{2} - \frac{9}{4} = \frac{6}{4} - \frac{9}{4} = -\frac{3}{4}$

$$\begin{bmatrix} -\frac{9}{4} & \frac{27}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\frac{1}{4}v_1 - \frac{3}{4}v_2 = 0$
$\frac{1}{4}v_1 = \frac{3}{4}v_2$
$3v_2 = v_1$
Let $v_1 = 3$
$v_2 = 1$
$\therefore \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is the eigenvector

$\lambda_2 = -\frac{3}{4}$
$\frac{3}{2} - \left(-\frac{3}{4}\right) = \frac{6}{4} + \frac{3}{4} = \frac{9}{4}$

$$\begin{bmatrix} \frac{3}{4} & \frac{27}{4} \\ \frac{1}{4} & \frac{9}{4} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\frac{1}{4}v_1 + \frac{9}{4}v_2 = 0$
$\frac{1}{4}v_1 = \frac{-9}{4}v_2$
$v_1 = -9v_2$
$v_2 = -\frac{1}{9} - 9v_1$
let $v_1 = 9$
$\therefore \begin{bmatrix} 9 \\ -1 \end{bmatrix}$ is the eigenvector

c) The eigenvalues are $\frac{-3}{4}, \frac{9}{4}$

$\therefore$ the largest one in absolute value is $\frac{9}{4}$. Therefore, the geometric rate of increase is $\frac{9}{4}$ because $\frac{9}{4} > 1$ we know the population is growing.

d) $\lambda_1 = \frac{9}{4}$ has eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\therefore$ in the long run we have 3 juveniles for every adult

2.

a) A=$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix} = 0$

ad-bc=0

$(4-\lambda)(1-\lambda) - (-1)(2) = 0$

$4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\lambda = 2, 3$

$\lambda_1 = 2$

$[A - \lambda I]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 4-2 & -1 \\ 2 & 1-2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2v_1 - v_2 = 0$

$v_2 = 2v_1$

let $v_1 = 1$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the eigenvector

$\lambda_2 = 3$

$\begin{bmatrix} 4-3 & -1 \\ 2 & 1-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_1 - v_2 = 0$

$v_2 = v_1$

$\therefore \begin{bmatrix}1\\1\end{bmatrix}$ is the eigenvector

$u_1(0) = 0$
$u_2(0) = 3$

$\vec{u}(t) = C_1 v_1 e^{\lambda t} + C_2 v_2 e^{\lambda t}$
$\vec{u}(t) = C_1 \begin{bmatrix}1\\2\end{bmatrix} e^{2t} + C_2 \begin{bmatrix}1\\1\end{bmatrix} e^{3t}$
$\begin{bmatrix}0\\3\end{bmatrix} = C_1 \begin{bmatrix}1\\2\end{bmatrix} e^{0} + C_2 \begin{bmatrix}1\\1\end{bmatrix} e^{0}$
$0 = C_1(1) + C_2(1) \rightarrow\ C_1 = -C_2$
$3 = 2C_1 + C_2$

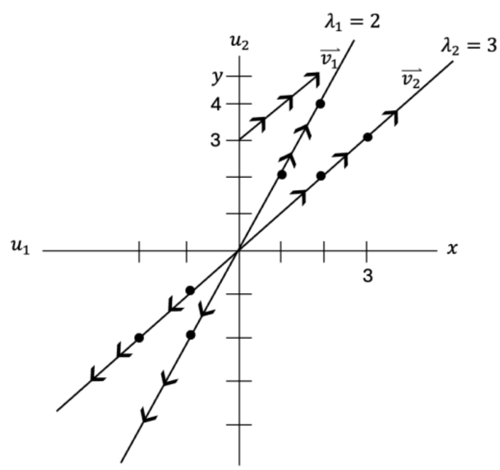
$\therefore 3 = 2(-C_2) + C_2$
$3 = -2C_2 + C_2$
$3 = -C_2$

$C_2 = -3$
$\therefore C_1 = -3$

$\vec{u}(t) = 3 \begin{bmatrix}1\\2\end{bmatrix} e^{2t} - 3 \begin{bmatrix}1\\1\end{bmatrix} e^{3t}$
$\overrightarrow{u_1}(t) = 3e^{2t} - 3e^{3t}$ and $\overrightarrow{u_2}(t) = 6e^{2t} - 3e^{3t}$

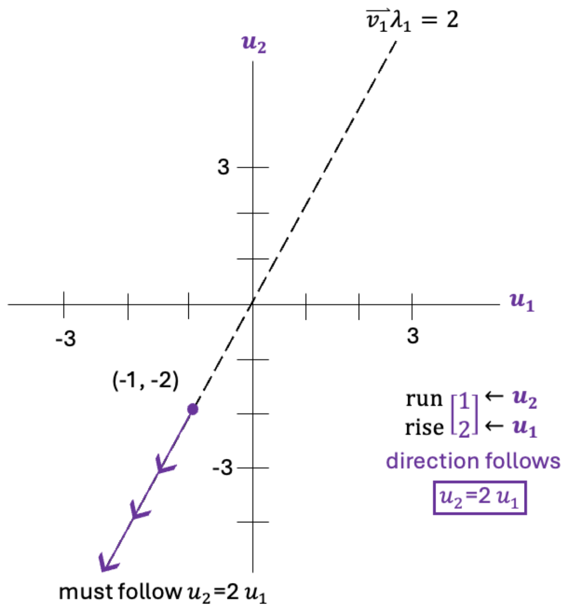
b) $\lambda_1 = 2$ (+=away from origin) $\lambda_2 = 3 = += $ (away from origin)

$\overrightarrow{v_1} = \begin{bmatrix}1\\2\end{bmatrix}$ $\overrightarrow{v_2} = \begin{bmatrix}1\\1\end{bmatrix}$

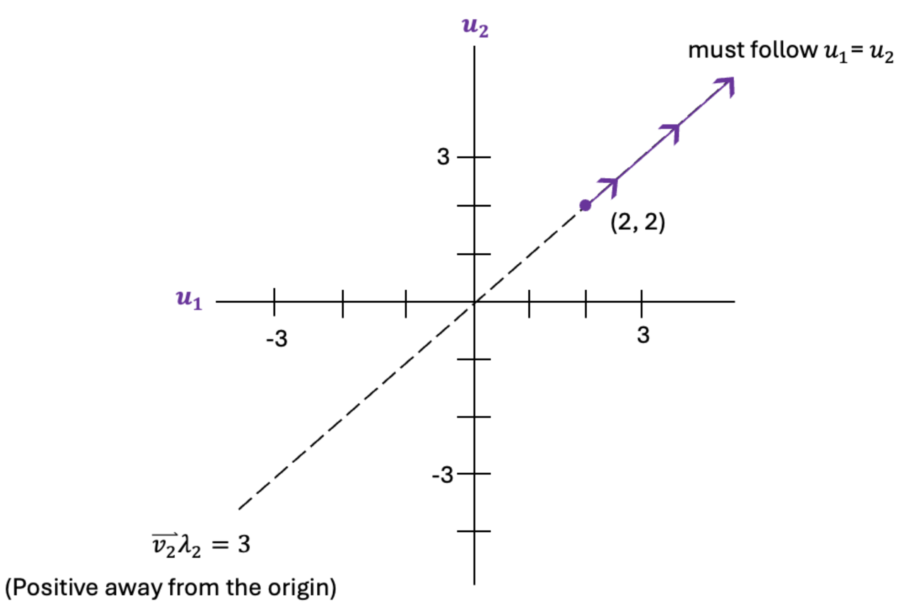


The trajectory of (0,3) can't cross $\overrightarrow{v_1} = \begin{bmatrix}1\\2\end{bmatrix}$
ie. the line $u_2 = 2u_1$
c)

$u_1(0) = -1$
$u_2(0) = -2$
$\vec{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$



d)
$u_1(0) = 2$
$u_2(0) = 2$
$\vec{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$



Practice Exam Questions

X1. Determine the solution to $\vec{u}'(t) = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}$ with initial condition

$\vec{u}(0) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$tr(A) = 4 + (-7) = -3$

$detA = ad - bc = 4(-7) - (-3)(6)$

$= -28 + 18 = -10$

$\lambda^2 - tr(A)\lambda + detA = 0$

$\lambda^2 + 3\lambda - 10 = 0$

$(\lambda + 5)(\lambda - 2) = 0$

$\lambda = -5, 2$

$$\begin{array}{c} \begin{matrix} \text{a} & \text{b} \end{matrix} \\ \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \\ \begin{matrix} \text{c} & \text{d} \end{matrix} \end{array}$$

$$v_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -3 \\ -5 - 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\overrightarrow{v_2} = \begin{bmatrix} -3 \\ 2 - 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The solution is

$$\vec{u}(t) = c_1 e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

t=0

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 e^0 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$3 = c_1 + 3c_2 \quad \boxed{1} \quad (x - 3)$

$-5 = 3c_1 + 2c_2 \quad \boxed{2}$

$-9 = -3c_1 - 9c_2$

$-5 = 3c_1 + 2c_2$

$Add \quad -14 = -7c_2$

$c_2 = 2 \quad into \ \boxed{1}$

$3 = c_1 + 3(2)$

$c_1 = 3 - 6 = -3$

$$\vec{u}(t) = -3e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The eigenvalues are real, one positive and one negative, so it is a saddle point.

X2. Determine the solution to $\vec{u}'(t) = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}$ with initial condition

$\vec{u}(0) = \begin{bmatrix} 6 \\ -8 \end{bmatrix}$

$tr(A) = 5 + (-2) = 3$

$detA = 5(-2) - (-3)(2) = -10 + 6 = -4$

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0$

$\lambda = 4, -1$

$$\begin{array}{cc} a & b \\ \multicolumn{2}{c}{\begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}} \\ c & d \end{array}$$

$$\overrightarrow{v_1} = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -3 \\ 4 - 5 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -3 \\ -1 - 5 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The solution is $u(t) = c_1 e^{4t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$t = 0 \quad \begin{bmatrix} 6 \\ -8 \end{bmatrix} = c_1 e^0 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$6 = 3c_1 + c_2 \quad \boxed{1} \quad \times -2$

$-8 = c_1 + 2c_2 \quad \boxed{2}$

$-12 = -6c_1 - 2c_2$

$-8 = c_1 + 2c_2$

$ADD \quad -20 = -5c_1$

$c_1 = 4$

Sub into $\boxed{1}$ $\quad 6 = 3(4) + c_2$

$c_2 = 6 - 12 = -6$

$\vec{u}(t) = 4e^{4t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} - 6e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

X3. Determine the solution to $\vec{u}'(t) = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$ with initial condition $\vec{u}(0) = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$

$$tr(A) = 2 + (-1) = 1$$
$$detA = 2(-1) - 2(5) = -2 - 10 = -12$$
$$\lambda^2 - \lambda - 12 = 0$$
$$(\lambda - 4)(\lambda + 3) = 0$$
$$\lambda = 4, \qquad \lambda = -3$$

$$\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix} \\ \begin{array}{cc} c & d \end{array} \end{array}$$

$$v_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ 4 - 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ -3 - 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

The solution is $\vec{u}(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

$$t = 0 \quad \begin{bmatrix} 10 \\ 3 \end{bmatrix} = c_1 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$10 = c_1 + 2c_2 \quad \boxed{1}$$
$$3 = c_1 - 5c_2 \quad \boxed{2}$$
$$SUBTRACT \quad 7 = 7c_2$$
$$c_2 = 1$$
$$sub\ c_2 = 1 \quad into \quad \boxed{1}$$
$$10 = c_1 + 2(1)$$
$$c_1 = 8$$
$$\vec{u}(t) = 8e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

X4. Determine the solution to $\vec{u}'(t) = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ with initial condition $\vec{u}(0) = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

$$tr(A) = 1 + 4 = 5$$
$$detA = 1(4) - 1(-2) = 4 + 2 = 6$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 2)(\lambda - 3) = 0$$
$$\lambda = 2\,, \qquad \lambda = 3$$

$$\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \\ \begin{array}{cc} c & d \end{array} \end{array}$$

$$\vec{v}_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 1 \\ 2 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The solution is $\vec{u}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$t = 0 \quad \begin{bmatrix} 4 \\ 7 \end{bmatrix} = c_1 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$4 = c_1 + c_2 \quad \boxed{1}$$
$$7 = c_1 + 2c_2 \quad \boxed{2}$$
$$Subtract \quad -3 = -c_2$$
$$c_2 = 3$$

Sub into $\boxed{1}$ $\quad 4 = c_1 + 3$

$$c_1 = 1$$

The solution is:

$$\vec{u}(t) = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

X5. Solve $\vec{u}'(t) = \begin{bmatrix} 2 & 5 \\ -2 & 4 \end{bmatrix} \vec{u}(t)\ with\ \vec{u}(0) = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

$$tr(A) = 2 + 4 = 6$$
$$detA = 2(4) - 5(-2) = 8 + 10 = 18$$
$$\lambda^2 - 6\lambda + 18 = 0$$
$$\lambda = \frac{-b \pm \sqrt{6^2 - 4ac}}{2a}$$
$$= \frac{6 \pm \sqrt{36 - 4(1)(18)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{-36}}{2}$$
$$= \frac{6 \pm 6i}{2}$$
$$= 3 \pm 3i$$

$a = 3 > 0 \quad \therefore$spiral away from origin

$$\begin{array}{cc} a & b \\ \multicolumn{2}{c}{\begin{bmatrix} 2 & 5 \\ -2 & 4 \end{bmatrix}} \\ c & d \end{array}$$

$\lambda_1 = 3 + 3i$ $\qquad\qquad$ $\lambda_2 = 3 - 3i$

$a = 3$

$b = 3$

$$\vec{v}_1 = \begin{bmatrix} \lambda - d \\ c \end{bmatrix} = \begin{bmatrix} 3 + 3i - 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 + 3i \\ -2 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} -1 + 3i \\ -2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ -2 \end{bmatrix}}_{\vec{x}} + i \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\vec{y}}$$

$$\vec{u}(t) = c_1 e^{at}(\cos(bt)\vec{x} - \sin(bt)\vec{y}) + c_2 e^{at}(\sin(bt)\vec{y}) + c_2 e^{at}(\sin(bt)\vec{x} + \cos(bt)\vec{y})$$

$$\therefore \vec{u}(t) = c_1 e^{3t}\left(\cos(3t)\begin{bmatrix} -1 \\ -2 \end{bmatrix} - \sin(3t)\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right) + c_2 e^{3t}(\sin(3t)\begin{bmatrix} -1 \\ -2 \end{bmatrix} + \cos(3t)\begin{bmatrix} 3 \\ 0 \end{bmatrix})$$

Substitute $\vec{u}(0) = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

$t = 0$

$$\begin{bmatrix} 7 \\ 4 \end{bmatrix} = c_1 e^0 \left[\cos 0 \begin{bmatrix} -1 \\ -2 \end{bmatrix} - \sin(0)\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right] + c_2 e^0 (\sin 0 \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \cos 0 \begin{bmatrix} 3 \\ 0 \end{bmatrix})$$

$$\begin{bmatrix} 8 \\ 4 \end{bmatrix} = c_1\left(\begin{bmatrix} -1 \\ -2 \end{bmatrix} - 0\right) + c_2(0 + \begin{bmatrix} 3 \\ 0 \end{bmatrix})$$

$$\begin{bmatrix} 8 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$8 = -c_1 + 3c_2 \quad \boxed{1}$

$4 = -2c_1 + 0 \quad \boxed{2}$

$c_1 = -2$ $\quad$ sub into $\boxed{1}$

$8 = -(-2) + 3c_2$

$8 - 2 = 3c_2$

$3c_2 = 6$

$c_2 = 2$

$$\vec{u}(t) = -2e^{3t}(\cos(3t)\begin{bmatrix} -1 \\ -2 \end{bmatrix} - \sin(3t)\begin{bmatrix} 3 \\ 0 \end{bmatrix}) + 2e^{3t}(\sin(3t)\begin{bmatrix} -1 \\ -2 \end{bmatrix} + \cos(3t)\begin{bmatrix} 3 \\ 0 \end{bmatrix})$$

A= $\begin{bmatrix} 2 & 5 \\ -2 & 4 \end{bmatrix}$

Since in the complex eigenvalues, the real part a=3>0, it is a spiral away from the origin.
Since the value of $a_{21} = -2 < 0$, it goes in a clockwise motion.

Long Method

$\lambda = 3 + 3i$

$A = \begin{bmatrix} 2 & 5 \\ -2 & 4 \end{bmatrix}$

$(A - \lambda I)\vec{v} = \vec{0}$

$\left(\begin{bmatrix} 2 & 5 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 3 + 3i$

$\begin{bmatrix} 2 - (3 + 3i) & 5 \\ -2 & 4 - (3 + 3i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 - 3i & 5 \\ -2 & 1 - 3i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2v_1 + v_2 - 3iv_2 = 0$

$-2v_1 = -v_2 + 3iv_2$

$v_1 = \dfrac{v_2 - 3iv_2}{2}$

Let $v_2 = 2$

$v_1 = \dfrac{2 - 3i(2)}{2}$

$v_1 = 1 - 3i$

$\therefore \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 - 3i \\ 2 \end{bmatrix}$ or $\times -1$

Or $\begin{bmatrix} -1 + 3i \\ -2 \end{bmatrix}$

X6. Solve $\overrightarrow{u'}(t) = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix} \vec{u}(t)$ $with\ \vec{u}(0) = \begin{bmatrix} 18 \\ 10 \end{bmatrix}$

$tr(A) = 3 + 1 = 4$

$detA = 3 + 13(5) = 68$

$\lambda^2 - 4\lambda + 68 = 0$

$\lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(68)}}{2(1)}$

$= \frac{4 \pm \sqrt{-256}}{2}$

$= \frac{4 \pm 16i}{2}$

$\lambda = 2 \pm 8i$

$$\begin{array}{cc} a & b \\ \multicolumn{2}{c}{\begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix}} \\ \text{c} & \text{d} \end{array}$$

$\lambda_1 = 2 + 8i$ $\qquad \lambda_2 = 2 - 8i$

$a = 2 \quad b = 8$

$a = 2 > 0$ spiral away from origin

$\vec{v}_1 = \begin{bmatrix} \lambda - d \\ c \end{bmatrix} = \begin{bmatrix} 2 + 8i - 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 + 8i \\ 5 \end{bmatrix}$

$= \underbrace{\begin{bmatrix} 1 \\ 5 \end{bmatrix}}_{\vec{x}} + i \underbrace{\begin{bmatrix} 8 \\ 0 \end{bmatrix}}_{\vec{y}}$

$\vec{u}(t) = c_1 e^{at}(\cos(bt)\vec{x} - \sin(bt)\vec{y}) + c_2 e^{at}(\sin(bt)\vec{x} + \cos(bt)\vec{y})$

$\vec{u}(t) = c_1 e^{2t}(\cos(8t) \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \sin(8t) \begin{bmatrix} 8 \\ 0 \end{bmatrix}) + c_2 e^{2t}(\sin(8t) \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \cos(8t) \begin{bmatrix} 8 \\ 0 \end{bmatrix})$

Substitute $\vec{u}(0) = \begin{bmatrix} 18 \\ 10 \end{bmatrix}$

$t = 0$

$\begin{bmatrix} 18 \\ 10 \end{bmatrix} = c_1 e^0(\cos 0 \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \sin 0 \begin{bmatrix} 8 \\ 0 \end{bmatrix}) + c_2 e^0(\sin 0 + \cos 0 \begin{bmatrix} 8 \\ 0 \end{bmatrix})$

$\begin{bmatrix} 18 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 8 \\ 0 \end{bmatrix}$

$18 = c_1 + 8c_2 \quad \boxed{1}$

$10 = 5c_1 \quad \boxed{c_1 = 2}$ $\quad sub\ into\ \boxed{1}$

$18 = 2 + 8c_2$

$\therefore 16 = 8c_2 \quad \boxed{c_2 = 2}$

$\vec{u}(t) = 2e^{2t}(\cos(8t) \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \sin(8t) \begin{bmatrix} 8 \\ 0 \end{bmatrix}) + 2e^{2t}(\sin(8t) \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \cos(8t) \begin{bmatrix} 8 \\ 0 \end{bmatrix})$

Does the spiral move in a clockwise or counter clockwise direction?

$A = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix}$ **$a = 2 > 0$ spiral away from origin. And, $a_{21} = 5 > 0$, so it is moving counterclockwise.**

Quiz 6: Practice on Sections H to L

1. $A\vec{x} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 6-8 \\ 18+16 \end{bmatrix} = \begin{bmatrix} -2 \\ 34 \end{bmatrix}$

   First element is = -2
   Second element is =34

2. $M\vec{y} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 2 & 4 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0(2)-1(3)+2(4) \\ 1(2)+2(3)+4(4) \\ 1(2)+(-2)(3)+3(4) \end{bmatrix}$

   $M\vec{y} = \begin{bmatrix} 0-3+8 \\ 2+6+16 \\ 2-6+12 \end{bmatrix} = \begin{bmatrix} 5 \\ 24 \\ 8 \end{bmatrix}$

   The first element is 5.

3. $A = \begin{bmatrix} 2 & 2 \\ 0 & 5 \end{bmatrix} \quad trA = 7 \qquad detA = 10$

   $\lambda^2 - 7\lambda + 10 = 0$

   $(\lambda - 2)(\lambda - 5) = 0$

   $\lambda = 2 \qquad\qquad \lambda = 5$

   $v_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} \qquad\qquad v_2 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix}$

   $= \begin{bmatrix} 2 \\ 2-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 2 \\ 5-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\overrightarrow{x_t} = c_1 v_1 {\lambda_1}^t + c_2 v_2 {\lambda_2}^t$$

Let t=0

$$\overrightarrow{x_t} = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}(2)^t + c_2\begin{bmatrix} 2 \\ 3 \end{bmatrix}(5)^t$$

$$\begin{bmatrix} 8 \\ 6 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}(2)^0 + c_2\begin{bmatrix} 2 \\ 3 \end{bmatrix}(5)^0$$

$$8 = c_1 + 2c_2$$

$$\underline{6 = 3c_2}$$

$$6 = 3c_2$$

$$c_2 = 2$$

$8 = c_1 + 2(2)$

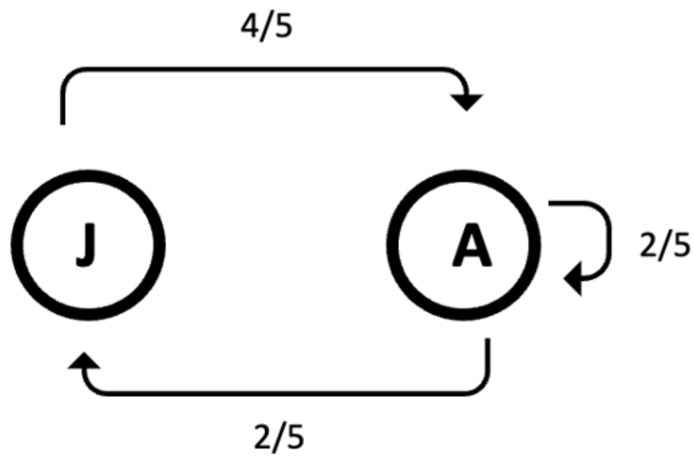
$8 = 4 + c_1$

$c_1 = 4$

$\overrightarrow{x_t} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix}(2)^t + 2\begin{bmatrix} 2 \\ 3 \end{bmatrix}(5)^t$

$\therefore x(t) = 4(2)^t + 4(5)^t$

$y(t) = 6(5)^t$



4.

$$\begin{array}{c|cc} & J & A \\ \hline J & 0 & 2/5 \\ A & 4/5 & 2/5 \end{array} \quad \begin{bmatrix} 0 & 2/5 \\ 4/5 & 2/5 \end{bmatrix}$$

$trA = \frac{2}{5}$ $\quad detA = 0 - \frac{8}{25} = -\frac{8}{25}$

$\lambda^2 - \frac{2}{5} - \frac{8}{25} = 0$

$\left(\lambda + \frac{2}{5}\lambda\right)\left(\lambda - \frac{4}{5}\right) = 0$

$\lambda = -\frac{2}{5}, \qquad \lambda = \frac{4}{5}$

$\overrightarrow{v_1} = \begin{bmatrix} \frac{2}{5} \\ -\frac{2}{5} - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\overrightarrow{v_2} = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} - 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x(\vec{t}) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} (-\frac{2}{5})^t + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} (\frac{4}{5})^t$

At $t = 0$ $\quad \begin{bmatrix} 4 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$4 = c_1 + c_2$

$\underline{7 = -c_1 + 2c_2}$

$11 = 3c_2$

$c_2 = \frac{11}{3}$

$4 = c_1 + \frac{11}{3}$

$c_1 = \frac{12}{3} - \frac{11}{3} = \frac{1}{3}$

$\therefore \vec{x}(t) = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} (-\frac{2}{5})^t + \frac{11}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} (\frac{4}{5})^t$

a) $Juvenile\ x1(t) = \frac{1}{3}(1)(-\frac{2}{5})^t + \frac{11}{3}(1)(\frac{4}{5})^t = \frac{1}{3}(-\frac{2}{5})^t + \frac{11}{3}(\frac{4}{5})^t$

b) $Adults\ x2(t) = \frac{1}{3}(-1)(-\frac{2}{5})^t + \frac{11}{3}(2)(\frac{4}{5})^t = \frac{-1}{3}(-\frac{2}{5})^t + \frac{22}{3}(\frac{4}{5})^t$

c) the population is shrinking, since as t approaches infinity, both (-2/5)^t and (4/5)^t will approach 0. NOTE: If one of the fractions was (8/5)^t or 2^t, etc. then as x approaches infinity, that term would approach infinity, so it would be growing over time

5. $a) \lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0$

$\lambda = 4,$ Positive $\therefore$moves away from $(0,0) \therefore \overrightarrow{v_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\lambda = -1$ negative goes toward $(0,0)$ $\therefore \overrightarrow{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

$$4 = 2c_1 - c_2$$
$$6 = 3c_1 + c_2$$
$$Add \quad 10 = 5c_1$$
$$c_1 = 2$$

$$4 = 2(2) - c_2$$
$$4 = 4 - c_2$$
$$-c_2 = 0$$
$$c_2 = 0$$

$$\therefore \begin{bmatrix} u_1(t) \\ u_1(t) \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t}$$
$$u_1(t) = 4e^{4t}$$
$$u_2(t) = 6e^{4t}$$

b) $\begin{bmatrix} -4 \\ 16 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$

$$-4 = 2c_1 - c_2$$
$$16 = 3c_1 + c_2$$
$$Add \quad 12 = 5c_1$$
$$c_1 = \frac{12}{5}$$
$$-4 = 2\left(\frac{12}{5}\right) - c_2$$
$$-4 = \frac{24}{5} - c_2$$
$$\frac{-20}{5} - \frac{24}{5} = -c_2$$
$$-\frac{44}{5} = -c_2$$
$$c_2 = \frac{44}{5}$$
$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \frac{12}{5} \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t} + \frac{44}{5} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$
$$u_1(t) = \frac{24}{5} e^{4t} - \frac{44}{5} e^{-t}$$
$$u_2(t) = \frac{36}{5} e^{4t} + \frac{44}{5} e^{-t}$$

6. $A = \begin{bmatrix} a & b \\ -3 & -6 \\ 5 & 8 \end{bmatrix}$

$\quad\ c \quad d$

$$tr(A) = -3 + 8 = 5$$

$$\begin{aligned}\det(A) &= ad - bc \\ &= -3(8) + 6(5) \\ &= -24 + 30 \\ &= 6\end{aligned}$$

$\lambda^2 - tr(A)\lambda + \det(A) = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\lambda = 2, \quad \lambda = 3$

$\overrightarrow{v_1} = \begin{bmatrix} -6 \\ 2-(-1) \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \end{bmatrix} \qquad \overrightarrow{v_2} = \begin{bmatrix} -6 \\ 3+3 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = c_1 \begin{bmatrix} -6 \\ 5 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t}$

$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} -6 \\ 5 \end{bmatrix} e^{0} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{0}$

$3 = -6c_1 - c_2$

$2 = 5c_1 + c_2$

$Add \quad 5 = -c_1$

$c_1 = -5$

$2 = 5(-5) + c_2$

$2 + 25 = c_2$

$c_2 = 27$

$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = -5 \begin{bmatrix} -6 \\ 5 \end{bmatrix} e^{2t} + 27 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t}$

$u(t) = 30e^{2t} - 27e^{3t}$

$v(t) = -25e^{2t} + 27e^{3t}$

Y. Predator and Prey Equations

Example 1. $-r + wr = 0$

$-r(l - w) = 0$

$r = 0, \ \ w = 1 \qquad sub\ into\ \boxed{2}$

$r = 0 \quad w - wr = 0 \qquad \therefore \binom{0{,}0}{r, w}\ is\ an\ equilibrium$

$w - 0 = 0$

$w = 0$

$w = 1 \quad w - wr = 0$

$1 - 1r = 0$

$1 = r$

$\therefore \binom{1{,}1}{r, w}$ is and equilibrium

NOTE: I tested points in different regions for practice!

Test point $(3{,}2)$ $\ w = 2 \quad r = 3$

$\frac{dr}{dt} = -r + wr = -3 + 2(3)$

$= 3 > 0 \quad \therefore right$

$\frac{dw}{dt} = w + wr \ = 2 - 2(3)$

$= -4 < 0 \quad \therefore down$

Test point $\binom{2, -1}{r, w}$

$\frac{dr}{dt} = -r + wr \ = -2 + (-1)(2)$

$= -4 < 0 \quad \therefore left$

$\frac{dw}{dt} = w - wr = -1 - (-1)(2) = -1 + 2$

$= 1 > 0 \quad \therefore up$

Test point $\binom{-1{,}2}{r. w}$

$\frac{dr}{dt} = -r + wr \ = 1 + 2(-1) = -1 < 0 \quad \therefore left$

$\frac{dw}{dt} = w - wr = 2 - 2(-1) = 4 > 0 \quad \therefore up$

Test point $\binom{-1, -1}{r,\ w}$

$\frac{dr}{dt} = -r + wr = 1 + (-1)(-1) = 2 > 0 \quad \therefore right$

$\frac{dw}{dt} = w - wr = -1 - (-1)(-1) = -1 - 1 = -2 < 0 \ \therefore down$

Test point $\binom{2, \frac{1}{2}}{r, w}$

$\frac{dr}{dt} = -r + wr = -2 + \frac{1}{2}(2) = -1 < 0 \quad \therefore left$

$\frac{dw}{dt} = w - wr = \frac{1}{2} - \frac{1}{2}(2) = \frac{1}{2} - 1 = -\frac{1}{2} < 0 \quad \therefore down$

Test point $\underset{r,w}{\left(\frac{1}{2},\frac{1}{2}\right)}$

$$\frac{dr}{dt} = -r + wr = -\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4} < 0 \quad \therefore left$$

$$\frac{dw}{dt} = w - wr = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} > 0 \ \therefore up$$

Test point $\underset{r,w}{\left(\frac{1}{2},2\right)}$

$$\frac{dr}{dt} = -r + wr = -\frac{1}{2} + 2\left(\frac{1}{2}\right) = -\frac{1}{2} + 1 = \frac{1}{2} > 0 \quad \therefore right$$

$$\frac{dw}{dt} = w - wr = 2 - 2\left(\frac{1}{2}\right) = 2 - 1 = 1 > 0 \ \therefore up$$

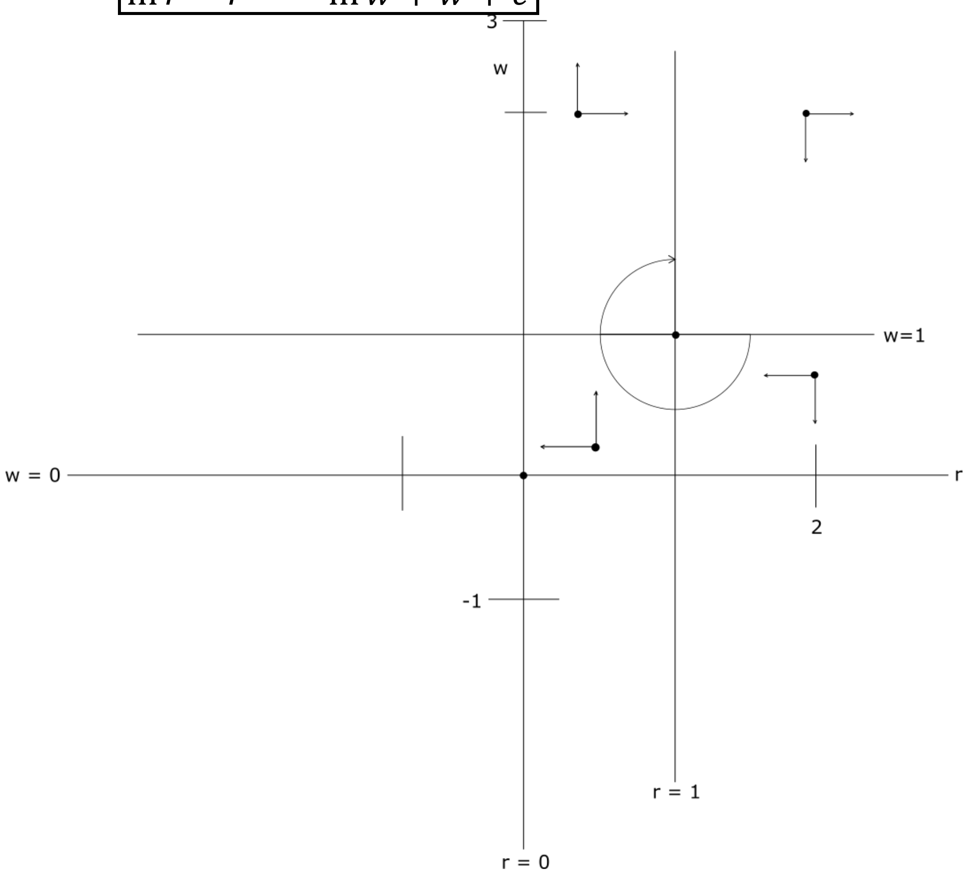
Integrate to find the equation

$$\frac{dr}{dw} = \frac{-r+wr}{w-wr} = \frac{r(-1+w)}{w(1-r)}$$

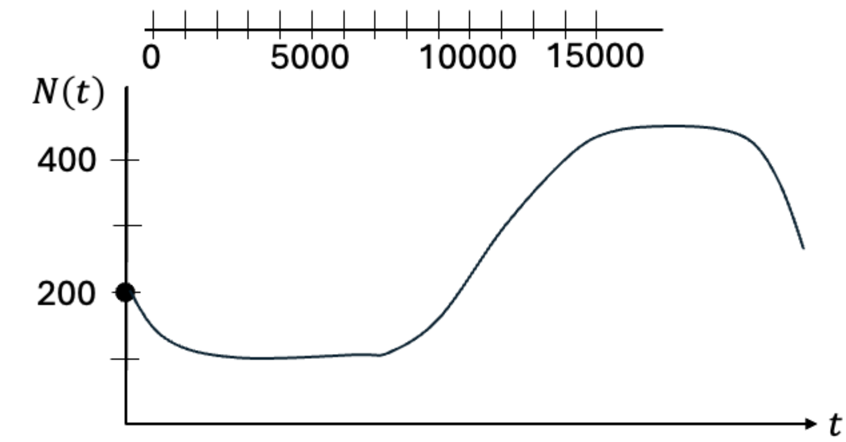
$$\int \frac{(1-r)dr}{r} = \int \left(\frac{-1+w}{w}\right) dw$$

$$\int \left(\frac{1}{r} - 1\right) dr = \int \left(\frac{-1}{w} + 1\right) dw$$

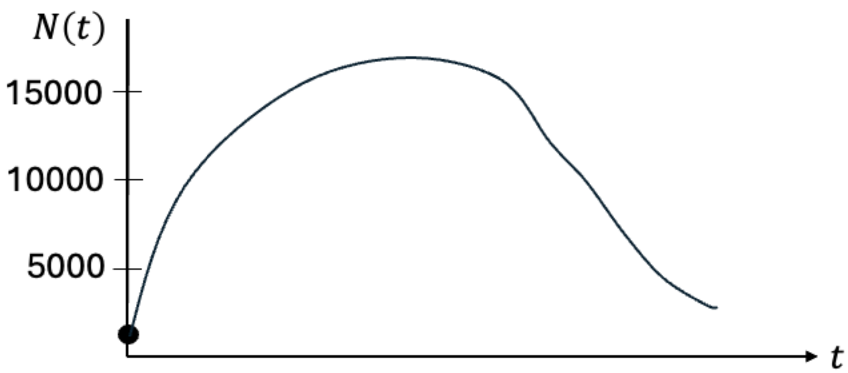
$$\boxed{\ln r - r = -\ln w + w + c}$$



Example 2.



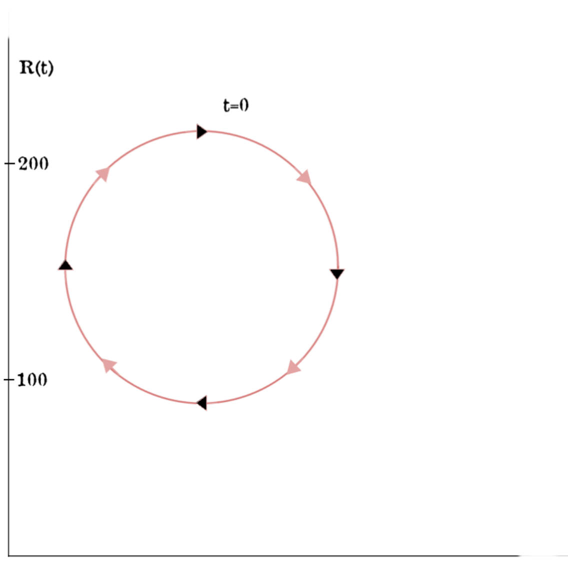
Ladybugs



Aphids

Practice Exam Questions on Predator and Prey Equations

Y1. See graphs



W(t)

Y2. $x' = x\underset{a-\alpha y}{\overset{prey}{\left(\quad \right)}}$ $\quad y' = y\underset{-b+Bx}{\overset{predator}{\left(\quad \right)}}$

a, b, α, B all positive constants

Find equation

$\boxed{1}$ $x' = 0 \quad x(a - \alpha y) = 0$

$x = 0 \quad a - \alpha y = 0$

$a = \alpha y$

$y = \frac{a}{\alpha}$

$\boxed{2}$ $y' = 0 \quad y(-b + Bx) = 0$

$sub\ x = 0 \quad y(-b + 0) = 0$

$-by = 0$

$y = 0$

(0,0) is an equilibrium

$sub\ y = \dfrac{a}{\alpha}\ into\ \boxed{2}$

$\frac{a}{\alpha}(-b + Bx) = 0$

$\frac{-ab}{\alpha} + \frac{aBx}{\alpha} = 0$

$\frac{aBx}{\alpha} = \frac{ab}{\alpha}$

$Bx = b$

$x = \frac{b}{B} \qquad \therefore \left(\frac{b}{B}, \frac{a}{\alpha}\right)$ is another equilibrium

Test point $\left(\frac{2b}{B}, \frac{2a}{\alpha}\right)$

x, y

$x' = x(a - \alpha y) = \frac{2b}{B}\left(a - \alpha\left(\frac{2a}{\alpha}\right)\right)$

$= \frac{2ba}{B} - \frac{4ab\ \alpha}{B\alpha} = \frac{-2ab}{B} \quad < 0 \quad \therefore left$

$y' = y(-b + Bx) = \frac{2a}{\alpha}\left(-b + B\left(\frac{2b}{B}\right)\right)$

$= \frac{-2ab}{\alpha} + \frac{4ab}{\alpha} = \frac{2ab}{\alpha} > 0 \quad \therefore up$

Test point $\left(\frac{b}{2B}, \frac{a}{2\alpha}\right)$

$x = \frac{b}{2B} \qquad y = \frac{a}{2\alpha}$

$x' = x(a - \alpha y)$

$= \frac{b}{2B}\left(a - \alpha\left(\frac{a}{2\alpha}\right)\right)$

$$= \frac{ab}{2B} - \frac{b\alpha a}{4B\alpha} \quad = \frac{2ab-ab}{4B} = \frac{ab}{4B} > 0 \quad \therefore right$$

$$y' = y(-b+Bx)$$
$$= \frac{a}{2\alpha}\left[-b + B(\frac{b}{2B})\right]$$
$$= \frac{-ab}{2\alpha} + \frac{abB}{4\alpha B}$$
$$= \frac{-2ab+ab}{4\alpha} = \frac{-ab}{4\alpha} < 0 \quad \therefore down$$

Test point $\left(\frac{b}{2B}, \frac{2a}{\alpha}\right)$
$x\ ,y$

$$x' = x(a-\alpha y)$$
$$= \frac{b}{2B}\left(a - \alpha\left(\frac{2a}{\alpha}\right)\right)$$
$$= \frac{ab}{2B} - \frac{2ab}{2B}$$
$$= \frac{-ab}{2B} < 0 \quad \therefore left$$

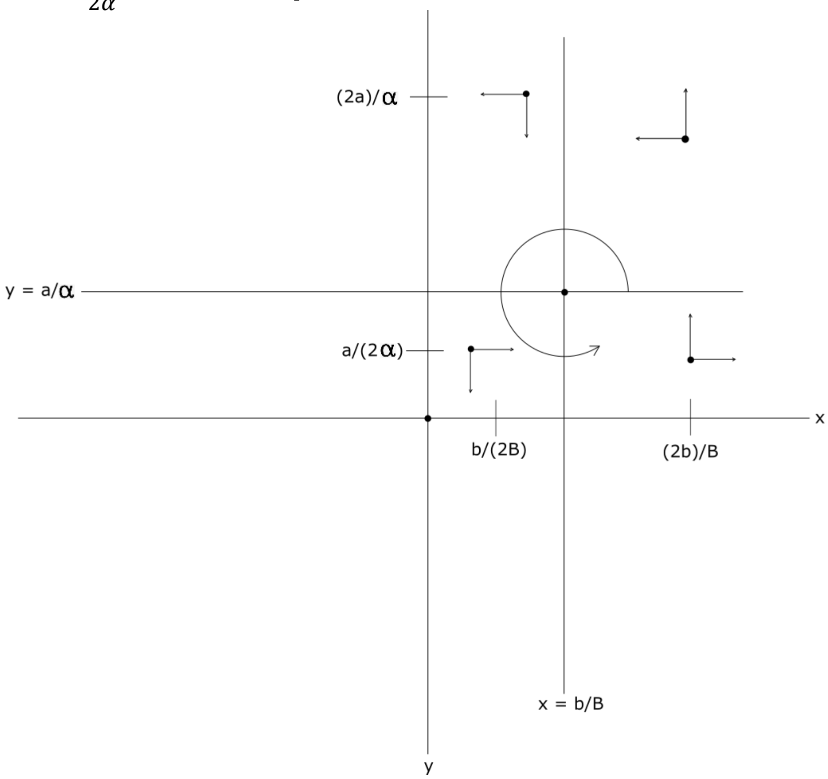
$$y' = y(-b+Bx)$$
$$= \frac{2a}{\alpha}\left[-b + B(\frac{b}{2B})\right]$$
$$= \frac{2a}{\alpha}\left[-b + \frac{b}{2}\right]$$
$$= \frac{-2ab}{\alpha} + \frac{2ab}{2\alpha}$$
$$= \frac{-4ab+2ab}{2\alpha}$$
$$= \frac{-2ab}{2\alpha}$$
$$= \frac{-ab}{\alpha} < 0 \quad \therefore down$$

Test point $\left(\frac{2b}{B}, \frac{a}{2\alpha}\right)$
x, y

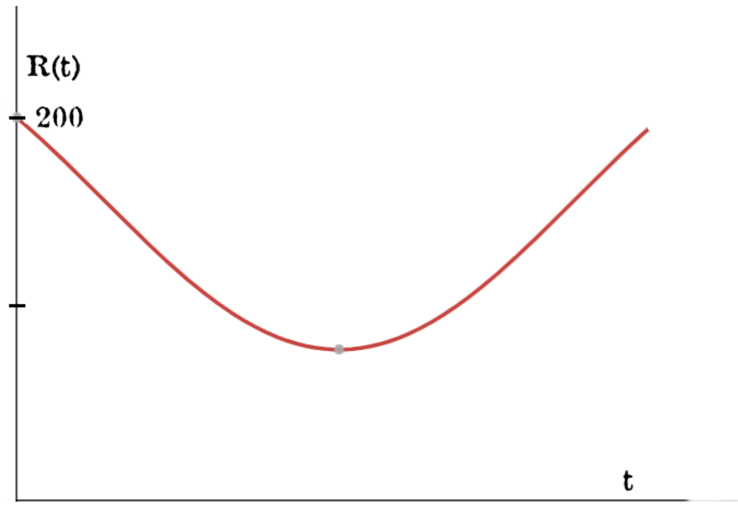
$$x' = x(a-\alpha y)$$
$$= \frac{2b}{B}\left(a - \alpha\left(\frac{a}{2\alpha}\right)\right)$$
$$= \frac{2ab}{B} - \frac{2ab}{2B}$$
$$= \frac{ab}{B} > 0 \quad \therefore right$$

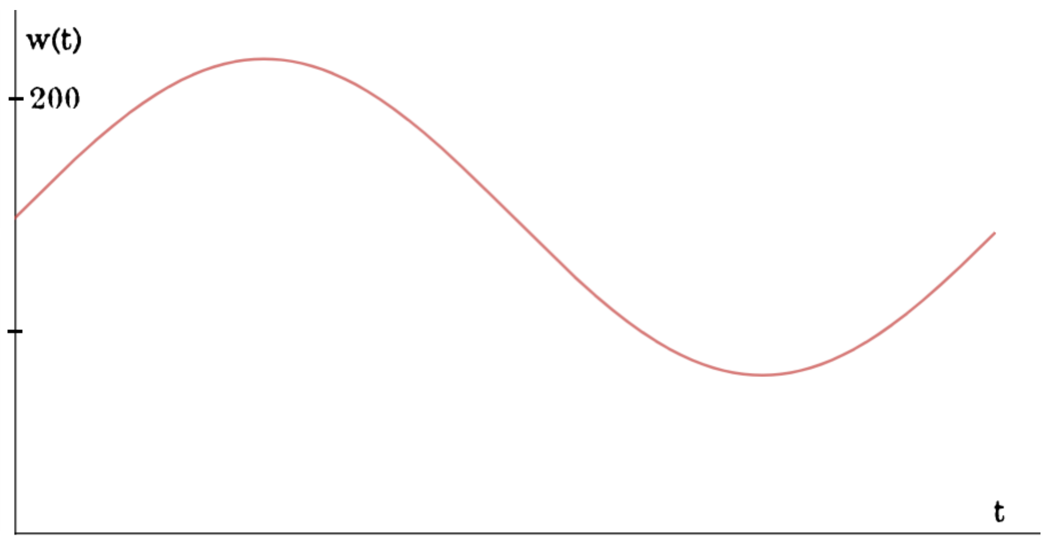
$$y' = y(-b+Bx)$$

$$= \frac{a}{2\alpha}\left[-b + B\left(\frac{2b}{B}\right)\right]$$
$$= \frac{-ab}{2\alpha} + \frac{2ab}{2\alpha}$$
$$= \frac{ab}{2\alpha} > 0 \qquad \therefore up$$



Y3.





Z. Phase Plane Analysis

Example 1. Lotka-Volterra Predator-Prey Equations

(R, W)

$rR - aRW = 0$ $\boxed{1}$ $\qquad - kW + bRW = 0$ $\boxed{2}$

$R(r - aW) = 0$

$\boxed{R = 0}$ $or\ \ r - aW = 0$

$r = aW \quad or$ $\boxed{W = \frac{r}{a}}$ R nullclines are R=0, W=r/a

From equation 2: sub in R=0

$-kW + bRW = 0$ $\boxed{2}$

-kW+0=0

$\boxed{W = 0}$ Equilibrium (0,0)

Sub $W = \frac{r}{a}$ into equation 2:

$-kW + bRW = 0$

$-k\left(\frac{r}{a}\right) + bR\left(\frac{r}{a}\right) = 0$

$-kr + bRr = 0$

$bRr = kr$

$R = \frac{kr}{br} = \boxed{\frac{k}{b}}$ W nullclines are W=0 and R=k/b

Equilibrium (k/b, r/a)

$\therefore equilibrium \quad \left(\frac{k}{b}, \frac{r}{a}\right) \quad and \quad (0,0)$

You can also just do R nullclines from the first equation and N nullclines from the second equation and graph the lines and see where they meet. Where an R Nullcline meets an N nullcline is an equilibrium. Make sure you don't use where an N nullcline meets another N nullcline as an equilibrium or where an R nullcline meets another R nullcline.

Test point $\begin{pmatrix} \frac{2k}{b}, \frac{2r}{a} \\ R,\ \ W \end{pmatrix}$

$\frac{dR}{dt} = rR - aRW$

$= r\left(\frac{2k}{b}\right) - a\left(\frac{2k}{b}\right)\left(\frac{2r}{a}\right)$

$= \frac{2rk}{b} - \frac{4rk}{b}$

$= \frac{-2rk}{b} < 0 \qquad \therefore left$

$\frac{dW}{dt} = -kW + bRW$

$= -k\left(\frac{2r}{a}\right) + b\left(\frac{2k}{b}\right)\left(\frac{2r}{a}\right)$

$$= \frac{-2kr}{a} + \frac{4kr}{a}$$
$$= \frac{2kr}{a} > 0 \qquad \therefore up$$

Test point $\begin{pmatrix} \frac{2k}{b}, \frac{r}{2a} \\ R, W \end{pmatrix}$

$$\frac{dR}{dt} = rR - aRW$$
$$= r\left(\frac{2k}{b}\right) - a\left(\frac{2k}{b}\right)\left(\frac{r}{2a}\right)$$
$$= \frac{2rk}{b} - \frac{rk}{b}$$
$$= \frac{rk}{b} > 0 \qquad \therefore right$$

$$\frac{dW}{dt} = -kw + bRW$$
$$= -k\left(\frac{r}{2a}\right) + b\left(\frac{2k}{b}\right)\left(\frac{r}{2a}\right)$$
$$= \frac{-kr}{2a} + \frac{kr}{a}$$
$$= \frac{-kr+2kr}{2a}$$
$$= \frac{kr}{2a} > 0 \qquad \therefore up$$

Test point $\begin{pmatrix} \frac{-2k}{b}, \frac{r}{2a} \\ R, W \end{pmatrix}$

$$\frac{dR}{dt} = rR - aRW$$
$$= r\left(\frac{-2k}{b}\right) - a\left(\frac{-2k}{b}\right)\left(\frac{r}{2a}\right)$$
$$= \frac{-2rk}{b} + \frac{rk}{b}$$
$$= \frac{-kr}{b} < 0 \qquad \therefore left$$

$$\frac{dW}{dt} = -kW + bRW$$
$$= -k(\frac{r}{2a}) + b\left(\frac{-2k}{b}\right)\left(\frac{r}{2a}\right)$$
$$= \frac{-kr}{2a} - \frac{kr}{a}$$
$$= \frac{-kr}{2a} - \frac{2kr}{2a}$$
$$= \frac{-3kr}{2a} < 0 \qquad \therefore down$$

Test point $\begin{pmatrix}\frac{-2k}{b}, \frac{2r}{a} \\ R, W\end{pmatrix}$

$$\frac{dR}{dt} = rR - aRW$$
$$= r\left(\frac{-2k}{b}\right) - a\left(\frac{-2k}{b}\right)\left(\frac{2r}{a}\right)$$
$$= \frac{-2rk}{b} + \frac{4rk}{b}$$
$$= \frac{2rk}{b} > 0 \qquad \therefore right$$
$$\frac{dW}{dt} = -kW + bRW$$
$$= -k\left(\frac{2r}{a}\right) + b\left(\frac{-2k}{b}\right)\left(\frac{2r}{a}\right)$$
$$= \frac{-2kr}{a} - \frac{4kr}{a}$$
$$= \frac{-6kr}{a} < 0 \qquad \therefore down$$

NOTE: I included extra test points for practice again!

Test point $\begin{pmatrix}\frac{k}{2b}, \frac{2r}{a} \\ R, W\end{pmatrix}$

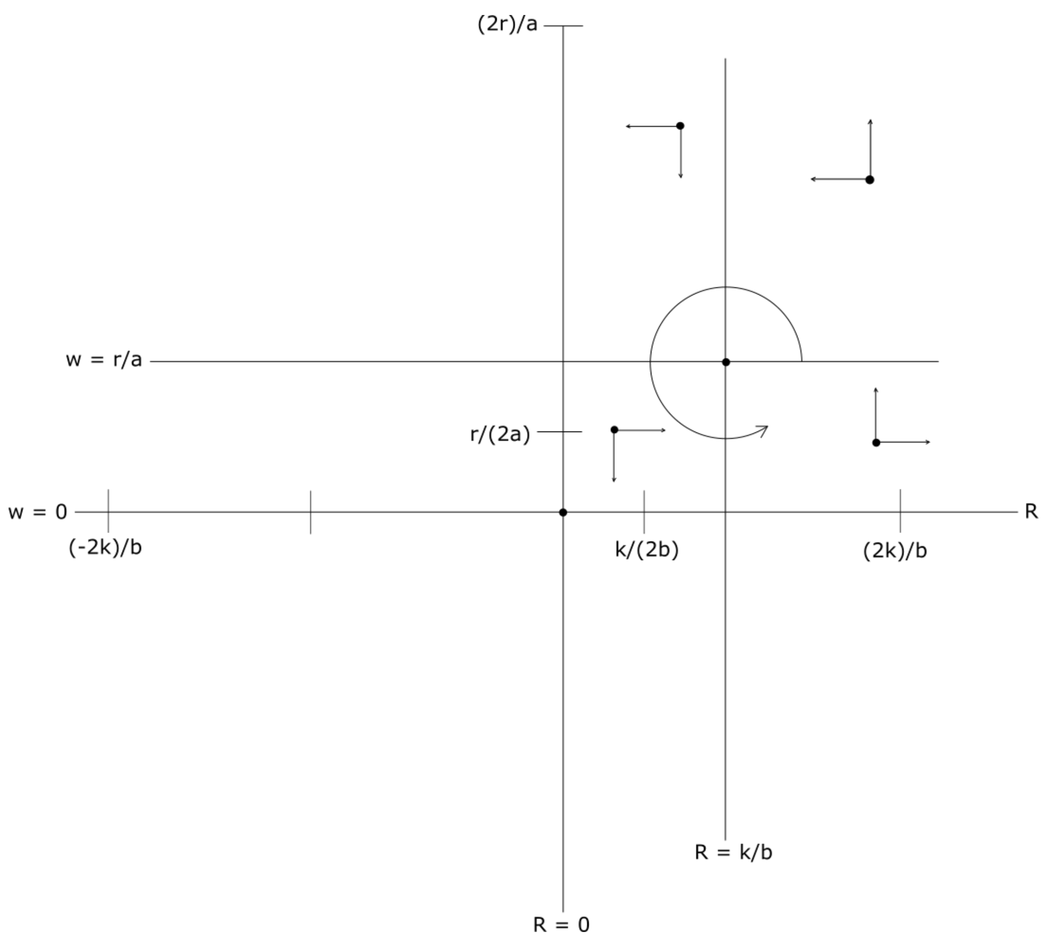
$$\frac{dR}{dt} = rR - aRW$$
$$= r\left(\frac{k}{2b}\right) - a\left(\frac{k}{2b}\right)\left(\frac{2r}{a}\right)$$
$$= \frac{kr}{2b} - \frac{kr}{b}$$
$$= \frac{kr-2kr}{2b}$$
$$= \frac{-kr}{2b} < 0 \qquad \therefore left$$

$$\frac{dW}{dt} = -kW + bRW$$
$$= -k\left(\frac{2r}{a}\right) + b\left(\frac{k}{2b}\right)\left(\frac{2r}{a}\right)$$
$$= \frac{-2kr}{a} + \frac{kr}{a}$$
$$= \frac{-kr}{a} < 0 \qquad \therefore down$$

Test point $\begin{pmatrix} \frac{k}{2b}, \frac{r}{2a} \\ R, W \end{pmatrix}$

$$\begin{aligned}
\frac{dR}{dt} &= rR - aRW \\
&= r\left(\frac{k}{2b}\right) - a\left(\frac{k}{2b}\right)\left(\frac{r}{2a}\right) \\
&= \frac{kr}{2b} - \frac{kr}{4b} \\
&= \frac{2kr - kr}{4b} \\
&= \frac{kr}{4b} > 0 \qquad \therefore right
\end{aligned}$$

$$\begin{aligned}
\frac{dW}{dt} &= -kW + bRW \\
&= -k\left(\frac{r}{2a}\right) + b\left(\frac{k}{2b}\right)\left(\frac{r}{2a}\right) \\
&= \frac{-kr}{2a} + \frac{kr}{4a} \\
&= \frac{-2kr + kr}{4a} \\
&= \frac{-kr}{4a} < 0 \qquad \therefore down
\end{aligned}$$



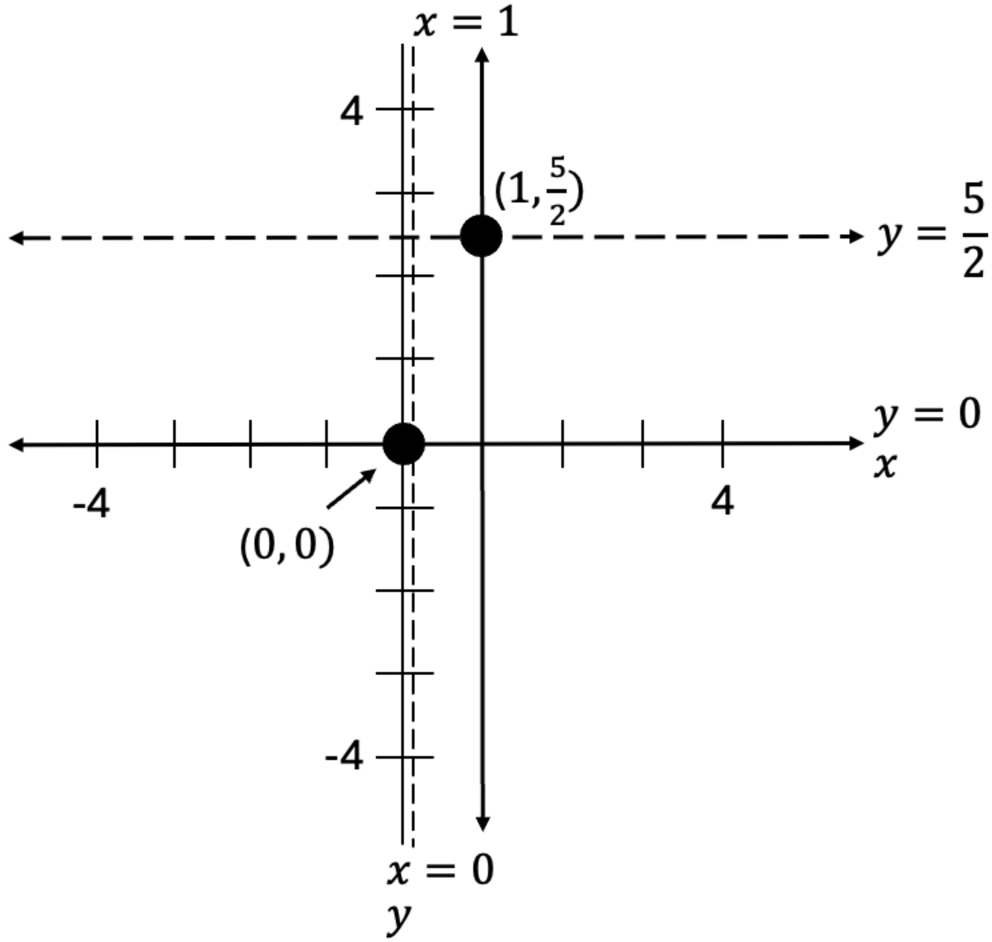
Example 2.

Set dx/dt=0 and we get the x nullclines:

$$5x - 2xy = 0$$
$$\text{Factor}\, x(5 - 2y) = 0$$
$$x = 0 \text{ or } y = \frac{5}{2}$$

Set dy/dt=0 and we get the y nullclines:

$$-y + xy = 0$$
$$y(-1 + x) = 0$$
$$y=0 \text{ or } x=1$$



Equilibrium at (0,0) and (1, 5/2)

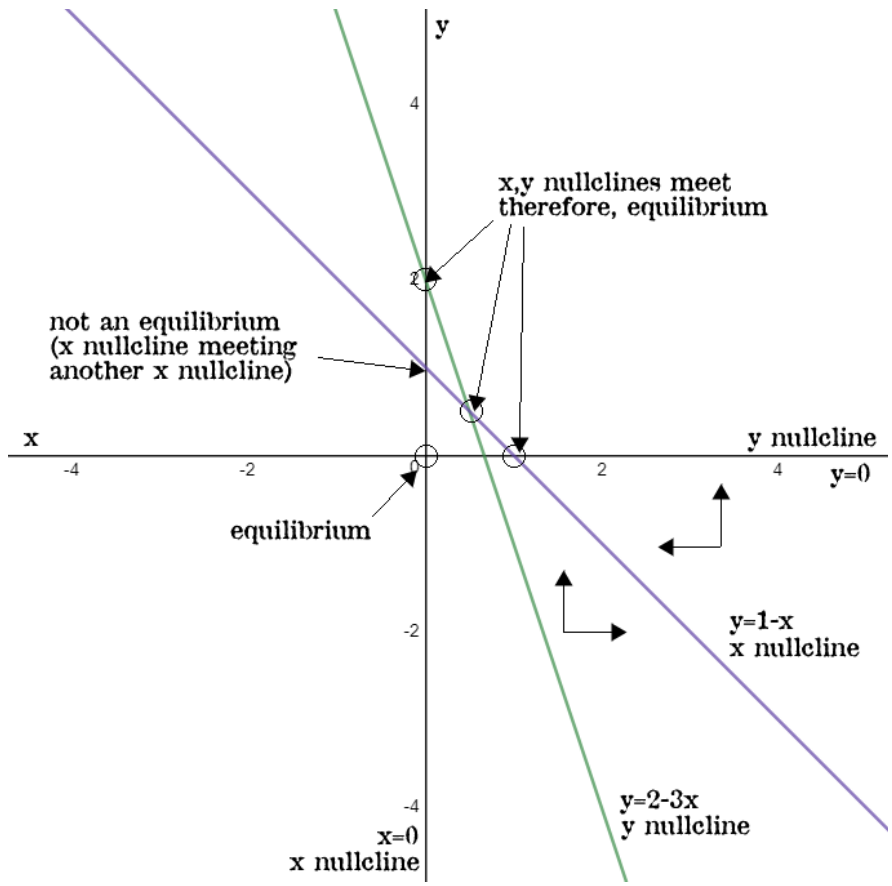
Practice Exam Questions on Phase Plots

Z1. $\underline{x - nullclines}$

$$\frac{dx}{dt} = 0$$
$$x(1-x) - xy = 0$$
$$x - x^2 - xy = 0$$
$$x(1 - x - y) = 0$$
$$x = 0 \quad or \quad 1 - x - y = 0$$
$$\therefore y = 1 - x \quad or \quad y - 1 = x$$

$\underline{y - nullclines}$

$$\frac{dy}{dt} = 0$$
$$2y\left(1 - \frac{y}{2}\right) - 3xy = 0$$
$$2y - y^2 - 3xy = 0$$
$$y(2 - y - 3x) = 0$$
$$y = 0 \quad or \quad 2 - y - 3x = 0$$
$$\therefore 2 - 3x = y \quad or \quad y = 2 - 3x$$



Equilibrium (0,2)

(0,0) (1/2, 1/2) and (1,0)

Test point (4,-1)

dx/dt=4(1-4)-4(-1)= -8<0 left
dy/dt= 2(-1)(1)=9>0 up
So, sketch the direction of motion on the graph

Z2. See graph on next page

$\underline{x - nullclines}$

$\frac{dx}{dt} = 0$

$2x\left(1 - \frac{x}{2}\right) - xy = 0$

$2x - x^2 - xy = 0$

$x(2 - x - y) = 0$

$x = 0 \quad or \quad 2 - x - y = 0$

$y = 2 - x$

$\underline{y - nullclines}$

$\frac{dy}{dt} = 0$

$0 = 3y\left(1 - \frac{y}{3}\right) - 2xy$

$0 = 3y - y^2 - 2xy$

$0 = y(3 - y - 2x)$

$y = 0 \quad or \quad 3 - y - 2x = 0$

$3 - 2x = y$

Test point (4,2)

$$\frac{dx}{dt} = 2(4)\left(1 - \frac{4}{2}\right) - 4(2) = -16 < 0$$

$$\frac{dy}{dt} = 3(2)\left(1 - \frac{2}{3}\right) - 2(4)(2) = 2 - 16 < 0$$

Test point (-1,1)

$$\frac{dx}{dt} = 2(-1)\left(1 + \frac{1}{2}\right) - (-1)(1) = -2\left(\frac{3}{2}\right) + 1 = -2 < 0$$

$$\frac{dy}{dt} = 3(1)\left(1 - \frac{1}{3}\right) - 2(-1)(1) = 3\left(\frac{2}{3}\right) + 2 = 4 > 0$$

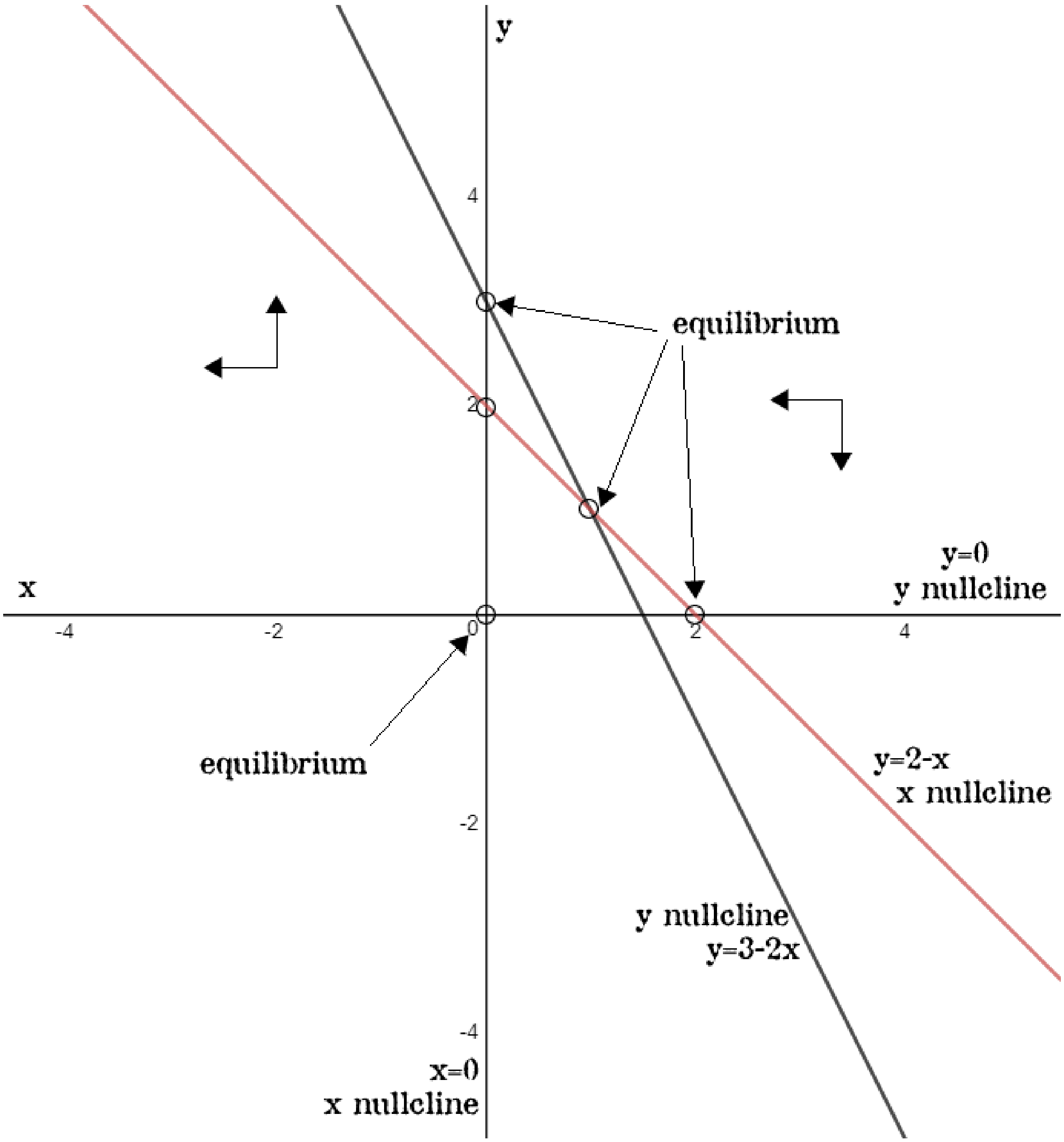
Equilibrium algebraically:

Sub. $x = 0 \quad into\ y = 3 - 2x\ and\ get\ y = 3$. So, (0,3) is an equilibrium.

Sub. y=0 into y=2-x and get x=2. So, (2,0) is an equilibrium

Set y=2-x equal to y=3-2x and get 2-x=3-2x and 2x-x=3-2 and x=1 and subst. x=1 into y=3-2x and get y=3-2(1)=1 and the other equilibrium is (1,1).

We also have an equilibrium at (0,0) from the two equations.



You can also get the equilibrium by graphing the equations you get when you set dx/dt= 0 and dy/dt=0, ie. called the x and y nullclines.
Equilibrium at (0,0) (2,0) (0,3) and (1,1).

Z3. x nullclines, set dx/dt=0

$$x(-x-y+1)=0$$
$$x=0\ and\ y=-x+1$$

Z4. x nullclines, set dx/dt=0

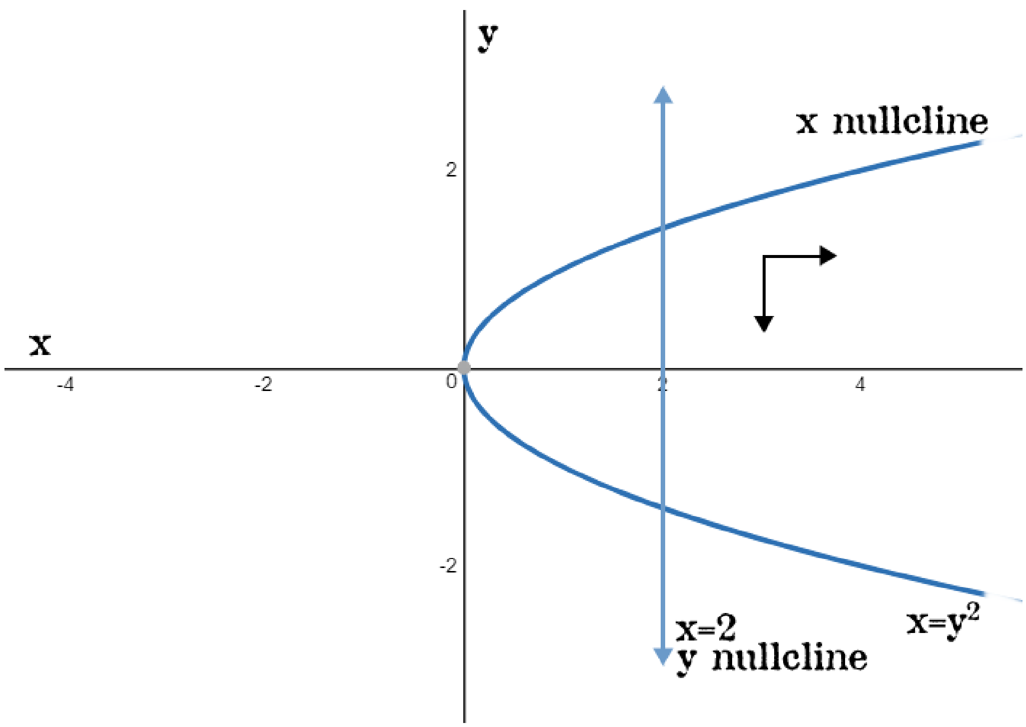
$$x-y^2=0$$
$$x=y^2$$

y null clines, set dy/dt=0

$$2-x=0\ or\ x=2$$

Equilibrium:
Substitute x=2 into $x=y^2$ and get $x=\pm\sqrt{2}$
So, the equilibrium are: $(2,\sqrt{2})\ and\ (2,-\sqrt{2})$

At test point (3,1)
dx/dt= 3-1=2>0 right
dy/dt=2-3=-1<0 down



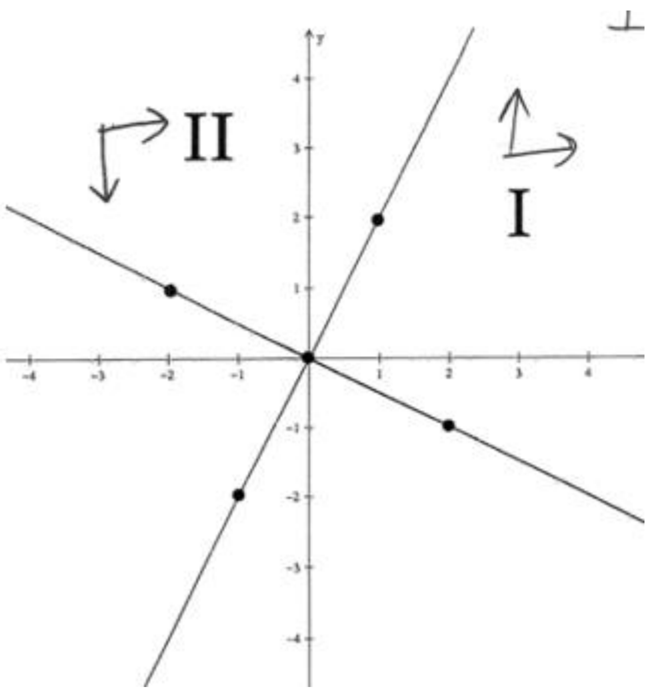
Z5.

$I\ test\ pt\ (3,1)$

$$\frac{dx}{dt} = 3 + 2(1) = 5 > 0$$
$$\frac{dy}{dt} = 2(3) - 1 = 6 > 0$$

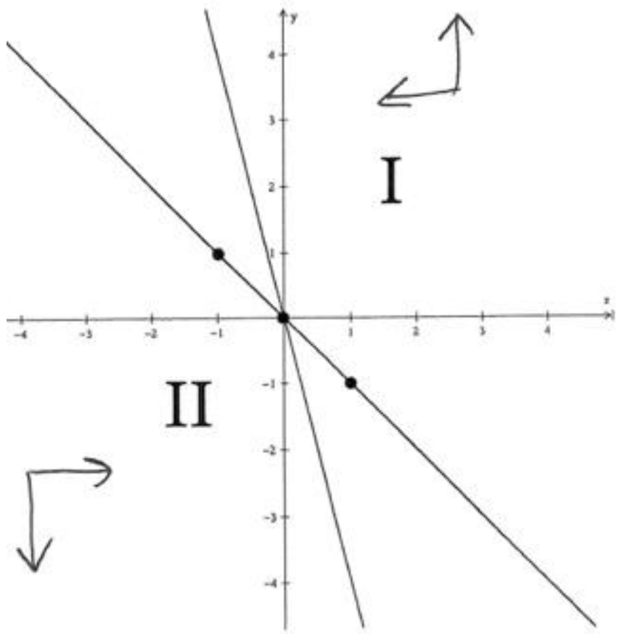
$II\ test\ \ (-1,3)$

$$\frac{dx}{dt} = -1 + 2(3) = 5 > 0$$
$$\frac{dy}{dt} = 2(-1) - 3 = -5 < 0$$



Z6.

. Consider the graph of the system $\frac{dx}{dt} = -x - y$ and $\frac{dy}{dt} = 4x + y$. Find the direction of tion at I and II below.



I test pt $(2,1)$

$\frac{dx}{dt} = -2-1 = -3 < 0$

$\frac{dy}{dt} = 4(2)+1 = 9 > 0$

II test pt. $(-2,-3)$

$\frac{dx}{dt} = -2-(-3) = 1 > 0$

$\frac{dy}{dt} = 4(-2)+(-3) < 0$

Z7. $\underline{x - nullclines}$

$\frac{dx}{dt} = 0$

$3x + 2y - 2 = 0$

$2y = -3x + 2$

$y = -\frac{3}{2}x + 1$

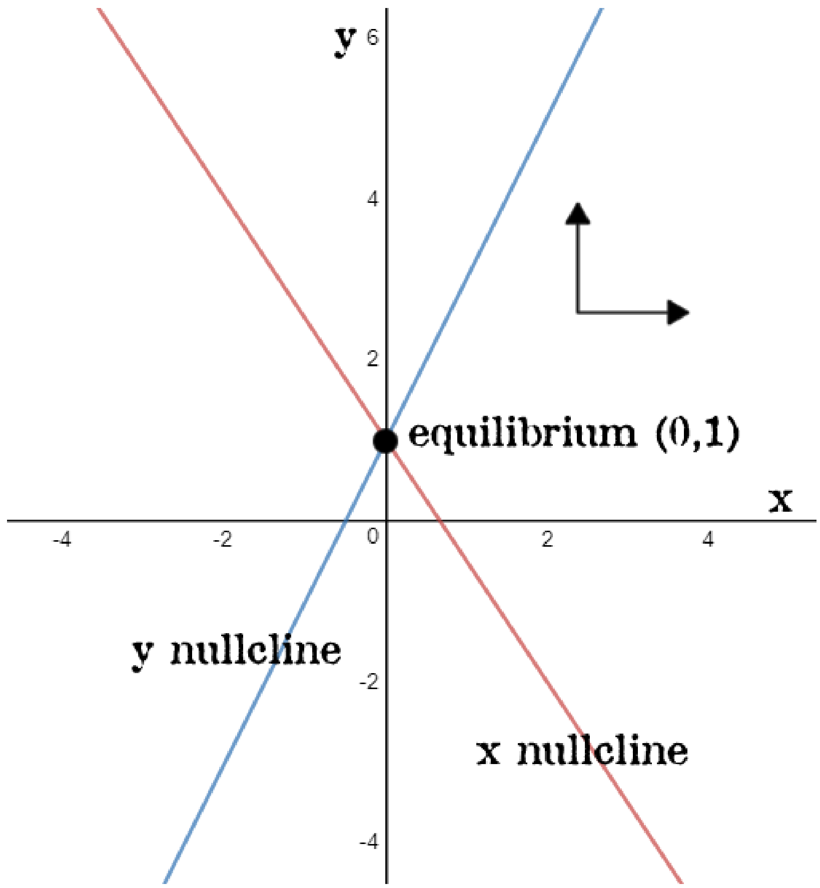
$\underline{y - nullclines}$

$\frac{dy}{dt} = 0$

$2x - y + 1 = 0$

$2x + 1 = y$

$y = 2x + 1$



Find equilibrium algebraically:

Set y=y

$-\frac{3}{2}x + 1 = 2x + 1$

multiply by 2

$-3x + 2 = 4x + 2$

$7x = 0$

$x = 0$ substitute x=0 into either equation and get y=2(0)+1 =1

$Equilibrium\ is\ (0,1)$.

Test point (3,1)

dx/dt=3x+2y-2=3(3)+2(1)-2=9>0 right

dy/dt=2x-y+1=2(3)-1+1=6 >0 up

AA. Practice Final Exam Solutions

AA1. $N = No(\frac{1}{2})^{\frac{t}{h}}$

$\therefore N = 100(\frac{1}{2})^{\frac{t}{20}}$

AA2. $v(t) = e^{\int P(t)dt}$

$= e^{\int \frac{1}{t-2}dt} = e^{\ln(t-2)} = t-2$

$\frac{d}{dt} = (v(t)(u)) = v(t)q(t)$

$\int \frac{d}{dt} = (t-2)(u) = (t-2)(4)$

$(t-2)u = \frac{4t^2}{2} - 8t + c$

$u(1) = 3 \quad (1-2)(3) = 2(1)^2 - 8(1) + c$

$c = -3 - 2 + 8$

$\mathrm{c} = 3$

$(t-2)u = 2t^2 - 8t + 3$

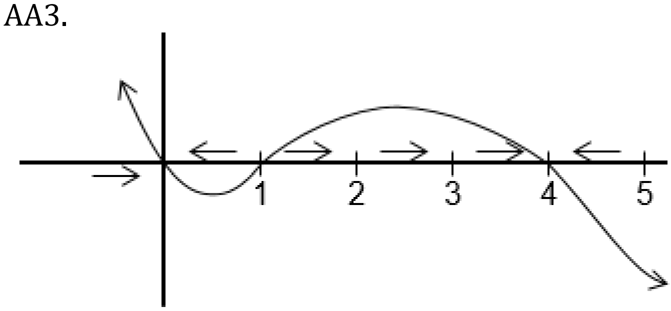
$u(4) = ?$

$(4-2)u = 2(4)^2 - 8(4) + 3$

$2u = 32 - 32 + 3$

$u = \frac{3}{2}$

AA3.



$N = 0, 1, 4$ equilibrium

$N = \frac{1}{2} \quad \frac{dN}{dt} = \frac{1}{2}\left(\frac{1}{2} - 1\right)(4 - \frac{1}{2}) < 0$

$N = 2 \quad \frac{dN}{dt} = 2(2-1)(4-2) > 0$

$N = 5 \quad \frac{dN}{dt} = 5(5-1)(4-5) < 0$

$\therefore \mathit{It\ will\ go\ to}\ 4$

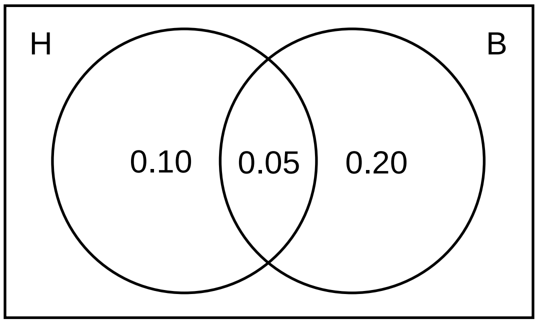
AA4. $\boxed{E}$ is false. It is only true if E, F are independent.

AA5. Let $H = heart\ disease \quad B = high\ blood\ pressure$

$$\Pr(H) = 0.15 \quad \Pr(B) = 0.25 \quad \Pr(H \cap B) = 0.05$$

$$\Pr(H\ or\ B\ but\ not\ both) = 0.10 + 0.20$$

$$= 0.30 \quad or \quad \frac{30}{100}$$



b) $\Pr(H \cap B^c) = \Pr(H) - \Pr(H \cap B)$

$$= 0.15 - 0.05$$

$$= 0.10 \quad \text{SEE VENN DIAGRAM}$$

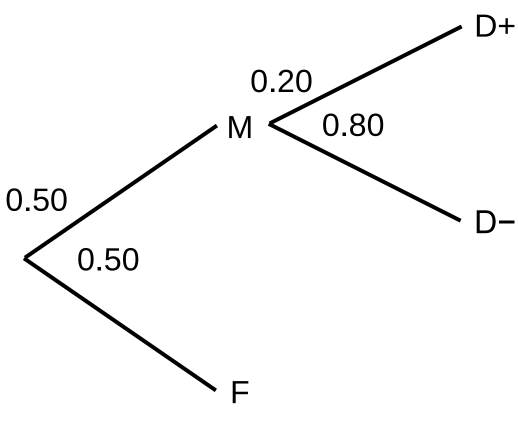
c) $\Pr(B^c/H) = \frac{\Pr(B^c \cap H)}{\Pr(H)} = \frac{0.10}{0.15}$

$$= \frac{10}{15} = \boxed{\frac{2}{3}}$$

d) $\Pr(H/B) = \frac{\Pr(H \cap B)}{\Pr(B)} = \frac{0.05}{0.25}$

$$= \frac{5}{25} = \boxed{\frac{1}{5}}$$

AA6.



$\Pr(M \cap D^+) = \Pr(M) \times \Pr(D^+/M)$

$$= 0.5 \times 0.20$$

$$= \frac{1}{2}\left(\frac{2}{10}\right) = \boxed{\frac{1}{10}}$$

AA7. a)

x	Pr (x)
1	0.4
2	0.3
3	0.3
	$\overline{1}$

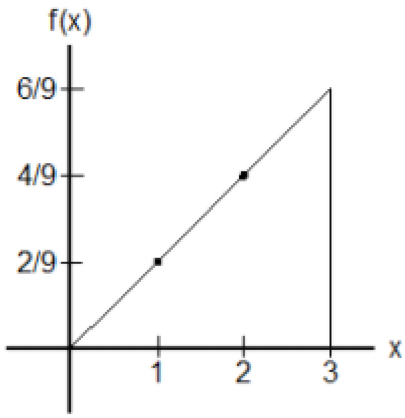
$$E(x) = \sum x \text{ Pr } (x)$$
$$= 1(0.4) + 2(0.3) + 3(0.3)$$
$$= 0.4 + 0.6 + 0.9$$
$$= \boxed{1.9}$$

b) $Var(x) = \sum x^2 \text{ Pr}(x) - [E(x)]^2$
$$= 1^2(0.4) + 2^2(0.3) + 3^2(0.3) - 1.9^2$$
$$= 0.4 + 1.2 + 2.7 - 3.61$$
$$= 0.69$$
$$SD(x) = \sqrt{0.69} = 0.83$$

AA8.a) $d(x) = F'(x) = \frac{2x}{9} = \frac{2}{9}x$
$$f(x) = \frac{2}{9}x$$

$$\text{Pr}(0 \leq x \leq 1) = \frac{b \times h}{2}$$
$$= \frac{(1)(\frac{2}{9})}{2}$$
$$= \frac{2}{18} = \frac{1}{9}$$

b) $E(x) = \int_{-\infty}^{\infty} x\, f(x) dx$
$$= \int_0^3 x\left(\frac{2}{9}x\right) dx$$
$$= \int_0^3 \frac{2}{9}x^2 dx$$
$$= \frac{2}{9}\left[\frac{x^3}{3}\right]_0^3$$
$$= \frac{2}{9}\left[\frac{3^3}{3} - 0\right]$$
$$= \frac{2}{9}\left[\frac{27}{3}\right]$$
$$= \frac{2}{9}[9]$$
$$= 2$$



AA9. $n = 25 \quad \bar{x} = 5 \quad s^2 = 20$

$$t = \frac{\bar{x} - E(x)}{\sqrt{\frac{s^2}{n}}}$$

$$= \frac{5-0}{\sqrt{\frac{20}{25}}} = \frac{5}{\frac{\sqrt{20}}{5}} = \frac{5\times5}{\sqrt{20}} = \frac{25}{\sqrt{20}}$$

AA10. $\bar{x} = 30 \quad s^2 = 25$

$$36 - ?s = 30$$
$$36 - s? = 30$$
$$36 - 30 = 5?$$
$$5? = 6$$
$$? = \frac{6}{5}$$

AA11. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+2+3 \\ 8+5+6 \end{bmatrix} = \begin{bmatrix} 7 \\ 19 \end{bmatrix}$

$2 \times 3 \;\; 3 \times 1 \qquad Ans\ 2 \times 1$

AA12. invertible if $ad - bc \neq 0$

$$ad - bc = 2(-6) - (3)(-4)$$
$$= -12 + 12$$
$$= 0$$

$\therefore$ *not invertible since* $ad - bc = 0$

AA13. no inverse if $ad - bc = detA = 0$

$$\begin{array}{ccccc} 3 & 0 & 1 & 3 & 0 \\ 0 & k & 2 & 0 & k \\ 1 & k+1 & 1 & 1 & k+1 \end{array}$$

$left = k + 6(k+1) + 0$ $\quad$ $right = 3k + 0 + 0$

$= 7k + 6$ $\quad$ $= 3k$

$$detA = R - L = 0$$
$$3k - (7k + 6) = 0$$
$$3k - 7k - 6 = 0$$
$$-4k = 6$$
$$k = -\frac{6}{4} = -\frac{3}{2}$$

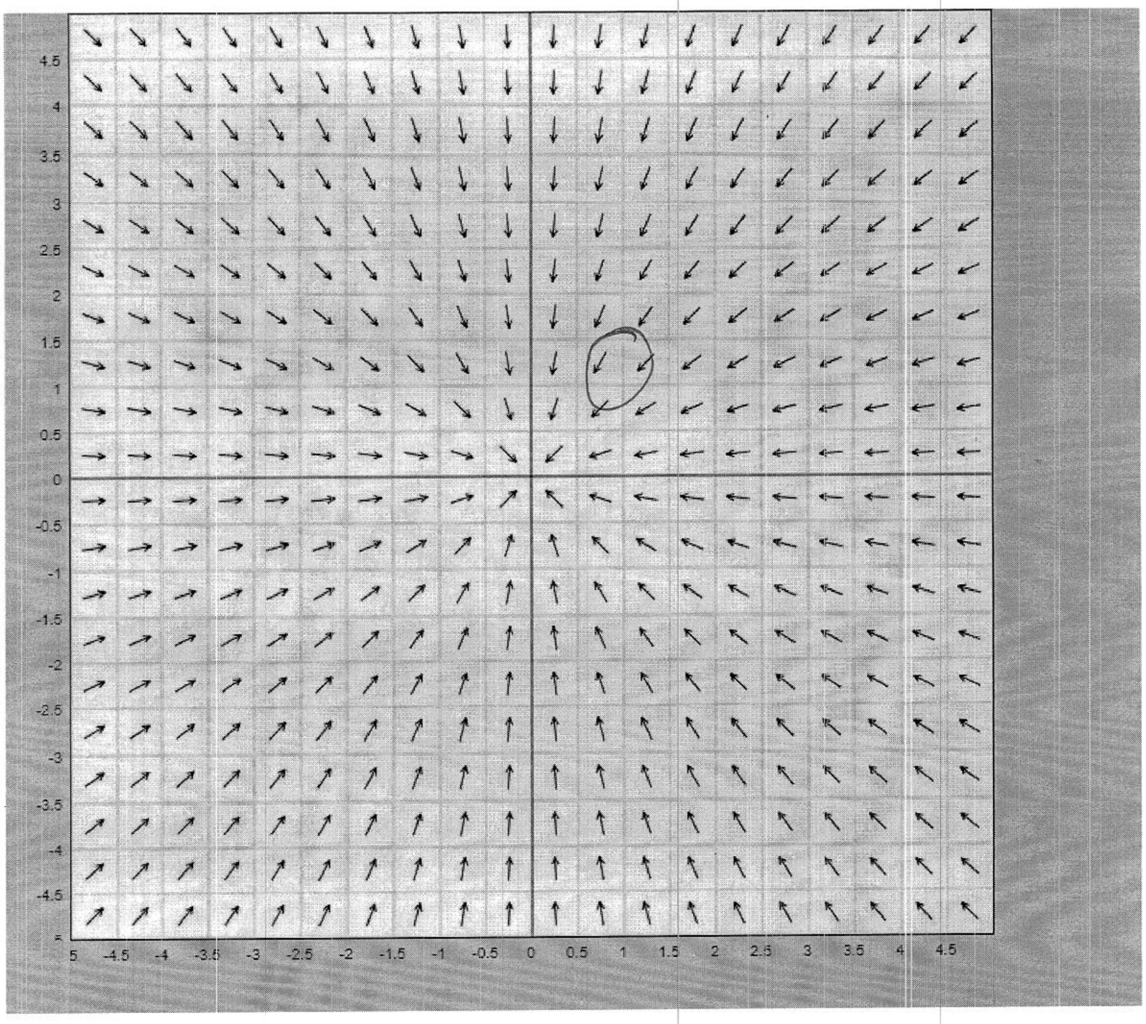
AA14.

$$\begin{array}{c} \\ J \\ A \\ adult \end{array} \begin{array}{c} \begin{array}{ccc} \text{J} & \text{A} & \text{adult} \end{array} \\ \begin{bmatrix} 0 & 0.7 & 2.5 \\ 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \end{bmatrix} \end{array}$$

AA15. a) $= 1$ $\quad$ b) $= 3$

$c) = 2$ $\quad$ d) $= 4$

15.

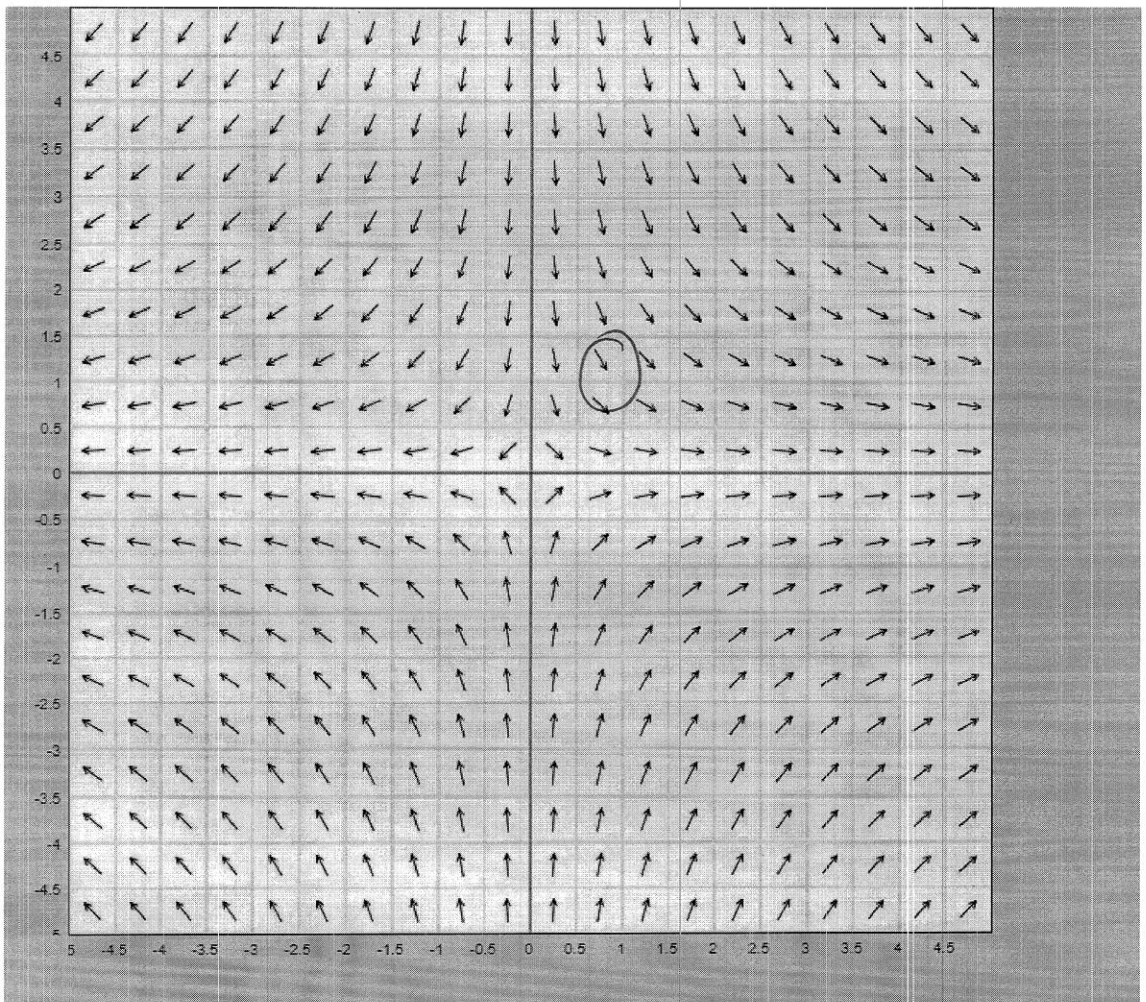


$$x' = -x$$
$$y' = -y$$

$$F\begin{bmatrix}1\\1\end{bmatrix} = \begin{array}{cc}\begin{bmatrix}-1\\-1\end{bmatrix} & \begin{matrix}\text{run}\\\text{rise}\end{matrix}\end{array}$$

left + down

15.



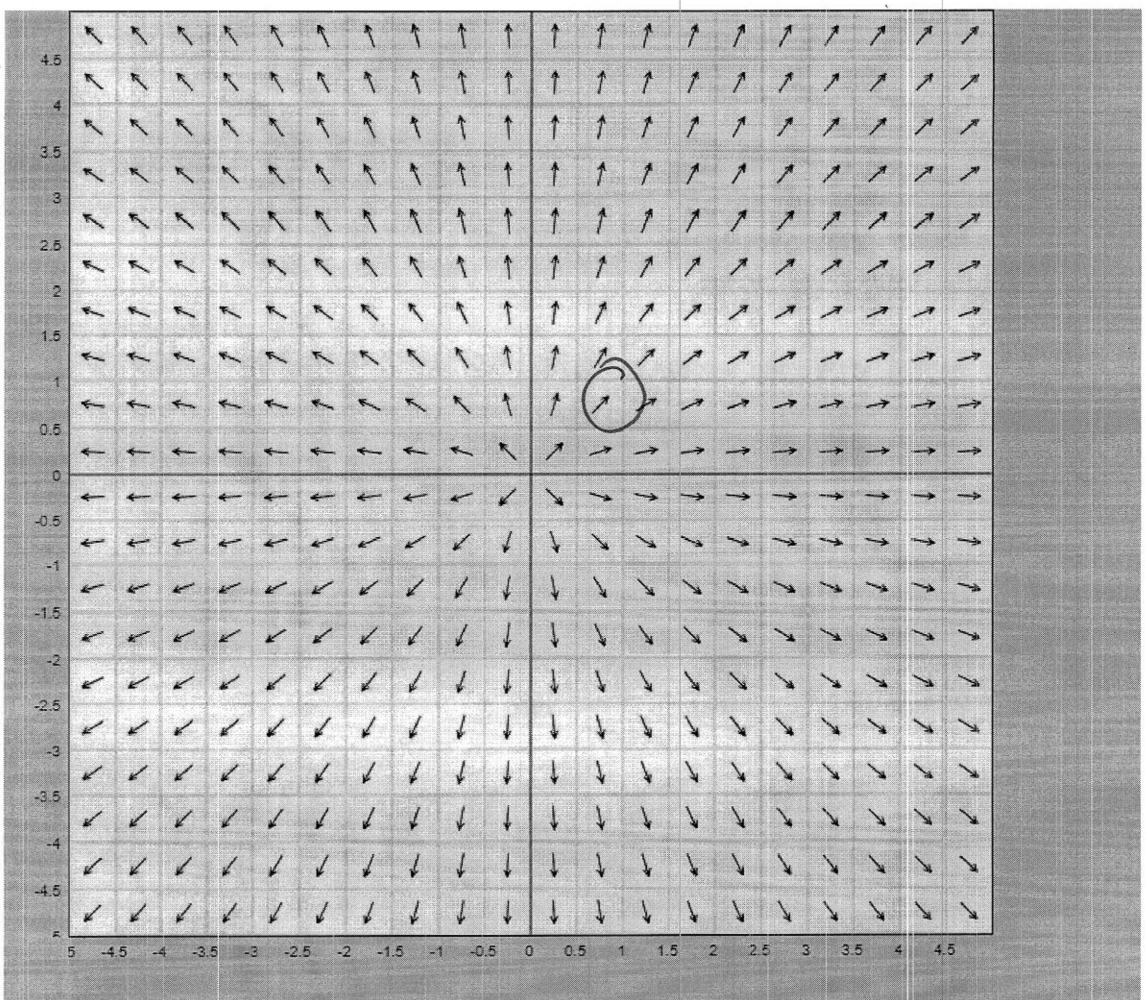
$x' = x$

$y = -y$

$F\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ run rise

right & down

15.

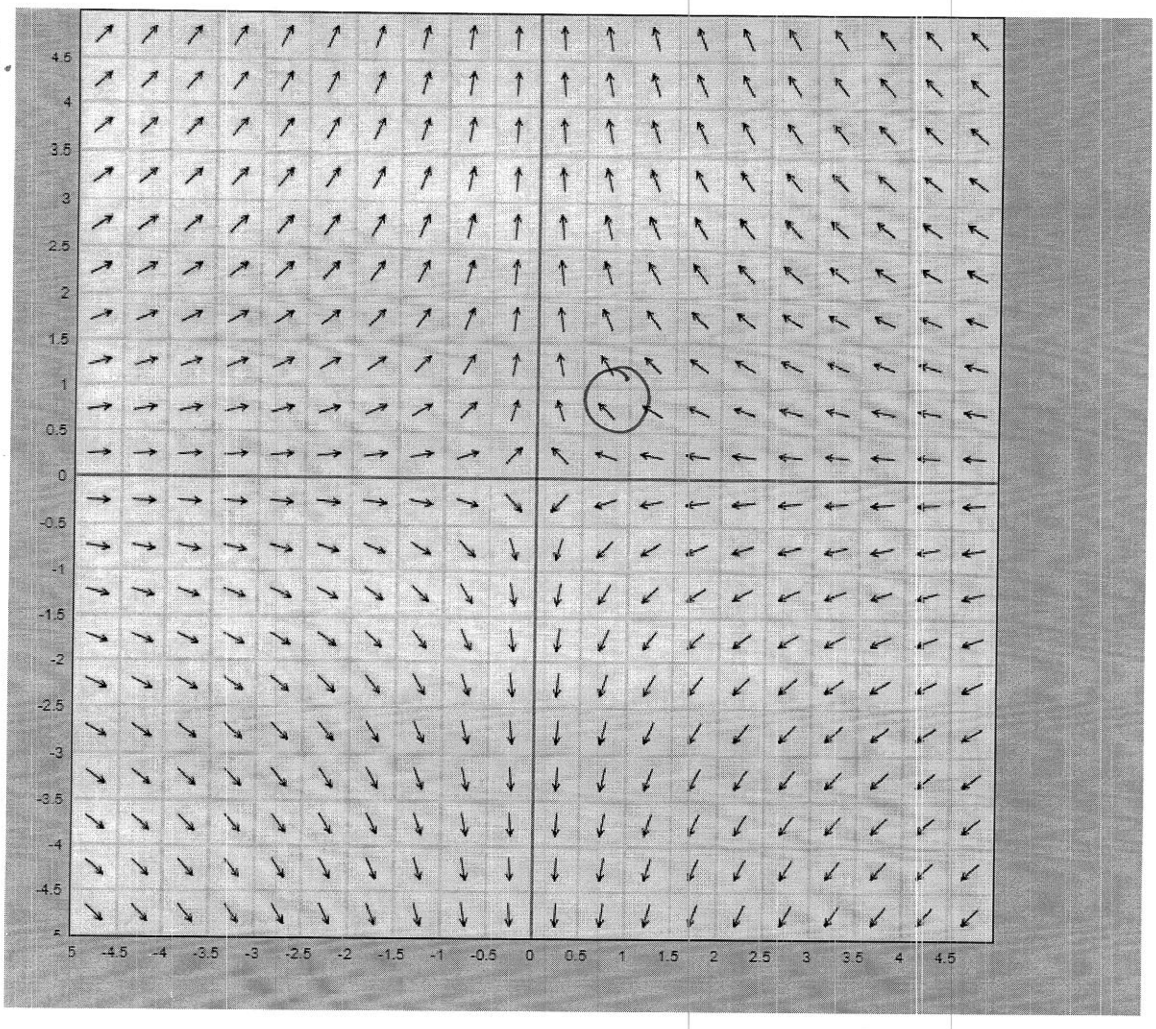


$x' = x$

$y' = y$

$F\begin{bmatrix}1\\1\end{bmatrix} \overset{\text{run}}{\underset{\text{rise}}{=}} \begin{bmatrix}1\\1\end{bmatrix}$ right + up

15.



$x' = -x$
$y = y$

left + up

$$F\begin{bmatrix}1\\1\end{bmatrix} = \begin{array}{c l}\begin{bmatrix}-1\\1\end{bmatrix} & \begin{array}{l}\text{run}\\\text{rise}\end{array}\end{array}$$

↖

AA16. $tr(A) = -6 + 5 = -1$

$detA = -6(5) - 3(4) = -30 - 12 = -42$

$\lambda^2 - tr(A)\lambda + detA = 0$

$\lambda^2 + \lambda - 42 = 0$

$(\lambda + 7)(\lambda - 6) = 0$

$\lambda = -7,6$

$\lambda = -7$ $\qquad$ $\lambda = 6$

$v_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix}$ $\qquad$ $v_2 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix}$

$= \begin{bmatrix} 3 \\ -7 - (-6) \end{bmatrix}$ $\qquad$ $= \begin{bmatrix} 3 \\ 6 + 6 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ $\qquad$ $= \begin{bmatrix} 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

AA17. $tr(A) = 2 + 4 = 6$

$detA = 2(4) - (5)(-2) = 8 + 10 = 18$

$\lambda^2 - tr(A)\lambda + detA = 0$

$\lambda^2 - 6\lambda + 18 = 0$

$a = 1 \quad b = -6 \quad c = 18$

$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{6 \pm \sqrt{36 - 4(1)(18)}}{2(1)}$

$= \frac{6 \pm \sqrt{-36}}{2}$

$= \frac{6 \pm 6i}{2}$

$= 3 \pm 3i$

$v_1 = \begin{bmatrix} \lambda - d \\ c \end{bmatrix} = \begin{bmatrix} 3 + 3i - 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 3i - 1 \\ -2 \end{bmatrix}$

AA18. from AA16, $\lambda = 7, -6$

$$\vec{v}_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 3 \\ 7-(-6) \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \end{bmatrix}$$
$$\vec{v}_2 = \begin{bmatrix} 3 \\ -6+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\overrightarrow{x_t} = c_1\overrightarrow{v_1}(\lambda_1)^t + c_2\overrightarrow{v_2}(\lambda_2)^t$$

$$\overrightarrow{x_t} = c_1\begin{bmatrix} 3 \\ 13 \end{bmatrix}(7)^t + c_2\begin{bmatrix} 3 \\ 0 \end{bmatrix}(-6)^t$$

At $t = 0$ $\begin{bmatrix} 9 \\ 13 \end{bmatrix} = c_1\begin{bmatrix} 3 \\ 13 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$9 = 3c_1 + 3c_2$
$13 = 13c_1$
$c_1 = 1$

$9 = 3c_1 + 3c_2$
$9 = 3(1) + 3c_2$
$6 = 3c_2$
$c_2 = 2$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 1\begin{bmatrix} 3 \\ 13 \end{bmatrix}(7)^t + 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}(-6)^t$$

AA19.

$$\begin{array}{c|c} & \begin{matrix} P & B & H \end{matrix} \\ \begin{matrix} P \\ B \\ H \end{matrix} & \begin{bmatrix} 3/5 & 1/2 & 0 \\ 1/5 & 0 & 1/2 \\ 1/5 & 1/2 & 1/2 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 3/5 & 1/2 & 0 \\ 1/5 & 0 & 1/2 \\ 1/5 & 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 3/5\,x + 1/2\,y \\ 1/5\,x + 1/2\,z \\ 1/5\,x + 1/2\,y + 1/2\,z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

From $\boxed{1}$ $3/5\,x + 1/2\,y = x$

$1/2\,y = 2/5\,x$

$10(1/2\,y) = 10(2/5\,x)$

$5y = 4x$

$y = 4/5\,x$

Let $x = 5$ $then$ $y = 4/5\,(5)$ $\therefore y = 4$

From $\boxed{2}$ $1/5\,x + 1/2\,z = y$

$1/5\,(5) + 1/2\,z = 4$

$1 + 1/2\,z = 4$

$1/2\,z = 3$ $\therefore z = 6$

$$\begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix} \begin{matrix} \leftarrow P \\ \leftarrow B \\ \leftarrow H \end{matrix}$$

$\therefore \Pr(pasta) = \frac{5}{5+4+6} = \frac{5}{15} = \frac{1}{3}$

AA20. $A = \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 2 \\ 6 & -3 \end{bmatrix} \\ \begin{array}{cc} c & d \end{array} \end{array}$

$tr(A) = 1 + (-3) = -2$

$detA = 1(-3) - 2(6) = -15$

$\lambda^2 + 2\lambda - 15 = 0$

$(\lambda + 5)(\lambda - 3) = 0$

$\lambda = -5,\ 3$

$\lambda_1 = -5$

$v_1 = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ -5 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\lambda_2 = 3$

$v_2 = \begin{bmatrix} 2 \\ 3 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{u}(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

$\vec{u}(t) = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$

At $t = 0$,

$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{0} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{0}$

$2 = c_1 + c_2$

$-1 = -3c_1 + c_2$

$subtract\ \ 3 = 4c_1$

$c_1 = \frac{3}{4}$

$2 = c_1 + c_2$

$2 = \frac{3}{4} + c_2$

$c_2 = \frac{8}{4} - \frac{3}{4} = \frac{5}{4}$

$\therefore \vec{u}(t) = \frac{3}{4} \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-5t} + \frac{5}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$

$u_1(t) = \frac{3}{4} e^{-5t} + \frac{5}{4} e^{3t}$ $\qquad$ $u_2(t) = \frac{-9}{4} e^{-5t} + \frac{5}{4} e^{3t}$

a b

AA21. $A = \begin{array}{c} \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix} \\ c \qquad d \end{array}$

$tr(A) = 3 + 1 = 4$
$detA = 3 + 13(5) = 68$
$\lambda^2 - 4\lambda + 68 = 0$
$\lambda = \frac{4 \pm \sqrt{16 - 4(1)(68)}}{2(1)}$
$= \frac{4 \pm \sqrt{-256}}{2} = \frac{4 \pm \sqrt{16}i}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i \leftarrow \quad a = 2 \quad b = 2$
$\overrightarrow{v_1} = \begin{bmatrix} \lambda - d \\ c \end{bmatrix} = \begin{bmatrix} 2 + 2i - 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 + 2i \\ 5 \end{bmatrix}$
$\vec{v} = \underset{\vec{x}}{\begin{bmatrix} 1 \\ 5 \end{bmatrix}} + i \underset{\vec{y}}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$

$a = 2, b = 2 \quad \vec{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$
$u(t) = e^{at} k_1(0)(\cos(bt)\vec{x} - \sin(bt)\vec{y}) + e^{at} k_2(0)(\sin(bt)\vec{x} + \cos(bt)\vec{y}))$

$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = e^{2t} k_1(0)(\cos(2t) \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \sin(2t) \begin{bmatrix} 2 \\ 0 \end{bmatrix} + e^{2t} k_2(0)(\sin(2t) \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \cos(2t) \begin{bmatrix} 2 \\ 0 \end{bmatrix})$

Sub $t = 0$,
$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = k_1(0) \begin{bmatrix} 1 \\ 5 \end{bmatrix} + k_2(0) \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$2 = k_1(0) + 2k_2(0)$
$1 = 5k_1(0) + 0$

$k_1(0) = \frac{1}{5}$

$2 = \frac{1}{5} + 2k_2(0)$
$\frac{10}{5} - \frac{1}{5} = 2k_2(0)$
$\frac{9}{5} = 2k_2(0)$
$k_2(0) = \frac{9}{10}$

$k_1(0) = \frac{1}{5}$ & $k_2(0) = \frac{9}{10}$

$u_1(t) = e^{2t} k_1(0)(\cos(2t) - 2\sin(2t)) + e^{2t} k_2(0)(\sin(2t) + 2\cos(2t))$
$u_1(t) = \frac{1}{5} e^{2t}(\cos(2t) - 2\sin(2t)) + \frac{9}{10} e^{2t}(\sin(2t) + 2\cos(2t))$
$= \frac{2}{10} e^{2t} \cos(2t) + \frac{18}{10} e^{2t} \cos(2t) - \frac{4}{10} e^{2t} \sin(2t) + \frac{9}{10} e^{2t} \sin(2t))$
$= 2e^{2t} \cos(2t) + \frac{1}{2} e^{2t} \sin(2t)$
$= e^{2t}(2\cos(2t) + \frac{1}{2}\sin(2t))$

Best of Luck on The Exam!!!