MATH 1600 Final Exam Booklet Solutions (Fall 2024)

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Midterm Material

1. Vectors

Example 1.1. Given A=(2,-1) and B=(5,2), find vector $\overrightarrow{AB}$ and draw this vector in standard position.

Solution:

$\overrightarrow{AB} = b - a = (5,2) - (2,-1) = (3,3)$

Example 1.2.

a) $\vec{u} + \vec{v} = (-1,-2,3) + (-2,-1,1) = (-3,-3,4)$

b) $= 3(-1,-2,3) - 5(-2,-1,1)$

$= (-3,-6,9) + (10,5,-5) = (7,-1,4)$

Example 1.3 Solve for vector $\vec{x}$ in terms of vectors $\vec{a}$ and $\vec{b}$:

$\vec{b} + 3(\vec{x} - 4\vec{a}) = 5(\vec{x} + \vec{a}) - (\vec{a} - \vec{b})$

Solution:

$\vec{b} + 3(\vec{x} - 4\vec{a}) = 5(\vec{x} + \vec{a}) - (\vec{a} - \vec{b})$

$\vec{b} + 3\vec{x} - 12\vec{a} = 5\vec{x} + 5\vec{a} - \vec{a} + \vec{b}$

$3\vec{x} - 5\vec{x} = 4\vec{a} + \vec{b} - \vec{b} + 12\vec{a}$

$-2\vec{x} = 16\vec{a}$

$\vec{x} = -8\vec{a}$

1.5 Homework on Chapter 1

1. Solve for the vector $\vec{x}$ in terms of the vectors $\vec{a}$ *and* $\vec{b}$

a) $\vec{x} - 3\vec{a} - 3\vec{b} = 6(\vec{x} - 5\vec{b})$

$\vec{x} - 3\vec{a} - 3\vec{b} = 6\vec{x} - 30\vec{b}$

$\vec{x} - 6\vec{x} = -30\vec{b} + 3\vec{a} + 3\vec{b}$

$-5\vec{x} = -27\vec{b} + 3\vec{a}$

$\vec{x} = \frac{27}{5}\vec{b} - \frac{3}{5}\vec{a}$

b) $\vec{x} + 3\vec{a} - 3\vec{b} = 4(\vec{x} + \vec{a} + \vec{b})$

$\vec{x} + 3\vec{a} - 3\vec{b} = 4\vec{x} + 4\vec{a} + 4\vec{b}$

$\vec{x} - 4\vec{x} = 4\vec{a} - 3\vec{a} + 4\vec{b} + 3\vec{b}$

$-3\vec{x} = \vec{a} + 7\vec{b}$

$\vec{x} = -\frac{1}{3}\vec{a} - \frac{7\vec{b}}{3}$

2. a) $-\vec{v} + 3\vec{w} = = -(-2,-1,1) + 3(5,-1,-2)$

$= (2,1,-1) + (15,-3,-6) = (17,-2,-7)$

b) $3\vec{u} - \vec{v} + 2\vec{w} = 3(-1,-2,3) - (-2,-1,1) + 2(5,-1,-2)$

$= (-3,-6,9) + (2,1,-1) + (10,-2,-4)$

$= (9,-7,4)$

2. Products, Distance, etc.

Example 2.1. Given the following vectors: $\vec{u} = (-1, -2, 3), \vec{v} = (-2, -1, 1)$, find $\vec{u} \cdot \vec{v}$.

$\vec{u} \cdot \vec{v} = (-1, -2, 3) \cdot (-2, -1, 1) = (-1)(-2) + (-2)(-1) + (3)(1) = 2 + 2 + 3 = 7$

*** Recall, a dot product ALWAYS gives you a number or scalar answer!! Cross product always gives you a VECTOR answer, ie. (1,2,4) etc.

Example 2.2. If $\vec{u} = (-1, -2, 3)$, find its magnitude:

Solution: $\|\vec{u}\| = \sqrt{(-1)^2 + (-2)^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

Example 2.3. Find a unit vector in the same direction as $\vec{u} = (1, 3)$

Solution:

$\|\vec{u}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$

So, a unit vector in the same direction is $\frac{\vec{u}}{\|\vec{u}\|} = \frac{(1,3)}{\sqrt{10}} = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$

Example 2.4. Find all the value of k for which the vector $\vec{u} = (k, 2)$ has magnitude equal to $\sqrt{10}$.

Solution:

$\sqrt{k^2 + 2^2} = \sqrt{10}$

$k^2 + 4 = 10$

$k^2 = 6$

$k = \pm\sqrt{6}$

Example 2.5. Find all values of c such that the vector $\left(c, \frac{c}{3}, \frac{c}{2}\right)$ *is a unit vector*.

Solution:

$$\sqrt{c^2 + (\frac{c}{3})^2 + (\frac{c}{2})^2 = 1}$$

$$c^2 + \frac{c^2}{9} + \frac{c^2}{4} = 1$$

$$\frac{36c^2}{36} + \frac{4c^2}{36} + \frac{9c^2}{36} = 1$$

$$\frac{49c^2}{36} = 1$$

$$49c^2 = 36$$

$$c^2 = \frac{36}{49}$$

$$c = \pm\sqrt{\frac{36}{49}} = \pm\frac{6}{7}$$

Example 2.6. Given the following vectors: $\vec{u} = (-1, -2, 3)$ *and* $\vec{v} = (-2, -1, 1)$, find the distance between them.

Solution:

$$\text{d}(\vec{u}, \vec{v}) = \sqrt{(-2+1)^2 + (-1+2)^2 + (1-3)^2} = \sqrt{1+1+4} = \sqrt{6}$$

Example 2.7. Given $\vec{u}=(-1,-1,-2)$ and $\vec{v}=(1,-3,2)$, find the cosine of the angle between the two vectors. Simplify your answer.

Solution:

$$\begin{aligned}\cos\theta &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\\ &= \frac{(-1,-1,-2)\cdot(1,-3,2)}{\sqrt{(-1)^2+(-1)^2+(-2)^2}\sqrt{(1)^2+(-3)^2+(2)^2}}\\ &= \frac{-1+3-4}{\sqrt{1+1+4}\sqrt{1+9+4}}\\ &= \frac{-2}{\sqrt{6}\sqrt{14}}\\ &= \frac{-2}{\sqrt{84}}\end{aligned}$$

Although this value is correct, we can actually simplify it further.

$$\begin{aligned}\cos\theta &= \frac{-2}{\sqrt{84}}\\ &= \frac{-2}{\sqrt{4}\sqrt{21}}\\ &= -\frac{2}{2\sqrt{21}}\\ &= -\frac{1}{\sqrt{21}}\end{aligned}$$

Example 2.8. Find the value of k for which the vectors (14,k,k) and (4,k,15) are orthogonal.

Solution:

$$(14,k,k)\cdot(4,k,15)=0$$

$$56+k^2+15k=0$$

$$k^2+15k+56=0$$

$$(k+7)(k+8)=0$$

$$k=-7,-8$$

Example 2.9. a) Determine whether the angle between $\vec{u} = [1,2,3,-1]$ and $\vec{v} = [5,6,2,1]$ is acute, obtuse or a right angle.

Solution:

$\vec{u} = [1,2,3,-1]$ and $\vec{v} = [5,6,2,1]$

$\vec{u} \cdot \vec{v} = [1,2,3,-1] \cdot [5,6,2,1]$

$= 5 + 12 + 6 - 1$

$= 22 > 0$

$\therefore \theta$ is acute

b) Find the angle between $\vec{u}$ *and* $\vec{v}$

Solution:

$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||||\vec{v}||}$

$||\vec{u}|| = \sqrt{1^2 + 2^2 + 3^2 + (-1)^2}$

$= \sqrt{1 + 4 + 9 + 1}$

$= \sqrt{15}$

$||\vec{v}|| = \sqrt{5^2 + 6^2 + 2^2 + 1^2}$

$= \sqrt{25 + 36 + 4 + 1}$

$= \sqrt{66}$

$$\cos\theta = \frac{22}{\sqrt{15}\sqrt{66}}$$

$$\boldsymbol{\theta = cos^{-1}\left(\frac{22}{\sqrt{15}\sqrt{66}}\right)}$$

Example 2.10. Find the projection of $\vec{v}$ onto $\vec{u}$ where $\vec{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Solution:

$$proj_u v = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right)\vec{u}$$

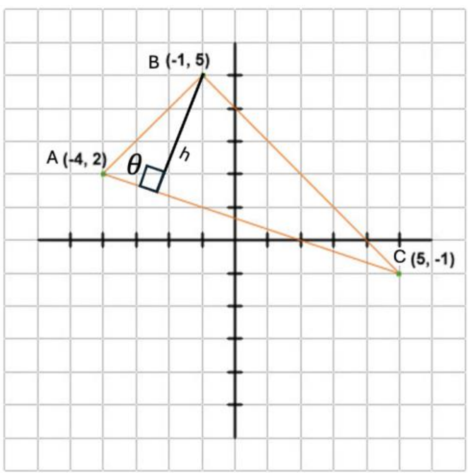
$$=\left(\frac{(3,-1)\cdot(5,4)}{(3,-1)\cdot(3,-1)}\right)(3,-1)$$

$$=\left(\frac{15-4}{9+1}\right)(3,-1)$$

$$=\frac{11}{10}(3,-1)$$

$$=\left(\frac{33}{10},\frac{-11}{10}\right)$$

Example 2.11. Find the area of the triangle below:



1

$$\sin\theta = \frac{h}{\|\overrightarrow{AB}\|}$$

$$h = \|\overrightarrow{AB}\| \sin\theta$$

2

$$\text{Area} = \frac{1}{2}\|\overrightarrow{AC}\|h$$

$$= \frac{1}{2}\|\overrightarrow{AC}\|\|\overrightarrow{AB}\| \sin\theta$$

$$\overrightarrow{AC} = [5,-1]-[-4,2]$$
$$= [9,-3]$$
$$\overrightarrow{AB} = [-1,5]-[-4,2]$$
$$= [3,3]$$

$$\left\|\overrightarrow{AC}\right\| = \sqrt{9^2+(-3)^2}$$
$$= \sqrt{81+9}$$
$$= \sqrt{90}$$

$$\left\|\overrightarrow{AB}\right\| = \sqrt{3^2+3^2}$$
$$= \sqrt{18}$$
$$\cos\theta = \frac{(9,-3)\cdot(3,3)}{\sqrt{90}\sqrt{18}}$$

$$\cos\theta = \frac{27-9}{\sqrt{90}\sqrt{18}} = \frac{18}{\sqrt{9}\sqrt{10}\sqrt{18}} = \frac{18}{3\sqrt{10}\sqrt{18}}$$
$$\cos\theta = \frac{6}{\sqrt{180}}$$
$$\cos\theta = \frac{6}{\sqrt{9}\sqrt{20}}$$
$$= \frac{6}{3\sqrt{4}\sqrt{5}}$$
$$= \frac{2}{2\sqrt{5}}$$

$$\cos\theta = \frac{1}{\sqrt{5}}$$

$$\cos^2\theta = \frac{1}{5}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\therefore \sin^2\theta = 1 - \cos^2\theta$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$
$$\sin\theta = \sqrt{1-1/5}$$
$$\sin\theta = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

$$\therefore \text{Area} = \tfrac{1}{2}\left\|\overrightarrow{AC}\right\|\left\|\overrightarrow{AB}\right\|\sin\theta$$

$A = \frac{1}{2}\sqrt{90}\sqrt{18}\left(\frac{2}{\sqrt{5}}\right)$

$A = \frac{1}{\not{2}}\sqrt{9}\sqrt{10}\,\sqrt{9}\sqrt{2}\left(\frac{\not{2}}{\sqrt{5}}\right)$

$A = 3(3)\frac{\sqrt{10}\,\sqrt{2}}{\sqrt{5}}$

$A = 9\sqrt{2}\sqrt{2} = 9 \times 4 = 36 \text{ units}^2$

<u>Example 2.12.</u> Find the area of the triangle between the points A(1,-2,5) B(0,-1,4) and C(1,0,4).

<u>Solution:</u> : $\vec{u} = B - A = (-1, 1, -1) \text{ and } \vec{v} = C - A = (0, 2, -1)$

METHOD 1: $\vec{u} \cdot \vec{v} = 0 + 2 + 1 = 3$

$||\vec{u}|| = \sqrt{(1)^2 + 1^2 + (-1)^2} = \sqrt{3}$

$||\vec{v}|| = \sqrt{0^2 + 2^2 + (-1)^2} = \sqrt{5}$

$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\,||\vec{v}||}$

$\cos\theta = \frac{3}{\sqrt{3}\sqrt{5}} = \frac{3}{\sqrt{15}}$

$\sin\theta = \sqrt{1 - \cos^2\theta}$

$\sin\theta = \sqrt{1 - \left(\frac{3}{\sqrt{15}}\right)^2}$

$= \sqrt{1 - \frac{9}{15}}$

$= \sqrt{\frac{15}{15} - \frac{9}{15}}$

$= \sqrt{\frac{6}{15}}$

$A = \frac{1}{2}\,||\vec{u}||\,||\vec{v}||\,\sin\theta$

$A = \frac{1}{2}\,\sqrt{3}\,\sqrt{5}\sqrt{\frac{6}{15}} = \frac{1}{2}\frac{\sqrt{15}\,\sqrt{6}}{\sqrt{15}}$ $\qquad A = \frac{1}{2}\,\sqrt{6}$

METHOD 2: $\vec{u} \cdot \vec{v} = 3$ (from Method 1)

$\vec{u} \cdot \vec{u} = (-1,1,-1) \cdot (-1,1,-1) = 1 + 1 + 1 = 3$

$\text{proj}_u\, \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\, \vec{u}$

$= \frac{3}{3}(-1,1,-1)$

$= (-1,1,-1)$

$A = \frac{1}{2}\, \left| |\vec{u}| \right| \left| \vec{v} - \text{proj}_u\, \vec{v} \right|$

$A = \frac{1}{2}\sqrt{3}\ ||(0,2,-1) - (-1,1,-1)||$

$= \frac{1}{2}\sqrt{3}\ ||(1,1,0)||$

$= \frac{1}{2}\sqrt{3}\sqrt{1^2 + 1^2 + 0^2}$

$= \frac{1}{2}\sqrt{3}\sqrt{2} = \frac{\sqrt{6}}{2}$

*** Cross Product is ALWAYS a <u>Vector</u>**

The cross product is **orthogonal** to both vectors $\vec{u}$ and $\vec{v}$.

<u>Example 2.13.</u> Find the cross product where $\vec{u} = (1,2,1)$ and $\vec{v} = (-1,-2,4)$.

<u>Solution:</u> Write out the vectors twice, with u first if you are finding $\vec{u} \times \vec{v}$

$$\begin{matrix} 1 & 2 & 1 & 1 & 2 & 1 \\ -1 & -2 & 4 & -1 & -2 & 4 \end{matrix}$$

$(8 - (-2)),\ -1 - 4,\ -2 + 2)$

$\vec{u} \times \vec{v} = (10, -5, 0)$

Example 2.14. Find the cross product $\vec{u} \times \vec{v}$ where $\vec{u}$=(3,-3,-1) and $\vec{v}$=(-1,-1,2).

Solution:

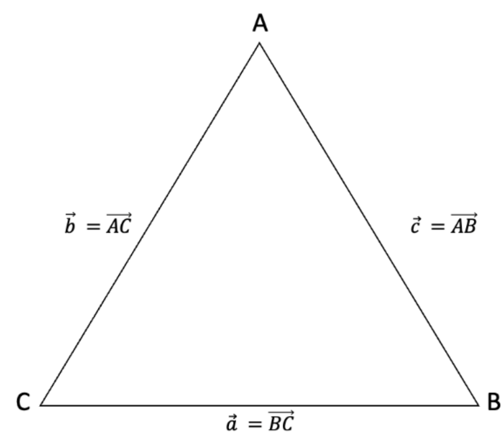
$$\begin{matrix} 3 & -3 & -1 & 3 & -3 & -1 \\ -1 & -1 & 2 & -1 & -1 & 2 \end{matrix}$$

$$\vec{u} \times \vec{v} = (-6-1,\ 1-6, -3-3)$$

$$= (-7, -5, -6)$$

Example 2.15. If $A = (4,5)$ $B = (6,1)$ and $\mathrm{C} = (10,3)$ determine if ΔABC is a right Δ.

Solution: **If the dot product of any two vectors is 0, there is a right angle in the triangle.**



$\overrightarrow{AB} = B - A = (6,1) - (4,5) = (2, -4)$

$\overrightarrow{AC} = (6, -2)$

$\overrightarrow{BC} = (10,3) - (6,1) = (4,2)$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = (2, -4) \cdot (4,2)$

$= 8 - 8 = 0$

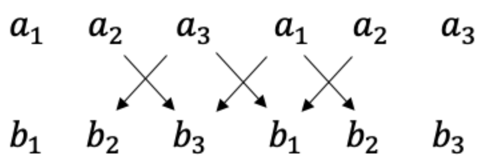
$\therefore\ \overrightarrow{AB} \perp \overrightarrow{BC}$

$\therefore\ \Delta ABC$ is a right Δ

Example 2.16. For any two vectors $\vec{a}$ and $\vec{b}$, prove that the cross product of $\vec{a}$ and $\vec{b}$ is orthogonal to $\vec{b}$.

Solution: $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$ if $\vec{a} \times \vec{b}$ is orthogonal to $\vec{b}$

$\vec{a} \times \vec{b} = (a_1, a_2, a_3) \times (b_1, b_2, b_3)$



$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

$(\vec{a} \times \vec{b}) \cdot \vec{b}$

$= (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \cdot (b_1, b_2, b_3)$

$= a_2b_3b_1 - a_3b_2b_1 + a_3b_1b_2 - a_1b_3b_2 + a_1b_2b_3 - a_2b_1b_3$

=0 ∴ orthogonal

Example 2.17. Which of the following make sense where c is a scalar and $\vec{u}, \vec{v}$, and $\vec{w}$ in R^3.

a) $\|\vec{u} \cdot \vec{v}\|$

b) $\vec{u} \cdot (3\vec{v} - 5\vec{w})$

c) $\vec{u} \cdot (\vec{v} \cdot \vec{w})$

d) $c \cdot (\vec{u} - \vec{v})$

Remember, you can only do cross product in R^3.

a) is undefined as once you do dot product, you get a scalar or a number and you can't do the magnitude of a number, only a vector

b) is defined since we can subtract the vectors first in the brackets and then we can dot product that vector with the vector $\vec{u}$ in front

c) is undefined since once we do the brackets first, we dot product to get a scalar or a number and we can't do the dot product of vector $\vec{u}$ with a scalar like 5.

d) is undefined since we can subtract the two vectors in brackets and get another vector but then we can't do the dot product of a scalar, c, and a vector. We can only do the dot product of two vectors.

2.13 Homework on Chapter 2

1. Find the projection of the vector (0,1,6) onto the vector (4,-1,2).

$$proj_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$$

$$= \frac{(0,1,6)\cdot(4,-1,2)}{(4,-1,2)\cdot(4,-1,2)}(4,-1,2)$$

$$= \frac{0-1+12}{16+1+4}(4,-1,2)$$

$$= \frac{11}{21}(4,-1,2)$$

2. Find the distance between the vectors $\vec{u} = (1,3,2) and\ \vec{v} = (-1,4,5)$.

$$d\ u,v = \sqrt{(-1-1)^2 + (4-3)^2 + (5-2)^2}$$

$$= \sqrt{4+1+9} = \sqrt{14}$$

3. Find the area of the triangle between the points P(3,-1,4) Q(1,-1,3) and R(4,-3,2).

Step 1 $\vec{u} = q - p = (1,-1,3) - (3,-1,4) = (-2,0,-1)$

$\vec{v} = r - p = (4,-3,2) - (3,-1,4) = (1,-2,-2)$

Step 2
$$\begin{matrix} -2 & 0 & -1 & -2 & 0 & -1 \\ 1 & -2 & -2 & 1 & -2 & -2 \end{matrix}$$

$$\vec{u}\times\vec{v} = (0-2, -1-4,\ 4-0) = (-2,-5,4)$$

Step 3 $A = \frac{1}{2}\ \|(-2,-5,4)\|$

$$= \frac{1}{2}\sqrt{(-2)^2 + (-5)^2 + 4^2}$$

$$= \frac{1}{2}\sqrt{4+25+16}$$

$$= \frac{1}{2}\sqrt{45}$$

4. Find the area of the triangle with vertices A(3,-1,2) B(1,3,1) and C(1,2 3).

$\vec{u} = b - a = (-2, 4, -1)$

$\vec{v} = c - a = (-2, 3, 1)$

$$\vec{u} \times \vec{v} \quad \begin{matrix} -2 & 4 & -1 & -2 & 4 & -1 \\ -2 & 3 & 1 & -2 & 3 & 1 \end{matrix}$$

$$= (4 + 3, 2 + 2, -6 + 8) = (7,4,2)$$

$$A = \frac{1}{2}\|(7,4,2)\| = \frac{1}{2}\sqrt{7^2 + 4^2 + 2^2}$$

$$= \frac{1}{2}\sqrt{49 + 16 + 4} \quad = \frac{1}{2}\sqrt{69}$$

5. Find the area between points $A(3, -1,4)$ $Q(1, -1,3)$ and R$(4, -3,2)$

<u>Method 1</u>

$\vec{u} = (-2,0, -1)$

$\vec{v} = (1, -2, -2)$

$||\vec{u}|| = \sqrt{(-2)^2 + 0^2 + (-1)^2} = \sqrt{5}$

$||\vec{v}|| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$

$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||}$

$\cos\theta = \frac{(-2,0,-1)\cdot(1,-2,-2)}{(3)\sqrt{5}}$

$= \frac{-2+0+2}{(3)\sqrt{5}} = 0$

$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1} = 1$

$\therefore A = \frac{1}{2}\left(\sqrt{3}\right)\left(\sqrt{5}\right)(1) = \frac{3\sqrt{5}}{2}$

<u>Method 2</u>

$\text{proj}_u\, \vec{v} = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{v}}\, \vec{u}$

$= 0$ since $\vec{u} \cdot \vec{v} = 0$

$\therefore A = \frac{1}{2}\, ||\vec{u}|| ||\vec{v} - 0|| \qquad A = \frac{1}{2}\sqrt{5}(3) = \frac{3\sqrt{5}}{2}$

6. If $\vec{u}$=(5,4k,3,6) and $\vec{v}$=(-2,2,-3,k) are orthogonal, find k.

A. -1/14	B. 19/14	C. -19/14	D. none of the above

$$(5,4k,3,6)\cdot(-2,2,-3,k)=0$$

$$-10+8k-9+6k=0$$

$$14k=19$$

$$k=19/14$$

Therefore, the answer is B).

7. Find all values of c such that the vector $\vec{u}=(2c,3c,c,0)$is a unit vector.

$$\sqrt{(2c)^2+(3c)^2+c^2+0^2}=1$$

$$4c^2+9c^2+c^2=1$$

$$14c^2=1$$

$$c^2=\frac{1}{14} \qquad c=\pm\sqrt{\frac{1}{14}}=\pm\frac{1}{\sqrt{14}}$$

8. If $\vec{v}=k\vec{u}$ is a unit vector in the opposite direction to the vector $\vec{u}=(3,3,3,3)$, what is the value of k?

A. 6	B.-6	C.3	D. -3	E. -1/6

$$||\vec{u}||=\sqrt{3^2+3^2+3^2+3^2}$$

$$=\sqrt{9+9+9+9}=\sqrt{36}=6$$

$\vec{v}=k(3,3,3,3)=\left(\frac{-3}{6},\frac{-3}{6},\frac{-3}{6},\frac{-3}{6}\right)$ ←unit vector in opposite direction

$$\therefore k=\frac{-1}{6}$$

Therefore, the answer is E).

9. Let $\vec{u}, \vec{v}, and\ \vec{w}\ be\ vectors\ in\ R^3\ and\ let\ c\ and\ d\ be\ scalars.$

Which of the following operations are defined?

i) $c\vec{u} + d\vec{v}$

ii) $\vec{u} + (c\vec{w})$

iii) $\vec{u} \cdot (\vec{v} \cdot \vec{w})$

iv) $c + \vec{w}$

A. i), ii) and iii) only	B. i) and ii) only	C. i), ii) and iv) only	D. i) and iii) only	E. none of the above

i) defined, ie. you can do it ii) defined iii) not defined as once you do the dot product in the brackets, you can't dot a vector with the scalar you got in the brackets iv) no defined as you can't add a scalar (number) with a vector

Therefore, the answer is B).

10. Let $\vec{u} = (2, -1, 6) and\ \vec{v} = (2,2,2) be\ vectors. Find\ cos\theta.$

$$\vec{u} \cdot \vec{v} = (2, -1, 6) \cdot (2,2,2)$$

$$= 4 - 2 + 12 = 14$$

$$||\vec{u}|| = \sqrt{2^2 + (-1)^2 + 6^2} = \sqrt{4 + 1 + 36} = \sqrt{41}$$

$$||\vec{v}|| = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12}$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\ ||\vec{v}||} = \frac{14}{\sqrt{41}\sqrt{12}}$$

11. Let θ be the angle between vectors $\vec{u} = (1, -4), \vec{v} = (k, 3). If\ cos\ \theta = 0, find\ k.$

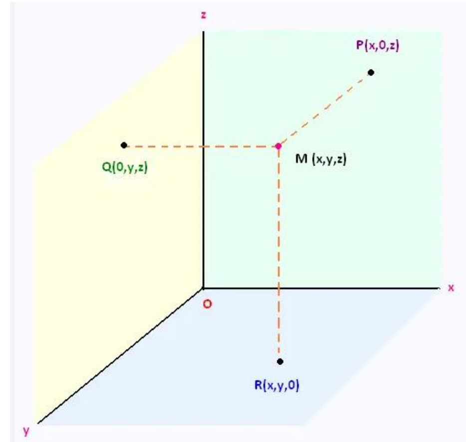
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\ \ ||\vec{v}||} = 0$$

$$\frac{(1,-4)\cdot(k,3)}{\sqrt{1^2+(-4)^2}\ \sqrt{k^2+3^2}} = 0$$

$$k - 12 = 0$$

$$k = 12$$

12. Find the angle between the diagonals on two adjacent sides of a cube with side lengths 2.



$\vec{u} \cdot \vec{v} = [0,2,2] \cdot [2,0,2]$

$= 0 + 0 + 4 = 4$

$||\vec{u}|| = \sqrt{0^2 + 2^2 + 2^2} = \sqrt{8}$

$||\vec{v}|| = \sqrt{2^2 + 0^2 + 2^2} = \sqrt{8}$

$\therefore \cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||} = \frac{4}{\sqrt{8}\sqrt{8}} = \frac{4}{8}$

$\cos\theta = \frac{1}{2}$

$\therefore \theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta = 60°$ (from special triangle's)

13. Find all value(s) of *k* such that the expression $(2,-1,1)+k(-4,1,-2)$ is a unit vector.

First, let's simplify the expression:

$(2,-1,1)+k(-4,1,-2)=(2-4k,-1+k,1-2k)$

To be a unit vector, we require that $\|\vec{v}\|=\sqrt{x^2+y^2+z^2}=1$ or equivalently $x^2+y^2+z^2=1^2=1$. The second equation is easier to solve because the square root is gone.

So, let's figure out what value(s) of *k* make this true.

$$\begin{aligned} x^2+y^2+z^2 &= (2-4k)^2+(-1+k)^2+(1-2k)^2 \\ &= (2-4k)(2-4k)+(-1+k)(-1+k)+(1-2k)(1-2k) \\ &= (4-16k+16k^2)+(1-2k+k^2)+(1-4k+4k^2) \\ &= 6-22k+21k^2 \\ &= 1 \end{aligned}$$

$21k^2-22k+5=0$

Use the quadratic formula.

a = 21

b = -22

c = 5

$$\begin{aligned} k &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{22\pm\sqrt{22^2-4(21)(5)}}{2(21)} \\ &= \frac{22\pm\sqrt{484-420}}{42} \\ &= \frac{22\pm\sqrt{64}}{42} \\ &= \frac{22\pm 8}{42} \\ &= \frac{30}{42},\frac{14}{42} \\ &= \frac{5}{7},\frac{1}{3} \end{aligned}$$

Therefore, if *k = 1/3* or *5/7*, then the expression produces a **unit vector**.

14. $\vec{u} \times \vec{v}$ *is only defined in* R^3

i) defined ii) defined iii) undefined iv) undefined v) undefined

Therefore, the answer is D).

How would your answer differ if it were in R^3? iii) and iv) would be undefined since when you do the brackets first, you get a number since it is dot product and then you can't dot or cross a vector with a number, so you can't do it!

15. Find the cross product $\vec{v} \times \vec{u}$ where $\vec{u}$=(5,2,-1) and $\vec{v}$=(-1,2,3).

$$\vec{v} \times \vec{u} \quad \begin{matrix} -1 & 2 & 3 & -1 & 2 & 3 \\ 5 & 2 & -1 & 5 & 2 & -1 \end{matrix}$$

$$= (-2-6,\ \ 15-1,\ \ -2-10)$$

$$= (-8, 14, -12)$$

16. $\vec{u} = b - a = (-1,\ 1, -1)$

$$\vec{v} = c - a = (0,2,-1)$$

$$\begin{matrix} -1 & 1 & -1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 0 & 2 & -1 \end{matrix}$$

$$\vec{u} \times \vec{v} = (-1+2, 0-1, -2-0)$$

$$= (1, -1, -2)$$

$$A = \frac{1}{2}\ \|(1, -1, -2)\|$$

$$A = \frac{1}{2}\sqrt{1^2 + (-1)^2 + (-2)^2}$$

$$= \frac{1}{2}\sqrt{6} \quad or \quad \frac{\sqrt{6}}{2}\ units^2$$

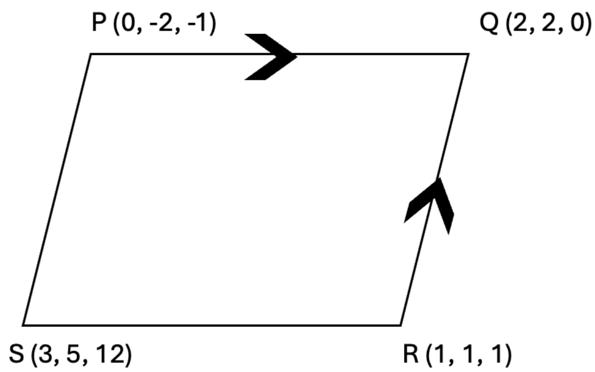
17. $\vec{u} \times \vec{v}$ $\begin{matrix} 1 & 3 & -1 & 1 & 3 & -1 \\ -2 & 4 & -1 & -2 & 4 & -1 \end{matrix}$

$= (-3+4, 2+1, \ 4+6)$

$= (1,3,10)$

$A = \|(1,3,10)\| = \sqrt{1+9+100} = \sqrt{110}\, units^2$

18. P(0,-2,-1) Q(2,2,0) R (1,1,1) and S(3,5,2)



$\vec{u} = q - p = (2, 4, 1)$

$\vec{v} = q - r = (1,1,-1)$

$\vec{u} \times \vec{v}$ $\begin{matrix} 2 & 4 & 1 & 2 & 4 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \end{matrix}$

$= (-4-1, 1+2, 2-4)$

$A = \|(-5,3,-2)\|$

$= \sqrt{(-5)^2 + 3^2 + (-2)^2} = \sqrt{25+9+4} = \sqrt{38}\, units^2$

3. Lines and Planes

Example 3.1. Find the normal form for the line containing the point P(7, 8, 9) with normal vector $\vec{n}$=(1, 2, 3).

Solution:

$\vec{n} \cdot (x - p) = 0$

$(1,2,3) \cdot \big(x - (7,8,9)\big) = 0$ or $(1,2,3) \cdot (x - 7, x - 8, x - 9) = 0$

Example 3.2. Find the equation in general form (standard equation) each case below.

a) containing the point P(4, 5, 6) with normal vector $\vec{n}$=(1, 2, 3).

Solution:

Step 1. Find d.

$d = \vec{n} \cdot p = (1,2,3) \cdot (4,5,6) = 4 + 10 + 18 = 32$

Step 2. Write $ax + by + cz = \vec{n} \cdot p$ where $\vec{n}$=(a,b,c)

In this question, $\vec{n}$=(1,2,3) so a=1, b=2 and c=3

Therefore, the equation in standard form is: x+ 2y + 3z = 32

OR don't use d at all and go straight to ax+by+cz=$\vec{n} \cdot p$

b) containing the point P(0, -1, 2) with $\vec{n}$=(5, 3, 4)

Solution:

$a = 5, \quad b = 3, \quad c = 4$

$5x + 3y + 4z = (5,3,4) \cdot (0, -1,2)$

$5x + 3y + 4z = 0 - 3 + 8$

$5x + 3y + 4z = 5$

Example 3.3. Give a vector form equation for the line passing through P(1,2,3) and parallel to the vector $\vec{d}$ =(4,5,6).

Solution:

$\vec{x} = p + t\vec{d}$

$$x = (1,2,3) + t(4,5,6)$$

$$(x, y, z) = (1,2,3) + t(4,5,6)$$

If we solve for x, y and z, separately, we get the PARAMETRIC EQUATIONS for the line.

(x,y,z)= (1,2,3) + (4t, 5t, 6t)= (1+4t, 2+5t, 3+6t)

$$x = 1 + 4t$$

$$y = 2 + 5t$$

$$z = 3 + 6t$$

Example 3.4. Find a vector form equation and the parametric equations for the line passing through the point P(-1,4,3) and parallel to the vector $\vec{d}$ = (5,0,-1).

Solution:

$\vec{x} = p + t\vec{d}$

$(x, y, z) = (-1,4,3) + t(5,0,-1)$

$x = -1 + 5t$

$y = 4$

$z = 3 - t$

Example 3.5. Find a vector form equation for the line passing through the points P(-1,4,3) and

Solution:

Q(-4,5,-6).

$P(-1,4,3) \quad Q\ (-4,5,-6)$

$\vec{d} = \overrightarrow{PQ} = Q - P = (-4,5,-6) - (-1,4,3)$
$\vec{d} = (-3,1,-9)$

$\vec{x} = P + t\vec{d}$
$\vec{x} = (-1,4,3) + t(-3,1,-9)$

Example 3.6. Find the vector form and parametric form equations of the plane that contains the point P(5,1,2) and has normal vector (1,3,2).

Solution:

$P\ (5,1,2) \quad \vec{n}\ (1,3,2)$

General $ax + by + cz = \vec{n} \cdot p$

$1x + 3y + 2z = (1,3,2) \cdot (5,1,2)$

$x + 3y + 2z = 12$

NOTE: To find points, just use any x,y,z that make the left side equal to the right side, which is 12 in this question.

$P\ (5,1,2)\ \ Q = (12,0,0)\ \ R = (0,0,6)$

$$\vec{u} = Q - P = (12,0,0) - (5,1,2) = (7,-1,-2)$$

$$\vec{v} = R - P = (0,0,6) - (5,1,2) = (-5,-1,4)$$

$$\vec{x} = p + s\vec{u} = t\vec{v}$$

Vector form of plane → $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} + s\begin{bmatrix} 7 \\ -1 \\ -2 \end{bmatrix} + t\begin{bmatrix} -5 \\ -1 \\ 4 \end{bmatrix}$

Parametric $\begin{cases} x = 5 + 7s - 5t \\ y = 1 - s - t \\ z = 2 - 2s + 4t \end{cases}$

Example 3.7. Find the distance from point Q(1,3) to the line 2x - 4y = 8.

Solution:

a=2 and b= -4 from the normal of the general equation.

Also, c=8 from this equation

From Point Q(1,3), $x_0 = 1 \; and \; y_0 = 3$.

$$d(P,L) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}} = \frac{|(2)(1) + (-4)(3) - 8|}{\sqrt{2^2 + (-4)^2}} = \frac{18}{\sqrt{20}} = \frac{18}{\sqrt{4}\sqrt{5}}$$

The distance is $\frac{9}{\sqrt{5}}$ units.

Example 3.8. Find the distance from point Q (2,5) to the line L: $(x, y) = (-1,4) + t(1, -1)$

Solution:

Method 1

$d(P, L) = d(v, proj_d v)$

Step 1.

$\vec{d} = (1, -1)$

$\vec{v} = q - p = (2,5) - (-1,4) = (3,1)$

Step 2. Find the projection of $\vec{v}$ onto $\vec{d}$ using

$proj_d v = \frac{\vec{d} \cdot \vec{v}}{\vec{d} \cdot \vec{d}} \vec{d} = \frac{(1,-1) \cdot (3,1)}{(1,-1) \cdot (1,-1)} (1, -1) = \frac{2}{2} (1, -1) = (1, -1)$

Step 3. Find $\vec{v}$ - $proj_d v$

(3,1) - (2,-2) = (1,-1)

Step 4. Find d(P,L)=$\|v - proj_d v\|$=$\|(3,1) - (1, -1)\| = \|(2,2)\| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Method 2

An easier way to do this is:

Distance from a Point Q(x_0, y_0) to a Line is $\boldsymbol{d}(P, L) = \frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}$ where ax+by=c.

Note: in R^2, if $\vec{d} = (1, -1)$ then $\vec{n} = (1, 1)$ to find $\vec{n}$ from $\vec{d}$ in R^2 (or vice versa)

4. Switch the sign of y -coordinate and
4. Switch x and y

From point Q(2,5), $x_0 = 2$ *and* $y_0 = 5$.

ax + by=$\vec{n} \cdot p$

x+y=$(1,1) \cdot (-1,4)$

x+y = 3

$$d(P, L) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}} = \frac{|(1)(2) + (1)(5) - 3|}{\sqrt{1^2 + 1^2}} = \frac{4}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

<u>Example 3.9.</u> Find the distance from the point Q(1,3,5) to the plane P:5x + y - 3z = 0.

From the plane,$\vec{n} = (5,1,-3)$, so a=5, b=1 and c= -3. The right side of the equation is d=0.

Also, Q=(1,3,5) so $x_0 = 1$, $y_0 = 3$ *and* $z_0 = 5$.

$$d(Q,P)=\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}} = \frac{|(5)(1)+(1)(3)+(-3)(5)-0|}{\sqrt{5^2+1^2+(-3)^2}} = \frac{|8-15|}{\sqrt{35}} = \frac{7}{\sqrt{35}} = \mathbf{\frac{7\sqrt{35}}{35}}\ \boldsymbol{or}\ \mathbf{\frac{\sqrt{35}}{5}}$$

<u>Example 3.10.</u> One plane P_1 has equation 4x - 2y - 7z = 3. Determine if another plane P_2 : 4x + y + 2z = 2 is parallel, perpendicular, identical or none of these.

Solution:

$\overrightarrow{n_1} = (4, -2, -7)$

$\overrightarrow{n_2} = (4, 1, 2)$

Check to see if their dot-product is zero.

$\overrightarrow{n_1} \cdot \overrightarrow{n_2} = (4, -2, -7) \cdot (4, 1,2) = 16 - 2 - 14$=0

Therefore, the planes 4x - 2y + 7z = 3 and 4x + y + 2z = 2 are perpendicular.

Example 3.11. Two planes have equations 5x - 2y + 3z = 3 and 10x - 4y + 6z = 6. Determine if they are parallel, perpendicular, identical, or none of these.

Solution:

$\overrightarrow{n_1} \cdot \overrightarrow{n_2} \neq 0$, so they are not perpendicular planes.

$\overrightarrow{n_1} = (5, -2,3)$ and $\overrightarrow{n_2} = (10, -4,6)$

Notice that the normal vectors are scalar multiples. ie. $2\overrightarrow{n_1} = \overrightarrow{n_2}$

Therefore, the planes are parallel. Check for identical:

The constant terms on the right are a multiple of 2 as well, so they are identical planes.

Example 3.12. Find a general form equation of a line through (2,3) and perpendicular to the line 5x - 4y =2.

Solution:

Perpendicular to $5x - 4y = 2$ $\overrightarrow{n_1} = (5, -4)$ $\overrightarrow{d_1} = (4,5)$

General (Standard) form $ax + by = \vec{n} \cdot p$

$\overrightarrow{n_2} = \overrightarrow{d_1} = (4,5)$ since perpendicular

$\therefore 4x + 5y = (4,5) \cdot (2,3)$

$4x + 5y = 8 + 15$

$4x + 5y = 23$

Example 3.13. Find the vector form of the equation of a line through point (1,2) and parallel to the line $(-2,6)\cdot\big(x-(1,-6)\big)=0$.

Solution:

$(-2,6)\cdot\big(x-(1,-6)\big)=0.$

This equation is in normal form with:

$\overrightarrow{n_1}=(-2,6) \quad \therefore\ \overrightarrow{d_1}=(-6,-2)\ or\ (6,2)$

Vector form is: $\vec{x}=p+t\vec{d}$

Since our new line is parallel to the original line, we have:

$\overrightarrow{d_2}=\overrightarrow{d_1}=(6,2)$

$\vec{x}=(1,2)+t(6,2)$

Example 3.14. Find the vector form of the equation of a line through point (1,2) and perpendicular to the line$(-2,6)\cdot\big(x-(1,-6)\big)=0$.

Solution:

$(-2,6)\cdot\big(x-(1,-6)\big)=0.$

This equation is in normal form with:

$\overrightarrow{n_1}=(-2,6)$

Vector form is: $\vec{x}=p+t\vec{d}$

Since our new line is perpendicular to the original line, we have:

$\overrightarrow{d_2}=\overrightarrow{n_1}=(-2,6)$

$\vec{x}=(1,2)+t(-2,6)$

Example 3.15. Find the point of intersection of the line (x,y,z)=(5,-1,-2)+t(1,2,3) with the plane $3x + 2y + z = 21$.

Solution:

$$x = 5 + t, \quad y = -1 + 2t, \quad z = -2 + 3t$$

$$3x + 2y + z = 21$$

$$3(5 + t) + 2(-1 + 2t) + (-2 + 3t) = 21$$

$$15 + 3t - 2 + 4t - 2 + 3t = 21$$

$$10t + 11 = 21$$

$$10t = 21 - 11$$

$$10t = 10$$

$$t = 1$$

$$\therefore x = 5 + 1 = 6$$

$$y = -1 + 2(1) = 1$$

$$z = -2 + 3(1) = 1 \qquad \therefore POI\ (6,1,1)$$

Example 3.16. Find the point of intersection of the two lines given below:

$$x = 1 + t \quad y = 3 - 4t$$

$$x = 2 + 4s \quad y = 3 - 2s$$

Solution: At the point of intersection, the x-values are equal for both lines and so are the y-values. So, we set them equal:

$$x = x \qquad y = y$$

$$2 + 4s = 1 + t \qquad 3 - 2s = 3 - 4t$$

$$4s - t = -1 \;\boxed{1} \qquad -2s + 4t = 0 \;\boxed{2}$$

Now, write the equations on top of each other and eliminate one of the variables:

$$4s - t = -1$$

$$-2s + 4t = 0 \quad (\times 2)$$

$$4s - t = -1$$

$$-4s + 8t = 0$$

Add $7t = -1$

$$t = -\frac{1}{7}$$

Now, substitute the value of t back into the parametric form equations of the line.

$$x = 1 - \frac{1}{7} = \frac{7}{7} - \frac{1}{7} = \frac{6}{7}$$

$$y = 3 - 4\left(\frac{-1}{7}\right) = \frac{21}{7} + \frac{4}{7} = \frac{25}{7} \quad \therefore POI\ \left(\frac{6}{7}. \frac{25}{7}\right)$$

Example 3.17.

You can find the normal using the cross product or you can use matrices:

Remember, your normal can be any multiple of what you get for the cross product as well!!

Method 1: Use Cross product to find the normal vector

$$\begin{array}{lcccccc} \vec{u} & 1 & 1 & -1 & 1 & 1 & -1 \\ \vec{v} & 2 & 1 & 1 & 2 & 1 & 1 \end{array}$$

$\vec{n} = \vec{u} \times \vec{v} = (1+1, -2-1, 1-2)$

$\vec{n} = (2, -3, -1)$

$ax + by + cz = \vec{n} \cdot p$

$2x - 3y - z = (-2, 3, 1) \cdot (1, 2, 3)$

$2x - 3y - z = -7$

Method 2: Use Determinants to find the normal vector

$$\begin{array}{lccc} \vec{u} & 1 & 1 & -1 \\ \vec{v} & 2 & 1 & 1 \end{array}$$

Using determinant=$ad - bc$

$$\left(\det\begin{bmatrix}1 & -1 \\ 1 & 1\end{bmatrix}, \boxed{-}\det\begin{bmatrix}1 & -1 \\ 2 & 1\end{bmatrix}, \det\begin{bmatrix}1 & 1 \\ 2 & 1\end{bmatrix}\right)$$

$\therefore \vec{n} = (2, -3, -1)$

$ax + by + cz = \vec{n} \cdot p$

$2x - 3y - z = (-2, 3, 1) \cdot (1, 2, 3)$

$2x - 3y - z = -7$

Method 3: Use matrices to find the normal

We solve the homogenous system:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \text{R2-2R1} \rightarrow \text{R2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \text{R2 x -1} \rightarrow \text{R2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \text{R1 – R2} \rightarrow \text{R1}$$

$x \quad y \quad z$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

Let $z = t$

$x = -2t$

$y = 3t$

$\vec{n} = (x, y, z) = (-2t, 3t, t) = t(-2, 3, 1)$

General form:

$ax + by + cz = \vec{n} \cdot p$

$-2x + 3y + z = (-2, 3, 1) \cdot (1, 2, 3)$

$-2x + 3y + z = -2 + 6 + 3$

$-2x + 3y + z = 7$

or $2x - 3y - z = -7$

3.12 Homework on Chapter 3

1. Find the normal form for the line containing the point P(0, -1, 2) with $\vec{n}$=(5, 3,64).

$$(5,3,64)\cdot\big(x-(0,-1,2)\big)=0$$

2. Find the equation in general form (standard equation) containing the point P(2, 2, -1) with $\vec{n}$= (1, 3, -2)

$a=1, \quad b=3, \quad c=-2$

$$1x+3y-2z=(1,3,-2)\cdot(2,2,-1)$$

$$x+3y-2z=2+6+2$$

$$x+3y-2z=10$$

3. Find the parametric equations for the line passing through the point P(2,-3,4) and parallel to the vector $\vec{d}=$ (4,1, -2).

$$(x,y,z)=(2,-3,4)+t(4,1,-2)$$

$$x=2+4t$$

$$y=-3+t$$

$$z=4-2t$$

4. Find a set of parametric equations for the line passing through the two points P(1,4,6) and Q(3,4,7).

$$\vec{d}=q-p=(2,0,1)$$

$$\vec{x}=p+t\vec{v}$$

$$(x,y,z)=(1,4,6)+t(2,0,1)$$

$$x=1+2t \qquad y=4 \qquad z=6+t$$

5. Find the vector equation of a plane through the points P(5,3,4) Q(1,-1,2) and R(3,4,6).

First, find the direction vectors $\vec{u} = q - p$ and $\vec{v} = r - p$

$\vec{u} = (1,-1,2) - (5,3,4) = (-4,-4,-2)$

$\vec{v} = r - p = (3,4,6) - (5,3,4) = (-2,1,2)$

Using point P, we get the equation: $x = (5,3,4) + s(-4,-4,-2) + t(-2,1,2)$

6. Find the distance from point Q(1,2,-1) to the plane P: 4x - 2y + 3z = 0

From the plane, a=4, b= -2 and c=3 and d=0. From point Q, $x_0 = 1$, $y_0 = 2$, and $z_0 = -1$

$$\mathrm{d}(\mathrm{Q},\mathrm{P}) \frac{|(4)(1) + (-2)(2) + (3)(-1) - 0|}{\sqrt{4^2 + (-2)^2 + 3^2}} = \frac{3}{\sqrt{29}}$$

7. Find the distance from Q(5,4) to the line (x,y)=(2,6) + t(1,-1).

Step 1.

The point P on the line is P=(2,6). Point Q=(5,4) and the direction vector is $\vec{d} = (1,-1)$

$\vec{v} = q - p = (5,4) - (2,6) = (3,-2)$

Step 2.

Find the projection of $\vec{v}$ onto $\vec{d}$ using

$\mathrm{proj}_{\mathrm{d}}\mathrm{v} = \frac{\vec{d}\cdot\vec{v}}{\vec{d}\cdot\vec{d}}\vec{d} = \frac{(1,-1)\cdot(3,-2)}{(1,-1)\cdot(1,-1)}(1,-1) = \frac{5}{2}(1,-1) = (\frac{5}{2}, -\frac{5}{2})$

Step 3. Find $\vec{v}$ - $\mathrm{proj}_{\mathrm{d}}\mathrm{v}$

$(3,-2) - \left(\frac{5}{2}, -\frac{5}{2}\right) = \left(\frac{6}{2} - \frac{5}{2}, -\frac{4}{2} + \frac{5}{2}\right) = (\frac{1}{2}, \frac{1}{2})$

Step 4. Find $\mathrm{d}(\mathrm{Q},\mathrm{L}) = \|v - proj_d v\| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

Method 2:

From Point Q(5,4), $x_0 = 5 \ and \ y_0 = 4$.

From the line, $\vec{d} = (1, -1)$, so the normal is $\vec{n}$=(1,1) and a=1, b=1.

The equation is ax + by=$\vec{n} \cdot p$

x-y = (1,1) · (2,6)

x – y = 8.

$$d = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}} = \frac{|(1)(5) + (1)(4) - 8|}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}} \ or \frac{\sqrt{2}}{2}.$$

8. Find the vector form and parametric equations of a plane through points P(2,1,4) Q(1,-1,0) and R(1,3,4).

$\vec{x} = p + s\vec{u} + t\vec{v}$

$\vec{u} = q - p = (1, -1, 0) - (2,1,4) = (-1, -2, -4)$

$\vec{v} = r - p = (1,3,4) - (2,1,4) = (-1,2,0)$

Therefore, the vector form of the equation is:

$\vec{x} = (2,1,4) + s(-1, -2, -4) + t(-1,2,0)$

And, the parametric equations are: x=2 - s - t, y=1 -2s + 2t and z=4 -4s

9. Find the general form equation of the plane passing through the point P(1,2,1) with normal

$\vec{n}$=(-1,3,4).

$-x + 3y + 4z = (-1,3,4) \cdot (1,2,1)$

$-x + 3y + 4z = -1 + 6 + 4$

$-x + 3y + 4z = 9$

10. Give the parametric equations of the line passing through the points (1,2,3) and (3,1,-5).

$\vec{d} = q - p = (2, -1, -8)$

$\vec{x} = (1,2,3) + t(2, -1, -8)$

$x = 1 + 2t$

$y = 2 - t$

$z = 3 - 8t$

NOTE: If you used point Q, your parametric equations would look different, but they would still be correct

11. Which point lies on the plane 4x –y + z =10?

A. (1,-2,1)	B. (2,0,2)	C. (2,-2,1)	D. (3,1,1)	E. None of the above

See which point gives $LS = RS$ Check (2,0,2)

$LS = 4(2) - 0 + 2 = 10$

$RS = 10 \qquad \therefore LS = RS$

Therefore, the answer is B).

12. Find a vector form equation through P (1,2,4) and parallel to a line with parametric equations

x=-3s+2, y=4s+5 and z=2s-7.

The direction vector is the numbers in front of s, so $\vec{d} = (-3,4,2)$ and point P is (1,2,4)

So, the equation is $\therefore \vec{x} = p + td$

$\vec{x} = (1,2,4) + t(-3,4,2)$

13. Two planes have equations π_1: 4x - 2y + 6z = 3 and π_2:6x - 3y + 9z = 4.5. Determine if they are parallel, perpendicular, identical, or none of these.

$\vec{n}_1 = (4, -2,6)$ $\quad\quad$ $\vec{n}_2 = (6, -3,9)$

$$\vec{n}_1 \times \frac{3}{2} = \vec{n}_2 \quad and \;\; 3 \times \frac{3}{2} = 4.5$$

∴The constant terms on the right follow the same multiple as the normals ∴They're identical planes

14. Two planes have equations π_1: 4x - 2y +2z = 1 and π_2: 5x - 3y - 15z = 5. Determine if they are parallel, perpendicular, identical, or none of these.

$$(4, -2,2) \cdot (5, -3, -15) = 20 + 6 - 30 \neq 0$$

∴not perpendicular

$\vec{n}_1$ is not a multiple of $\vec{n}_2$ ∴not identical or parallel

Therefore, the answer is none of these.

15. Find the distance between the parallel lines:

$$\ell_1: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + s \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

and

$$\ell_2: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Since the lines are parallel, choose arbitrary points Q on ℓ_1 and P on ℓ_2,

Let $Q = (1,2)$ and P $= (5,3)$

The direction vector of $\ell_1$and ℓ_2 is $\vec{d} = (-3,4)$

Now, $\vec{v} = \overrightarrow{PQ} = q - p = (1, 2) - (5, 3)$

$\vec{v} = (-4, -1)$

$$\text{proj}_{\vec{d}}\, \vec{v} = \left(\frac{\vec{d} \cdot \vec{v}}{\vec{d} \cdot \vec{d}}\right) \vec{d}$$

$$= \frac{(-3, 4) \cdot (-4, -1)}{(-3, 4) \;\cdot (-3, 4)} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$= \frac{12-4}{9+16}\begin{bmatrix}-3\\4\end{bmatrix}$

$= \frac{8}{25}\begin{bmatrix}-3\\4\end{bmatrix}$

∴ the distance between the lines is

$||\ \vec{v} - \text{proj}_{\vec{d}}\ \vec{v}||$

$= \left\|\begin{bmatrix}-4\\-1\end{bmatrix} - \frac{8}{25}\begin{bmatrix}-3\\4\end{bmatrix}\right\|$

$= \left\|\begin{bmatrix}-4\\-1\end{bmatrix} + \begin{bmatrix}\frac{24}{25}\\\frac{32}{25}\end{bmatrix}\right\|$

$= \left\|\begin{bmatrix}-\frac{100}{25}\\-\frac{25}{25}\end{bmatrix} + \begin{bmatrix}\frac{24}{25}\\\frac{32}{25}\end{bmatrix}\right\|$

$= \left\|\begin{bmatrix}-\frac{76}{25}\\\frac{7}{25}\end{bmatrix}\right\| = \left\|\frac{1}{25}\begin{bmatrix}-76\\7\end{bmatrix}\right\|$

$= \frac{1}{25}\sqrt{(-76)^2 + 7^2} = \frac{\sqrt{5825}}{25}$

Note: you would need to use a calculator for the last step

16. Find the distance between these parallel planes:

$P_1: 2x + y - 3z = 0$ and $P_2: 2x + y - 3z = 2$

Since P_1 and P_2 are parallel, choose any arbitrary point on $P_1, Q = (0,0,0)$ and find $d(Q, P_2)$.

P_2 has equation $2x + y - 3z = 2$, so a=2, b=1, c=-3 and d=2. Since Q=(0,0,0) we know

$x_0 = y_0 = z_0 = 0$

$\therefore d\ (P_1, P_2) = d(\ Q, P_2)$

$= \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 +\ b^2 +\ c^2}}$

$= \frac{|2(0) + 1(0) - 3(0) - 2|}{\sqrt{2^2 +\ 1^2 +\ (-3)^2}}$

$d\ (P_1, P_2) = \frac{|-2|}{\sqrt{4+1+9}} = \frac{2}{\sqrt{14}}$ or $\frac{2\sqrt{14}}{14} = \frac{\sqrt{14}}{7}$

17. Find the acute angle between planes

$P_1: x + y + 2z = 0$ and $P_2: 2x + y - 3z = 0$

$P_1: \overrightarrow{n_1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad P_2: \overrightarrow{n_2} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$

$\therefore \cos\theta = \dfrac{\overrightarrow{n_1} \cdot \overrightarrow{n_2}}{\| \overrightarrow{n_1} \| \| \overrightarrow{n_2} \|}$

$= \dfrac{(1)(2) + (1)(1) + 2(-3)}{\sqrt{1^2 + 1^2 + 2^2}\,\sqrt{2^2 + 1^2 + (-3)^2}}$

$= \dfrac{-3}{\sqrt{6}\,\sqrt{14}}$

$\cos\theta = \dfrac{-3}{\sqrt{84}} \qquad \theta = \cos^{-1}\left(\dfrac{-3}{\sqrt{84}}\right)$

18. Find the point of intersection of the line x=(1,2,3) + t(-1,2,4) with the plane x - 2y + 4z = 20.

Step 1. The parametric equations are x=1-t and y= 2 + 2t and z= 3 + 4t

Step 2. Substitute them into the plane x - 2y + 4z = 20.

(1-t) -2(2+2t) +4(3+4t) = 20

1 – t – 4 – 4t +12 + 16t = 20

11t + 9 = 20

11t = 11

t=1

Step 3. Substitute t=1 into the parametric equations above.

x= 1 - t = 1- (1)= 0

y= 2+2t =2 + 2(1)= 4

z= 3 + 4t = 3 + 4(1)=7

So, the point of intersection is (0,4 ,7).

19. Find the point of intersection of the line x=(1,-1,0) + t(1,2,1) with the plane 3x – y + 2z = 10.

The parametric form equations of the line are: $x = 1 + t \quad y = -1 + 2t \quad z = t$

Substitute them into the plane to find t.

3x – y + 2z = 10

$3(1 + t) - (-1 + 2t) + 2(t) = 10$

$3 + 3t + 1 - 2t + 2t = 10$

$3t + 4 = 10$

$3t = 6$

$t = \frac{6}{3} = 2$ Now, substitute t back into the parametric equations to find the point of intersection.

$$x = 1 + t = 1 + 2 = 3 \quad y = -1 + 2t = -1 + 2(2) = 3 \quad z = t = 2$$

The point of intersection is (3,3,2).

20. Find the point of intersection of the two lines below:

$$x = -1 + 3t \qquad\qquad x = 1 + 2s$$

$$y = 3 - 2t \qquad\qquad y = 4 - s$$

$$x = x \qquad\qquad y = y$$

$$1 + 2s = -1 + 3t \qquad\qquad 4 - s = 3 - 2t$$

$$2s - 3t = -2 \qquad\qquad -s + 2t = -1$$

$$2s - 3t = -2$$

$$-s + 2t = -1 \quad \times 2$$

$$\overline{2s - 3t = -2}$$

$$-2s + 4t = -2$$

$$\overline{Add \qquad t = -4}$$

$x = -1 + 3(-4) = -13$ and y=$3 - 2t = 3 - 2(-4) = 11$. The point of intersection is (-13,11).

21. x=3 + s x=2 + t

y=3- 2s y=3- 4t

z=s z=t - 1

**** If there is an answer that says "no point of intersection", you want to make sure you check your final point of intersection in BOTH LINES to make sure it is the same for x, y AND z in both sets of parametric equations. If it isn't, the answer is "no point of intersection".**

We first start by dealing with just the first two variables, x and y as before:

$$x = x \qquad\qquad y = y$$

$$3 + s = 2 + t \qquad\qquad 3 - 2s = 3 - 4t$$

$$s - t = -1 \qquad\qquad -2s + 4t = 0$$

$s - t = -1 \quad \times 2$

$-2s + 4t = 0$

$2s - 2t = -2$

$-2s + 4t = 0$

$Add \quad 2t = -2$

$t = -1$

$s - t = -1$

$s - (-1) = -1$

$s + 1 = -1$

$s = -2$

Now, substitute the value of s into the equations involving s:

$x = 3 + s = 3 + 2 = 1$

$y = 3 - 2s = 3 - 2(-2) = 7$

$z = s = -2$

The point of intersection is (1,7,-2).

Now, substitute the value of t into the equations involving t:

$x = 2 + t = 2 - 1 = 1$

$y = 3 - 4t = 3 - 4(-1) = 7$

$z = t - 1 = -1 - 1 = -2$

$\therefore POI\ (1,7,-2)$

NOTE: This is the same point of intersection we got on the last page!!

22. Find the point R on ℓ that is closest to $Q = (2, 5)$ and $\ell: \vec{x} = (-1, 4) + t(1, -1)$

$\vec{v} = \overrightarrow{PQ}$ let $\vec{w} = \overrightarrow{PR} = Proj_d \vec{v}$

$\vec{v} = P + t\vec{d}$ and our line is $\vec{x} = (-1, 4) + t(1, -1)$

$\therefore P = (-1, 4)$ and $\vec{d} = (1, -1)$

Then, $\vec{v} = \overrightarrow{PQ} = q - p = \begin{bmatrix} 2 \\ 5 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Also, $\vec{w} = proj_d \vec{v} = \frac{\vec{d} \cdot \vec{v}}{\vec{d} \cdot \vec{d}} \vec{d}$

$$\vec{w} = \frac{(1, -1) \cdot (3, 1)}{(1, -1) \cdot (1, -1)} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\vec{r} = \vec{P} + \overrightarrow{PR} = \vec{P} + proj_d \vec{v}$

$= \vec{P} + \vec{w}$

$\vec{r} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$\therefore$ the point R on ℓ that is closest to Q is $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

4. Systems of Equations

Example 4.1. i) and iii) are linear...answer is D

Example 4.2. (3,5+s, t,3+s+t) is a solution with 2 parameters, so it is a plane of intersection.

Example 4.3. (1,2+t,t) is a solution with only 1 parameter, so it is a line of intersection.

Example 4.4. (2,3,4) is a solution with 0 parameters, so it is a point of intersection.

Example 4.5. Which of the following are in row-reduced echelon form?

If it is not, explain why. If it is row-reduced echelon form, solve the system.

a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$

Solution:

yes, it is the identity matrix, so it is a unique solution POI (2,3,4)

b) $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$

Solution:

no, the row of 0's must be in the bottom

c) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$

Solution:

yes...infinitely many solutions

the parameter is $x_4 = t$ (not a leading 1)

From the first row, we get $x_1 = -2t$

From the second row, we get $x_2 = 3$

and from the last row, we get $x_3 + t = 0 \; or \; x_3 = \; -t.$

The solution is (-2t, 3 -t, t) $where \; t \in R$, a line of intersection (one parameter)

d) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right]$

Solution:

yes, it has infinitely many solutions...let y=t be the parameter

The first row gives x+2t = 2 and we get x= 2 - 2t and the second gives z=5.

The solution is (2-2t, t, 5) $where\ t \in R$.

e) $\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right]$

Solution:

yes, and since the last row is 0=2, there is no solution

f) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Solution:

yes, and it has no solution even though row 4 says infinitely many because row 3 says no solution, so there is no solution to the whole system

g) $\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 3 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$

Solution:

yes, and there are 3 leading 1's, so 6 -3 = 3 parameters

Let $x_2 = r$ and $x_4 = s$ and $x_5 = t$

from the first row, we get $x_1 + 2r + 3s + t = 1\ and\ x_1 = 1 - 2r - 3s - t$

from the second row we get $x_3 + 2s + 2t = 3\ and\ x_3 = 3 - 2s - 2t$

and from the last row we get $x_6 = 1$.

The solution is (1-2r-3s-t, r, 3-2s-2t, s, t, 1 $where\ r, s, t \in R$.

Example 4.6. Solve the following system using Gauss-Jordan Elimination.

3x + 3y = 15

2x + 3y = 13

Solution:

$\left[\begin{matrix}3 & 3\\2 & 3\end{matrix}\middle|\begin{matrix}15\\13\end{matrix}\right]$ R1 × 1/3 →R1

$\left[\begin{matrix}1 & 1\\2 & 3\end{matrix}\middle|\begin{matrix}5\\13\end{matrix}\right]$ R_2 - $2R_1$→R_2

$\left[\begin{matrix}1 & 1\\0 & 1\end{matrix}\middle|\begin{matrix}5\\3\end{matrix}\right]$ R_1- R_2→R_1

$\left[\begin{matrix}1 & 0\\0 & 1\end{matrix}\middle|\begin{matrix}2\\3\end{matrix}\right]$

The POI is (2,3). This is a unique solution and a consistent system ie. has a solution.

Example 4.7. Solve using Gauss-Jordan Elimination.

12x - 3y = 6

-16x + 4y = -8

Solution:

$\left[\begin{matrix}12 & -3\\-16 & 4\end{matrix}\middle|\begin{matrix}6\\-8\end{matrix}\right] R_2 \times \frac{1}{4} \to R_2$

$\left[\begin{matrix}12 & -3\\-4 & 1\end{matrix}\middle|\begin{matrix}6\\-2\end{matrix}\right] R_1 \times 1/12 \to R_1$

$\left[\begin{matrix}1 & \frac{-1}{4}\\-4 & 1\end{matrix}\middle|\begin{matrix}\frac{1}{2}\\-2\end{matrix}\right] R_2 + 4R_1 \to R_2$

$\left[\begin{matrix}1 & \frac{-1}{4}\\0 & 0\end{matrix}\middle|\begin{matrix}\frac{1}{2}\\0\end{matrix}\right]$

Infinitely many solutions   Let y=t

$$x - \frac{1}{4}t = \frac{1}{2}$$

$$x = \frac{1}{2} + \frac{1}{4}t$$

The solution is $\left(\frac{1}{2} + \frac{1}{4}t, t\right) where\ t \in R$. It is a consistent system ie. has a solution.

Example 4.8. Solve using Gauss-Jordan Elimination.

4x - 8y = 6

x - 2y = 5

Solution:

$\left[\begin{array}{cc|c} 4 & -8 & 6 \\ 1 & -2 & 5 \end{array}\right]$ $R_1 \leftrightarrow R_2$

$\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 4 & -8 & 6 \end{array}\right]$ $R_2 - 4R_1 \rightarrow R_2$

$\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 0 & -12 \end{array}\right]$ $R_2 - 2R_1 \rightarrow R_2$

The last line says 0 0 / # so there is no solution

Example 4.9. Solve using Gauss-Jordan Elimination.

5x + 4y - z = 0

20y - 6z = 22

2z = 6

Solution:

$\left[\begin{array}{ccc|c} 5 & 4 & -1 & 0 \\ 0 & 20 & -6 & 22 \\ 0 & 0 & 2 & 6 \end{array}\right]$ $R_1 \times 1/5 \rightarrow R_1$

$R_2 \times 1/20 \rightarrow R_2$ and $R_3 \times 1/2 \rightarrow R_3$

$\left[\begin{array}{ccc|c} 1 & 4/5 & -1/5 & 0 \\ 0 & 1 & -3/10 & 11/10 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$R_2 + 3/10\, R_3 \rightarrow R_2$

$\left[\begin{array}{ccc|c} 1 & 4/5 & -1/5 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$R_1 - 4/5\, R_2 \rightarrow R_1$

$\left[\begin{array}{ccc|c} 1 & 0 & -1/5 & -8/5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$R_1 + 1/5\, R_3 \rightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The point of intersection is (-1,2,3)...exactly one solution

Solve: 5x + 4y - z = 0

20y - 6z = 22

2z = 6

From the last equation 2z=6, we get z=3. We can substitute this into 20y-6z = 22 and get:

20y - 6z = 22

20y – 6(3)=22

20y=22+18

20y=40

y=2

Lastly, we substitute y=2 and z=3 into the first equation to get:

5x +4y – z = 0

5x +4(2) – 3 =0-

5x +8-3=0

5x= -5

x= -1

And once again, we get the solution (-1,2,3).

Example 4.10. Solve using Gauss-Jordan Elimination.

4x -2y + 2z = 2

3x + 2y - 4z = 4

-6x + 3y - 3z =2

Solution:

$$\left[\begin{array}{ccc|c} 4 & -2 & 2 & 2 \\ 3 & 2 & -4 & 4 \\ -6 & 3 & -3 & 2 \end{array}\right] R_1 \times 1/4 \to R1$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 1/2 \\ 3 & 2 & -4 & 4 \\ -6 & 3 & -3 & 2 \end{array}\right] R_2 - 3R_1 \to R_2 \quad , R_3 + 6R_1 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 7/2 & -11/2 & 5/2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$ There is no solution since 0=5 is not true.

Example 4.11. Solve the following system using Gauss-Jordan Elimination:

$$2x_1 - 4x_2 + 6x_3 + 2x_4 = -6$$

$$2x_1 - x_2 + 3x_3 - x_4 = 0$$

Solution:

$$\left[\begin{array}{cccc|c} 2 & -4 & 6 & 2 & -6 \\ 2 & -1 & 3 & -1 & 0 \end{array}\right] R_1 \div 2 \to R1$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & -3 \\ 2 & -1 & 3 & -1 & 0 \end{array}\right] R_2 - 2R_1 \to R2$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & -3 \\ 0 & 3 & -3 & -3 & 6 \end{array}\right] R_2 \div 3 \to R2$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & -3 \\ 0 & 1 & -1 & -1 & 2 \end{array}\right] R_1 \div +2R2 \to R1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

Let x_3=s and x_4=t

x_1= 1 - s + t

x_2= 2 + s + t

The solution is (1 -s + t, 2+s+t, s, t) $where s, t \in R$...two parameters...therefore a plane of intersection

Example 4.12. Find the value of "k" so that the following system of equations has no solution, exactly one solution and infinitely many solutions.

$$4x - 2y = 6$$

$$12x + 2ky = 4$$

Solution:

$$\left[\begin{array}{cc|c} 4 & -2 & 6 \\ 12 & 2k & 4 \end{array}\right]$$

$R_2 - 3R1 \rightarrow R_2$

$$\left[\begin{array}{cc|c} 4 & -2 & 6 \\ 0 & 2k+6 & -14 \end{array}\right]$$

If k= -3, we get 0 0 /-14 which means no solution

If k≠ -3, we can row-reduce the matrix to get the identity matrix, so there is exactly one solution.

Since all k values except k=-3 give the identity, it is impossible to get a row of zeros...So, there is "no value of k" that will result in infinitely many solutions ie. 0 0/ 0

Example 4.13. $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 3 & 3 & 0 \\ 0 & 1 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & c^2-36 & c+6 \end{array}\right]$

For what value of c does the matrix above have:

a) a 3-parameter family of solutions? b) a 2-parameter family of solutions?

c) no solution? d) exactly one solution?

Solution:

a) If c= -6, we get 0 0 0 0 0 /0 which is infinitely many solutions...this will have two leading 1's and therefore, 5-2=3 parameters

b) If c≠6, -6, we can get 3 leading 1's and therefore 5-3=2 parameters

c) If c=6, we get: 0 0 0 0 0 /6 which means no solution

d) no value of c....we can't get a 5x5 identity since there are only three rows

Example 4.14. Find the value of k for which the matrix $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & k+1 & k-5 & 6 \end{array}\right]$ has a unique solution.

A. k= -1	B. k= 5	C. k= -1, 5	D. no value of k	E. all values of k

Solution:

A unique solution means we get the identity and with 4 columns we would need at least 4 rows to get a 4x4 identity. So, a unique solution is impossible. ie. D. is the solution, no value of k

Example 4.15. Find the value of k for which the matrix $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & k+1 & k-5 & 6 \end{array}\right]$ has a one parameter family of solutions, ie. one free variable.

A. k= -1	B. k= 5	C. k= -1, 5	D. no value of k	E. all values of k

Solution: It is the same matrix as the last example, so we know that a unique solution is impossible. For no solution we would need all 0's before the line and then the 6 after, ie. 0 0 0 0/ 6 but there is no value of k that will make both k+1 and k-5 equal to zero at the same time, so no solution is also impossible. Therefore, the system ALWAYS has infinitely many solutions and the answer is E.

Example 4.16. Find the value of k for which the matrix has no solution, infinitely many solutions and a unique solution.

$$\left[\begin{array}{cc|c} 1 & -3 & k \\ k & 1 & 4 \end{array}\right]$$

Solution: $R_2 - kR_1 \to R_2$

$$\left[\begin{array}{cc|c} 1 & -3 & k \\ 0 & 1+3k & 4-k^2 \end{array}\right]$$

infinitely many is impossible since k= -1/3 makes the left side 0 and

k=2,-2 makes the right side 0

no solution k= -1/3

unique k$\neq -1/3$

Example 4.17. Find the value of k for which there is a 2-parameter family of solutions:

$$\left[\begin{array}{cccc|c} 1 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & k-4 & k^2-16 \\ 0 & 0 & 0 & k^2-9 & k-3 \end{array}\right]$$

Solution:

$\#\, parameter = n - r = number\ of\ columns - rank$

$\#\, parameter = n - r = 4\ columns - 2\ leading\ 1's$

$= 2$

$\therefore$ *this means the last row must be all* $0's$ (since we already have two leading 1's)

$\therefore k^2 - 9 = 0 \quad and \quad k - 3 = 0 \qquad \therefore k = 3$ in order to have 2 parameters (free variables).

Example 4.18. Find the rank of each matrix.

a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

Solution:

R2-2R1→ $R2\ and$ R3-R1 → $R3$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ in RREF

So, the rank is 1....only count the non-zero 1's...same as the number of leading 1's

b) $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Solution:

do R3-R2 $\rightarrow R3$

$B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ do R1-2R2 $\rightarrow R1$

$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

rank=2

Example 4.19. $A = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 3 & 0 \end{array}\right]$ represents a homogeneous system because b=0 vector.

The system of equations 3x + 4y = 0 is a homogeneous system.

2x + 3y = 0

4.15 Homework on Chapter 4

1. Which of the following equations are not linear?

$i) 2x + 4y^2 + \sqrt{z} = 15$

$ii) -2x + 3z = 3y - 11$

$iii) xy - y + 2z + 3w = 6$

A. i) only	B. ii) only	C) iii) only	D. i) and iii) only	E. ii) and iii) only

i) and iii)...answer D

2. Which of the following equations are linear in x, y and z? Note: a, b and c ϵ R

$i) ax + b^2 y + \frac{1}{b} z = 12$

$ii) ax^2 + bz + \frac{c}{y} = 17$

$iii) \frac{a}{x} + \frac{z}{c} + yc = 12$

A. i) only	B. ii) only	C) iii) only	D. i) and iii) only	E. none of them

i) yes ii) no iii) no...answer is A

3. Which of the following are in row-reduced echelon form?

If it is not, explain why. If it is row-reduced echelon form, solve the system.

a) $\left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

Yes, and the last row indicates no solution, so it is inconsistent.

b) $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$

no, as the third leading 1 (last row) has a 1 above it and we can only have 0's above and below leading 1's

c) $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

it is a unique solution because it has the 2x2 identity. The solution is (0,2).

4. Write the augmented matrix for the system of equations below.

$x - 2y = 4 - 3z$

$x + 8 - 3y = 2z$

$y + 2z = 3$

Make sure the equations are in the proper form first, with all of the variables on the left and the constant terms on the right.

$x - 2y + 3z = 4$

$x - 3y - 2z = -8$

$y + 2z = 3$

The augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 1 & -3 & -2 & -8 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

5. Write the augmented matrix for the system of equations below.

$x + 2y = 6$

$x \quad = 3$

$-3 + x = y$

$$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 1 & 0 & 3 \\ 1 & -1 & 3 \end{array}\right]$$

6. Find the value of "k" so that the following system of equations has:

$5x - y = 2$

$6x + ky = 6$

a) no solution

b) infinitely many solutions

c) a unique solution (exactly one solution)

$\left[\begin{array}{cc|c} 5 & -1 & 2 \\ 6 & k & 6 \end{array}\right]$ $R_2 - R_1 \rightarrow R2$...if you want to avoid fractions, you must do this in several steps.

$\left[\begin{array}{cc|c} 5 & -1 & 2 \\ 1 & k+1 & 4 \end{array}\right]$ $R_1 \leftrightarrow R_2$

$\left[\begin{array}{cc|c} 1 & k+1 & 4 \\ 5 & -1 & 2 \end{array}\right]$ $R_2 - 5R_1 \rightarrow R2$

$\left[\begin{array}{cc|c} 1 & k+1 & 4 \\ 0 & -5k-6 & -18 \end{array}\right]$

a) if -5k-6=0 then -5k=6 and k= -6/5 and this will result in no solution

b) there is no value of k that will result in 0 0/ 0

c) Any other value of k can be row-reduced to the identity matrix...exactly one solution

If k≠ -6/5 there is exactly one solution

7. Find the value of "k" so that the following system of equations has:

$$2x - y = 2$$

$$3x + ky = 3$$

a) no solution

b) infinitely many solutions

c) a unique solution (exactly one solution)

$$\left[\begin{array}{cc|c} 2 & -1 & 2 \\ 3 & k & 5 \end{array}\right] \quad R_1 \div 2 \rightarrow R1$$

$$\left[\begin{array}{cc|c} 1 & -1/2 & 1 \\ 3 & k & 3 \end{array}\right] \quad R_2 - 3R_1 \rightarrow R_2$$

$$\left[\begin{array}{cc|c} 1 & -1/2 & 1 \\ 0 & k + \frac{3}{2} & 0 \end{array}\right]$$

a) there is no value of k that will result in no solution

b) if k=-3/2 or -1.5 there will be a row of zeros...infinitely many solutions

Any other value of k can be row-reduced to the identity matrix...exactly one solution

c) If $k \neq -3/2$ there is exactly one solution

8. $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & k^2 - 16 & 4 + k \end{array}\right]$

For what value of k does the matrix above have:

a) infinitely many solutions?

b) no solution?

c) exactly one solution?

If k= -4 we get a row of zeros

If k=4, we get 0 0 0/ 8...no solution

If $k \neq 4, -4$ we get a unique solution

9. Solve using Gauss-Jordan Elimination. (Find the RREF and solve)

4x + 2y = 18   3x - y = 16

$\left[\begin{array}{cc|c} 4 & 2 & 18 \\ 3 & -1 & 16 \end{array}\right] R_1 \div 4 \rightarrow R_1$

$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{9}{2} \\ 3 & -1 & 16 \end{array}\right] R_2 - 3R_1 \rightarrow R_2$

$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & \frac{-5}{2} & \frac{5}{2} \end{array}\right] R_2 \times (-\frac{2}{5}) \rightarrow R_2$

$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & 1 & -1 \end{array}\right] R_1 - \frac{1}{2}R_3 \rightarrow R_1$

$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1 \end{array}\right]$ The solution is (5,-1). There is a solution (unique), so it is consistent.

10. Solve using Gauss-Jordan Elimination. (Find the RREF and solve)

2x - 6y + 6z = -8

4x + 6y - 2z = 30

4x - 3y - z = 19

$\left[\begin{array}{ccc|c} 2 & -6 & 6 & -8 \\ 2 & 3 & -1 & 15 \\ 4 & -3 & -1 & 19 \end{array}\right]$ Divide row 2 by 2

$\left[\begin{array}{ccc|c} 1 & -3 & 3 & -4 \\ 2 & 3 & -1 & 15 \\ 4 & -3 & -1 & 19 \end{array}\right]$

R2-2R1 $\rightarrow R2$

R3-4R1 $\rightarrow R3$

$\left[\begin{array}{ccc|c} 1 & -3 & 3 & -4 \\ 0 & 9 & -7 & 23 \\ 0 & 9 & -13 & 35 \end{array}\right]$

R2÷ 9 $\rightarrow R2$

R3-R2 $\rightarrow R3$

$\left[\begin{array}{ccc|c} 1 & -3 & 3 & -4 \\ 0 & 1 & -7/9 & 23/9 \\ 0 & 0 & -6 & 12 \end{array}\right]$

R3÷ $-6 \rightarrow R3$

$\left[\begin{array}{ccc|c} 1 & -3 & 3 & -4 \\ 0 & 1 & -7/9 & 23/9 \\ 0 & 0 & 1 & -2 \end{array}\right]$

R2+7/9R3 $\rightarrow R2$

$\left[\begin{array}{ccc|c} 1 & -3 & 3 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$

R1-3R3 $\rightarrow R1$

$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$

R1+3R2 $\rightarrow R1$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$ The solution is (5, 1, -2). This is a unique solution and the system is consistent.

11. Solve using Gauss-Jordan Elimination. (Find the RREF and solve)

x + y - 3z = 4

2x + y - z = 2

6x + 4y - 8z = 12

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 6 & 3 & -8 & 12 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 6 & 3 & -8 & 12 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 0 & -1 & 5 & -6 \\ 0 & -3 & 10 & -12 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

R2-2R1 → $R2$ -R2→ $R2$ R1-R2→ $R1$

R3-6R1 → $R3$ R3-3R2 → $R3$

There is a row of zeros in the matrix. Therefore, there are infinitely many solutions.

We need to introduce a parameter or free variable.

Let z=t. From the second row, we get: y - 5z = 6

y - 5t = 6

y=6+5t

From the first row, we get: x +2t=-2

x=-2-2t

The solution is (-2-2t, 6+5t, t) $where\ t \in R$....intersection is a line since there is one parameter.

The system has a solution (infinitely many), so it is consistent system of linear equations.

12. Solve this system of linear equations:

$$2x + 2y + 2z = 2$$

$$3y + 3z = 3$$

$$4y + 4z = 8$$

$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 0 & 3 & 3 & 3 \\ 0 & 4 & 4 & 8 \end{array}\right]$ R1÷ 2 → $R1$ and R3÷ 4 → $R3$ and R2÷ 3 → $R3$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right]$ R3 – R2 →R3 $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$

...we don't need to finish!

From the last row, we can see that there is no solution. So, we don't need to row-reduce any further. If they asked for the RREF, we would have to do R1- R2 → $R1$. The system has no solution, so the system is inconsistent.

13. Row-reduce and write the solution:

$$\left[\begin{array}{ccccc|c} 2 & 4 & 2 & 0 & 2 & 2 \\ 1 & 1 & -1 & 1 & 2 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 2 & 4 & 2 & 0 & 2 & 2 \\ 1 & 1 & -1 & 1 & 2 & 2 \end{array}\right]$$ Divide row 1 by 2:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 1 & 2 & 2 \end{array}\right]$$ R2 - R1→R2

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & -1 & -2 & 1 & 1 & 1 \end{array}\right]$$ -R2→R2

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 & -1 & -1 \end{array}\right]$$ R1 - 2R2→R1

$$\left[\begin{array}{ccccc|c} 1 & 0 & -3 & 2 & 3 & 3 \\ 0 & 1 & 2 & -1 & -1 & -1 \end{array}\right]$$

So, there are leading 1's in the x_1 and x_2 columns and so there must be 3 parameters

Let x_3=r, x_4=s and x_5=t

From the second row, we get $1x_2+2r - 1s - t = -1$ and solving for x_2:

$x_2=-1 - 2r + s + t$

From the first row, we get $x_1 - 3r + 2s + 3t = 3$ and solving for x_1 we get:

$x_1=3+3r - 2s - 3t$

(3+3r-2s-3t, -1-2r+s+t, r,s,t) $where\ r, s, t \in R$

14. Row-reduce and write the solution: $\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 4 & 0 & 2 \\ 1 & 1 & -1 & 1 & 2 & 2 \end{array}\right]$

$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 4 & 0 & 2 \\ 1 & 1 & -1 & 1 & 2 & 2 \end{array}\right]$ Divide row 2 by 2:

$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 1 & 2 & 2 \end{array}\right]$ R3 - R1→R3

$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & -3 & 0 & 1 & 2 \end{array}\right]$ R1 - R2→R1

$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & -3 & 0 & 1 & 2 \end{array}\right]$ R3÷-3→R3

$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1/3 & -2/3 \end{array}\right]$ R2 - R3→R2 $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 2 & 1/3 & 5/3 \\ 0 & 0 & 1 & 0 & -1/3 & -2/3 \end{array}\right]$ R1-R3→R1

$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 4/3 & -1/3 \\ 0 & 1 & 0 & 2 & 1/3 & 5/3 \\ 0 & 0 & 1 & 0 & -1/3 & -2/3 \end{array}\right]$

The first three columns have leading 1's...so, there are two parameters representing the 4th and 5th columns

Let x_4=s and x_5=t

The last row gives us....x_3 - 1/3t= -2/3 or x_3= -2/3 + 1/3 t

The second row gives us...x_2 + 2s + 1/3 t = 5/3 or x_2= 5/3 - 2s - 1/3 t

The first row gives us...x_1 -s + 4/3 t = -1/3 or x_1= -1/3 + s - 4/3 t

The solution is a plane since there are two parameters and it is:

(-1/3 + s - 4/3 t, 5/3 - 2s - 1/3t, -2/3 + 1/3t, s, t) $where\ s, t \in R$

15. Describe the region that is the intersection of the following planes:

NOTE: You have to row-reduce first!

x+y+2z=-2

3x -y + 14z = 6

2x + 4y = -10

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 3 & -1 & 14 & 6 \\ 2 & 4 & 0 & -10 \end{array}\right] R_2 - 3R_1 \to R_2 \left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & -4 & 8 & 12 \\ 2 & 4 & 0 & -10 \end{array}\right] R2 \div -4 \to R2 \text{ and } R3 - 2R1 \to R3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 2 & -4 & -6 \end{array}\right] R3 - 2R2 \to R3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] R1 - R2 \to R1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

infinitely many solutions

parameters= n – r = 3 – 2 = 1

Since there is only 1 parameter, it is a line of intersection

Let z=t

The solution (not required) is:

from the first-row x+4t=1 and x=1-4t

from the second-row y-2t=-3 and y=-3 +2t

(1-4t, -3 +2t, t) $where\ t \in R$

16. Describe the region that is the intersection of the following planes:

2x - 2y - 6z = 2

6x - 6y - 18z= 6

-4x + 4y + 12z = -4

$\left[\begin{array}{ccc|c} 2 & -2 & -6 & 2 \\ 6 & -6 & -18 & 6 \\ -4 & 4 & 12 & -4 \end{array}\right]$ R1÷2→R1

$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 1 \\ 6 & -6 & -18 & 6 \\ -4 & 4 & 12 & -4 \end{array}\right]$ R2 - 6R1→R2 and R3 + 4R1→R3

$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ This is in row-reduced form

There is one leading one (column) and two other columns representing parameters.

Since we have 2 parameters, the region is a plane.

The solution (not required) is:

Let y=s and z=t

The intersection is a plane, since we have 2 parameters.

The solution (not asked for) is:

From the first row, we get x - y - 3z = 1

and with the parameters, we get x -s - 3t =1 or x=1+s+3t

The solution is (1+s+3t, s, t) $where\ s, t \in R$

17. If A + B are square matrices of some dim or slice, is $(A+B)^2 = A^2 + 2AB + B^2$?

In general, no:

$(A+B)^2 = (A+B)(A+B)$

$= A^2 + AB + BA + B^2$

$AB \neq BA$ for many matrices A and B

If AB = BA (A, B are commutative then it is true)

18. Solve by back substitution.

a) $2x + y + 3z = 8$ (1)

$2y - z = 1$ (2)

$3z = 3$ (3)

Substitute $z=1$ from (3) into equation (2).

$2y - z = 1$

$2y - 1 = 1$

$2y = 2$

$y=1$

Substitute $y=1$ and $z=1$ into equation (1)

$2x+ y + 3z = 8$

$2x + 1 + 3(1)=8$

$2x =4$

$x=2$

The solution is (2,1,1).

b) $2x - y + z = 13$ (1)

$y - 4z = 2$ (2)

$3z = 9$ (3)

From equation (3), we get $z=3$.

Substitute $z=3$ into equation (2): $y - 4z = 2$

$$y - 4(3)=2$$

$$y=14$$

Substitute $y=14$ and $z=3$ into equation (1):

$$2x - y + z = 13$$

$$2x - 14 +3 = 13$$

$$2x = 13-3+14$$

$$2x=24$$

$$x=12$$

The solution is (12, 14, 3).

19. Find the value of k so that the system of equations has

a) no solution,

b) exactly one solution and

c) infinitely many solutions.

$$\left[\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 2 & 2k & 2 & 2 \\ 3 & 3 & 3k & 3 \end{array}\right]$$ R1$\leftrightarrow R3$ and R2$\div 2 \rightarrow R2$ and divide row 3 by 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{array}\right]$$ R2-R1 $\rightarrow R2$ and R3 -kR1 $\rightarrow R3$

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & -1+k & -k+1 & 0 \\ 0 & -k+1 & -k^2+1 & -k+1 \end{array}\right]$$ R3+R2 $\rightarrow R3$

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & -1+k & -k+1 & 0 \\ 0 & 0 & -k^2-k+2 & -k+1 \end{array}\right]$$

factor $-k^2 - k + 2 = 0$

$-(k^2 + k - 2) = 0$

$-(k+2)(k-1) = 0$

k=-2, 1

If k=1...

the last row becomes 0 0 0 / 0 infinitely many solutions

If k= -2...

the last row becomes 0 0 0/3 no solution

If k$\neq 1, -2$ there is a unique (exactly one) solution

20. $\left[\begin{array}{ccc|c} 2 & 4 & -6 & 8 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right]$ Divide row 1 by 2:

$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right]$ R2-3R1 → $R2$ and R3-4R1 → $R3$

$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & -7 & a^2-2 & a-14 \end{array}\right]$ R3+7R2 → $R3$

$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$

If a= -4, we get 0 = -8 which means no solution

If a=4, we get 0 = 0 which means infinitely many solutions

Therefore, there is a unique solution if a≠4, -4

21. Row reduce and find the solution: $3x_2 - 6x_3 + 6x_4 + 4x_5 = -5$

$$3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9$$

$$x_1 - 3x_2 + 4x_3 - 3x_4 + 2x_5 = 5$$

$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$

R3-3R2 → R3

$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$ REF

R2-R3 → R2

$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$

R1-2R3 → R1

$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$

R1 + 3R2 → R1

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{array}\right]$$

R1 ←→ R3

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

R2-3R1 → R2

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

R2 ÷ 2 → R2

$$\begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{ccccc} & & s & t & \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$\left[\begin{array}{ccccc|c} \boxed{1} & 0 & -2 & -3 & 0 & -24 \\ 0 & \boxed{1} & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{array}\right] \text{RREF}$$

Let $x_3 = s, x_4 = t$

$x_1 - 2s - 3t = -24$

$\therefore x_1 = -24 + 2s + 3t$

$x_2 - 2s + 2t = -7$

$x_2 = -7 + 2s - 2t$

$x_5 = 4$

Solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -24 + 2s + 3t \\ -7 + 2s - 2t \\ s \\ t \\ 4 \end{bmatrix} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix} + s\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$where\ s, t \in R$$

22.
$$\begin{bmatrix}1 & 1 & -3\\2 & 1 & -1\\6 & 3 & -8\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & -3\\2 & 1 & -1\\6 & 3 & -8\end{bmatrix} \quad \begin{bmatrix}1 & 1 & -3\\0 & -1 & 5\\0 & -3 & 10\end{bmatrix} \begin{bmatrix}1 & 1 & -3\\0 & 1 & -5\\0 & 0 & 0\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 2\\0 & 1 & -5\\0 & 0 & 0\end{bmatrix}$$

R2-2R1 → $R2$ -R2→ $R2$ R1-R2→ $R1$

R3-6R1 → $R3$ R3-3R2 → $R3$

The rank is 2. i.e. 2 non-zero rows in RREF

23. $\left[\begin{array}{cc|c}1 & -3 & k\\k & 1 & 4\end{array}\right]$

$R_2 - kR_1 \to R_2$

$$\left[\begin{array}{cc|c}1 & -3 & k\\0 & 1+3k & 4-k^2\end{array}\right]$$

infinitely many is impossible since k= -1/3 makes the left side 0 and

k=2,-2 makes the right side 0

no solution k= -1/3

unique k$\neq -1/3$

24. $\# \, parameter = n - r = number\ of\ columns - rank$

$\# \, parameter = n - r = 4\ columns - 2\ leading\ 1's$

$= 2$

$\therefore$ *this means the last row must be all* $0's$ (since we already have two leading 1's)

$\therefore k^2 - 9 = 0 \quad and \quad k - 3 = 0 \qquad \therefore k = 3$

25. $\left[\begin{array}{ccc|c}1 & 0 & 0 & 3\\0 & 1 & 1 & 2k\\0 & 5 & k & 4\end{array}\right]$ $R3 - 5R2 \to R3$

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 3\\0 & 1 & 1 & 2\quad k\\0 & 0 & k-5 & 4-10k\end{array}\right]$$

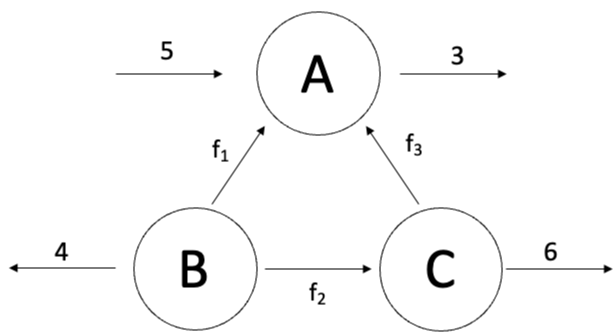
If k=5, we get 0 0 0/ -46 which means no solution

If k$\neq$ 5, we get a unique solution

There is no value of k that will make the bottom row a full row of 0's, so infinitely many solutions is impossible.

5. Network Analysis

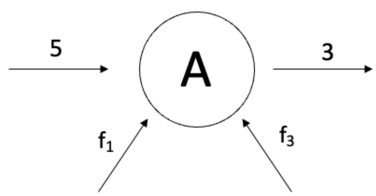
Example 5.1. Find all solutions $\vec{f} = (f_1, f_2, f_3) \in R^3$ to the following network flow problem.



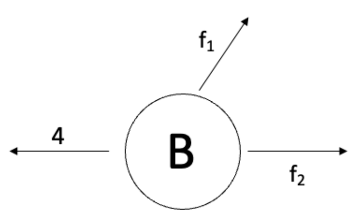
Solution:

$f_1+f_3+5=3$

$\therefore f_1+f_3=-2$ **1**

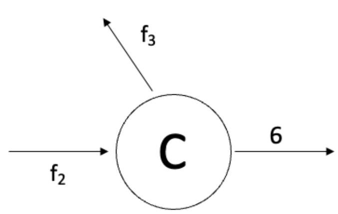


$f_1 + f_2 = 4$ **2**



$f_2 = f_3 + 6$

$f_2 - f_3 = 6$ **3**



Matrix form

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 6 \end{array}\right] R_2 - R_1 \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 6 \\ 0 & 1 & -1 & 6 \end{array}\right] R_3 - R_2 \rightarrow R_3$$

$f_1 \quad f_2 \quad f_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $f_3 = t \ (t \in R)$ parameter

$f_1 + t = -2 \ \therefore f_1 = -2 - t$

$f_2 - t = 6 \qquad f_2 = 6 + t$

$\therefore$ solution is $\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} -2 - t \\ 6 + t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \ where \ t \in R$

Example 5.2.

If the solutions are:

$f_1 = 16 - t$

$f_2 = 6 - t$

$f_3 = 21 + t$

$f_4 = t$

Flows are positive in real-life examples, so we get constraints for t

Since $f_4 = t$ $\boxed{t \geq 0}$ $\therefore f_4 \geq 0$

From $f_2 = 6 - t$, $6 - t \geq 0$ $\therefore 6 \geq t$ or $\boxed{t \leq 6}$

The other constraints won't make any restrictions smaller than what we already have:

$\therefore$ we get $0 \leq t \leq 6$

and

Since $f_1 = 16 - t$ so, $\boxed{10 \leq f_1 \leq 16}$

Substitute $t = 0, t = 6$

$f_2 = 6 - t$ so, $\boxed{0 \leq f_2 \leq 6}$

Substitute $t = 0, t = 6$

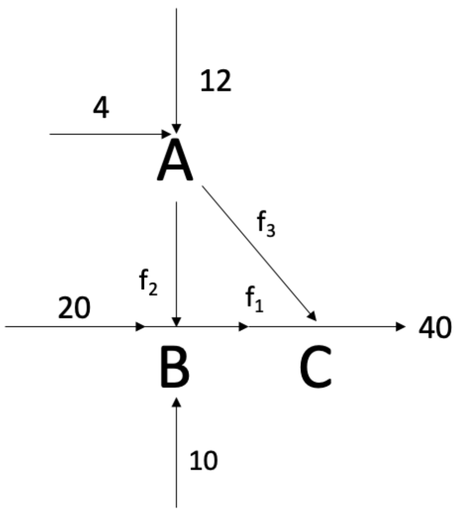
$f_3 = 21 + t$ so, $\boxed{21 \leq f_3 \leq 27}$

Substitute $t = 0, t = 6$

$f_4 = t$ $\boxed{0 \leq f_4 \leq 6}$

5.3 Homework on Chapter 5

1. Find all solutions $\vec{f} = (f_1, f_2, f_3) \in R^3$ to the following network flow problem.



A. $12 + 4 = f_2 + f_3$ $\quad \therefore f_2 + f_3 = 16$ **1**
B. $10 + f_2 + 20 = f_1$ $\quad \therefore f_1 - f_2 = 30$ **2**
C. $f_1 + f_3 = 40$ **3**

Matrix form

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 16 \\ 1 & -1 & 0 & 30 \\ 1 & 0 & 1 & 40 \end{array}\right]$$ R_1 switch R_2

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 30 \\ 0 & 1 & 1 & 16 \\ 1 & 0 & 1 & 40 \end{array}\right]$$ $R_3 - R_1 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 30 \\ 0 & 1 & 1 & 16 \\ 0 & 1 & 1 & 10 \end{array}\right]$$ $R_3 - R_2 \rightarrow R_3$

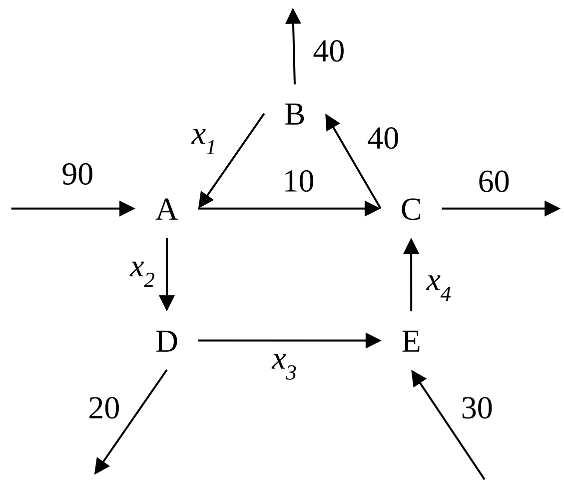
$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 30 \\ 0 & 1 & 1 & 16 \\ 0 & 0 & 0 & -6 \end{array}\right]$$ $R_1 - R_2 \rightarrow R_1$

$\therefore no$ solution

$$\begin{array}{ccc} f_1 & f_2 & f_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 46 \\ 0 & 1 & 1 & 16 \\ 0 & 0 & 0 & -6 \end{array}\right]$$

2.



Consider the network of streets with intersections A, B, C, D, and E above. The arrows indicate the direction of traffic flow along the *one-way streets*, and the numbers refer to the *exact* number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.

a) Write down a system of linear equations which describes the traffic flow.

b) Solve the system of linear equations.

<u>Solution:</u>

a) We can set up a system of linear equations based on each intersection:

A: $90 + x_1 = x_2 + 10$

B: $40 = x_1 + 40$

C: $x_4 + 10 = 60 + 40$

D: $x_2 = 20 + x_3$

E: $x_3 + 30 = x_4$

Simplifying, by putting variables on the left and numbers on the right, we get

A: $x_1 - x_2 = -80$

B: $x_1 = 0$

C: $x_4 = 90$

D: $x_2 - x_3 = 20$

E: $x_3 - x_4 = -30$

Matrix form:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -80 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 90 \\ 0 & 1 & -1 & 0 & 20 \\ 0 & 0 & 1 & -1 & -30 \end{bmatrix}$$

Gauss time!

$$\begin{bmatrix} \boxed{1} & -1 & 0 & 0 & -80 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 90 \\ 0 & 1 & -1 & 0 & 20 \\ 0 & 0 & 1 & -1 & -30 \end{bmatrix} \xrightarrow{R_2 \to -R_1 + R_2} \begin{bmatrix} \boxed{1} & -1 & 0 & 0 & -80 \\ 0 & \boxed{1} & 0 & 0 & 80 \\ 0 & 0 & 0 & 1 & 90 \\ 0 & 1 & -1 & 0 & 20 \\ 0 & 0 & 1 & -1 & -30 \end{bmatrix}$$

$$\xrightarrow[R_4 \to -R_2 + R_4]{R_1 \to R_2 + R_1} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 80 \\ 0 & 0 & 0 & 1 & 90 \\ 0 & 0 & -1 & 0 & -60 \\ 0 & 0 & 1 & -1 & -30 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_5} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 80 \\ 0 & 0 & \boxed{1} & -1 & -30 \\ 0 & 0 & -1 & 0 & -60 \\ 0 & 0 & 0 & 1 & 90 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_3 + R_4} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 80 \\ 0 & 0 & \boxed{1} & -1 & -30 \\ 0 & 0 & 0 & -1 & -90 \\ 0 & 0 & 0 & 1 & 90 \end{bmatrix}$$

$$\xrightarrow{R_4 \to -R_4} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 80 \\ 0 & 0 & \boxed{1} & -1 & -30 \\ 0 & 0 & 0 & \boxed{1} & 90 \\ 0 & 0 & 0 & 1 & 90 \end{bmatrix}$$

$$\xrightarrow[R_5 \to -R_4 + R_5]{R_3 \to R_4 + R_3} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 80 \\ 0 & 0 & \boxed{1} & 0 & 60 \\ 0 & 0 & 0 & \boxed{1} & 90 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the RREF matrix, we can see that

$x_1 = 0$

$x_2 = 80$

$x_3 = 60$

$x_4 = 90$

6. Linear Independence

Example 6.1.
Are the following vectors linearly independent or dependent?

$$\{(1,2),(4,2),(7,1)\}$$

Solution:

$$span\{(1,2),(4,2),(7,1)\}=(0,0)$$
$$a(1,2)+b(4,2)+c(7,1)=(0,0)$$

$$a+4b+7c=0$$
$$2a+2b+c=0$$

$$\left[\begin{array}{ccc|c}1 & 4 & 7 & 0\\ 2 & 2 & 1 & 0\end{array}\right]\xrightarrow{R_2-2R_1\to R_2}\left[\begin{array}{ccc|c}1 & 4 & 7 & 0\\ 0 & -6 & -13 & 0\end{array}\right]$$

$$\xrightarrow{\frac{R_2}{(-6)}\to R_2}\left[\begin{array}{ccc|c}1 & 4 & 7 & 0\\ 0 & 1 & -13/6 & 0\end{array}\right]$$

$$\xrightarrow{R_1-4R_2\to R_1}\left[\begin{array}{ccc|c}1 & 0 & -5/3 & 0\\ 0 & 1 & 13/6 & 0\end{array}\right]$$

Since the third column is a ***free variable*** (parameter), this indicates that the vectors are ***linearly dependent***.

Example 6.2.

Is the set $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ where $S \in M_{22}$ linearly independent or dependent?

Solution:

Using matrices,

$$C_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + C_3 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Find $C_1, C_2,$ and C_3

$C_1 + C_3 = 0$

$C_1 + C_2 = 0$

$C_2 = 0$

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$ R2-R1 → R2

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$ R3-R2 → R3

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ R2+R3 → R2

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ R1-R3 → R1

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

∴ ***since $\boldsymbol{C_1 = C_2 = C_3 = 0}$ the matrices are linearly independent***

Example 6.3.

Is the vector $\vec{w}$=(2,3,-7,3) in the linear span of $\vec{v}_1$=(2,1,0,3), $\vec{v}_2$= (3,-1,5,2) and $\vec{v}_3$= (-1,0,2,1)?

Solution:

In other words, can you make w from a combination of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$?

(2,3,-7,3)= c_1(2,1,0,3)+ c_2(3,-1,5,2) + c_3 (-1,0,2,1)

From the first entry: 2=2 c_1+3c2- c_3

From the second entry: 3= c_1- c2: c_1=3+ c_2

From the third entry: -7=5 c_2+2c_3 c_3= -7/2-5c_2/2

Sub back into the first equation

2=2(3+ c_2) +3 c_2- (-7/2-5 c_2/2)

-15/2=15 C_2/2

C_2= -1

c_1 =3+ c_2=3+(-1)=2

c_1=2

c_3= -7/2 – 5/2 c_2= - 7/2 – 5/2 (-1)= -2/2 = -1

c_3= -1

Check to make sure this works for the fourth entry:

3=3c_1+2c_2+ c_3

LS=3 RS=3(2)+2(-1)+ (-1) =3

OR use matrices:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & 7 \end{array}\right] \ldots RREF \ldots \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$ and we get $\boldsymbol{c_1 = 2, c_2 = -1 \text{ and } c_3 = -1}$.

This works, so w is in the linear span of v_1, v_2 and v_3 because w can be created by a linear combination of the three vectors.

$$\vec{w} = 2\vec{v}_1 - \vec{v}_2 - \vec{v}_3$$

Example 6.4. Is vector b in the span of the columns of matrix A?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Solution:

b is in the span of the columns of A if the system $Ax = b$ has a solution

Ie. If $x + 2y = 5$

$3x + y = 2$ has a solution

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 1 & 2 \end{array}\right] \rightarrow \ row\ reduce$$

$$R2 - 3R1 \rightarrow R2 \quad \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -13 \end{array}\right]$$

$$R2 \div (-5) \rightarrow R2 \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 13/5 \end{array}\right]$$

$$R1 - 2R2 \rightarrow R1 \quad \left[\begin{array}{cc|c} 1 & 0 & -1/5 \\ 0 & 1 & 13/5 \end{array}\right] \quad 5 - 2\left(\frac{13}{5}\right) = \frac{25}{5} - \frac{26}{5} = -\frac{1}{5}$$

$$\therefore x = -\frac{1}{5}, y = \frac{13}{5}$$

$A\begin{bmatrix} -\frac{1}{5} \\ \frac{13}{5} \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Therefore, yes b is in the span of columns of A. (since it is a multiple of A)

Example 6.5. Are the vectors $\vec{u}\,(2,-1,1)$ $\vec{v}(3,-4,-2)$ and $\vec{w}(5,-10,-8)$ linearly independent?

Solution:

$c_1\vec{u}+c_2\vec{v}+c_3\vec{w}=\vec{0}$

$2C_1+3C_2+5C_3=0$

$-C_1-4C_2-10C_3=0$

$C_1-2C_2-8C_3=0$

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}\left|\begin{matrix}0\\0\\0\end{matrix}\right] \ldots\text{RRREF}\ldots \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\left|\begin{matrix}0\\0\\0\end{matrix}\right]$$

Let $C_3=t$ (parameter or free variable)

$\therefore\ C_1-2t=0\quad C_1=2t$

$C_2+3t=0\quad C_2=-3t$

$\therefore$ the solution is NOT the trivial one $C_1=C_2=C_3=0$. $\therefore$ vectors are linearly dependent.

OR use rankA=2<3=n, so they are linearly dependent.

Example 6.6.

Let $A_1 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

Is $B = \begin{bmatrix} 4 & 5 \\ -1 & 7 \end{bmatrix}$ a linear combination of A_1, A_2 *and* A_3?

Solution:

We want to find scalars C_1, C_2 *and* C_3 such that

$C_1A_1 + C_2A_2 + C_3A_3 = B$

$$C_1 \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} + C_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + C_3 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} C_2 + C_3 & C_1 + C_3 \\ -2C_1 + C_3 & C_2 + 2C_3 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -1 & 7 \end{bmatrix}$$

$C_2 + C_3 = 4$
$C_1 + C_3 = 5$
$-2C_1 + C_3 = -1$
$C_2 + 2C_3 = 7$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 5 \\ -2 & 0 & 1 & -1 \\ 0 & 1 & 2 & 7 \end{array}\right] R_1 \leftrightarrow R_2 \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 4 \\ -2 & 0 & 1 & -1 \\ 0 & 1 & 2 & 7 \end{array}\right] R_3 + 2R_1 \rightarrow R_3 \text{ and } R_4 - R_2 \rightarrow R_4$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 1 & 3 \end{array}\right] R_3 \div 3 \rightarrow R_3 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right] R_2 - R_3 \rightarrow R_2 \text{ and } R_4 - R_3 \rightarrow R_4$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \therefore C_1 = 2,\ C_2 = 1,\ C_3 = 3$$

Therefore, B is a linear combination of A_1, A_2 *and* A_3.

B= $2A_1 + A_2 + 3A_3$

Example 6.7. Determine if the matrices are linearly independent.

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix} \; B = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} \; C = \begin{bmatrix} -2 & -1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \; D = \begin{bmatrix} -1 & -3 \\ 1 & 8 \\ 3 & 4 \end{bmatrix}$$

Solution:

We need to find the constants so that: $C_1A + C_2B + C_3C + C_4D = 0$

$$C_1\begin{bmatrix} 0 & 1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix} + C_2\begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} + C_3\begin{bmatrix} -2 & -1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} + C_4\begin{bmatrix} -1 & -3 \\ 1 & 8 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_2 - 2C_3 - C_4 = 0$$

$$C_1 - C_3 - 3C_4 = 0$$

$$4C_1 + 2C_2 + C_4 = 0$$

$$2C_1 + 4C_2 + 2C_3 + 8C_4 = 0$$

$$-C_1 + C_2 + 3C_4 = 0$$

$$C_2 + C_3 + 4C_4 = 0$$

$$\begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & -2 & -1 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 4 & 2 & 0 & 1 & 0 \\ 2 & 4 & 2 & 8 & 0 \\ -1 & 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 4 & 0 \end{array}\right] \ldots\text{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The solution is:

$C_1 = C_2 = C_3 = C_4 = 0$ ∴ the set of matrices is linearly independent

(only the trivial solution)

6.5 Homework on Chapter 6

1.Determine whether the set $S = \{(1,2,1,1),(2,0,2,2),(2,2,2,0),(2,4,0,2)\}$ is linearly dependent.

Step 1:

Write the vectors in the set as the columns of a matrix.

We'll call this matrix *A*. We find: $A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 0 & 2 & 4 \\ 1 & 2 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}$.

Step 2:

Reduce the matrix *A* into row-reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 0 & 2 & 4 \\ 1 & 2 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -4 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, since matrix *A* reduces to the identity matrix, the set must be linearly independent.

2. Determine whether each of the following sets of vectors is linearly independent:

a) $\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}-1\\5\end{bmatrix}$ Since neither vector is a multiple of the other, they are NOT dependent

∴the set is linearly independent

b) $\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$

Find C_1, C_2 *and* C_3

$$C_1\begin{bmatrix}0\\1\\1\end{bmatrix} + C_2\begin{bmatrix}0\\1\\-1\end{bmatrix} + C_3\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$C_3 = 0$$
$$C_1 + C_2 = 0$$
$$C_1 - C_2 + C_3 = 0$$

Since $C_3 = 0$

We have: $C_1 + C_2 = 0$

$\underline{C_1 - C_2 = 0}$

Add $2C_1 = 0$

$C_1 = 0$

$\therefore C_1 = C_2 = C_3 = 0$ they are linearly independent

(only the trivial solution)

c) $\begin{bmatrix}1\\3\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\3\\3\end{bmatrix}$

$$\left[\begin{array}{ccc|c}1 & 1 & -1 & 0\\3 & 1 & 3 & 0\\0 & -1 & 3 & 0\end{array}\right] \qquad \left[\begin{array}{ccc|c}1 & 1 & -1 & 0\\0 & -2 & 6 & 0\\0 & -1 & 3 & 0\end{array}\right]$$

$R2 - 3R1 \rightarrow R2$ $\qquad R2 \div (-2) \rightarrow R2$

$$\left[\begin{array}{ccc|c}1 & 1 & -1 & 0\\0 & 1 & -3 & 0\\0 & -1 & 3 & 0\end{array}\right] \quad \left[\begin{array}{ccc|c}1 & 1 & -1 & 0\\0 & 1 & -3 & 0\\0 & 0 & 0 & 0\end{array}\right] \quad \left[\begin{array}{ccc|c}1 & 0 & 2 & 0\\0 & 1 & -3 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

$R3 + R2 \rightarrow R3$ $\qquad R1 - R2 \rightarrow R1$ $\qquad$ *infinitely many solutions*

$C_3 = t$

$C_1 = -2t$

$C_2 = 3t$ ∴ linearly dependent

$(-2t, 3t, t)$ where $t \in R$.

3. a) Are the following vectors linearly independent or dependent?

$$\{(1,0,2),(1,0,1),(0,1,2)\}$$

$$span\{(1,0,2),(1,0,1),(0,1,2)\}=(0,0,0)$$
$$a(1,0,2)+b(1,0,1)+c(0,1,2)=(0,0,0)$$

$$a+b=0$$
$$c=0$$
$$2a+b+2c=0$$

$$a+b=0 \rightarrow b=-a$$
$$2a+b=0$$
$$c=0$$

$$2a-a=0$$
$$a=0, b=0$$

$$\therefore a=b=c=0$$

Since the only solution is the trivial solution, the vectors are linearly independent.

b) Is the set $W = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ where $W \in M_{22}$ linearly independent or dependent?

$$\begin{aligned} spanW &= span\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\} \\ &= a\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} a + c &= 0 \\ a + b &= 0 \\ c &= 0 \\ a + b &= 0 \end{aligned}$$

$$\therefore a = b = c = 0$$

Since the only solution is the trivial solution, the vectors are linearly independent.

4. Given the vectors $\vec{v}_1 = (1,2,3)$, $\vec{v}_2 = (1,1,1)$, $\vec{v}_3 = (1,1,0)$, and $\vec{v}_4 = (9,14,16)$, express $\vec{v}_4$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

Identify scalars *a, b, c* such that:

$\vec{v}_4 = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$

$(9,14,16) = a(1,2,3) + b(1,1,1) + c(1,1,0)$

$(9,14,16) = (a+b+c, 2a+b+c, 3a+b)$

Thus:

1. $a + b + c = 9$
2. $2a + b + c = 14$
3. $3a + b = 16$

Eqn.2-Eqn1 $\Rightarrow a = 5$

Substitute $a = 5$ into Eqn. 3 $\Rightarrow b = 1$

Substitute $a = 5$, $b = 1$ into Eqn. 2 $\Rightarrow c = 3$

Therefore:

$\vec{v}_4 = 5\vec{v}_1 + \vec{v}_2 + 3\vec{v}_3$

OR

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 1 & 1 & 14 \\ 3 & 1 & 0 & 16 \end{array}\right] RREF \ldots \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

5. Do the vectors v_1=(2,-1,3), v_2= (4,1,2) and v_3= (8,-1,8) span R^3?

In other words can any vector w=(x,y,z) be made from a combination of v_1, v_2 and v_3?
$(x,y,z) = a_1(2,-1,3) + a_2(4,1,2) + a_3(8,-1,8)$

From the first entry: $x = 2a_1 + 4a_2 + 8a_3$ $\quad a_1 = x/2 – 2a_2 – 4a_3$
From the second entry: $y = -a_1 + a_2 – a_3$ $\quad y = - (x/2 – 2a_2 – 4a_3) + a_2 – a_3$
$y = -x/2 + 3a_2 + 3a_3$
$3a_2 = y – x/2 - 3a_3$
$a_2 = y/3 + x/6 - a_3$
From the third entry: $z = 3a_1 + 2a_2 + 8a_3$
$z = (3x/2 - 6 (y/3 + x/6 - a_3) – 12a_3) + (2y/3 + x/3 – 2a_3) + 8a_3$
$z = 3x/2 - 2y – x + 6a_3 – 12a_3 + 2y/3 + x/3 + 6a_3$
$z – 5x/6 + 4y/3 = 0a_3$

There is no value of a_3 that will work therefore these vectors do not span R^3.

6. Find all values of k for which the set of vectors $\{(1,0,1),(0,3,2),(k,k,3)\}$ is linearly independent.

We'll use the test for linear dependence.

First, place the vectors in the set as the columns of a vector. We find:

$$\begin{bmatrix} 1 & 0 & k \\ 0 & 3 & k \\ 1 & 2 & 3 \end{bmatrix}$$

Now we row-reduce this vector as follows:

$$\begin{bmatrix} 1 & 0 & k \\ 0 & 3 & k \\ 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & k \\ 0 & 3 & k \\ 1 & 2 & 3-k \end{bmatrix} \to \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & \frac{k}{3} \\ 1 & 2 & 3-k \end{bmatrix} \to \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & \frac{k}{3} \\ 0 & 0 & 3-k-\left(\frac{2k}{3}\right) \end{bmatrix}$$

$\to \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & k/3 \\ 0 & 0 & \frac{9-5k}{3} \end{bmatrix}$ do R3 – R1→ $R3, R3-2R2 \to R3)$

Rough work: $3-k-\frac{2k}{3}=\frac{9}{3}-\frac{3k}{3}-\frac{2k}{3}=\frac{9-5k}{3}$

In order for the vectors above to be linearly dependent, their corresponding matrix must reduce to the identity matrix. In order for the above matrix to be reduced to the identity matrix, the third row must have a leading 1. This leading 1 can then be manipulated to "clear" the entries above it.

In order for this entry to be made into a leading 1, it cannot be zero.

Thus, we find the condition:

$$\frac{9-5k}{3} \neq 0$$

$9 - 5k \neq 0$

$$\Rightarrow k \neq \frac{9}{5}.$$

Thus, the set of vectors is linearly independent for $k \in \Re$ such that $k \neq \frac{9}{5}$.

7. Let $s = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a linearly independent set of vectors in a vector space V. Which of the following sets are also linearly independent?

A. $\{\mathbf{0}, \mathbf{u} + \mathbf{v}, \mathbf{v} - \mathbf{w}\}$
B. $\{2\mathbf{u}, -\mathbf{v}, 3\mathbf{w}\}$
C. $\{\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{w}\}$

(1) B and C
(2) All
(3) A and C
(4) C only
(5) A and B

The answer is (1): B and C.

Statement A is NOT true since $k_1\mathbf{0} + k_2(\mathbf{u} + \mathbf{v}) + k_3(\mathbf{v} - \mathbf{w}) = 0 \Leftrightarrow k_2\mathbf{u} + (k_3 + k_2)\mathbf{v} - k_3\mathbf{w} = 0$. But **u, v,** and **w** are linearly independent. Hence $k_2 = 0$ and $k_3 = 0$ but k_1 can be either 0 or non-zero number. Therefore, the set $\{\mathbf{0}, \mathbf{u} + \mathbf{v}, \mathbf{v} - \mathbf{w}\}$ are linearly dependent. In general, any set containing the vector **0** is linearly dependent, for the same reason.

Statement B is True since $k_1(2\mathbf{u}) + k_2(-\mathbf{v}) + k_3(3\mathbf{w}) = 0 \Leftrightarrow 2k_1\mathbf{u} - k_2\mathbf{v} + 3k_3\mathbf{w} = 0$. But **u, v,** and **w** are linearly independent. Therefore, $2k_1 = 0$, $-k_2 = 0$, and $3k_3 = 0$. Or $k_1 = k_2 = k_3 = 0$.

Statement C is True since $k_1\mathbf{u} + k_2(\mathbf{u} + \mathbf{v}) + k_3(\mathbf{u} + \mathbf{w}) = 0$

$\Leftrightarrow (k_1 + k_2 + k_3)\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = 0 \Leftrightarrow k_1 + k_2 + k_3 = 0,\ k_2 = 0, \text{ and } k_3 = 0 \Rightarrow k_1 = k_2 = k_3 = 0.$

8. For what values of k is $M = \begin{bmatrix} 1 & 4 \\ k-1 & 1 \end{bmatrix}$ in the vector space W = span $\left\{ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ k^2 & 4 \end{bmatrix} \right\}$?

(1) $k = 2$ and $k = -1$
(2) $k = 0$
(3) $k = -2$ and $k = -1$
(4) $k = 2$
(5) $k = 2$ and $k = -2$

The answer is (1): $k = 2$ and $k = -1$.

$\begin{bmatrix} 1 & 4 \\ k-1 & 1 \end{bmatrix} = a\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 4 & 1 \\ k^2 & 4 \end{bmatrix}$ where a, b are numbers.

$\begin{bmatrix} 1 & 4 \\ k-1 & 1 \end{bmatrix} = \begin{bmatrix} a+4b & b-a \\ a+bk^2 & a+4b \end{bmatrix} \Rightarrow$

b=4+a

a+ 4b = 1

a +4 (4+a)=1

a+16+4a =1

5a = - 15

a= -3

So, b= 4 + (-3) = 1

$$\begin{cases} a+4b=1, b-a=4 \\ a+bk^2 = k-1, a+4b=1 \end{cases} \Rightarrow \begin{cases} b=1, a=-3 \\ k^2-k-2=0 \end{cases} \Rightarrow \begin{cases} k=-1 \\ k=2 \end{cases}.$$

$a + bk^2 = k - 1$

$-3 + k^2 = k - 1$

$k^2 - k - 2 = 0$

(k-2)(k-1)=0 and we get k=2, - 1.

9. Show that $R^2 = span\left(\begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}2\\-2\end{bmatrix}\right)$.

We must show that for any vector

$$\begin{bmatrix}a\\b\end{bmatrix}, \quad x\begin{bmatrix}2\\2\end{bmatrix} + y\begin{bmatrix}2\\-2\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

for some $x\ and\ y$.

$$\left[\begin{array}{cc|c}2 & 2 & a\\2 & -2 & b\end{array}\right] \quad row\ reduce \quad R1 \div 2 \to R1$$

$$R2 - R1 \to R2$$

$$\left[\begin{array}{cc|c}1 & 1 & a/2\\0 & -4 & b-a\end{array}\right] \quad R2 \div (-4) \to R2$$

$$\left[\begin{array}{cc|c}1 & 1 & a/2\\0 & 1 & \frac{b-a}{-4}\end{array}\right] \quad R1 - R2 \to R1$$

Or $\left[\begin{array}{cc|c}1 & 1 & a/2\\0 & 1 & \frac{a-b}{4}\end{array}\right]$ $\qquad \frac{a}{2} - \frac{a-b}{4}$

$$\left[\begin{array}{cc|c}1 & 0 & \frac{a+b}{4}\\0 & 1 & \frac{a-b}{4}\end{array}\right] \qquad = \frac{2a-a+b}{4}$$

$$= \frac{a+b}{4}$$

$$\therefore \frac{a+b}{4}\begin{bmatrix}2\\2\end{bmatrix} + \frac{a-b}{4}\begin{bmatrix}2\\-2\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

10. If $P_1 = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}$. Find a linear combination of P_1 and P_2 given above, which is equal to $\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$.

We need to find a and b so that

$$a\begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} + b\begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

So, we must have

$$\tfrac{4}{5}a + \tfrac{1}{5}b = 6$$
$$\tfrac{2}{5}a - \tfrac{2}{5}b = 2$$
$$\tfrac{1}{5}a + \tfrac{4}{5}b = 3$$

multiply the first equation by 5 and the second equation by 5 as well

4 a + b = 30

2 a – 2b = 10 (divide by 2)

Adding the equations below:

$$4a + b = 30$$

so $\underline{a - b = 5}$

$$5a = 35 \Rightarrow a = 7,\ b = a - 5 = 2$$

Thus,

$$7\begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} + 2\begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

11. Is $C = \begin{bmatrix} 5 & 2 \\ 5 & 6 \end{bmatrix}$ a linear combination of $A_1, A_2 \& A_3$?

Find $C_1, C_2 \& C_3$:

$$C_1 \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} + C_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} C_2 + C_3 & C_1 + C_3 \\ -2C_1 + C_3 & C_2 + 2C_3 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 5 & 6 \end{bmatrix}$$

$C_2 + C_3 = 5$

$C_1 + C_3 = 2$

$-2C_1 + C_3 = 5$

$C_2 + 2C_3 = 6$

$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 5 \\ 1 & 0 & 1 & 2 \\ -2 & 0 & 1 & 5 \\ 0 & 1 & 2 & 6 \end{array}\right]$ R1 ←→ R2

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ -2 & 0 & 1 & 5 \\ 0 & 1 & 2 & 6 \end{array}\right]$ R3+2R1 → R3 (5+2(2)) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 3 & 9 \\ 0 & 1 & 2 & 6 \end{array}\right]$ R3÷ 3→ R3

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 6 \end{array}\right]$ R4-R2 → R4

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$ R4-R3 → R4

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -2 \end{array}\right]$ No Solution

∴ C is NOT a linear combination of A_1, A_2 and of A_3

12. Determine if matrices $A = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix}$ $C = \begin{bmatrix} -16 & -6 \\ -12 & 34 \end{bmatrix}$ are linearly independent.

$$C_1 \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} + C_2 \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix} + C_3 \begin{bmatrix} -16 & -6 \\ -12 & 34 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$2C_1 - 4C_2 - 16C_3 = 0$

$-C_2 - 6C_3 = 0$

$-3C_1 - 12C_3 = 0$

$C_1 + 5C_2 + 34C_3 = 0$

$$\left[\begin{array}{ccc|c} 2 & -4 & -16 & 0 \\ 0 & -1 & -6 & 0 \\ -3 & 0 & -12 & 0 \\ 1 & 5 & 34 & 0 \end{array}\right] \text{RREF}\ldots \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ let } C_3 = t$$

$C_1 + 4t = 0 \;\therefore C_1 = -4t$

$C_2 + 6t = 0 \;\therefore C_2 = -6t$

$\therefore$ solution is $\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} -4t \\ -6t \\ t \end{bmatrix}$

$$= t \begin{bmatrix} -4 \\ -6 \\ 1 \end{bmatrix}$$

$where \quad t \epsilon R$

$\therefore$ the set of matrices A, B, C is linearly <u>dependent</u> (not just the solution $C_1 = C_2 = C_3 = 0$)

13. a) $c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -8 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$

$4c_1 - 8c_2 = 6$

$c_1 - 2c_2 = 5$

Row reduce:

$\left[\begin{matrix} 4 & -8 \\ 1 & -2 \end{matrix}\middle|\begin{matrix} 6 \\ 5 \end{matrix}\right]$ $C_1 \leftrightarrow C_2$

$\left[\begin{matrix} 1 & -2 \\ 4 & -8 \end{matrix}\middle|\begin{matrix} 5 \\ 6 \end{matrix}\right]$ $C_2 - 4C_1 \rightarrow C_2$

$\left[\begin{matrix} 1 & -2 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 5 \\ -12 \end{matrix}\right]$ $C_2 – 2C_1 \rightarrow C_2$

∴ Since there is <u>no solution</u>; $\vec{v}$ is not a linear combination of vectors $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$

$b)\ c_1 \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 20 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 22 \\ 6 \end{bmatrix}$

$5c_1 + 4c_2 - c_3 = 0$

$20c_2 - 6c_3 = 22$

$2c_3 = 6$

∴ from the last equation $c_3 = 3$

Using back substitution

$20c_2 - 6c_3 = 22$

$20c_2 - 6(3) = 22$

$20c_2 = 22 + 18$

$20c_2 = 40$

$c_2 = 2$

$5c_1 + 4c_2 - c_3 = 0$

$5c_1 + 4(2) - (3) = 0$

$5c_1 + 8 - 3 = 0$

$5c_1 = -5$

$c_1 = -1$

$\therefore \vec{v}$ is a linear combination of $\overrightarrow{u_1}, \overrightarrow{u_2}$ and $\overrightarrow{u_3}$

and $\vec{v} = -\overrightarrow{u_1} + 2\overrightarrow{u_2} + 3\overrightarrow{u_3}$

14. Recall, b is in the span of the columns of A if and only if $Ax = b$ has a solution.

$3x + 3y = 15$

$2x + 3y = 13$

$\left[\begin{matrix} 3 & 3 \\ 2 & 3 \end{matrix}\middle|\begin{matrix} 15 \\ 13 \end{matrix}\right]$ R1 × 1/3 →R1

$\left[\begin{matrix} 1 & 1 \\ 2 & 3 \end{matrix}\middle|\begin{matrix} 5 \\ 13 \end{matrix}\right]$ R_2 - $2R_1$→R_2

$\left[\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} 5 \\ 3 \end{matrix}\right]$ R_1- R_2→R_1

$\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} 2 \\ 3 \end{matrix}\right]$

There is a unique solution. AX=b, so:

$\therefore A\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 13 \end{bmatrix}$ and yes, b is in the span of A.

7. Matrices, Matrix Operations, and Matrix Algebra

Example 7.1. State the dimension of each matrix.

a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ dim 2×1

b) $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 3 & 2 \end{bmatrix}$ dim is 2×3

c) $\begin{bmatrix} 3 & 5 & 4 \end{bmatrix}$ dim is 1×3

d) $\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 3 & 6 \end{bmatrix}$ dim is 3×2

e) $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 4 & 3 & 6 \\ 5 & 4 & 7 \end{bmatrix}$ dim is 4×3

Example 7.2.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}.$$

Example 7.3.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Scalar Multiplication:

Multiply every entry in the matrix by the scalar *k*.

Example 7.4. $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & -1 \\ 3 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & -2 \\ 2 & 3 & 2 \\ 1 & -1 & 3 \end{bmatrix}$

Find 2A - 3B

Solution:

$$2\begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & -1 \\ 3 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 & -2 \\ 2 & 3 & 2 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -11 & 10 \\ -2 & -5 & -8 \\ 3 & 5 & -5 \end{bmatrix}$$

Example 7.5. Find the transpose of each matrix.

b) B= $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Solution:

$$B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

c) C= $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$

Solution:

$$C^T = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Example 7.6.

Solution:
a) 3x2 multiplied by 2x2

Possible and answer is 3x2
$$\begin{bmatrix} 0+9 & 0+6 \\ 2+3 & 1+2 \\ 6-3 & 3-2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 5 & 3 \\ 3 & 1 \end{bmatrix}$$

b) 1x3 multiplied by 3x1
Possible and answer is 1x1
$[-4 + 10 + 12] = [18]$

c) 2x4 multiplied by 3x3

Since 4≠3, this product is NOT possible.

d) 3x1 multiplied by a 1x3, gives a 3x3

$$\begin{bmatrix} -1 & -3 & -4 \\ 2 & 6 & 8 \\ 1 & 3 & 4 \end{bmatrix}$$

<u>Example 7.7.</u> Consider the following matrices *A*, *B*, *C*, and *D*:

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 4 & -5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 2 & -5 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 4 \\ -1 & 3 \end{bmatrix}$$

Compute each matrix sum or product (if possible) or explain why an expression is undefined.

a) $B - 3A$
b) CD
c) $C^T D$

<u>Solution:</u>

a) $B - 3A = \begin{bmatrix} 7 & -5 & 1 \\ 2 & -5 & -3 \end{bmatrix} + (-3)\begin{bmatrix} 1 & 0 & -3 \\ 4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -5 & 1 \\ 2 & -5 & -3 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 9 \\ -12 & 15 & -6 \end{bmatrix} = \begin{bmatrix} 4 & -5 & 10 \\ -10 & 10 & -9 \end{bmatrix}$

b) $CD = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 17 \\ -11 & -6 \end{bmatrix}$

c) $C^T D = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -1 \\ 7 & 18 \end{bmatrix}$

Example 7.8. Use matrix column representation of the product to write each column of AB as a linear combination of the columns of A, where:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \; B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

Solution:

The column vectors of B are

$$b_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad b_2 \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \quad b_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

So, the matrix column representation is

$$Ab_1 = 1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 3 \end{bmatrix}$$

$$Ab_2 = 2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$Ab_3 = -1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 7 & 2 & 1 \\ 8 & 0 & 5 \\ 3 & 2 & 2 \end{bmatrix}$$

Example 7.9. If $A = \begin{bmatrix} 4 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 3 & 0 \end{bmatrix}$

then

$$Ab_1 = \begin{bmatrix} 4 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ 2 \end{bmatrix} \text{ and } Ab_2 = \begin{bmatrix} 4 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Therefore, $AB = [Ab_1 \,\vdots\, Ab_2] = \begin{bmatrix} 17 & 2 \\ 2 & -2 \end{bmatrix}$.

Notice here that the matrix-column representation of AB enables us to write each column of AB as a linear combination of the columns of A with the coefficients being the entries of B.

For example,

$$\begin{bmatrix} 17 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Suppose A is $m \times n$ and B is $n \times r$, so the product AB exists.

If we partition A in terms of its row vectors, as

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{bmatrix}$$

Then

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{bmatrix} \quad B = \begin{bmatrix} A_1 B \\ A_2 B \\ \cdot \\ \cdot \\ \cdot \\ A_m B \end{bmatrix}$$

Once again, this result is a direct consequence of the definition of matrix multiplication. The form on the right is called the ***row-matrix representation*** of the product.

Example 7.10. Using the row-matrix representation, compute AB

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 3 & 0 \end{bmatrix}$$

We compute

$$A_1B = \begin{bmatrix} 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 2 \end{bmatrix} \text{ and } A_2B = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \end{bmatrix}$$

Therefore, $AB = \begin{bmatrix} A_1B \\ \hline A_2B \end{bmatrix} = \begin{bmatrix} 17 & 2 \\ \hline 2 & -2 \end{bmatrix}$, as before.

The definition of the matrix product AB uses the natural partition of A into rows and B into columns, this form might well be called the ***row-column representation*** of the ***column-row representation*** of the product.

In this case, we have

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \text{ and } B = \begin{bmatrix} B_1 \\ \hline B_2 \\ \hline \vdots \\ \hline B_n \end{bmatrix}$$

so

$$AB = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} B_1 \\ \hline B_2 \\ \hline \vdots \\ \hline B_n \end{bmatrix} = a_1B_1 + a_2B_2 + \cdots + a_nB_n \qquad (2)$$

Here, note that the sum resembles a dot product expansion; the difference is that the individual terms are matrices, not scalars. Let's make sure that this makes sense. Each term, a_iB_i is the product of an $m \times 1$ and a $1 \times r$ matrix. Thus, each a_iB_i is an $m \times r$ matrix – the same size as AB. The products a_iB_i are called ***outer products***, and (2) is called the ***outer product expansion*** of AB.

Poole, D. (2014). *Linear Algebra: A Modern Introduction* (4th Edition). Cengage.

Example 7.11. Let A=$\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}$. Find A^2 and A^3 and A^{45}.

Solution:

$A^2 = \begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 9\end{bmatrix}$...the 2,2 entry is 3^2=9

$A^3 = AA^2 = \begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 9\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 27\end{bmatrix}$...the 2,2 entry is 3^3=27

$A^{45} = \begin{bmatrix}1 & 0\\0 & 3^{45}\end{bmatrix}$

Example 7.12. If A is a 2x4 matrix, B^T is a 4x3 matrix and C is a 2x3 matrix, then A^TC + B is:

A. undefined	B. a 3x4	C. a 2x4	D. a 2x3	E. none of the above

Solution:

A^T is a 4x2

A^TC is a 4x2 2x3 which gives a 4x3

so the final answer is a 4x3 + 3x4 which is not defined and the answer is A.

Example 7.13. Given the matrices $E=\begin{bmatrix}1 & 0 & 2\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ and $A=\begin{bmatrix}1 & 5 & 6 & 0\\8 & 4 & 7 & -1\\2 & -5 & 1 & 7\end{bmatrix}$, determine EA, and explain (in words) the action of the elementary matrix on matrix *A*.

Solution:

$$EA=\begin{bmatrix}5 & -5 & 8 & 14\\8 & 4 & 7 & -1\\2 & -5 & 1 & 7\end{bmatrix}$$

Row 2 and Row 3 are unchanged and can be copied directly from matrix *A*.

Row 1 is a linear combination of $R_1 + 2R_3$ from *A*.

<u>Example 7.14.</u> Given A=$\begin{bmatrix} 3 & 2 & -1 \\ 4 & 2 & 1 \end{bmatrix}$, find Ae_3 and e_2A.

<u>Solution:</u>

$Ae_3 = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ∴ this is the 3rd column of A

$$2 \times \underbrace{3 \quad 3} \times 1 \qquad Ans\ \ 2 \times 1$$

$e_2A = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \end{bmatrix}$ ∴ this is the 2nd row of A

$$1 \times \underbrace{2 \quad 2} \times 3 \qquad Ans\ \ 1 \times 3$$

Example 7.15. If $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a+c & b+d \end{bmatrix}$

$BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} b & a+b \\ d & c+d \end{bmatrix}$

Since $AB = BA$

$\begin{bmatrix} c & d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} b & a+b \\ d & c+d \end{bmatrix}$

$\therefore \quad c = b \ \boxed{1} \qquad d = a + b \ \boxed{2}$

$a + c = d \ \boxed{3} \qquad b + d = c + d \ \boxed{4}$

From $\boxed{1}$ $b = c$

From $\boxed{2}$ and $\boxed{3}$

$a + b = d$

$a + c = d$

$\therefore \ a + b = a + c$

$\therefore \ b = c$ (same as $\boxed{1}$)

From $\boxed{2}$ $d = a + b$ substitute into $\boxed{3}$

$a + c = a + b$

$\therefore c = b$ (same again!)

From $\boxed{4}$ $b + d = c + d$

$\therefore b = c$ too!

$\therefore c = b$ and $d = a + b$

$\therefore B = \begin{bmatrix} a & b \\ b & a+b \end{bmatrix}$

Example 7.16. Consider the following matrix:

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Determine:

a) A^3
b) A^{-2}

Solution:

Since *A* is a diagonal matrix, this will greatly simplify our calculations.

a) $$A^3 = \begin{bmatrix} -8 & 0 & 0 \\ 0 & 216 & 0 \\ 0 & 0 & 64 \end{bmatrix}$$

b) $$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & 0 & 0 & 0 \\ 0 & 1/d_2 & 0 & 0 & 0 \\ 0 & 0 & 1/d_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1/d_n \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/a_1 & 0 & 0 \\ 0 & 1/a_2 & 0 \\ 0 & 0 & 1/a_3 \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} 1/{a_1}^2 & 0 & 0 \\ 0 & 1/{a_2}^2 & 0 \\ 0 & 0 & 1/{a_3}^2 \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{36} & 0 \\ 0 & 0 & \frac{1}{16} \end{bmatrix}$$

7.11 Homework on Chapter 7

1. Find each of the following, given matrices A and B.

A=$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix}$ and B=$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

a) AB

$$AB = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}=\begin{bmatrix} 4 & 0 & -1 \\ 9 & 2 & 3 \\ 2 & 1 & -2 \end{bmatrix}$$

b) $A^T B^T$

$$A^T B^T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 3 & -1 \\ 5 & 10 & 2 \end{bmatrix}$$

2. Let A=$\begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix}$. Then, A^2 equals:

A. $\begin{bmatrix} 2 & 8 \\ 6 & -2 \end{bmatrix}$	B. $\begin{bmatrix} 1 & 16 \\ 9 & 1 \end{bmatrix}$	C. $\begin{bmatrix} 13 & 0 \\ 0 & 11 \end{bmatrix}$	D. $\begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$	E. none of the above

$$A^2 = \begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$$

The answer is d).

3. Let matrix B=$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, find the (1,2) entry of B^{35}.

A. 1	B. 0	C. 35	D. $\begin{bmatrix} 1 & 35 \\ 0 & 1 \end{bmatrix}$	E. none of the above

$$B^2 = B \times B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$B^3 = B^2 \times B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$B^{35} = \begin{bmatrix} 1 & 35 \\ 0 & 1 \end{bmatrix}$$

The (1,2) entry is 35.

4. Let A be a 3x4 matrix, B be a 3x3 matrix and C^T be a 4x3 matrix.

Then, which of the following is not defined?

i) AC^T	ii) $A+4C^TB$	iii) AC^TB

A. i) only	B. ii) and iii) only	C. i) and ii) only	D. ii) and iii) only	E. none of the above

i) AC^T=3x4 by 4x3 = 3x3=defined

ii) A +4C^TB= 3x4 + (4x3) by (3x3)

= 3x4 + 4x3

=undefined

iii) 3x4 (4x3) by (3x3)

=3x3 by 3x3

= 3x3

The answer is E). ii only is undefined

5. Let $A = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Which of the following is defined?

A) B + 2A

B) A^T + 5B

C) $2AB^T$ + 4A

D) A^2 + 7A

E) B^TA + 5A

A) B+2A ...no, matrices must be the same dimension to add them

B) A^T +5B ...no

C) $2AB^T$ +4A= 3x1 = no

D) A^2 +7A= you can't square matrices that aren't square, so not defined

E) B ^{T}A+5A=3x3 3x1 + 3x1= 3x1 + 3x1 Yes this is defined

Therefore, E) is defined.

6. Let A= $\begin{bmatrix} 1 & 0 \\ 4 & k \end{bmatrix}$ and let I be the 2x2 identity matrix. Find the value of k for which A^2=I.

A. k= 1	B. k= -1	C. k=1,-1	D. $k \neq \pm 1$	E. none of the above

$\begin{bmatrix} 1 & 0 \\ 4 & k \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 4+4k & k^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The only value that works for both the (2,1) and (2,2 positions is k= -1. NOTE: k=1 doesn't give the 0 in the (2,1) entry for the identity

The answer is B).

7. Find matrix X such that X-2A= -4B.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$

$X - 2A + 4B = 0$

$X - 2\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + 4\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = 0$

$X + \begin{bmatrix} -2 & -4 \\ -6 & -10 \end{bmatrix} + \begin{bmatrix} -4 & 0 \\ 8 & 12 \end{bmatrix} = 0$

$X + \begin{bmatrix} -6 & -4 \\ 2 & 2 \end{bmatrix} = 0$

$X = \begin{bmatrix} 6 & 4 \\ -2 & -2 \end{bmatrix}$

8. If A, B and C are matrices of appropriate size, which one of the following rules does NOT generally hold?

1) $(AB)(BA) = AB^2A$
2) $(ABC)^t = C^tB^tA^t$
3) $(A+B)C = AC+BC$
4) $(A^tB)^t(A+C) = AB^tA + AB^tC$
5) $(CB)(AB^t)^t = CB^2A^t$

4) doesn't hold $(A^tB)^t(A+C) = AB^tA + AB^tC$

$(A^tB)^t(A+C) = \left(B^t(A^t)^t\right)(A+C) = (B^tA)(A+C)$

Since $B^tA \neq AB^t$, $(B^tA)(A+C) \neq (AB^t)(A+C)$.

Therefore, $(A^tB)^t(A+C) \neq AB^tA + AB^tC$.

9. Let R= $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Find $(R-3I)^2$, *where I is the* $3x3$ *identity matrix.*

$$\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)^2$$

$$= \begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}^2$$

$$= \begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & -2 \\ 0 & 4 & -8 \\ 0 & 0 & 4 \end{bmatrix}$$

10. If M=$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 4\\5 & 6 & 0\end{bmatrix}$ and N=$\begin{bmatrix}a & b & 5\\20 & c & -4\\-5 & 4 & 1\end{bmatrix}$ and MN is the identity matrix, find the value of b.

We are given that MN=I=$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

So, we have...

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 4\\5 & 6 & 0\end{bmatrix}\begin{bmatrix}a & b & 5\\20 & c & -4\\-5 & 4 & 1\end{bmatrix} = \begin{bmatrix}1 & \mathbf{0} & 0\\0 & 1 & 0\\0 & \mathbf{0} & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 4\\5 & 6 & 0\end{bmatrix}\begin{bmatrix}a & b & 5\\20 & c & -4\\-5 & 4 & 1\end{bmatrix} = \begin{bmatrix}1 & \mathbf{0} & 0\\0 & 1 & 0\\0 & \mathbf{0} & 1\end{bmatrix}$$

Find a row and column that can be multiplied to find "b"...

1,2 entry= first row of M multiplied by the second column of N

$\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}b\\c\\4\end{bmatrix}$=0 since the 1,2 entry of matrix I, the 3x3 identity is 0

$b + 2c + 12 = 0$

b= -12 – 2c

3,2 entry= third row of M multiplied by the second row of N

$\begin{bmatrix}5 & 6 & 0\end{bmatrix}\begin{bmatrix}b\\c\\4\end{bmatrix}$ =0 since the 3,2 entry of matrix I, is 0

this gives us 5b+6c + 0 = 0 substitute b= -12 – 2c

and get:

5(-12 – 2c)+6c = 0

-60 – 10c + 6c = 0

-4c=60

c= -15

b= -12 – 2c = -12 -2(-15) = -12 + 30 = 18

11. Find each product, if possible:

a) $\begin{bmatrix} 0 & 3 \\ 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

Possible and answer is 3x2

$$\begin{bmatrix} 0+9 & 0+6 \\ 2+3 & 1+2 \\ 6-3 & 3-2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 5 & 3 \\ 3 & 1 \end{bmatrix}$$

b) $\begin{bmatrix} 4 & 5 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} =$

Possible and answer is 1x1

$[-4 + 10 + 12] = [18]$

c) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 3 & 5 \\ 2 & 2 & 6 \end{bmatrix} =$

2x4 multiplied by 3x3

Since 4≠3, this product is NOT possible.

d) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}$

3x1 multiplied by a 1x3, gives a 3x3

$$\begin{bmatrix} -1 & -3 & -4 \\ 2 & 6 & 8 \\ 1 & 3 & 4 \end{bmatrix}$$

12. Consider the following matrices *A*, *B*, *C*, and *D*:

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 4 & -5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 2 & -5 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 4 \\ -1 & 3 \end{bmatrix}$$

Compute each matrix sum or product (if possible) or explain why an expression is undefined.

a) $-2A$
b) AC
c) $A^T C$

a) $$-2A = -2\begin{bmatrix} 1 & 0 & -3 \\ 4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -6 \\ -8 & 10 & -4 \end{bmatrix}$$

b) A*C* is undefined because *A* is a 2 x **3** matrix and *C* is a **2** x 2 matrix. Notice that the number of columns in *A* does not match the number of rows in *C*. Therefore, the matrices are not size compatible for matrix multiplication.

c) $$A^T C = \begin{bmatrix} 1 & 4 \\ 0 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -10 & 11 \\ 15 & -10 \\ -12 & -5 \end{bmatrix}$$

13. $$A^2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$$

14. $A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ find A^{11}

$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$A \times A$

$$A^3 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$A^2 \times A$

$$A^4 = A^3 \times A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A^5 = A^4 \times A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^6 = A^5 \times A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^7 = A^6 \times A = I \times A = A$$

$$A^8 = A^7 \times A = A \times A = A^2$$

$$A^9 = A^8 \times A = A^2 \times A = A^3$$

$$A^{10} = A^9 \times A = A^3 \times A = A^4$$

$$A^{11} = A^{10} \times A = A^4 \times A = A^5$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

15. Note: this is $a_1B_1 + a_2B_2 + a_3B_3$

$a_1B_1 + a_2B_2 + a_3B_3$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 2 \\ 9 & 0 & 3 \\ 3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 & 1 \\ 8 & 0 & 5 \\ 3 & 2 & 2 \end{bmatrix}$$

16. a) We have the block structure

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Now, the blocks A_{12}, A_{21}, B_{12}, and B_{21} are all zero blocks, so any products involving those blocks are zero. So, matrix AB reduces to:

$$AB = \begin{bmatrix} A_{11}B_{11} & 0 \\ 0 & A_{22}B_{22} \end{bmatrix}$$

$$A_{11}B_{11} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$$

$$A_{22}B_{22} = \begin{bmatrix} 3 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = [3(2) - 4(2)]$$

$$\therefore AB = \begin{bmatrix} 2 & 4 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

b) We have the block structure

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Now, $B_{21}$is the zero matrix, so products involving that blocks are zero. Further, $A_{12}, A_{21},$ and B_{11} are all the identity matrix I_2. So this product simplifies to

$$AB = \begin{bmatrix} A_{11}I_2 & A_{11}B_{12} + I_2B_{22} \\ I_2I_2 & I_2B_{12} + A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{11}B_{12} + B_{22} \\ I_2 & B_{12} + A_{22}B_{22} \end{bmatrix}$$

$$A_{11}B_{12} + B_{22} = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & 3 \end{bmatrix}$$

$$B_{12} + A_{22}B_{22} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

Therefore,

$$AB = \begin{bmatrix} 1 & -2 & 0 & 2 \\ 4 & 3 & 4 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

8. Matrix Inverses

Example 8.1. Find the b_{12} position of the inverse of B=$\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$

Solution: $B^{-1} = \frac{1}{(1)(-1)-(4)(2)} \begin{bmatrix} -1 & -4 \\ -2 & 1 \end{bmatrix} = \frac{-1}{9} \begin{bmatrix} -1 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/9 & 4/9 \\ 2/9 & -1/9 \end{bmatrix}$

b_{12}= the number in the 1st row and 2nd column of the inverse matrix= 4/9

Example 8.2. 2x+3y=6 x-7y=10

Matrix Form: $\begin{bmatrix} 2 & 3 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$ Solution: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -7 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 10 \end{bmatrix}$

Example 8.3. Find the inverse of each matrix.

a) A=$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 2 & 5 & -1 \end{bmatrix}$

Solution:

$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 2 & 5 & -1 & 0 & 0 & 1 \end{array}\right]$ R3 -2R1→ $R3$ and R2(-1) → $R2$

$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right]$ R3 – R2 → $R3$

$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right]$ R1 + R3→ $R1$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right]$ R1 – 2R2 → $R1$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right]$

$A^{-1} = \begin{bmatrix} -1 & 3 & 1 \\ 0 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$

Example 8.4. Given the following system of equations, find the solution.

$ax_1+bx_2+cx_3=7$ $dx_1+ex_2+fx_3=3$ $gx_1+hx_2+jx_3=2$

You are also given that: $\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & j & 0 & 0 & 1 \end{array}\right]$ row reduces to $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & -1 & 6 & 2 \end{array}\right]$

Solution:

$b = \begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix} \quad X = A^{-1}b$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 4 \\ -1 & 6 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix}$$

∴ x1 = 1(7) +2(3) -1(2)=7+6-2=11

∴ x2=7 +3 +8=18 and ∴ x3= - 7 +18 +4=15

The solution is (11,18,15).

Example 8.5. Use the inverse of the coefficient matrix to solve the system.

x + 4y = 3

2x + 5y = 6

The coefficient matrix is:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$$

Solution:

$$A^{-1} = \frac{1}{(1)(5)-(4)(2)} \begin{bmatrix} 5 & -4 \\ -2 & 1 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 5 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & \frac{4}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

The solution is X=A^-1^b

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & \frac{4}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Example 8.6. Solve for x:

$B^{-1}(AX)^{-1} = (B^{-1}A^3)^2$

Solution:

$B^{-1}(AX)^{-1} = (B^{-1}A^3)^2$

$[(AX)B]^{-1} = (B^{-1}A^3)^2$…use the property $(AB)^{-1} = B^{-1}A^{-1}$

$[((AX)B)^{-1}]^{-1} = [(B^{-1}A^3)^2]^{-1}$…take the inverse of both sides

$(AX)B = [(B^{-1}A^3)(B^{-1}A^3)]^{-1}$

$AXB = (B^{-1}A^3)^{-1}(B^{-1}A^3)^{-1}$

$AXB = A^{-3}(B^{-1})^{-1}A^{-3}(B^{-1})^{-1}$

$AXB = A^{-3}BA^{-3}B$

$A^{-1}AXB = A^{-1}A^{-3}BA^{-3}B$…multiply on the left of both sides by A^{-1}

$IXB = A^{-1}A^{-3}BA^{-3}B$

$XB = A^{-4}BA^{-3}B$

$XBB^{-1} = A^{-4}BA^{-3}BB^{-1}$…multiply on the right of both sides by B^{-1}

$XI = A^{-4}BA^{-3}I$

$X = A^{-4}BA^{-3}$

Example 8.7.

Solution:

We row reduce A:

$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $R1 \leftrightarrow R2$ $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $R2 - 2R1 \to R2$

$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ $R2 \times -1 \to R2$ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $R1 - 2R2 \to R1$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$\therefore A$ is invertible and can be written as a product of elementary matrices.

$R1 \leftrightarrow R2$ $\quad R2 - 2R1 \to R2$ $\quad R2 \times -1 \to R2$

$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ $\quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$R1 - 2R2 \to R1$ $\quad E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$E_4 E_3 E_2 E_1 A = I_2$

$\therefore A = (E_4 E_3 E_2 E_1)^{-1} = {E_1}^{-1} {E_2}^{-1} {E_3}^{-1} {E_4}^{-1}$

$$A = \underset{E^{-1}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \underset{{E_2}^{-1}}{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}} \underset{{E_3}^{-1}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \underset{{E_4}^{-1}}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}$$

Example 8.8. Find the value of k for which $\begin{bmatrix} k & 10 \\ 10 & k \end{bmatrix}$ has no inverse.

A. 0	B. 10	C. -10	D. 100	E. none of the above

Solution:

There is no inverse if detA=0 or if ad-bc=0.

(k)(k)-(10)(10)=0

$k^2 = 100$

k=10, -10

The answer is e).

Example 8.9. Find the value of x for which the matrix $\begin{bmatrix} 1 & 0 & x \\ 0 & 3 & 4 \\ -1 & 3 & 1 \end{bmatrix}$ is non-invertible?

A. 3	B. -3	C. 0	D. 1	E. none of the above

Solution:

$[A|I] \to [I|A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & x & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ -1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & x & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & 3 & x+1 & 1 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & x & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & 0 & x-3 & 1 & -1 & 1 \end{array}\right]$$

R3+R1 $\to R3$ R3 - R2$\to R3$

If the matrix is invertible, we get the identity matrix I when we row-reduce.

If it is not invertible, we get a row of 0's and can't get the identity matrix...

From the last row, x - 3=0, and x=3.

The answer is a).

8.6 Homework on Chapter 8

1. Find $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}^{-1}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \text{ R3-R1} \rightarrow \text{R3 } \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right] \text{ R1-2R2} \rightarrow \text{R1, R3+2R2} \rightarrow \text{R3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -4 & -1 & 2 & 1 \end{array}\right] \text{ R3} \div (-4) \rightarrow \text{R3 } \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/4 & -1/2 & -1/4 \end{array}\right] \text{ R2+R3} \rightarrow \text{R2, R1-6R3} \rightarrow \text{R1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1 & 3/2 \\ 0 & 1 & 0 & 1/4 & 1/2 & -1/4 \\ 0 & 0 & 1 & 1/4 & -1/2 & -1/4 \end{array}\right]$$

Therefore,

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1/2 & 1 & 3/2 \\ 1/4 & 1/2 & -1/4 \\ 1/4 & -1/2 & -1/4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & -1 \\ 1 & -2 & -1 \end{bmatrix}$$

2. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$. Show that *A* is invertible and find A^{-1}.

Form the augmented matrix and reduce:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \text{ R2-3R1} \rightarrow \text{R2 } \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -3 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \text{ R3+R2} \rightarrow \text{R3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -3 & 1 & 0 \\ 0 & 0 & -1 & -3 & 1 & 1 \end{array}\right] \text{ R3 x -1} \rightarrow \text{R3 } \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -3 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right] \text{ R1-R2} \rightarrow \text{R1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right]$$

It follows that A is invertible and that $A^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ 6 & -2 & -3 \\ 3 & -1 & -1 \end{bmatrix}$.

3. Find the entry in the 1st row and 2nd column of the inverse of $A=\begin{bmatrix}1&0&1\\0&2&1\\1&1&1\end{bmatrix}$.

(1) 2
(2) 0
(3) −2
(4) −1
(5) 1

Solution: (4) −1

Using the inversion algorithm,

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&2&1&0&1&0\\1&1&1&0&0&1\end{array}\right] \text{R3-R1} \rightarrow \text{R3} \left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&2&1&0&1&0\\0&1&0&-1&0&1\end{array}\right] \text{R2-2R3} \rightarrow \text{R2}$$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&0&1&2&1&-2\\0&1&0&-1&0&1\end{array}\right] \text{R1-R2} \rightarrow \text{R1} \left[\begin{array}{ccc|ccc}1&0&0&-1&-1&2\\0&0&1&2&1&-2\\0&1&0&-1&0&1\end{array}\right] \text{R2} \leftrightarrow \text{R3}$$

$$\left[\begin{array}{ccc|ccc}1&0&0&-1&-1&2\\0&1&0&-1&0&1\\0&0&1&2&1&-2\end{array}\right]$$

Therefore, $A^{-1}=\begin{bmatrix}-1&-1&2\\-1&0&1\\2&1&-2\end{bmatrix}$ and its (1, 2)th entry is −1.

4. Find $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1}$ if it exists (use the inversion algorithm)

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \text{R1} \leftrightarrow \text{R2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 \end{array}\right] \text{R3-2R1} \rightarrow \text{R3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & 1 & 0 & -2 \end{array}\right] \text{R1-R2} \rightarrow \text{R1, R3+2R2} \rightarrow \text{R3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right]$$

$$\text{R1+R3} \rightarrow \text{R1, R2 - R3} \rightarrow \text{R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right]$$

So $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 2 \\ 1 & 2 & -2 \end{bmatrix}$.

5. Consider the following matrix: $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & -9 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & -8 \end{bmatrix}$

Determine A^2 and A^{-3}

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 \\ 0 & 0 & 81 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 64 \end{bmatrix}$$

$$A^{-3} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1/125 & 0 & 0 & 0 \\ 0 & 0 & -1/729 & 0 & 0 \\ 0 & 0 & 0 & 1/27 & 0 \\ 0 & 0 & 0 & 0 & -1/512 \end{bmatrix}$$

6. a) If. A=$\begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$, $find\ the\ inverse\ of\ A.$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2(1)-(-2)(-3)}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1/4 & -1/2 \\ -3/4 & -1/2 \end{bmatrix}$$

b) Find the solution to the system of equations 2x – 2y = 6 and -3x +y = 4.

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 6+8 \\ 18+8 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 14 \\ 26 \end{bmatrix} = \begin{bmatrix} -7/2 \\ -13/2 \end{bmatrix}$$

7. If A^{-1}= $\begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix}$, where A=$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and the system of linear equations is:

ax + by + cz = 5

dx + ey + fz = 1

gx + hy + iz= -1, then the solution to the system is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5-1+0 \\ 15+2-1 \\ 10+1-5 \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \\ 6 \end{bmatrix}$$

8. If $\begin{bmatrix} a & d & 1 \\ b & e & 3 \\ c & f & -2 \end{bmatrix}$ $is\ the\ inverse\ of$ $\begin{bmatrix} 4 & 0 & x \\ 2 & -2 & y \\ 0 & 3 & z \end{bmatrix}, find\ y.$

$\begin{bmatrix} a & d & 1 \\ b & e & 3 \\ c & f & -2 \end{bmatrix}\begin{bmatrix} 4 & 0 & x \\ 2 & -2 & y \\ 0 & 3 & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ since they are inverses of each other. We need to find y which is in the third column, but it is too difficult since x and z are also in the third column and there are too many unknowns. But, if AB=I, then BA=I as well, so switch the order of the matrices.

$$\begin{bmatrix} 4 & 0 & x \\ 2 & -2 & y \\ 0 & 3 & z \end{bmatrix}\begin{bmatrix} a & d & 1 \\ b & e & 3 \\ c & f & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, y is in the second row of the first matrix and we can dot product it with the third column of the second matrix and set it equal to the (2,3) entry of I which is a 0.

2(1) + (-2)(3) + (y)(-2) = 0

2 -6 – 2y = 0

-4 =2y

y= -2

9. Let *A* and *B* be invertible matrices. Which of the following is always true?

A. $(A-B)(A+B) = A^2 - B^2$
B. $(ABA^{-1})^3 = AB^3A^{-1}$
C. $(A^{-1}B)^{-1} = B^{-1}A$
D. $A+B$ can be written as a product of elementary matrices

(1) A, C and D
(2) B and C
(3) A, B and C
(4) A and D
(5) B, C and D

(2) B and C

Statement A: $(A-B)(A+B) = A^2 - BA + AB - B^2$. $AB \neq BA$ even when *A* and *B* are invertible. So $(A-B)(A+B) \neq A^2 - B^2$ and statement A is not always true.

Statement B:

$$(ABA^{-1})^3 = ABA^{-1}ABA^{-1}ABA^{-1} = ABIBIBA^{-1} = ABBBA^{-1} = AB^3A^{-1}$$

Therefore, statement B is true.

Statement C: $(A^{-1}B)^{-1} = B^{-1}(A^{-1})^{-1} = B^{-1}A$. So statement C is true.

Statement D: $A+B$ does not have to be invertible just because *A* and *B* are invertible. When $A+B$ is not invertible, $A+B$ cannot be written as a product of elementary matrices. So, this statement is not always true.

10. Solve for x:

$$A^{-1}(BX)^{-1} = (A^2B^{-1})^2$$

$$A^{-1}(BX)^{-1} = (A^2B^{-1})^2$$

$$[(BX)A]^{-1} = (A^2B^{-1})^2$$

$$\left[\left((BX)A\right)^{-1}\right]^{-1} = [(A^2B^{-1})^2]^{-1}$$

$$(BX)A = [(A^2B^{-1})(A^2B^{-1})]^{-1}$$

$$(BX)A = (A^2B^{-1})^{-1}(A^2B^{-1})^{-1}$$

$$(BX)A = (B^{-1})^{-1}(A^2)^{-1}(B^{-1})^{-1}(A^2)^{-1}$$

$$(BX)A = (BA^{-2})(BA^{-2})$$

$$B^{-1}BXA = (B^{-1}B)A^{-2}BA^{-2}$$

$$I(XA) = IA^{-2}(BA^{-2})$$

$$I(XA) = A^{-2}(BA^{-2})$$

$$XA = A^{-2}(BA^{-2})$$

$$XAA^{-1} = A^{-2}(BA^{-2})(A^{-1})$$

$$XI = A^{-2}BA^{-3}$$

$$X = A^{-2}BA^{-3}$$

11. Express $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ as a product of elementary matrices and write A^{-1} as a product of elementary matrices.

$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ R1←→ R2 $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ R2-3R1 → R2

$\begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}$ R2 ÷ (−4) → R2 $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ R1-2R2 → R1

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

∴ A is invertible and can be written as a product of elementary matrices

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ R1 ←→ R2 ∴ $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ R2-3R1 → R2 ∴ $E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ R2 ÷ (−4) → R2 ∴ $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ R1-2R2 → R1 ∴ $E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

∴ $E_4 E_3 E_2 E_1 A = I_2$

∴ $A = (E_4 E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$

and $A^{-1} = E_4 E_3 E_2 E_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

12. Find the inverse

B= $\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ R3 +R1→ $R3$ and R2 (-1) → $R2$

$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 1 \end{array}\right]$ R3 + R2 → $R3$

$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$ R2 + 2R3→ $R2$

$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -3 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$ R1 – 2R3→ $R1$

$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 2 & -2 \\ 0 & 1 & 0 & 2 & -3 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$ R1 + R2→ $R1$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2 & -3 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$

$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

13.a) Find the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$

$[A|I] \rightarrow [I|A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] R3 \times \frac{1}{2} \rightarrow R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] R2 + 3R3 \rightarrow R2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 3/2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] R1 - R3 \rightarrow R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 1 & 3/2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1/2 \end{bmatrix}$$

b) $B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & b & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $b \neq 0$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right] R3 \times \frac{1}{4} \rightarrow R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/4 \end{array}\right] R2 \times \frac{1}{b} \rightarrow R2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/b & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/4 \end{array}\right] R1 + 2R3 \rightarrow R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1/b & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/4 \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

c) $\left[\begin{array}{cccc|cccc} 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] R1 \leftrightarrow R3$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] R2 - 3R1 \to R2, R4 + R3 \to R4$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & -9 & 3 & 0 & 1 & -3 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] R2 \leftrightarrow R3$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 4 & -9 & 3 & 0 & 1 & -3 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] R2 \times -1 \to R2, R3 + 4R2 \to R3$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -5 & 3 & 4 & 1 & -3 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] R3 \times (-\frac{1}{5}) \to R3$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3/5 & -4/5 & -1/5 & 3/5 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] R4 - 2R3 \to R4, R2 + R3 \to R2$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3/5 & -9/5 & -1/5 & 3/5 & 0 \\ 0 & 0 & 1 & -3/5 & -4/5 & -1/5 & 3/5 & 0 \\ 0 & 0 & 0 & 1/5 & 13/5 & 2/5 & -6/5 & 1 \end{array}\right] R4 \times 5 \to R4$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3/5 & -9/5 & -1/5 & 3/5 & 0 \\ 0 & 0 & 1 & -3/5 & -4/5 & -1/5 & 3/5 & 0 \\ 0 & 0 & 0 & 1 & 13 & 2 & -6 & 5 \end{array}\right] R3 + \tfrac{3}{5}R4 \to R3, R2 + \tfrac{3}{5}R4 \to R2$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 6 & 1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 7 & 1 & -3 & 3 \\ 0 & 0 & 0 & 1 & 13 & 2 & -6 & 5 \end{array}\right] R1 - 3R3 \to R1$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & -21 & -3 & 10 & -9 \\ 0 & 1 & 0 & 0 & 6 & 1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 7 & 1 & -3 & 3 \\ 0 & 0 & 0 & 1 & 13 & 2 & -6 & 5 \end{array}\right] R1 + R2 \to R1$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -15 & -2 & 7 & -6 \\ 0 & 1 & 0 & 0 & 6 & 1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 7 & 1 & -3 & 3 \\ 0 & 0 & 0 & 1 & 13 & 2 & -6 & 5 \end{array}\right]$$

14. If A and B are invertible square matrices with $AB = I$, show $B = A^{-1}$:

Multiply on the left by A^{-1}:

$A^{-1}AB = A^{-1}I$

$IB = A^{-1}$

$\therefore B = A^{-1}$

Since $A^{-1}A = I$ and $AI = A$ for all A

9. Subspaces, Bases, Column Space, and Row Space

Example 9.1.

Show that the set of vectors of $\Re^3$ whose middle component is zero is a subspace of $\Re^3$.

Solution:

Strategy: we're going to need to show that the requirements of subspaces are met by all of the vectors that have the form described in the question.

Step 1:

First, we need to find a way to describe the vectors that fit the description given above, i.e. we need an expression for a general vector in the subspace. We know that the middle component must be zero, but the first and last component can be any element of $\Re$.

Therefore, a general vector in this set will look like:

$\vec{v} = (a, 0, b)$, where $a, b \in \Re$.

We'll call S the set whose vectors have the form $\vec{v} = (a, 0, b)$, where $a, b \in \Re$.

Step 2:

First, we need to check that the set S is closed under addition.

This means that we need to make sure that any time we add two vectors that belong to the set, the resultant vector will also belong to the set.

Let's see what happens when we add two general vectors that belong to S:

For every pair of vectors of S, $\vec{v}_1 = (a, 0, b)$ and $\vec{v}_2 = (c, 0, d)$, the sum will have the form:

$\vec{v}_1 + \vec{v}_2 = (a + c, 0, b + d)$

This vector will still belong to S (since the middle component is still zero).

Step 3:

Next, we need to check that the set S is closed under scalar multiplication.

This means that we need to make sure that any time we multiply any vector in the set by a scalar quantity, the resultant vector will also belong to the set.

Let's see what happens to a general vector that belongs to S, when we multiply it by a scalar:

When we multiply any vector $\vec{v} = (a, 0, b)$ from the set S by a scalar, it will have the form:

$k\vec{v} = k(a, 0, b) = (ka, 0, kb)$. This vector will still belong to S (since the middle component is still zero). Therefore, the set S is indeed a subspace of $\Re^3$.

Example 9.2.

Determine whether the set S of vectors of $\Re^3$ whose components are all even numbers is a subspace of $\Re^3$.

Solution:

No, this set is not a subspace of $\Re^3$.

We can prove this by showing that the set is not closed under scalar multiplication.

Take the element (2,2,2) that belongs to the set S.

Multiply this vector by the scalar $\frac{1}{2} \in \Re$.

The resultant vector: $\frac{1}{2}(2,2,2) = (1,1,1)$ does not belong to S.

Therefore, S is not a subspace.

Example 9.3.

Determine if the following set of vectors is a subspace of $\mathbb{R}^3$.

$T = \{(x, y, z) \in \mathbb{R}^3 | xy - z = 0\}$

Solution:

Vector Addition

$\vec{u} = (a, b, c) \quad ab - c = 0 \quad \vec{u} \in T$

$\vec{v} = (d, e, f) \quad de - f = 0 \quad \vec{v} \in T$

$$\vec{u} + \vec{v} = (a, b, c) + (d, e, f)$$
$$= (a + d, b + e, c + f)$$

Check restriction:

$$xy - z = (a + d)(b + e) - (c + f)$$
$$= ab + bd + ae + de - c - f$$
$$= (ab - c) + (de - f) + bd + ae$$
$$= 0 + 0 + bd + ae$$
$$\neq 0$$

$\therefore T$ is not closed under vector addition.

Since T failed one of the tests, T is not a subspace of $\mathbb{R}^3$.

Example 9.4.

Show that the set S = { (5,0,0), (0,6,0), (0,0,1) } is a basis for $\Re^3$.

Solution: In order to show that this set constitutes a basis for $\Re^3$, we need to show that it spans $\Re^3$ and that it is linearly independent.

Step 1: Show that S spans $\Re^3$.

Any vector in $\Re^3$ has the form (a,b,c), where a,b,c are real numbers.

We need to show that any vector of that form can be written as a linear combination of the vectors in S.

i.e. We need to show that there exist real numbers x,y,z such that the following expression holds:

$$(a,b,c) = x(5,0,0) + y(0,6,0) + z(0,0,1) = (5x,6y,z)$$

Equating the variables:

$5x = a$

$6y = b$

$z = c$

In matrix form:

$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ …RREF $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which has rank 3, so the column vectors span $\Re^3$.

We could also say that since the determinant of the matrix is nonzero, so we know that there is a unique solution for which the above system of equations holds.

Therefore, S spans $\Re^3$.

Step 2:

Note that the above equation can be transformed into RREF of the following form. Therefore, the set is independent. (only the trivial solution)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the set is a basis for $\Re^3$.

Example 9.5.

Let $W = span\{(1,0,0,1),(0,-1,-1,0),(0,0,0,1),(0,1,1,1)\}$ in $\mathbb{R}^4$. Find a basis of W which is a subset of the given spanning set.

Solution:

We are given a spanning set, but it is *implied* that the current spanning set has linearly dependent vectors. Let's check and see what vector(s) are linearly dependent, and remove it/them. Then, we will have a basis.

$$span\{(1,0,0,1),(0,-1,-1,0),(0,0,0,1),(0,1,1,1)\} = (0,0,0,0)$$

$$\begin{aligned} a &= 0 \\ -b+d &= 0 \\ -b+d &= 0 \\ a+c+d &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R4-R1\to R4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R2\times(-1)\to R2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R3+R2\to R3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Since the last column (represented by coefficient "*d*") has a free variable/parameter, it needs to be eliminated from the spanning set in order for the spanning set to be a *basis*.

Therefore, $W = span\{(1,0,0,1),(0,-1,-1,0),(0,0,0,1),\boxed{(0,1,1,1)}\}$

$$= span\{(1,0,0,1),(0,-1,-1,0),(0,0,0,1)\}$$

is a basis of W.

Example 9.6.

Consider $S = \{ A \in M_{22} \mid A = -A^T \}$.

a) Find a spanning set for *S*.
b) Find a basis for *S*.
c) Determine *dim S*.
d) Extend your basis in c) to a basis for M_{22}.

Solution:

a) First, we need to simplify the matrix *A*.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in \mathbb{R}$

$-A^T = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix}$

$A = -A^T$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix}$

$a = -a \rightarrow 2a = 0$

$b = -c$

$c = -b$

$d = -d \rightarrow 2d = 0$

$\therefore A = b \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$

Then, we can write the simplified matrix *A* as a span:

$$A = b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$= span \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

This is a spanning set!

Since the set *S* has a spanning set, it is a vector space.

A spanning set for *S* is $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$.

b) A basis for S is $\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$. We must state the following:

The proposed basis is 1) a spanning set and 2) linearly independent (since a single vector must be independent from *itself*).

c) There is 1 vector in the basis, so *dim S = 1*.

d) We need to add **3 vectors** to our basis to extend it to a basis of M_{22}.

First, we can "plug the holes" that are in the basis vector. I propose that we add the following two vectors to our set:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We need to add another vector…

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ (note that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ would also work)

Let's combine the four of them together, and check if they are linearly independent.

Our proposed basis for $\mathbb{R}^4$: $\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$

Check for linear independence:

$$span\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\} = a\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} b &= 0 \\ a &= 0 \\ -a + d &= 0 \\ c &= 0 \end{aligned}$$

$$\therefore a = b = c = d = 0$$

Since the only solution is the trivial solution, the vectors are linearly independent.

Example 9.7.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 1 \\ 3 & 10 & 13 \\ 0 & 8 & 2 \end{bmatrix}$$

a) Find bases for the row space, column space, and null space of *A*.

b) What is *dim row A*, *dim col A*, and *dim null space of A*?

Solution:

$$\begin{bmatrix} \boxed{1} & 2 & 4 \\ 0 & 4 & 1 \\ 3 & 10 & 13 \\ 0 & 8 & 2 \end{bmatrix} \xrightarrow{R3-3R1 \to R3} \begin{bmatrix} \boxed{1} & 2 & 4 \\ 0 & 4 & 1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{bmatrix}$$

$$\xrightarrow{R2 \div 4 \to R2} \begin{bmatrix} \boxed{1} & 2 & 4 \\ 0 & \boxed{1} & 1/4 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{bmatrix}$$

$$\xrightarrow[R3-4R2 \to R3]{\substack{R1-2R2 \to R1 \\ R4-8R2 \to R4}} \begin{bmatrix} \boxed{1} & 0 & 7/2 \\ 0 & \boxed{1} & 1/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$null\ A = \left[\begin{array}{ccc|c} 1 & 0 & 7/2 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x + 7/2z = 0$$
$$y + 1/4z = 0$$
$$z = t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -7/2 \\ -1/4 \\ 1 \end{bmatrix}$$

$\{[1, 0, 7/2], [0, 1, 1/4]\}$ is a basis for the row space of *A*.

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 10 \\ 8 \end{bmatrix} \right\}$ is a basis for the column space of *A*.

$\left\{ \begin{bmatrix} -7/2 \\ -1/4 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space or kernel of *A*. This is the solution of the system in RREF.

dim row A=2

dim col A=2

dim null space = 1

Example 9.8.

Let $A = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 1 & 3 & 0 \\ 2 & 1 & 3 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}$. What is the dimension of the row space?

Solution:

The dimension of the row space is the same as the rank of the matrix. …RREF

$$\begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 1 & 3 & 0 \\ 2 & 1 & 3 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{R1\leftarrow\rightarrow R2} \begin{bmatrix} \boxed{1} & 1 & 3 & 0 \\ 0 & 1 & 0 & -3 \\ 2 & 1 & 3 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow[R4-R1\rightarrow R4]{R3-2R1\rightarrow R1} \begin{bmatrix} \boxed{1} & 1 & 3 & 0 \\ 0 & \boxed{1} & 0 & -3 \\ 0 & -1 & -3 & 2 \\ 0 & -1 & -3 & 2 \end{bmatrix}$$

$$\xrightarrow[\substack{R3+R2\rightarrow R3 \\ R4+R2\rightarrow R4}]{R1-R2\rightarrow R1} \begin{bmatrix} \boxed{1} & 0 & 3 & 3 \\ 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & -3 & -1 \end{bmatrix}$$

$$\xrightarrow{R3\div(-3)\rightarrow R3} \begin{bmatrix} \boxed{1} & 0 & 3 & 3 \\ 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & \boxed{1} & 1/3 \\ 0 & 0 & -3 & -1 \end{bmatrix}$$

$$\xrightarrow[R1-3R3\rightarrow R1]{R4+3R3\rightarrow R4} \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & \boxed{1} & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $rank\, A = 3$, $\dim row A = 3$.

Example 9.9.

Given that the RREF of the standard matrix for $T: \mathbb{R}^5 \to \mathbb{R}^4$ is

$$\begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 5/2 \\ 0 & 1 & 2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for the row space, the column space, and null space of the standard matrix.

Solution:

A basis for the row space is $\{[1 \quad 0 \quad 3 \quad 0 \quad 5/2],[0 \quad 1 \quad 2 \quad 0 \quad 3/2],[0 \quad 0 \quad 0 \quad 1 \quad 1]\}$. These are the rows with leading 1's in the RREF.

A basis for the column space is $\left\{\begin{bmatrix} 3 \\ -2 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 3 \end{bmatrix}\right\}$. These are the columns in the original matrix A where elementary columns in the RREF.

To find a basis for the null space, we need to solve the augmented system $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 5/2 & 0 \\ 0 & 1 & 2 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We can write out the general solution from the above matrix to get

$x_1 + 3x_3 + \frac{5}{2}x_5 = 0$

$x_2 + 2x_3 + \frac{3}{2}x_5 = 0$

$x_3 = s, s \in \mathbb{R}$

$x_4 + x_5 = 0$

$x_5 = t, t \in \mathbb{R}$

Then, rewriting our solution in terms of the free variables, we get

$$x_1 = -3s - \frac{5}{2}t$$
$$x_2 = -2s - \frac{3}{2}t$$
$$x_3 = s$$
$$x_4 = -t$$
$$x_5 = t$$

We can then write our solution in vector form to get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5/2 \\ -3/2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Therefore, a basis for the null space is $\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/2 \\ -3/2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Example 9.10. Show vector $\vec{v}$ is in span (B) and find the coordinate vector $[\vec{v}]_B$.

B= $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\vec{v} = \begin{bmatrix} 5 \\ 9 \\ -2 \end{bmatrix}$

Solution:

We solve:

$$C_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ -2 \end{bmatrix}$$

$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 3 & 0 & 9 \\ 0 & 1 & -2 \end{array}\right]$ RREF... R2-3R1 → R2

$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 3 & -6 \\ 0 & 1 & -2 \end{array}\right]$ R2 ÷3→ R2

$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{array}\right]$ R3-R2 → R3 $\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$

R1+R2 → R1 $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$…This is the RREF.

$\therefore\ C_1 = 3, C_2 = -2$

$$3 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ -2 \end{bmatrix}$$

∴ the coordinate vector is $[\vec{v}]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

<u>Example 9.11.</u> Show vector $\vec{v}$ is in span (B) and find the coordinate vector $[\vec{v}]_B$.

B= $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\vec{v} = \begin{bmatrix} 5 \\ 9 \\ -2 \end{bmatrix}$

<u>Solution:</u>

We solve:

$$C_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ -2 \end{bmatrix}$$

$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 3 & 0 & 9 \\ 0 & 1 & -2 \end{array}\right]$ RREF... R2-3R1 → R2

$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 3 & -6 \\ 0 & 1 & -2 \end{array}\right]$ R2 $\times$ 1/3→ R2

$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{array}\right]$ R3-R2 → R3 $\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$

R1+R2 → R1 $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$…This is the RREF.

9.7 Homework on Chapter 9

1. $S = \{A \in M_{22} \mid A^2 = A\}$ is a subset of the vector space M_{22}. Is S a subspace of M_{22}? Justify.

To determine whether or not a subset is a subspace, one has to verify three things using the operations that are defined on the vector space:

- It must be closed under addition (i.e. $x + y \in S$ whenever $x, y \in S$).
- It must be closed under scalar multiplication (i.e. $ax \in S$ whenever $x \in S$ and $a \in F$, the underlying field).
- The zero vector of the vector space must belong to the subset. (This one follows from the other two above)

We can see that this subset is not a subspace because it fails to satisfy the condition about closure under addition. To justify, simply find any counter-example.

A counter-example:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is in S since } A_1^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A_1$$

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ is in S since } A_2^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A_2$$

$$\text{But } A_3 = A_1 + A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \text{ is not in S since } A_3^2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \neq A_3$$

2. Determine if the following set of vectors is a subspace of $\mathbb{R}^3$.

$T = \{(x, y, z) \in \mathbb{R}^3 | xy - z = 0\}$

Vector Addition

$\vec{u} = (a, b, c) \quad ab - c = 0 \quad \vec{u} \in T$

$\vec{v} = (d, e, f) \quad de - f = 0 \quad \vec{v} \in T$

$\vec{u} + \vec{v} = (a, b, c) + (d, e, f)$

$= (a + d, b + e, c + f)$

Check restriction:

$xy - z = (a + d)(b + e) - (c + f)$

$= ab + bd + ae + de - c - f$

$= (ab - c) + (de - f) + bd + ae$

$= 0 + 0 + bd + ae$

$\neq 0$

$\therefore T$ is not closed under vector addition.

<u>Since T failed one of the tests, T is not a subspace of</u> $\mathbb{R}^3$**.**

3. Determine if the following set of vectors is a subspace of $\mathbb{R}^3$.

$W = \{(x, y, z) \in \mathbb{R}^3 | 2x - 3y + 4z = 0\}$

Vector Addition

$\vec{u} = (a, b, c) \quad ab - c = 0 \quad \vec{u} \in W$

$\vec{v} = (d, e, f) \quad de - f = 0 \quad \vec{v} \in W$

$\vec{u} + \vec{v} = (a, b, c) + (d, e, f)$

$= (a + d, b + e, c + f)$

Check restriction:

$2x - 3y + 4z = 2(a + d) - 3(b + e) + 4(c + f)$

$= 2a + 2d - 3b - 3e + 4c + 4f$

$= (2a - 3b + 4c) + (2d - 3e + 4f)$

$= 0 + 0$

$= 0$

$\therefore W$ is closed under vector addition.

Scalar Multiplication

$k \in R$

$k\vec{u} = k(a, b, c)$

$= (ka, kb, kc)$

check restriction:

$2x - 3y + 4z = 2(ka) - 3(kb) + 4(kc)$

$= 2ka - 3kb + 4kc$

$= k(2a - 3b + 4c)$

$= k(0)$

$= 0$ for all k

$\therefore$ W is closed under scalar multiplication.

The zero vector is in W since 2(0)-3(0)+4(0)=0

Since W passed all of the tests, W is a subspace of $\mathbb{R}^3$.

4. Determine if the following set of vectors is a subspace of M_{22}.

$S = \{A \in M_{22} | A = A^T\}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in \mathbb{R}$

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$A = A^T$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$a = a$

$b = c$

$c = b$

$d = d$

$\therefore A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

Zero Vector

check restriction:

Is it possible for us to have a zero vector?

Yes!

If $a = b = d = 0$ (which are all real numbers), then $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$\therefore$ S contains a zero vector.

Vector Addition

$\vec{u} = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \quad a, b, d \in \mathbb{R} \quad \vec{u} \in S$

$\vec{v} = \begin{bmatrix} e & f \\ f & h \end{bmatrix} \quad e, f, h \in \mathbb{R} \quad \vec{v} \in S$

$\vec{u} + \vec{v} = \begin{bmatrix} a & b \\ b & d \end{bmatrix} + \begin{bmatrix} e & f \\ f & h \end{bmatrix}$

$= \begin{bmatrix} a+e & b+f \\ b+f & d+h \end{bmatrix}$

Check restriction:

$a + e \in \mathbb{R}$

$b + f \in \mathbb{R}$

$d + h \in \mathbb{R}$

$\therefore S$ is closed under vector addition.

Scalar Multiplication

$k \in \mathbb{R}$

$k\vec{u} = k \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

$= \begin{bmatrix} ka & kb \\ kb & kd \end{bmatrix}$

Check restriction:

$ka \in \mathbb{R}$

$kb \in \mathbb{R}$

$kd \in \mathbb{R}$

For all k

$\therefore S$ is closed under scalar multiplication.

Also, the zero matrix is in S.

Since S passed all of the tests, S is a subspace of M_{22}.

5. Let $A=\begin{bmatrix}1 & 2 & 4\\ 0 & 4 & 1\\ 3 & 10 & 13\\ 0 & 8 & 2\end{bmatrix}$

a) Find bases for the row space and the column space of *A*.

b) What are *dim row A and dim col A?*

$$\begin{bmatrix}\boxed{1} & 2 & 4\\ 0 & 4 & 1\\ 3 & 10 & 13\\ 0 & 8 & 2\end{bmatrix}\xrightarrow{R3-3R1\to R3}\begin{bmatrix}\boxed{1} & 2 & 4\\ 0 & 4 & 1\\ 0 & 4 & 1\\ 0 & 8 & 2\end{bmatrix}$$

$$\xrightarrow{R2\div 4\to R2}\begin{bmatrix}\boxed{1} & 2 & 4\\ 0 & \boxed{1} & 1/4\\ 0 & 4 & 1\\ 0 & 8 & 2\end{bmatrix}$$

$$\xrightarrow[R4-8R2\to R4]{\substack{R1-2R2\to R1\\ R3-4R2\to R3}}\begin{bmatrix}\boxed{1} & 0 & 7/2\\ 0 & \boxed{1} & 1/4\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$null\ A=\left[\begin{array}{ccc|c}1 & 0 & 7/2 & 0\\ 0 & 1 & 1/4 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$x+7/2z=0$$
$$y+1/4z=0$$
$$z=t$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=t\begin{bmatrix}-7/2\\ -1/4\\ 1\end{bmatrix}$$

$\{[1, 0, 7/2], [0, 1, 1/4]\}$ is a basis for the row space of *A*.

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 10 \\ 8 \end{bmatrix} \right\}$ is a basis for the column space of *A*.

$\left\{ \begin{bmatrix} -7/2 \\ -1/4 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space of *A*.

dim row A=2

dim col A =2

6. Let $A=\begin{bmatrix}0&1&0&-3\\1&1&3&0\\2&1&3&2\\1&0&0&2\end{bmatrix}$. What is the dimension of the row space?

The dimension of the row space is the same as the rank of the matrix. So… we Gauss…

$$\begin{bmatrix}0&1&0&-3\\1&1&3&0\\2&1&3&2\\1&0&0&2\end{bmatrix}\xrightarrow{R1\leftarrow\rightarrow R2}\begin{bmatrix}\boxed{1}&1&3&0\\0&1&0&-3\\2&1&3&2\\1&0&0&2\end{bmatrix}$$

$$\xrightarrow[R4-R1\rightarrow R4]{R3-2R1\rightarrow R3}\begin{bmatrix}\boxed{1}&1&3&0\\0&\boxed{1}&0&-3\\0&-1&-3&2\\0&-1&-3&2\end{bmatrix}$$

$$\xrightarrow[\substack{R3+R2\rightarrow R3\\R4+R2\rightarrow R4}]{R1-R2\rightarrow R1}\begin{bmatrix}\boxed{1}&0&3&3\\0&\boxed{1}&0&-3\\0&0&-3&-1\\0&0&-3&-1\end{bmatrix}$$

$$\xrightarrow{R3\div(-3)\rightarrow R3}\begin{bmatrix}\boxed{1}&0&3&3\\0&\boxed{1}&0&-3\\0&0&\boxed{1}&1/3\\0&0&-3&-1\end{bmatrix}$$

$$\xrightarrow[R4+3R3\rightarrow R4]{R1-3R3\rightarrow R1}\begin{bmatrix}\boxed{1}&0&0&2\\0&\boxed{1}&0&-3\\0&0&\boxed{1}&1/3\\0&0&0&0\end{bmatrix}$$

Since $rank\,A=3$, $\dim row A=3$.

7. Given that the RREF of the standard matrix for $T:\mathbb{R}^5 \to \mathbb{R}^4$ is

$$\begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 5/2 \\ 0 & 1 & 2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for the row space, the column space, and null space of the standard matrix.

A basis for the row space is $\{[1 \quad 0 \quad 3 \quad 0 \quad 5/2],[0 \quad 1 \quad 2 \quad 0 \quad 3/2],[0 \quad 0 \quad 0 \quad 1 \quad 1]\}$.

A basis for the column space is $\left\{\begin{bmatrix} 3 \\ -2 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 3 \end{bmatrix}\right\}$.

8. Let $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$ and $S = \{B \in M_{22} \mid AB = BA\}$.

a) Show that S is a subspace of M_{22}.
b) Find a basis for S.
c) What is *dim S*?
d) Find a matrix C such that $C \notin S$.
e) How could we know such a matrix C exists without explicitly finding it?
f) Extend your basis in (b) to a basis for M_{22}.

a) First, let's simplify the matrix *AB* based on the restriction given.

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+5c & b+5d \\ 5a+c & 5b+d \end{bmatrix}$$

$$BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} a+5b & 5a+b \\ c+5d & 5c+d \end{bmatrix}$$

$$AB = BA$$

$$\begin{aligned} a+5c &= a+5b \to b = c \\ b+5d &= 5a+b \to a = d \\ 5a+c &= c+5d \to a = d \\ 5b+d &= 5c+d \to b = c \end{aligned}$$

$$\therefore B = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

Vector Addition

$\vec{u} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad a, b \in \mathbb{R} \quad \vec{u} \in S$

$\vec{v} = \begin{bmatrix} e & f \\ f & e \end{bmatrix} \quad e, f \in \mathbb{R} \quad \vec{v} \in S$

$\vec{u} + \vec{v} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} + \begin{bmatrix} e & f \\ f & e \end{bmatrix}$

$= \begin{bmatrix} a+e & b+f \\ b+f & a+e \end{bmatrix}$

Check restriction:

$It\ has\ the\ form\ of\ matrix\ B$ and a+e and b+f∈ ℝ

∴ S is closed under vector addition.

Scalar Multiplication

$k \in \mathbb{R}$

$k\vec{u} = k \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

$= \begin{bmatrix} ka & kb \\ kb & ka \end{bmatrix}$

Check restriction:

$It\ has\ the\ form\ of\ matrix\ B$ and ka, kb ∈ ℝ

For all k

∴ S is closed under scalar multiplication.

Also, the zero matrix is in S. We just let a=b=0.

Since S passed all of the tests, S is a subspace of M_{22}.

Then,

$S = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \middle|\, a, b \in \mathbb{R} \right\}$

$= span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

b) A basis has two properties: it is a spanning set and its vectors are linearly independent. We already have a spanning set (see part (a)). Now we have to show that its vectors are linearly independent.

$$span\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$a\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+b\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$a=0$$
$$b=0$$

$$\therefore a=b=0$$

Since the only solution is the trivial solution, the vectors are linearly independent. Therefore, a basis for *S* is $\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\}$.

c) The basis in part (b) had 2 vectors. Therefore, *dim S = 2*.

d) Let's choose a very simple vector:

$$C=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

Now let's show that we cannot construct *C* from our basis for *S*.

$$span\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\}=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

$$a\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+b\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

$$a=1$$
$$b=0$$
$$a=0$$

Since there is a contradiction (*a = 1* AND *a = 0*), $C\notin S$.

e) Since *dim* M_{22} > *dim S* and *S* is a subspace of M_{22}, there are an infinite number of vectors that are contained in M_{22} (represented by this matrix *C*) that are not contained in *S*.

f) Since *dim S = 2* and *dim* $M_{22} = 4$, we need to add two linearly independent vectors to our basis to extend it to a basis of M_{22}.

Let's "plug some holes" in our basis vectors.

Proposed basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

Now let's check if these four vectors are linearly independent:

$$span\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} a + c &= 0 \\ b &= 0 \\ b + d &= 0 \\ a &= 0 \end{aligned}$$

$$\therefore a = b = c = d = 0$$

Since the only solution is the trivial solution, the vectors are linearly independent. Therefore, a basis for M_{22} is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

9. Determine whether the following statements are true or false. Justify your answer:

Determine which of the following sets of vectors are subspaces of $\mathbb{R}^n$ or M_{22} (where appropriate) using the subspace test.

i) $B = \{(x, y) \in \mathbb{R}^2 | 3xy = 0\}$

ii) $D = \{(x, y, z, w) \in \mathbb{R}^4 | x, y, z = 0, w \in \mathbb{R}\}$

iii) $G = \{A \in M_{22} | A = -A^T\}$

i) ***Vector Addition***

$$u = (a, b) \qquad 3ab = 0 \qquad u \in B$$
$$v = (c, d) \qquad 3cd = 0 \qquad v \in B$$

$$\begin{aligned} u + v &= (a, b) + (c, d) \\ &= (a + c, b + d) \end{aligned}$$

check restriction:

$$\begin{aligned} 3xy &= (a + c)(b + d) \\ &= 3ab + 3bc + 3ad + 3cd \\ &= 0 + 3bc + 3ad + 0 \\ &= 3bc + 3ad \\ &\neq 0 \end{aligned}$$

$\therefore$ B is not closed under vector addition.

<u>Since B failed one of the tests, B is not a subspace of R^2.</u>

ii) ***Vector Addition***

Vector Addition

$\vec{u} = (a, b, c, d) \quad a, b, c = 0, d \in \mathbb{R} \quad \vec{u} \in D$

$\vec{v} = (e, f, g, h) \quad e, f, g = 0, h \in \mathbb{R} \quad \vec{v} \in D$

Alternatively, can write

$\vec{u} = (0,0,0, a) \quad a \in \mathbb{R} \quad \vec{u} \in D$

$\vec{v} = (0,0,0, b) \quad b \in \mathbb{R} \quad \vec{v} \in D$

I will use this form in the solution. Either is correct.

$\vec{u} + \vec{v} = (0, 0, 0, a) + (0, 0, 0, b)$

$= (0 + 0, 0 + 0, 0 + 0, a + b)$

$= (0, 0, 0, a + b)$

Check restriction:

$x = 0 \qquad y = 0 \qquad z = 0$

$w = a + b \in \mathbb{R}$ ← since *a* and *b* are real numbers, their sum is also a real number.

$\therefore D$ is closed under vector addition.

Scalar Multiplication

$k \in \mathbb{R}$

$k\vec{u} = k(0, 0, 0, a)$

$= (0, 0, 0, ka)$

Check restriction:

$x = 0 \qquad y = 0 \qquad z = 0$

$w = ka \in \mathbb{R}$ ← since *k* and *a* are real numbers, their product is also a real number.

$\therefore D$ is closed under scalar multiplication.

Since D passed all of the tests, D is a subspace of $\mathbb{R}^4$**.**

iii)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in \mathbb{R}$

$-A^T = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

$A = -A^T$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

$a = -a \rightarrow 2a = 0$

$b = -c$

$c = -b$

$d = -d \rightarrow 2d = 0$

$\therefore A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$

Vector Addition

$\vec{u} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \quad b \in \mathbb{R} \quad \vec{u} \in G$

$\vec{v} = \begin{bmatrix} 0 & f \\ -f & 0 \end{bmatrix} \quad f \in \mathbb{R} \quad \vec{v} \in G$

$\vec{u} + \vec{v} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} + \begin{bmatrix} 0 & f \\ -f & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & b + f \\ -b - f & 0 \end{bmatrix}$

Check restriction:

$b + f \in \mathbb{R}$

$-b - f \in \mathbb{R}$

$\therefore G$ is closed under vector addition.

Scalar Multiplication

$k \in \mathbb{R}$

$$k\vec{u} = k\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & kb \\ -kb & 0 \end{bmatrix}$$

Check restriction:

$kb \in \mathbb{R}$

$-kb \in \mathbb{R}$

for all k

$\therefore G$ is closed under scalar multiplication.

<u>Since G passed all of the tests, G is a subspace of M_{22}.</u>

10. Let $U = \{(x, y, z) \in \mathbb{R}^3 | x - 4y - 2z = 0\}$

i) Find a spanning set for *U*.
ii) Find a basis for *U*.
iii) Determine *dim U*.
iv) Extend your basis in iii) to a basis for $\mathbb{R}^3$.

i)

$U = \{(x, y, z) \in \mathbb{R}^3 | x - 4y - 2z = 0\}$

$x - 4y - 2z = 0$

$x = 4y + 2z$

$(x, y, z) = (4y + 2z, y, z)$

$= y(4,1,0) + z(2,0,1)$

$= span\,\{(4,1,0), (2,0,1)\}$

The spanning set is $\{(4,1,0), (2,0,1)\}$.

ii) Let's show that the vectors in the spanning set are linearly independent.

$$span\{(4,1,0),(2,0,1)\} = (0,0,0)$$

$$a(4,1,0) + b(2,0,1) = (0,0,0)$$

$$4a + 2b = 0$$

$$a = 0$$

$$b = 0$$

Since the only solution is the trivial solution, the vectors are linearly independent. So, our spanning set is also a basis. Therefore, the basis is $\{(4,1,0),(2,0,1)\}$.

iii) The basis has two vectors in it, so *dim U = 2*.

iv) Many more answers are possible.

$\mathbb{R}^3 = span\{(4,1,0),(2,0,1),(0,0,1)\}$ or $\mathbb{R}^3 = span\{(4,1,0),(2,0,1),(0,1,0)\}$ are both valid given the instructions I have provided.

Remember that we also have to prove that these vectors are linearly independent to show that these are valid bases of $\mathbb{R}^3$. Here are the proofs for both sets of vectors that I have shown above.

Set 1:

$$span\{(4,1,0),(2,0,1),(0,0,1)\} = (0,0,0)$$
$$a(4,1,0)+b(2,0,1)+c(0,0,1) = (0,0,0)$$
$$4a+2b=0$$
$$a=0$$
$$b+c=0$$

$$a=b=c=0$$

Since the only solution is the trivial solution, the vectors are linearly independent. Therefore, a basis for $\mathbb{R}^3$ is $\{(4,1,0),(2,0,1),(0,0,1)\}$.

Set 2:

$$span\{(4,1,0),(2,0,1),(0,1,0)\} = (0,0,0)$$
$$a(4,1,0)+b(2,0,1)+c(0,1,0) = (0,0,0)$$
$$4a+2b=0$$
$$a+c=0$$
$$b=0$$

$$a=b=c=0$$

Since the only solution is the trivial solution, the vectors are linearly independent. Therefore, a basis for $\mathbb{R}^3$ is $\{(4,1,0),(2,0,1),(0,1,0)\}$.

11. Find a basis for $V = span\{(1,0,0),(2,0,1),(1,1,1),(0,0,2)\}$. Find *dim V*.

$$a(1,0,0) + b(2,0,1) + c(1,1,1) + d(0,0,2) = (0,0,0)$$

$$a + 2b + c = 0$$
$$c = 0$$
$$b + c + 2d = 0$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R1-2R2 \to R1} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -4 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R2-R3 \to R2]{R1-2R3 \to R1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Since *d* is a parameter, we will eliminate its corresponding vector from the spanning set to form a basis.

$V = span\{(1,0,0),(2,0,1),(1,1,1),\cancel{(0,0,2)}\}$

Therefore, a basis for *V* is $\{(1,0,0),(2,0,1),(1,1,1)\}$. The dimension of *V* is 3.

12. Consider the subspace W of $\mathbb{R}^3$ such that $W = span\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ where

$$\vec{v}_1 = (1,0,4), \quad \vec{v}_2 = (3,3,1), \quad \vec{v}_3 = (4,3,5), \quad and \quad \vec{v}_4 = (-1,-3,7)$$

i) Find a basis for W and describe W geometrically.
ii) Find the standard matrix for the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that col$(T) = W$.
iii) Find *null (T)*. Also, calculate *dim (null T) + dim (col T)*.

i) We are given a spanning set. So, let's check and see which vectors are linearly independent.

$$span\{v_1, v_2, v_3, v_4\} = av_1 + bv_2 + cv_3 + dv_4$$
$$a(1,0,4) + b(3,3,1) + c(4,3,5) + d(-1,-3,7) = (0,0,0)$$

$$a + 3b + 4c - d = 0$$
$$3b + 3c - 3d = 0$$
$$4a + b + 5c + 7d = 0$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 4 & -1 & 0 \\ 0 & 3 & 3 & -3 & 0 \\ 4 & 1 & 5 & 7 & 0 \end{array}\right] \xrightarrow{R3-4R1 \to R3} \left[\begin{array}{cccc|c} 1 & 3 & 4 & -1 & 0 \\ 0 & 3 & 3 & -3 & 0 \\ 0 & -11 & -11 & 11 & 0 \end{array}\right]$$

$$\xrightarrow{R2 \div 3 \to R2} \left[\begin{array}{cccc|c} 1 & 3 & 4 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -11 & -11 & 11 & 0 \end{array}\right]$$

$$\xrightarrow[R3+11R2 \to R3]{R1-3R2 \to R1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

ii) From the above matrix, we can see that there are free variables in columns 3 and 4, and fixed variables in columns 1 and 2. Therefore, the vectors corresponding to columns 1 and 2 are linearly independent.

A basis for W is therefore $\{v_1, v_2\} = \{(1,0,4), (3,3,1)\}$.

W is a plane through the origin in $\mathbb{R}^3$ with direction vectors (1,0,4) and (3,3,1).

We want $T(x,y) = W$. Simply put, the standard matrix will be $A = \begin{bmatrix} 1 & 3 \\ 0 & 3 \\ 4 & 1 \end{bmatrix}$, which is the column space (image) of W.

iii) The null space can be found by row-reducing the augmented standard matrix.

$$null\ A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 3 & 0 \\ 4 & 1 & 0 \end{bmatrix} \xrightarrow{R3-4R1\to R3} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 3 & 0 \\ 0 & -11 & 0 \end{array}\right]$$

$$\xrightarrow{R2\div 3\to R2} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & -11 & 0 \end{array}\right]$$

$$\xrightarrow[R3+11R2\to R3]{R1-3R2\to R1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

From the matrix above, we get $x = y = 0$.
Therefore, the nullity of A is {0}.

Using the rank-nullity theorem, we can determine the required dimensions.

dim(null T)=0 and dim (col T)= dimension of column space=2 (2 vectors)
$\therefore\ dim(null\ T) + dim(\mathrm{col}\ T) = n = 2$(number of columns)

13. Consider $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 4 & 8 & 10 \\ 2 & 4 & 0 \end{bmatrix}$.

i) Find bases for the row space, column space, and null space of *A*.

ii) What is *dim row A*, *dim col A*, and *dim null A*?

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & 0 & 2 \\ 4 & 8 & 10 \\ 2 & 4 & 0 \end{bmatrix} \xrightarrow[R4-2R1\to R4]{R3-4R1\to R3} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R2\div 2\to R2} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R3-6R2\to R3]{R1-R2\to R1} \begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$nullspace\ A = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x + 2y &= 0 \\ y &= t \\ z &= 0 \end{aligned}$$

$$\begin{aligned} x &= -2y \\ y &= t \\ z &= 0 \\ t &\in R \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

A basis for the row space is $\left\{\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\right\}$.

A basis for the column space is $\left\{\begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 10 \\ 0 \end{bmatrix}\right\}$

A basis for the null space is $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$ .The dimension of the row space is 2.

The dimension of the column space is 2. The dimension of the null space is 1.

14. Determine if $\vec{b} = \begin{bmatrix}1\\3\end{bmatrix}$ is in the column space of A and if $\vec{w} = \begin{bmatrix}-1 & 1 & 2\end{bmatrix}$ is in the row space of A where $A = \begin{bmatrix}1 & 0 & 1\\1 & 1 & -1\end{bmatrix}$

$\left[\begin{array}{ccc|c}1 & 0 & 1 & 3\\1 & 1 & -1 & 1\end{array}\right]$ R2-R1→ R2 $\left[\begin{array}{ccc|c}1 & 0 & 1 & 3\\0 & 1 & -2 & -2\end{array}\right]$

∴ since there is a solution (system is consistent)

∴ $\vec{b}$ is in col A.

$\left[\frac{A}{w}\right] = \left[\begin{array}{ccc}1 & 0 & 1\\1 & 1 & -1\\\hline -1 & 1 & 2\end{array}\right]$ R2-R1 → R2 R3+R1 → R3

$\left[\begin{array}{ccc}1 & 0 & 1\\0 & 1 & -1\\\hline 0 & 1 & 3\end{array}\right]$ R3-R2→ R3 $\left[\begin{array}{ccc}1 & 0 & 1\\0 & 1 & -2\\\hline 0 & 0 & 5\end{array}\right]$

R3÷5 → R3 $\left[\begin{array}{ccc}1 & 0 & 1\\0 & 1 & -2\\\hline 0 & 0 & 1\end{array}\right]$ R2+2R3 → R2 R1-R3 → R1

$\left[\begin{array}{ccc}1 & 0 & 0\\0 & 1 & 0\\\hline 0 & 0 & 1\end{array}\right]$

Note: if $\vec{w}$ is in row (A) then $\vec{w}$ is a linear combination of the rows of A, and if we row reduce $\left[\frac{A}{w}\right]$ we will get a matrix of the form $\left[\frac{A'}{0}\right]$

Here, we can't make the last row all 0's ∴ $\vec{w} \notin \text{row}(A)$.

15. Do the vectors $\begin{bmatrix}1\\2\\4\end{bmatrix}$ $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}5\\2\\1\end{bmatrix}$ form a basis for R^3?

Yes, if we row reduce, we see rank =3 with these vectors as columns (you can use rows too)

∴ they form a basis of R^3.

16. Consider the non-standard basis $B=(\vec{v}_1,\vec{v}_2)=(-\vec{e}_1+2\vec{e}_2, 2\vec{e}_1+\vec{e}_2)$ in $\mathbb{R}^2$.

Compute the following: $\begin{bmatrix}-2\\4\end{bmatrix}_B$, $[\vec{e}_1+3\vec{e}_2]_B$, and $\left[\frac{47}{4}\vec{v}_1+\frac{\pi}{6}\vec{v}_2\right]_B$.

First, let's find the numerical values of the vectors $\vec{v}_1$ and $\vec{v}_2$. This will make our computations easier:

$$\vec{v}_1=(-\vec{e}_1+2\vec{e}_2)=-\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$$
$$\vec{v}_2=(2\vec{e}_1+\vec{e}_2)=2\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$$

Now we can solve for the coordinates of each the desired vectors:

$$\begin{bmatrix}-2\\4\end{bmatrix}=c_1\vec{v}_1+c_2\vec{v}_2$$
$$\begin{bmatrix}-2\\4\end{bmatrix}=c_1\begin{bmatrix}-1\\2\end{bmatrix}+c_2\begin{bmatrix}2\\1\end{bmatrix}$$
$$c_1=2, c_2=0$$
$$\therefore \begin{bmatrix}-2\\4\end{bmatrix}_B=\begin{bmatrix}2\\0\end{bmatrix}$$

$$[\vec{e}_1+3\vec{e}_2]=c_1\vec{v}_1+c_2\vec{v}_2$$
$$=c_1(-\vec{e}_1+2\vec{e}_2)+c_2(2\vec{e}_1+\vec{e}_2)$$
$$c_1=1, c_2=1$$
$$\therefore [\vec{e}_1+3\vec{e}_2]_B=\begin{bmatrix}1\\1\end{bmatrix}$$

17. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and $\vec{v}_4$ be vectors in $\mathbb{R}^4$. Choose all of the correct statements.

i) $span(\vec{v}_1, \vec{v}_2)$ can be 1-dimensional.

ii) $sp(\vec{v}_1, \vec{v}_2 - \vec{v}_1, \vec{v}_3 - \vec{v}_2 - \vec{v}_1)$ can be 3-dimensional.

iii) $sp(\vec{v}_1, \vec{v}_2 - \vec{v}_1, \vec{v}_3 - \vec{v}_2 - \vec{v}_1, \vec{v}_3)$ can be 4-dimensional.

iv) $sp(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ can be 4-dimensional.

i) **Yes**. If $\vec{v}_1$ and $\vec{v}_2$ are scalar multiples (i.e., *linearly dependent*), then their span could be 1-dimensional.

ii) **Yes**. If the three vectors are *linearly independent*, then three unique linear combinations would be 3-dimensional.

iii) **No**. There needs to be **four** unique, linearly independent vectors in the span for this to be true. There are only **three** unique vectors in the span.

iv) **Yes**. If the four vectors are *linearly independent*, then the span would be 4-dimensional.

18. Show that the set S = { $(1,0,0),(1,1,0),(1,1,1)$ } is a basis for $\Re^3$.

In order to show that this set constitutes a basis for $\Re^3$, we need to show that it spans $\Re^3$ and that it is linearly independent.

Step 1: Show that S spans $\Re^3$.

$(a,b,c) = x(1,0,0) + y(1,1,0) + z(1,1,1) = (x + y + z, y + z, z)$

Equating the variables:

$x + y + z = a$

$y + z = b$

$z = c$

In matrix form:

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ …RREF $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which has rank 3, so the column vectors span $\Re^3$.

We could also say that since the determinant of the matrix is nonzero, so we know that there is a unique solution for which the above system of equations holds.

Therefore, S spans $\Re^3$.

Step 2:

Note that the above equation can be transformed into RREF of the following form. Therefore, the set is independent.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the set is a basis for $\Re^3$.

19. Given that the reduced row echelon form of the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & -1 & -10 \\ -1 & 1 & 3 & 2 & 7 \\ 2 & -2 & 1 & 2 & 14 \\ 3 & -3 & 4 & 1 & 2 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for each of the following: the row space of *A*, the column space of *A*, and the null space of *A*.

row space of *A*: Basis = $\{(1, -1, 0, 0, 1), (0, 0, 1, 0, -2), (0, 0, 0, 1, 7)\}$.

column space of *A*: Basis = $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}$

null space of *A*: Solving $A\mathbf{x} = \mathbf{0}$ is the equivalent to solving $R\mathbf{x} = \mathbf{0}$.

$$R\mathbf{x} = 0 \quad \Rightarrow \quad \begin{aligned} x_1 - x_2 + x_5 &= 0 \\ x_3 - 2x_5 &= 0 \\ x_4 + 7x_5 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= x_2 - x_5 \\ x_3 &= 2x_5 \\ x_4 &= -7x_5 \end{aligned}$$

Assign free variables x_2 and x_5 arbitrary values $r, s \in \mathbf{R}$, respectively, and the solution becomes

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} r - s \\ r \\ 2s \\ -7s \\ s \end{bmatrix} = r\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 2 \\ -7 \\ 1 \end{bmatrix}$$

Basis = $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ -7 \\ 1 \end{bmatrix} \right\}$.

20. Find a basis for the null space of $D = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -3 & 0 \end{bmatrix}$

By definition, the null space of *D* is the solution space of $D\vec{x} = \vec{0}$.

The augmented matrix of this equation in RREF is $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

assigning free variable $x_3 = t$, where $t \in \mathbf{R}$

we get $x_1 = -2t, x_2 = t, x_4 = 0$

So, the solution space is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, where $t \in \mathbf{R}$.

Therefore, a basis for the null space of *D* is $\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}$.

Appendix 1: Binary Numbers and Modulus

Example 1.1.
$[1,0,1,1,0] + [0,1,1,0,1] = [1,1,0,1,1]$

Example 1.2.
Calculate: $2 + 1 + 0 + 2 + 1 = 6 = 0\ in\ Z_3$ since $6 \div 3 = 2R0$

Example 1.3.
$\vec{u} = [2,2,0,2,1] + \vec{v} = [1,2,2,1,2] = [3,4,2,3,3] = [0,1,2,0,0]\ in\ Z_3$

Since $3 \div 3 = 1R0$ and $4 \div 3 = 1R1$

Example 1.4.

12/7 = 1 R5

So, -12 mod 7 = 7 – 5 = 2

If b>a>0, then a mod b =a

Example. 5 mod 8 = 5

Example 1.5.

Perform the operations:

a) $2 + 1 + 2 + 2$ in Z_3
$= 7 = 1$ in Z_3 $\quad 7 \div 3 = 2R1$

$b)$ $2x + 3 = 1$ in Z_5

$2x + 3 + 2 = 1 + 2$ — we need to add to 2 on the left, so it becomes 0 in Z_5

$2x = 3$ — Now, 3(2)=6 and $6 \div 5 = 1R1$, so we have 1x on the left

$(2)(3)x = (3)(3)$

$\therefore x = 9$ in Z_5 $\quad 9 \div 5 = 1R4$

$\therefore x = 4$ is the answer

c) (2)(3)(4)(3) in Z_4

(2)(3)(4)(3)= 72

72= 4(18) + 0

Therefore, the answer is 0.

Example 1.6.

a) $3[4,8,6]=[12,24,18]=[2,4,8]$

$12 \div 10 = 1R2$

$24 \div 10 = 2R4$

$18 \div 10 = 1R8$

b) $[6,8,4]+[5,5,5]$

$=[11,13,9]$ $\quad 11 \div 10 = 1R1$

$=[1,3,9]$ $\quad 13 \div 10 = 1R3$

$9 \bmod 10 = 9$

Example 1.7.

a) $6x = 6 \pmod 3$ mod 3 so try 0,1,2

$x=0 \quad 6(0)=0 \quad 0 \bmod 3 = 0$ not 6 $\therefore$ 0 not 6

$x=1 \quad 6(1)=6$

$x=2 \quad 6(2)=12 \quad 12 \div 3 = 4R0$

$\therefore$ only $\boxed{x=1}$ is a solution

b) $2x=1 \pmod 6$ try $0,1,2,3,4,5$

$x=0 \quad 2(0)=0 \quad 0 \bmod 6 = 0 \quad \therefore$ not 1

$x=1 \quad 2(1)=2 \quad 2 \bmod 6 = 2 \quad \therefore$ not 1

$x=2 \quad 2(2)=4 \quad 4 \bmod 6 = 4 \quad \therefore$ not 1

$\boxed{x=3} \quad 2(3)=6 \quad 6 \div 6 = 1R1 \quad \therefore \boxed{1}$

$x=4 \quad 2(4)=8 \quad 8 \div 6 = 1R2 \quad \therefore$ 2 not 1

$x=5 \quad 2(5)=10 \quad 10 \div 6 = 1R4 \quad \therefore$ 4 not 1

$\therefore$ the only solution is $x=3$

Example 1.8.

$= 1(2) + 2(4) + 3(5)$

$= 2 + 8 + 15$

$= 25 \bmod 5 \qquad 25 \div 5 = 5R0$

$= \boxed{0}$

Appendix 2: Row-Reducing and Solving Systems over Z_n and Inverse

Example 2.1. Solve over Z_3:

$x_1 + 2x_2 + 2x_3 = 0$

$x_1 + x_3 = 2$

$x_2 + 2x_3 = 2$

$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right]$ R2 + 2R1 → R2

Need to get 1+2=3 ∴ $3 \div 3 = 1R0 = 0$ in Z_3

$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right]$ $0 + 2(2) = 4 \div 3 = 1R1$, $1 + 2(2) = 5 \div 3 = 1R2$

R3 +2 R2 → R3

$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $2 + 2(0) = 2$

R1 + R2 → R1

$x \quad y \quad z$

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $2 + 2 = 4 \div 3 = 1R1$

$\therefore z = t$

$x + t = 2$ Here, we need to make "3t" on the left as that will be 0 in Z_3

$x + t + 2t = 2 + 2t$ or $x + 3t = 2 + 2t$ which means $x + 0 = 2 + 2t$ *and* $x = 2 + 2t$

$y + 2t = 2$ So, we get: $y + 2t + t = 2 + t$ *which means* $y + 0 = 2 + t$ *and* $y = 2 + t$

∴ Solution is

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + 2t \\ 2 + t \\ t \end{bmatrix}, t \in Z_3.$

*** in Z_p, there can <u>never</u> be infinitely many solutions. Here in Z_3, t can equal 0, 1, or 2 ∴ we get 3 solutions.**

If t=0, we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

If t=1, we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+2(1) \\ 2+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in Z_3

If t=2, we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+2(2) \\ 2+2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ in Z_3

$(6 \div 3 = 2R0)$

<u>Example 2.2.</u>

* Remember, all calculations are in Z_3

To get a 1, you need to multiply. To get a 0, you need to add a multiple of another row.

<u>Solution</u>:

$[A/I] \rightarrow [I/A^{-1}]$

$\left[\begin{matrix} 2 & 1 \\ 2 & 0 \end{matrix}\middle|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right]$ $R1 \times 2 \rightarrow R1$

$4 \div 3 = 1R1$

$\left[\begin{matrix} 1 & 2 \\ 2 & 0 \end{matrix}\middle|\begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix}\right]$ $6 \div 3 = 2R0$

$R2 + R1 \rightarrow R2$

$\left[\begin{matrix} 1 & 0 \\ 0 & 2 \end{matrix}\middle|\begin{matrix} 2 & 0 \\ 2 & 1 \end{matrix}\right]$ $R2 \times 2 \rightarrow R2$ $\left[\begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} 2 & 0 \\ 1 & 2 \end{matrix}\right]$ $R1 + R2 \rightarrow R1$ $\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} 0 & 2 \\ 1 & 2 \end{matrix}\right]$

$A^{-1} = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}$

Appendix 3: Code Vectors

Example 3.1.

Since we need an even number of "1's, the check digit would be 0 and the parity check code vector would be (1,0,1,0,0).

Example 3.2.

Since there are an even # of 1's, a single digit error could not have occurred.

The way this question is solved in some textbooks is very confusing. What you need to know is that when working with binary numbers, the first four numbers are: 0,1, 10, 100

You do the dot-product of the vector with the vector (1,1,1,...1).

$(1,0,1,0) \cdot (1,1,1,1) = 1 + 0 + 1 + 0$

Then, add the answer like you normally would.

Sum=2 **But, a sum of 2, means "0" in binary.

For any other sum, the answer is the last digit of the binary number.

Example 3.3. Find the check digit for the UPC code "883 929 048 81"

Solution:

3(8+3+2+0+8+1) + (8+9+9+4+8) + d

=3(22) + (38) + d

=66 + 38 + d

=104 + d

Now, decide what number d must be in order to get to the next multiple of 10.

The next multiple of 10 is 110, so d=6

The check digit is 6.

Example 3.4. a) The last digit is the check digit...d=6

3(0+6+8+0+4+0) + (0+3+3+1+5) + d OR (3,1,3,1,3,1,3,1,3,1,3,1)· (0, 0, 6, 3, 8, 3, 0,1,4, 5,0, 6)

=3(18) + 12 +6 = 0 + 0 + 18 +3 + 24 +3 +0 + 1+ 12 +5+ 0 +6

=54 + 12 + 6 =72

=72 which is NOT a multiple of 10, therefore there is an error in the code

b) The new code would be: " 060 383 014 506"

Check that it is correct:

3(0+0+8+0+4+0) +(6+3+3+1+5)+d OR do (3,1,3,1,3,1,3,1,3,1,3,1)· (0,6,0, 3,8,3, 0,1,4 ,5,0,6)

=3(12) + (18) + 6

=36 + 18 + 6

=60 which is a multiple of 10!!!!

Example 3.5.

To find the check digit:

The check vector is $\vec{c} = (10,9,8,7,6,5,4,3,2,1)$

If we let the ISBN number be vector $\vec{v}$, we get: $\vec{v} = (0,2,0,1,5,3,0,8,2,d)$

We now find $\vec{c} \cdot \vec{v}$ in Z_{11}

$(10,9,8,7,6,5,4,3,2,1) \cdot (0,2,0,1,5,3,0,8,2,d)$

=0+18 +0 +7 + 30+ 15+0+24+4+d

(now, without a calculator, it is difficult to add and divide by 11)

Now, look at these numbers and any multiples of 11 will become 0 in Z_{11}

= 0 + 11+ 7 + 7 + 11(2) + 8 + 11+ 4 + 11(2) + 2 + 4 + d

=0 + 7 + 7+ 8+ 4 +2+ 4+ d

=14 + 18+ d

=32 + d

=11(2) + 10 + d

=10 + d

Solve for d where: 10+ d = 11

d=1 Therefore, the check digit is 1.

Example 3.6. Show that the ISBN number can't be correct. ISBN= 300 640 615 2

Solution:

a) If the ISBN number is correct, then $\vec{d} \cdot \vec{v}$=0

(10,9,8,7,6,5,4,3,2,1) · (3,0,0,6,4,0,6,1,5,2)

=30 + 0 + 0 + 42 + 24 +0 +24 +3 +10 + 2

=11(2) + 8 + 11(3) + 9 + 11(2) + 2 + 11(2) + 2 + 3 + 10 + 2

=8 + 9 + 2 + 2 + 3 + 10 + 2

=21 + 15

=36

=11(3) + 3

=3

Therefore, since the result in Z_{11} is 3 and not 0, so it is incorrect.

b) Assume that the error was a transposition error involving the first two entries, find the correct ISBN-10 and prove that it is correct.

Solution:

ISBN=030 640 615 2

(10,9,8,7,6,5,4,3,2,1) · (0,3,0,6,4,0,6,1,5,2)

=0+27+0+42+24+0+24+3+10+2

=11(2)+5+11(3)+9+11(2)+2+11(2)+2+3+10+2

=5+9+2+2+3+10+2

=33

=11(3)+0

=0 Therefore, since the result is 0, it is a correct ISBN number.

Note: If they don't tell you which entries are switched, but just that two adjacent entries are switched, you would have to use trial and error. First, you would switch the first and second entries and calculate the dot product, then if it is not zero, move to second and third, etc.

Example 3.7.

$\vec{c} = [3, 2, 1] \qquad \vec{v} = [2, 3, d]$

$\vec{c} \cdot \vec{v} = [3, 2, 1] \cdot [2, 3, d]$

$= 3(2) + 2(3) + 1(d)$

$= 6 + 6 + d$

$= 12 + d \qquad (\text{mod } 4)$

For $\vec{c} \cdot \vec{v} = 0$, we get $d = 4$

i.e. $12 + 4 = 16 \qquad 16 \div 4 = 4R0$

Appendix 4: Complex Numbers

Example 4.1. Express $\frac{-2+3i}{3+7i}$ in the form $a + bi$

Solution: multiply by the conjugate

$$\frac{(-2+3i)}{(3+7i)}\frac{(3-7i)}{(3-7i)} = \frac{-6+14i+9i-21i^2}{9-21i+21i-49i^2}$$

$$= \frac{-6+23i-21(-1)}{9-49(-1)}$$

$$= \frac{15+23i}{58}$$

$$= \frac{15}{58} + \frac{23}{58}i$$

Example 4.2. Find the sum, product, and difference of $5 - 3i$ and $-2 + 8i$

Solution:

$(5 - 3i) + (-2 + 8i) = 3 + 5i$

$(5 - 3i) - (-2 + 8i) = 5 + 2 - 3i - 8i = 7 - 11i$

$(5 - 3i)(-2 + 8i) = -10 + 40i + 6i - 24i^2$

$= -10 + 40i - 24(-1)$

$= 14 + 46i$

Example 4.3. Find the absolute value of $4 + 5i$

Solution:

$|z| = |a + bi| = \sqrt{a^2 + b^2}$

Here,

$a = 4, b = 5$

$\therefore |z| = \sqrt{4^2 + 5^2}$

$= \sqrt{16 + 25}$

$= \sqrt{41}$

Note: argZ is not unique as if we add/subtract any multiple of 2π, we get another argument of Z.

But there is only one argument satisfying: $-\pi < 0 \leq \pi$

and this is called the principal argument and is denoted by ArgZ.

Example 4.4. Recall that $a = |z| \cos\theta$ and $b = |z| \sin\theta$.

The question states that the modulus is 2, which means $|z| = 2$,

and it also says that the argument is $\frac{-\pi}{6}$, so $\theta = -\frac{\pi}{6}$, subbing these values into the expression above, we get: $a = 2\cos\left(\frac{-\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

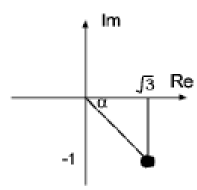
$$b = 2\sin\left(-\frac{\pi}{6}\right) = 2\left(-\frac{1}{2}\right) = -1$$

so, $z = a + bi$

$$z = \sqrt{3} - i$$

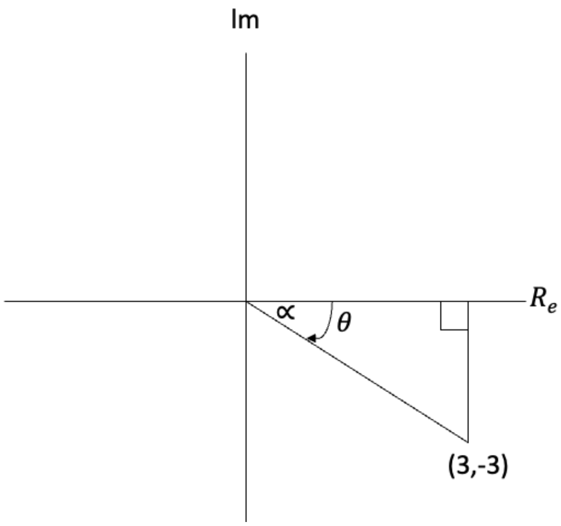
And now we represent this point on an argand diagram:

NOTE: If you took the angle clockwise from positive x around, it would be greater than π



Example 4.5. Write in polar form using its principal argument for $3 - 3i$.

$a = 3, b = -3$



$r = |z| = \sqrt{a^2 + b^2} = \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$

$\propto = \tan^{-1}\left|\frac{-3}{3}\right| = \tan^{-1}|-1| = \tan^{-1}(1)$

$\propto = \frac{\pi}{4} \quad \therefore \theta = \frac{-\pi}{4}$

$z = r\,(\cos\theta + i\sin\theta)$

$z = 3\sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right)$

Example. 4.6. Find the polar form of $zw, \frac{z}{w}$ and $\frac{1}{z}$.

a) $Z = -2 + 2i, \quad w = \sqrt{3} + i$

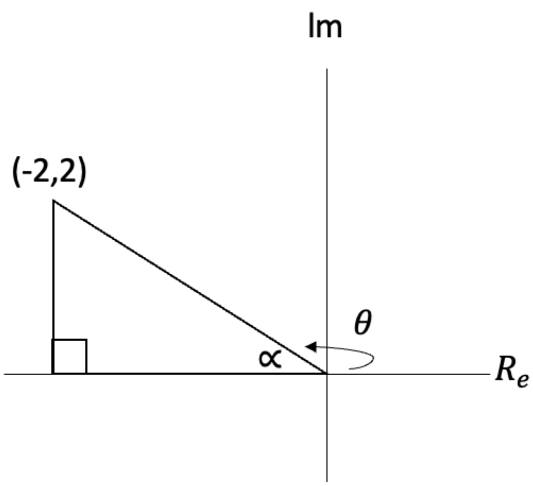
$a = -2, b = 2 \qquad a = \sqrt{3}, b = 1$

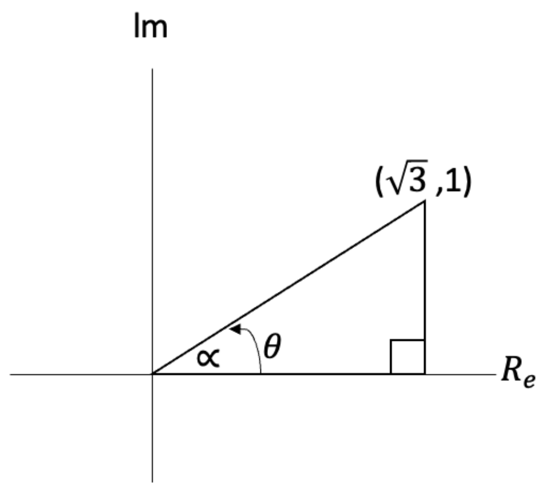
$r = \sqrt{(-2)^2 + 2^2} \qquad r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2}$

$r = \sqrt{8} = \sqrt{4}\sqrt{2} \qquad r = \sqrt{4}$

$r_1 = 2\sqrt{2} \qquad r_2 = 2$

$$zw = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i(\sin(\theta_1 + \theta_2))\right]$$





$\propto = \tan^{-1}\left|\frac{2}{-2}\right| = \tan^{-1}(1) = \frac{\pi}{4} \qquad \propto = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| \quad \propto = \frac{\pi}{6}$

$\therefore \theta_1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \qquad \theta_2 = \frac{\pi}{6}$

$\theta_1 + \theta_2 = \frac{3\pi}{4} + \frac{\pi}{6} = \frac{9\pi}{12} + \frac{2\pi}{12} = \frac{11\pi}{12}$

$$zw = 2\sqrt{2}(2)\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]$$

$$\frac{z}{w} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right] \text{ if Z } \neq 0$$

$$\theta_1 - \theta_2 = \frac{9\pi}{12} - \frac{2\pi}{12} = \frac{7\pi}{12}$$

$$\therefore \frac{z}{w} = \frac{2\sqrt{2}}{2}\left[\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right]$$

$$\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$\frac{1}{z} = \frac{1}{2\sqrt{2}}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right]$$

<u>Example 4.7.</u> In this case $n = 6$, thus

$$\left[\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right]^6 = \left[\cos\left(6 \cdot \frac{7\pi}{8}\right) + i\sin\left(6 \cdot \frac{7\pi}{8}\right)\right]$$

$$= \cos\left(\frac{42\pi}{8}\right) + i\sin\left(\frac{42\pi}{8}\right)$$

$$= \frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

Appendix 5: Markov Chains

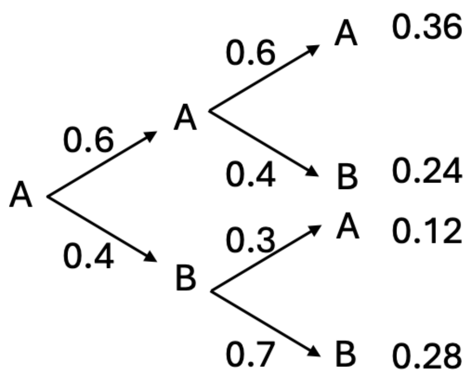
Example 5.1.

a)

$$P = \begin{matrix} & A & B \\ A & 0.6 & 0.3 \\ B & 0.4 & 0.7 \end{matrix}$$

$$\vec{x}_{t+1} = P\overrightarrow{x_t}$$

b)



The probability of switching from Brand A to Brand B after two months is 0.24+0.28=0.52 or 52%.

c) $\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}\begin{bmatrix} 60 \\ 40 \end{bmatrix}$

$$= \begin{bmatrix} 0.6(60) & 0.3(40) \\ 0.4(60) & 0.7(40) \end{bmatrix}$$

$$= \begin{bmatrix} 36 + 12 \\ 24 + 28 \end{bmatrix} = \begin{bmatrix} 48 \\ 52 \end{bmatrix}$$

∴ after 1 month 48 people will be using brand A and 52 people will be using brand B

d) $[I - P|\vec{O}]$

$I - P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$

$I - P = \begin{bmatrix} 0.4 & -0.3 \\ -0.4 & 0.3 \end{bmatrix}$

$\therefore [I - P|\vec{O}] = \left[\begin{matrix} 0.4 & -0.3 \\ -0.4 & 0.3 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$ $\qquad 0.4 = \frac{4}{10} = \frac{2}{5}$

$\therefore R1 \times \frac{5}{2} \to R1 \text{ and } \qquad R2 + R1 \to R2$

$\left[\begin{matrix} 1 & -3/4 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$ $\qquad -0.3 \times \frac{5}{2} = \frac{-3}{10} \times \frac{5}{2} = \frac{-15}{20} = \frac{-3}{4}$

$\therefore x_2 = t$

$x_1 - 3/4\, t = 0$

$x_1 = 3/4\, t$ Since we want a probability vector, it must add to 1:

$\therefore \frac{3}{4}t + t = 1$

$\frac{3}{4}t + \frac{4}{4}t = 1$

$\frac{7}{4}t = 1$

$t = \frac{4}{7}$

$\therefore x_2 = \frac{4}{7}$

$x_1 = \frac{3}{7}$

$\vec{x} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$. This is the long-term vector.

5.2 Homework on Appendix 5

1. a) $\quad H \quad L$

$$P = \begin{matrix} H \\ L \end{matrix}\begin{bmatrix} 0.70 & 0.80 \\ 0.30 & 0.20 \end{bmatrix}$$

b) $\begin{bmatrix} 0.7 & 0.8 \\ 0.3 & 0.2 \end{bmatrix}\begin{bmatrix} 200 \\ 100 \end{bmatrix}$

$= \begin{bmatrix} 0.7(200) + 0.8(100) \\ 0.3(200) + 0.2(100) \end{bmatrix}$

$= \begin{bmatrix} 220 \\ 80 \end{bmatrix}$

∴ there will be 220 high risk and 80 low risk drivers.

c) What is the long-term vector?

$\begin{bmatrix} 0.7 & 0.8 \\ 0.3 & 0.2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$[P - I]\vec{v} = \vec{O}$

$\begin{bmatrix} 0.7 & 0.8 \\ 0.3 & 0.2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.3 & 0.8 \\ 0.3 & -0.8 \end{bmatrix}$

$\left[\begin{matrix} -0.3 & 0.8 \\ 0.3 & -0.8 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$ → row reduce

$R2 - R1 \to R2$

$R1 \times \frac{-10}{3} \to R1$

$\quad x \quad\quad y$

$\left[\begin{matrix} 1 & -8/3 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$

$\frac{8}{10} \times \frac{-10}{3} = \frac{-8}{3}$

$y = t$

$x - 8/3t = 0$

$x = \frac{8}{3}t$ Since we want a probability vector, we must add up to 1: $\frac{8}{3}t + t = 1$

$\frac{8}{3}t + \frac{3}{3}t = 1$ *so,* $\frac{11}{3}t$ *=1 and t= 3/11.*

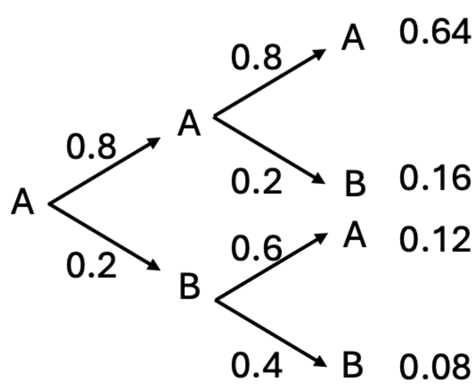
$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8/11 \\ 3/11 \end{bmatrix}$. This is the long-term vector.

2. a) $\quad A \quad B$

$P = \begin{matrix} A \\ B \end{matrix}\begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$

$\vec{x}_{t+1} = P\overrightarrow{x_t}$

b)



From the tree, the probability of switching from Brand A to Brand B after two months is 0.16+0.08=0.24 or 24%.

c) $\begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 50 \\ 150 \end{bmatrix} = \begin{bmatrix} 0.8(50) + 0.6(150) \\ 0.2(50) + 0.4(150) \end{bmatrix} = \begin{bmatrix} 40 + 90 \\ 10 + 60 \end{bmatrix} = \begin{bmatrix} 130 \\ 70 \end{bmatrix}$

$\begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 130 \\ 70 \end{bmatrix} = \begin{bmatrix} 0.8(130) + 0.6(70) \\ 0.2(130) + 0.4(70) \end{bmatrix} = \begin{bmatrix} 146 \\ 54 \end{bmatrix}$

After two months, 54 will be using Brand B and 146 using Brand A.

Homework on Appendices 1-4

1. If $\vec{u} = [1,0,1,1,0]$ and $\vec{v} = [0,1,1,0,1]$ are binary vectors, find $\vec{u} + \vec{v}$.

$\vec{u} = [0,1,1,0], \vec{v} = [1,1,1,1]$

$\vec{u} + \vec{v} = [1,0,0,1]$

2. Write out the addition and multiplication tables for Z_5

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

3. Perform the operations:
a)
$2 + 1 + 2 + 2$ in Z_3

$= 7 = 1$ in Z_3 $\quad\quad$ $7 \div 3 = 2R1$

b) $2(2 + 1 + 1)$in Z_3
$= 2(4) = 2(3(1) + 1)$ $\quad\quad$ $4 \div 3 = 1R1$ or do $\therefore 8 \div 3 = 2R2$

$= 2(1)$

$= 2$ in Z_3

c) $2 \cdot 2 \cdot 3$ in Z_4
$= 12$ $\quad\quad$ $12 \div 4 = 3R0$

$= 4(3) + 0$

$= 0$ in Z_4

4. Solve the equation or say no solution if there is not one.

(a) $x + 5 = 2$ in Z_6

$x + 5 + 1 = 2 + 1$ we add 1 on the left to get 5+1=6 which is 0 in Z_6…add 1 on the other side too!

$x + 0 = 3$ (in Z_6)

$x = 3$ in Z_6

(b) $2x = 3$ in Z_4

No solution since 2 times any # must be even, so we can't get a remainder of 3 when divided by

(c) $2x + 3 = 1$ in Z_5

$2x + 3 + 2 = 1 + 2$ we need to add to 2 on the left, so it becomes 0 in Z_5

$2x = 3$ Now, 3(2)=6 and $6 \div 5 = 1R1$, so we have 1x on the left

$(2)(3)x = (3)(3)$

$\therefore x = 9$ in Z_5 $9 \div 5 = 1R4$

$\therefore x = 4$ is the answer

5. Perform the indicated operations:

a) 2+2+2+2 in Z_3

To find the solution in Z_3, we first add the numbers up as we normally would.

2+2+2+2=8

Now, we need to write this number as multiples of 3 plus any remainder since we are dealing with Z_3 .

This remainder is our answer.

2+2+2+2=8=2(3) + 2

Therefore, the answer is 2.

b) 3(1+2+2) in Z_3

3(1+2+1)= 3 + 6 + 6 = 15

15= 3(5) + 0

Therefore, since the remainder is 0, when we divide by 3, the answer is 0.

c) (2)(3)(4)(3) in Z_4

(2)(3)(4)(3)= 72

72= 6(12) + 0

Therefore, the answer is 0.

d) (2)(3)(4)(2) in Z_5

(2)(3)(4)(2)= 48

48= 5(9) +3

Therefore, the answer is 3.

e) 3(3+3+2+1) in Z_4

3(3+3+2+1)= 3(9)= 27 or do 27$\div 5 = 6R3, \therefore 3\ is\ the\ answer$

27=4(6) + 3

Therefore, the answer is 3.

6. Perform the indicated operations:

a) 3 + 2+ 4 + 5 in Z_3

3+2+4+5=14

14=(3)(4) + 2 or do 14$\div 3 = 4R2, \therefore 2\ is\ the\ answer$

=2

The solution is 2 in Z_3

b) 3+2+4+4 in Z_4

3+2+4+4=13 or do 13$\div 4 = 3R1, \therefore 1\ is\ the\ answer$

13=4(3) + 1

=1

The solution is 1 in Z_4

c) 8(6+1+2+2) in Z_5

8(6+1+2+2)=8(12)

8(11)=88

88= 5(17) +3

=3 The solution is 3 in Z_5

7. -12 mod 5=? If a<0, then a mod b = b – remainder

Since 12/5 = 2 R2, -12 mod 7 = 5 – 2 = 3.

8. Yes, since 37 is a prime number any equation ax=b in Z_p with $a \neq 0$ has a unique solution in Z_p

9. Solve the following over Z_3:

$x + y = 2$

$2x + y = 1$

$\left[\begin{matrix} 1 & 1 \\ 2 & 1 \end{matrix}\middle| \begin{matrix} 2 \\ 1 \end{matrix}\right]$ R2 + R1 → R2 (to get a 0, we add a multiple of another row)

$\left[\begin{matrix} 1 & 1 \\ 0 & 2 \end{matrix}\middle| \begin{matrix} 2 \\ 0 \end{matrix}\right]$ R2 x 2 → R2 (to get a 1, we must multiply that row by a scalar)

$4 \div 3 = 1R1$

$\left[\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}\middle| \begin{matrix} 2 \\ 0 \end{matrix}\right]$ R1 +2R2 → R1

$\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle| \begin{matrix} 2 \\ 0 \end{matrix}\right]$ $2 + 2(0) = 2$

The unique solution is: $x = 2, y = 0$ or point of intersection (2,0).

10. Solve over Z_2:

$x_1 + x_2 + x_3 = 1$

$x_1 + x_2 = 1$

$x_2 + x_3 = 0$

$x_2 + x_4 = 0$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$ R2+R1→ R2

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$ R4+R3 → R4

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$ R2 ←→ R3

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$ R4 +R3 → R4

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$ R1 + R2 → R1

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$ R2 + R3 → R2

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

∴ the solution is $x_1 = 1$, $x_2 = x_3 = x_4 = 0$, a unique point of intersection (1,0,0,0).

11. Find the inverse of $A = \begin{bmatrix} 4 & 3 \\ 4 & 2 \end{bmatrix}$ if it exists over Z_5.

$[A/I]$→ $[I/A^{-1}]$

$\left[\begin{matrix} 4 & 3 \\ 4 & 2 \end{matrix}\middle|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right]$ R1x4→ R1

$\left[\begin{matrix} 1 & 2 \\ 4 & 2 \end{matrix}\middle|\begin{matrix} 4 & 0 \\ 0 & 1 \end{matrix}\right]$ R2+R1→ R2

$\left[\begin{matrix} 1 & 2 \\ 0 & 4 \end{matrix}\middle|\begin{matrix} 4 & 0 \\ 4 & 1 \end{matrix}\right]$ R2x4→ R2

$\left[\begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} 4 & 0 \\ 1 & 4 \end{matrix}\right]$ R1+3R2→ R1

2+3(1)

=5

$5 \div 5 = 1\text{R}0$

$\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} 2 & 2 \\ 1 & 4 \end{matrix}\right]$

4+3=7

7÷5=1R2

0+4(3)=12

12÷5=2R2

$\therefore A^{-1} = \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}$

NOTE: If the inverse doesn't exist, you will get a row of 0's on the left and be unable to row reduce to the identity matrix.

12. Find the Inverse over Z_3 if it exists where $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.

$[A/I]$→ $[I/A^{-1}]$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$ R2+2R1→ R2

1+2(2)=5

5÷3=1R2

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{array}\right]$ R2x2→ R2

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{array}\right]$ R1+R2 → R1

2x2=4

4÷3=1R1

$\left[\begin{array}{cc|cc} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{array}\right]$

$\therefore A^{-1} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$

NOTE: If the inverse doesn't exist, you will get a row of 0's on the left and be unable to row reduce to the identity matrix.

13. Find the check digit for the following ISBN number "186 197 271"

The check vector is $\vec{d} = (10,9,8,7,6,5,4,3,2,1)$ and vector $\vec{v}$ is defined as $\vec{v} = (1,8,6,1,9,7,2,7,1,d)$

$\vec{d} \cdot \vec{v} = (10,9,8,7,6,5,4,3,2,1) \cdot (1,8,6,1,9,7,2,7,1,d)$

=10+72+48+7+54+35+8+21+2+d

=10 + 11(6) + 6 + 11(4) + 4 + 7 + 11(4) + 10 + 11(3) +2 + 8 + 11 + 10+2+d

=10 + 6+4+7+10+2+8+10+2+d

=27 + 32+ d

=59 + d

=11(5) + 4 + d

Solve for d where: 4+d = 11

d=7

14. Find the check digit for the following ISBN number "019 852 663".

The check vector is $\vec{d} = (10,9,8,7,6,5,4,3,2,1)$ and vector $\vec{v} = (0,1,9,8,5,2,6,6,3,d)$

$\vec{d} \cdot \vec{v} = (10,9,8,7,6,5,4,3,2,1) \cdot (0,1,9,8,5,2,6,6,3,d)$

=0+9+72+56+30+10+24+18+6+d

=9 + 11(6) + 6 + 11(5)+1+11(2)+8+10+11(2)+2+11+7+6+d

=9+6+1+8+10+2+11+7+6+d

=24+36+d

=60+d

=11(5) +5 +d

=5+d

Solve for d where: 5+d=11

d=6

15. Calculate the check digit for the UPC "638 489 001 08"

3(odd) + even + d

=3(6+8+8+0+1+8) + (3+4+9+0+0) + d

=3(31) + 16 + d

=109 + d

109+ d= 110

d=1

Therefore, the check digit is 1.

16. $(3+4i)+(10-2i)$

$= 13 + 2i$

17. $(4+7i)(-2-3i)$

$= -8 - 12i - 14i - 21i^2$

$= -8 - 26i - 21(-1)$

$= 13 - 26i$

18. Evaluate and express in form $a + bi$

$$\frac{4-2i}{1+3i}$$

$$\frac{(4-2i)}{(1+3i)}\frac{(1-3i)}{(1-3i)} = \frac{4-12i-2i+6i^2}{1-3i+3i-9i^2}$$

$$= \frac{4-14i+6(-1)}{1-9(-1)}$$

$$= \frac{10-14i}{10} = 1 - \frac{7}{2}i$$

19. Find the absolute value of $2 + 3\sqrt{2i}$

$|z| = \sqrt{a^2 + b^2} \qquad a = 2 \quad b = 3\sqrt{2}$

$= \sqrt{2^2 + \left(3\sqrt{2}\right)^2}$

$= \sqrt{4 + 9(2)} = \sqrt{22}$

20.a) Find the polar form of $zw, \frac{z}{w}$ and $\frac{1}{w}$ where z= i – 1 and w =$2\sqrt{3} + 2i$.

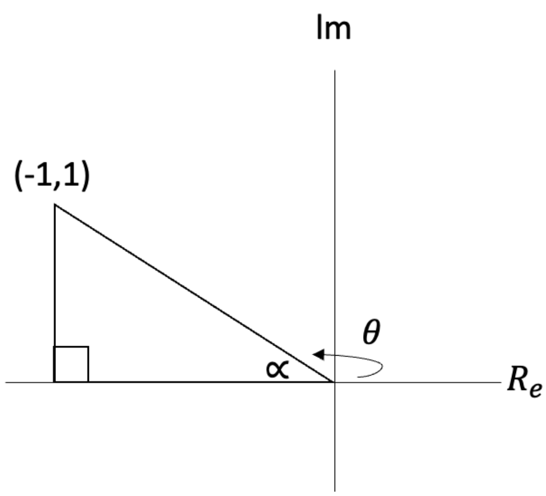
$z = -1 + i, \qquad w = 2\sqrt{3} + 2i$

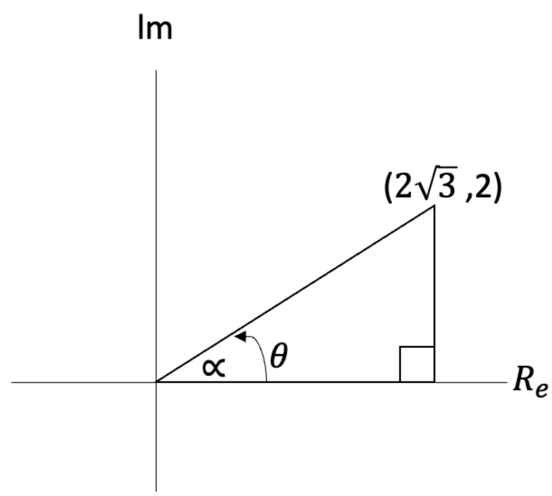
$a = -1, b = 1 \qquad a = 2\sqrt{3}, b = 2$

$r = \sqrt{(-1)^2 + 1^2} \qquad r = \sqrt{\left(2\sqrt{3}\right)^2 + 2^2}$

$r = \sqrt{4(3) + 4}$

$r_1 = \sqrt{2} \qquad r_2 = \sqrt{16} = 4$





$\propto = \tan^{-1}\left|\frac{1}{-1}\right| = \tan^{-1}(1) = \frac{\pi}{4}$

$\propto = \tan^{-1}\left|\frac{2}{2\sqrt{3}}\right| = \tan^{-1}\frac{1}{\sqrt{3}}$

$\therefore \theta_1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ $\propto = \frac{\pi}{6}$ $\theta_2 = \frac{\pi}{6}$

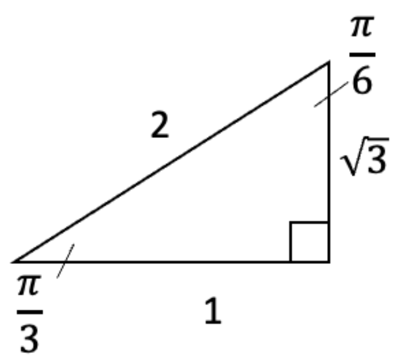
$\theta_1 + \theta_2 = \frac{3\pi}{4} + \frac{\pi}{6} = \frac{9\pi}{12} + \frac{2\pi}{12} = \frac{11\pi}{12}$

$zw = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i(\sin(\theta_1 + \theta_2))\right]$

$zw = 4\sqrt{2}\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]$

$\frac{z}{w} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$

$\frac{z}{w} = \frac{\sqrt{2}}{4}\left[\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right]$



$\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$

$\frac{1}{w} = \frac{1}{4}\left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]$

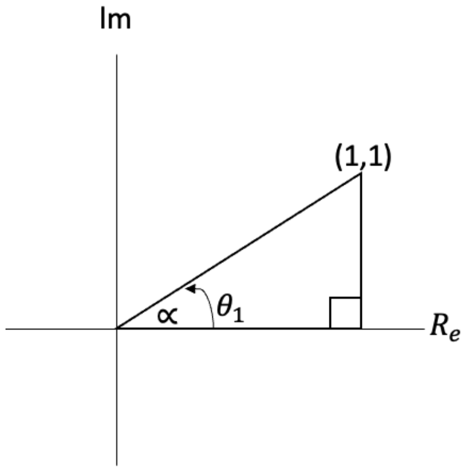
b) $z = 1 + i,$ $w = 2 - 2\sqrt{3}i$

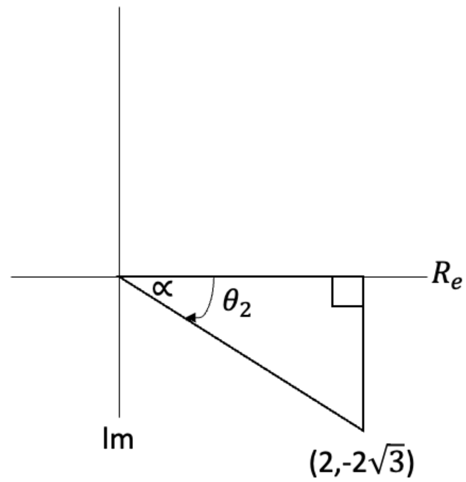
$a = 1, b = 1$ $a = 2, b = -2\sqrt{3}$

$r = \sqrt{1^2 + 1^2}$ $r = \sqrt{2^2 + \left(2\sqrt{3}\right)^2}$

$r = \sqrt{4 + 4(3)}$

$r_1 = \sqrt{2}$ $r_2 = \sqrt{16} = 4$





$\propto = \tan^{-1}\left|\frac{1}{1}\right| = \tan^{-1}(1) = \frac{\pi}{4}$

$\therefore \theta_1 = \frac{\pi}{4}$

$\propto = \tan^{-1}\left|\frac{-2\sqrt{3}}{2}\right| = \tan^{-1}\frac{\sqrt{3}}{1}$

$\propto = \frac{\pi}{3} \qquad \theta_2 = \frac{-\pi}{3}$

$\theta_1 + \theta_2 = \frac{\pi}{4} + \left(\frac{-\pi}{3}\right) = \frac{3\pi}{12} - \frac{4\pi}{12} = \frac{-\pi}{12}$

$\theta_1 - \theta_2 = \frac{\pi}{4} - \left(\frac{-\pi}{3}\right) = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{7\pi}{12}$

$zw = r_1 r_2\,[\cos(\theta_1 + \theta_2) + i(\sin(\theta_1 + \theta_2))]$

$zw = 4\sqrt{2}\left[\cos\left(\frac{-\pi}{12}\right) + i\sin\left(\frac{-\pi}{12}\right)\right]$

$\frac{z}{w} = \frac{r_1}{r_2}\,[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

$\frac{z}{w} = \frac{\sqrt{2}}{4}\left[\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right]$

$\frac{1}{w} = \frac{1}{r}(\cos\theta - i\sin\theta)$

$\frac{1}{w} = \frac{1}{4}\left[\cos\left(\frac{-\pi}{3}\right) - i\sin\left(\frac{-\pi}{3}\right)\right]$

21. Find the cube root of w if $w = -1 + i\sqrt{3}$

- First, we need to write *z* in its polar form, i.e. in the form

$$|w|\,(\cos\theta + i\sin\theta)$$

- To do this, we will find the argument and modulus, using the method explained in the previous section.

$$|w| = \sqrt{1+3} = 2$$

$$\alpha = \tan^{-1}\frac{|\sqrt{3}|}{|-1|} \quad \alpha = \frac{\pi}{3}$$

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

- In polar form our complex number is therefore:

$$w = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

- That’s most of the work done now. All we have left to do is to take the cube root of z, which means taking the cube root of the modulus and multiplying the angle by 1/3:

$$w^n = r^n(\cos n\theta + i\sin n\theta)$$

$$w^{1/3} = \left(2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right)^{1/3}$$

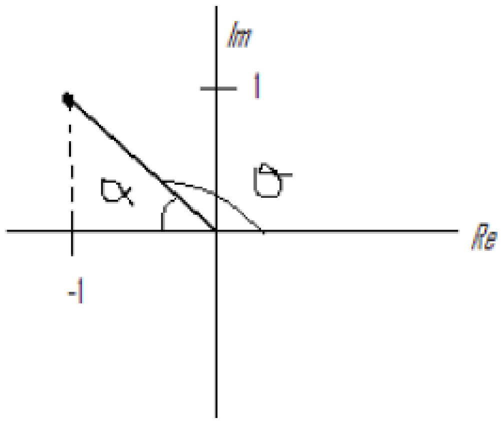
$$= 2^{1/3}\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)$$

22. Find the three cube roots of -1 + *i*.

Step 1: Find the modulus and principal argument of the complex number:

$$\text{modulant} = \sqrt{1+1} = \sqrt{2}$$

To find the principal argument, we plot the complex number on an argand diagram:



From the diagram, we see that

$$\alpha = \tan \frac{|1|}{|-1|} = \frac{\pi}{4}$$

Then,

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Step 2: We are looking for the cube root, so $n = 3$, and we will have to find $w_1 w_2 w_3$. Using the above formulas, we have

$$w_1 = |z|^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

$$= \left|\sqrt{2}\right|^{1/3}\left(\cos\frac{3\pi}{4\cdot 3} + i\sin\frac{3\pi}{4\cdot 3}\right)$$

$$= \sqrt{2}^{1/3}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$w_2 = |z|^{1/n}\left(\cos\frac{\theta + 2\pi}{n} + i\sin\frac{\theta + 2\pi}{n}\right)$$

$$= \sqrt{2}^{1/3}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

$$w_3 = |z|^{1/n}\left(\cos\frac{\theta + 4\pi}{n} + i\sin\frac{\theta + 4\pi}{n}\right)$$

$$= \sqrt{2}^{1/3}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$$

And we have found all three cube roots of this complex number.

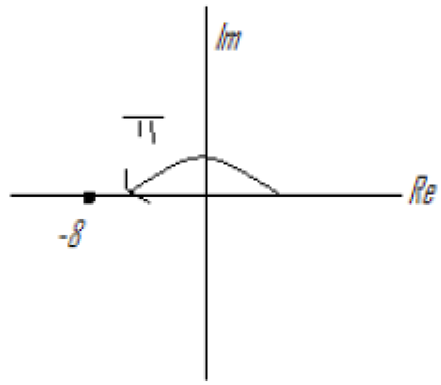
23. Find the cube roots of -8

$-8 = -8 + 0i$, so we have to find the modulus and argument of this complex number (note that though -8 is not a complex number, by writing it as $-8 + 0i$ we have converted it to a complex number). Once we find the argument and modulus, we write this complex number in its polar form, and use De Moivre's theorem to find the cube root.

Step 1: Find the modulus and argument

The modulus = $\sqrt{(-8)^2 + 0^2} = 8$

On an argand diagram, -8 looks like:



So, the argument is $\theta = \pi$ and the polar form is w=8 ($\cos\pi + isin\pi$)

Step 2: $n = 3$, and using the formula for *n*th root of complex numbers, we find $w_1 w_2 w_3$.

$$w_1 = |z|^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

$$w_1 = |8|^{1/3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$w_1 = |-2|\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$$

$$w_2 = 2\left(\cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}\right) = -2$$

$$w_3 = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 1 - i\sqrt{3}$$

24. What are the real and imaginary parts of the complex number $z = \dfrac{1-i}{2+i}$?

In order to express *z* in the form $z = a + bi$, we can multiply the numerator and denominator by the conjugate of 2 + *i*. This will essentially eliminate the complex terms in the denominator and allow us to obtain the standard form of *z*.

$$\frac{1-i}{2+i} = \frac{(1-i)\overline{(2+i)}}{(2+i)\overline{(2+i)}} = \frac{(1-i)(2-i)}{(2+i)(2-i)} = \frac{2-i-2i+i^2}{4-2i+2i-i^2} = \frac{2-3i-1}{4+1} = \frac{1-3i}{5} = \frac{1}{5} - \frac{3}{5}i = a + bi$$

So, the real part of *z* is $a = \dfrac{1}{5}$, and the imaginary part is $b = \dfrac{-3}{5}$.

25. Find polar form $(-\sqrt{3}, -\sqrt{-3})$

$a = -\sqrt{3} \quad b = -\sqrt{-3} = -\sqrt{3}$

$\tan \propto = \left|\frac{-\sqrt{3}}{-\sqrt{3}}\right|$

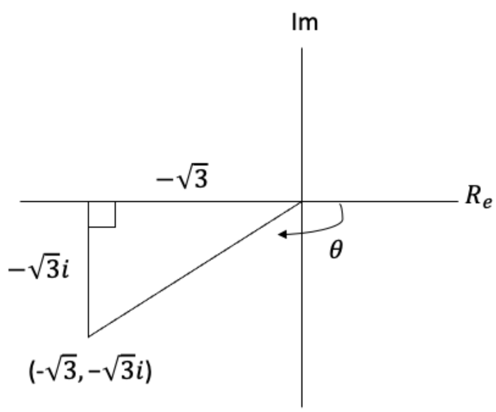
$\tan \propto = 1$

$\propto = \frac{\pi}{4}$

$r = \sqrt{\left(-\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^2}$

$= \sqrt{3+3}$

$= \sqrt{6}$



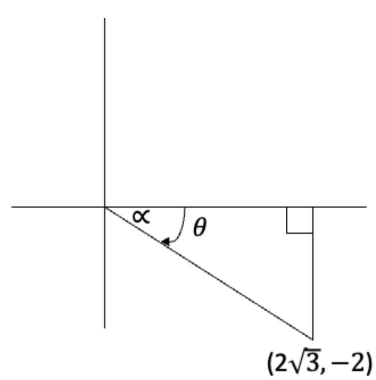
NOTE: If we take the angle and add pi, we get an angle greater than pi, so we do the following:

$\theta = \frac{\pi}{4} - \pi$

$\theta = \frac{\pi}{4} - \frac{4\pi}{4} = -\frac{3\pi}{4}$

$\therefore z = r(\cos\theta + i\sin\theta)$

$z = \sqrt{6}\left(\cos(-\frac{3\pi}{4}) + i\sin(-\frac{3\pi}{4})\right)$



26. Find the polar form of $z = -2i + 2\sqrt{3}$.

$z = 2\sqrt{3} - 2i$.

$a = 2\sqrt{3} \qquad b = -2$

$$r = \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2} = \sqrt{12+4} = 4$$

$$\propto = \tan^{-1}\left|\frac{-2}{2\sqrt{3}}\right|$$

$$\propto = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

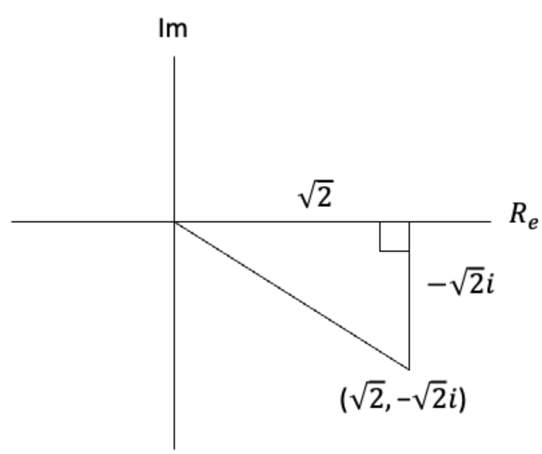
$$\propto = \frac{\pi}{6}$$

$$\theta = \left(-\frac{\pi}{6}\right)$$

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 4\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

27. Find the polar form of $Z = \sqrt{2} - \sqrt{2i}$



$a = \sqrt{2} \qquad b = -\sqrt{2}$

$$r = \sqrt{\left(\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2} = \sqrt{4} = 2$$

$$\propto = \tan^{-1}\left|\frac{-\sqrt{2}}{-\sqrt{2}}\right| = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{4} - \pi = \frac{\pi}{4} - \frac{4\pi}{4} = \frac{-3\pi}{4}$$

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 2\left(\cos\left(\frac{-3\pi}{4}\right) + isin\left(\frac{-3\pi}{4}\right)\right)$$

Post-Midterm Material

1. Linear Transformations

Example 1.1. Is T $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ x-2y \end{bmatrix}$ a linear transformation?

Is T $(c_1\overrightarrow{v_1} + c_1\overrightarrow{v_2}) = c_1T(\overrightarrow{v_1}) + c_2T(\overrightarrow{v_2})$

where $\overrightarrow{v_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \overrightarrow{v_2}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$$\mathrm{T}(c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2}) = T\left[c_1\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right]$$

$$= T\begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{bmatrix}$$

$$= \begin{bmatrix} (c_1x_1 + c_2x_2) + 2\,(c_1y_1 + c_2y_2) \\ (c_1x_1 + c_2x_2) - 2\,(c_1y_1 + c_2y_2) \end{bmatrix}$$

$$= \begin{bmatrix} c_1x_1 + 2c_1y_1 + c_2x_2 + 2\,c_2y_2 \\ c_1x_1 - 2c_1y_1 + c_2x_2 - 2\,c_2y_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1x_1 + 2c_1y_1 \\ c_1x_1 - 2c_1y_1 \end{bmatrix} + \begin{bmatrix} c_2x_2 + 2\,c_2y_2 \\ c_2x_2 - 2\,c_2y_2 \end{bmatrix}$$

$$= c_1\begin{bmatrix} x_1 + 2y_1 \\ x_1 - 2y_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 + 2\,y_2 \\ x_2 - 2\,y_2 \end{bmatrix} = c_1T(\overrightarrow{v_1}) + c_2T(\overrightarrow{v_2})$$

Example 1.2. Find the image of the vector [1,2] under this reflection.

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \therefore \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Example 1.3. Find the image of the vector [1,2] under this reflection.

$\therefore \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Example 1.4. Find the image of the vector [1,2] under this reflection.

$$\therefore \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Example 1.5. Find the image of the vector [1,2] under this reflection.

$$\therefore \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Example 1.6. Find the standard matrix for the transformation "reflection in the line $y = 2x$"

A direction vector is $\vec{d} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\therefore d_1 = 1 \quad d_2 = 2$

$$F_\ell(\vec{x}) = \frac{1}{1^2 + 2^2} \begin{bmatrix} 1^2 - 2^2 & 2(1)(2) \\ 2(1)(2) & -1^2 + 2^2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

Example 1.7. Find the image of the vector [1,2] under this projection.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Example 1.8. Find the projection onto the line $y = \frac{2}{3}x$.

A direction vector is $\vec{d} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, so $d_1 = 3\ and\ d_2 = 2$.

$P_\ell(\overrightarrow{e_1}) = \left(\frac{\vec{d}\cdot\overrightarrow{e_1}}{\vec{d}\cdot\vec{d}}\right)\vec{d}$

$= \frac{(3,2)\cdot(1,0)}{(3,2)\cdot(3,2)}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$= \frac{3}{13}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 9/13 \\ 6/13 \end{bmatrix}$

$P_\ell(\overrightarrow{e_2}) = \left(\frac{\vec{d}\cdot\overrightarrow{e_2}}{\vec{d}\cdot\vec{d}}\right)\vec{d}$

$= \frac{(3,2)\cdot(0,1)}{(3,2)\cdot(3,2)}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$= \frac{2}{13}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6/13 \\ 4/13 \end{bmatrix}$

$\therefore A = [P(\overrightarrow{e_1}) \quad P(\overrightarrow{e_2})] = \begin{bmatrix} 9/13 & 6/13 \\ 6/13 & 4/13 \end{bmatrix}$

Now, $d_1^2 = 3^2 = 9$ and $d_2^2 = 2^2 = 4$ and $d_1 d_2 = (3)(2) = 6, so$

$$A = \begin{bmatrix} \frac{d_1^2}{d_1^2+d_2^2} & \frac{d_1 d_2}{d_1^2+d_2^2} \\ \frac{d_1 d_2}{d_1^2+d_2^2} & \frac{d_2^2}{d_1^2+d_2^2} \end{bmatrix} = \begin{bmatrix} \frac{9}{9+4} & \frac{6}{9+4} \\ \frac{6}{9+4} & \frac{4}{9+4} \end{bmatrix} = \begin{bmatrix} 9/13 & 6/13 \\ 6/13 & 4/13 \end{bmatrix}$$

Example 1.9.

A rotation about the origin at an angle of θ in R^2 is a linear transformation from $R^2 \rightarrow R^2$.

$R_\theta(\vec{u} + \vec{v}) = R_\theta(\vec{u}) + R_\theta(\vec{v})$

$R_\theta(c\vec{u}) = cR_\theta(\vec{u})$

Example 1.10. Write the standard matrix for a rotation of 30° counterclockwise. Then, find the image of the vector (2,4) under this rotation.

Solution:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$[R_{30°}]\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \sqrt{3}-2 \\ 1+2\sqrt{3} \end{bmatrix}$$

Example 1.11.

Find $S \circ T: R^2 \to R^4$

Solution:

$$[S] = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } [T] = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 2 & 4 \end{bmatrix}$$

$$\therefore [S \circ T] = [S][T] = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 1 & -5 \\ -2 & 1 \\ 8 & 7 \end{bmatrix}$$

$$\therefore (S \circ T)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 1 & -5 \\ -2 & 1 \\ 8 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7x_1 + 8x_2 \\ x_1 - 5x_2 \\ -2x_1 + x_2 \\ 8x_1 + 7x_2 \end{bmatrix}$$

Example 1.12. Find the standard matrix for a 30° rotation in a clockwise direction.

Since a 30° clockwise rotation is the inverse of a 30° counterclockwise rotation, we get

$$[R_{-30°}] = [(R_{30})^{-1}] = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}^{-1}$$

…we could calculate the inverse using the 2x2 formula we talked about earlier!

Example 1.13.

Write the matrix representation (i.e., the standard matrix) of *T*.

Solution:

To find the standard matrix, we will apply the linear transformation to each vector in the standard basis for the domain ($\mathbb{R}^3$) to produce column vectors in the codomain ($\mathbb{R}^4$). Note that the linear system can be rewritten as

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7x+y-z \\ 8x+2y+z \\ -y+z \\ 9x+3z \end{bmatrix}$$

Then,

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 8 \\ 0 \\ 9 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

$$T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

Then, the standard matrix is $A = \left[T(e_1) \,|\, T(e_2) \,|\, T(e_3)\right] = \begin{bmatrix} 7 & 1 & -1 \\ 8 & 2 & 1 \\ 0 & -1 & 1 \\ 9 & 0 & 3 \end{bmatrix}$.

1.6 Homework on Chapter 1

1. A linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^3$ is given by

$$\begin{aligned} 2x_1 - 5x_2 + x_3 - 7x_4 - 2x_5 &= 8 \\ -6x_1 + 8x_3 - 2x_4 + 12x_5 &= 1 \\ -9x_1 + 9x_2 + 4x_3 + 5x_5 &= 2 \end{aligned}$$

a) Find the standard matrix of the transformation.

b) Compute $T(3,2,-6,1,-3)$ by direct substitution and by matrix multiplication.

Solution:

a) To find the standard matrix, we will apply the linear transformation to each vector in the standard basis for the domain ($\mathbb{R}^5$) to produce column vectors in the codomain ($\mathbb{R}^3$). Note that the linear system can be rewritten as

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_1 - 5x_2 + x_3 - 7x_4 - 2x_5 \\ -6x_1 + 8x_3 - 2x_4 + 12x_5 \\ -9x_1 + 9x_2 + 4x_3 + 5x_5 \end{bmatrix}$$

Then,

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -6 \\ -9 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 0 \\ 9 \end{bmatrix}$$

$$T\left(e_3\right)=T\left(\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\8\\4\end{bmatrix}$$

$$T\left(e_4\right)=T\left(\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-7\\-2\\0\end{bmatrix}$$

$$T\left(e_5\right)=T\left(\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-2\\12\\5\end{bmatrix}$$

Then, the standard matrix is $A=\left[T\left(e_1\right)|T\left(e_2\right)|T\left(e_3\right)|T\left(e_4\right)|T\left(e_5\right)\right]=\begin{bmatrix}2 & -5 & 1 & -7 & -2\\ -6 & 0 & 8 & -2 & 12\\ -9 & 9 & 4 & 0 & 5\end{bmatrix}$.

b) The linear transformation of the given vector can be determined by direct substitution:

$$T\begin{bmatrix} 3 \\ 2 \\ -6 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2(3)-5(2)+(-6)-7(1)-2(-3) \\ -6(3)+8(-6)-2(1)+12(-3) \\ -9(3)+9(2)+4(-6)+5(-3) \end{bmatrix} = \begin{bmatrix} -11 \\ -104 \\ -48 \end{bmatrix}$$

Or by matrix multiplication using the standard matrix:

$$T\begin{bmatrix} 3 \\ 2 \\ -6 \\ 1 \\ -3 \end{bmatrix} = A\begin{bmatrix} 3 \\ 2 \\ -6 \\ 1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -5 & 1 & -7 & -2 \\ -6 & 0 & 8 & -2 & 12 \\ -9 & 9 & 4 & 0 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ -6 \\ 1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2(3)-5(2)+1(-6)-7(1)-2(-3) \\ -6(3)+0(2)+8(-6)-2(1)+12(-3) \\ -9(3)+9(2)+4(-6)+0(1)+5(-3) \end{bmatrix}$$

$$= \begin{bmatrix} -11 \\ -104 \\ -48 \end{bmatrix}$$

2. a) Write the standard matrix for a rotation of 60° counterclockwise. Then, find the image of the point (2,4) under this rotation.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$[R_{60°}]\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{3} \\ \sqrt{3}+2 \end{bmatrix}$$

b) Find the standard matrix for a 60° rotation in a clockwise direction.

Since a 60° clockwise rotation is the inverse of a 60° counterclockwise rotation, we get

$$[R_{-60°}] = [(R_{60})^{-1}] = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}^{-1} = \frac{1}{\frac{1}{4}+\frac{3}{4}}\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3} =/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$$

(using the 2x2 inverse formula)

3. a) Write the standard matrix for a rotation of 45° counterclockwise. Then, find the image of the point (-2,4) under this rotation.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A = \begin{bmatrix} cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$[R_{45°}]\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{2}} - \frac{4}{\sqrt{2}} \\ \frac{-2}{\sqrt{2}} + \frac{4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{6}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{6\sqrt{2}}{2} \\ \frac{2\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -3\sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

b) Find the standard matrix for a 60° rotation in a clockwise direction.

Since a 60° clockwise rotation is the inverse of a 60° counterclockwise rotation, we get

$$[R_{-45°}] = [(R_{45})^{-1}] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{-1} = \frac{1}{\left(1/\sqrt{2}\,1/\sqrt{2} + 1/\sqrt{2}\,1/\sqrt{2}\right)}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(using the 2x2 inverse formula)

4. Consider the linear transformation $T: R^2 \to R^3$ defined by

$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ x_1 - x_2 \\ x_1 + 3x_2 \end{bmatrix}$ and the linear transformation $S: R^3 \to R^4$ defined by

$$S \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 + y_3 \\ y_2 - y_3 \\ y_1 + y_2 \\ y_1 + y_2 - y_3 \end{bmatrix}$$

Find $S \cdot T: R^2 \to R^4$

$[S] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ and $[T] = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}$

$$[S \cdot T] = [S][T] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 0 & -4 \\ 3 & -1 \\ 2 & -4 \end{bmatrix}$$

$\therefore (S \cdot T) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & -4 \\ 3 & -1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 + 3x_2 \\ -4x_2 \\ 3x_1 - x_2 \\ 2x_1 - 4x_2 \end{bmatrix}$

5. $\text{T} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\text{T} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

But T $\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right)$

$= T \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

6.

a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

c) $y = 3x$

Use the formula on p. 176

Reflection in any line y=mx:

The standard matrix that reflects a vector in ℓ is:

$$F_\ell(\vec{x}) = \frac{1}{d_1^2 + d_2^2}\begin{bmatrix} d_1^2 - d_2^2 & 2d_1d_2 \\ 2d_1d_2 & -d_1^2 + d_2^2 \end{bmatrix}$$

$$d = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \therefore d_1 = 1 \quad d_2 = 3$$

$$F_\ell(\vec{x}) = \frac{1}{1^2+3^2}\begin{bmatrix} 1^2 - 3^2 & 2(1)(3) \\ 2(1)(3) & -1^2 + 3^2 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 1 - 9 & 6 \\ 6 & -1 + 9 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} -8 & 6 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} -8/10 & 6/10 \\ 6/10 & 8/10 \end{bmatrix}$$

$$= \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$$

d) Projection onto the line $y = -x$

$\vec{d} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} d_1 = 1, d_2 = -1$

$$A = \begin{bmatrix} \frac{d_1^2}{d_1^2 + d_2^2} & \frac{d_1 d_2}{d_1^2 + d_2^2} \\ \frac{d_1 d_2}{d_1^2 + d_2^2} & \frac{d_2^2}{d_1^2 + d_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1^2}{1^2 + (-1)^2} & \frac{1(-1)}{1^2 + (-1)^2} \\ \frac{1(-1)}{1^2 + (-1)^2} & \frac{(-1)^2}{1^2 + (-1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

e) $A = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

f) $A = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

2. Determinants

Example 2.1. Find A_{23} where A= $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 4 & 0 & 3 & 2 \\ 3 & 1 & 2 & 1 \\ 2 & 2 & 1 & 3 \end{bmatrix}$

Solution:

Delete second row and third column

$$A_{23} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Example 2.2. Find B_{12} where B= $\begin{bmatrix} 2 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

Delete the first row and second column

$$B_{12} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

Example 2.3. Find the 2,3 minor and 2,3 cofactor of A= $\begin{bmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 3 & -1 \end{bmatrix}$

Solution:

2,3 minor= det A_{23}= determinant of the matrix leftover when you remove the 2nd row and 3rd column

det $\begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$= ad - bc = (2)(3) - (2)(0) = 6

2,3 cofactor= $(-1)^{2+3}$ (2,3 minor)= (-1)(6)= -6

Example 2.4. Find the 1,2 minor and the 1,2 cofactor of B=$\begin{bmatrix} 2 & 2 & 3 \\ 3 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

Solution:

Delete the first row and second column

1,2 minor = det $\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} = 0 - 2 = -2$

1,2 cofactor= $(-1)^{1+2}(-2) = 2$

Example 2.5. Find det A in each case:

a) $A = [2]$

Solution:

det A = 2

b) B=$\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$

Solution:

det B= ad-bc=(1)(3) - (5)(2)= 3-10 = -7

c) C=$\begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix}$

Solution:

det C = $(-1)^{1+1}$ (1) det$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ +$(-1)^{1+2}$(3)det$\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$+$(-1)^{1+3}$(5)det$\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$

det C=(1)(1)(2-0) + (-1)(3)(2+2) + (1)(5)(0+2)

det C=2 - 12 + 10=0

Now, we can do the same question, but expand along the SECOND column, since it has a zero!!!

det C = $(-1)^{1+2}$ (3) det$\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$ +$(-1)^{2+2}$(1)det$\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$+$(-1)^{1+3}$(5)det$\begin{bmatrix} 1 & 5 \\ -2 & 2 \end{bmatrix}$

= (-3)(2+2)+(1)(2+10)

= -12+12 = 0

Example 2.6. Find the determinant of each of the following:

a) A=[7]

Solution:

det A=7

b) B=$\begin{bmatrix} 2 & -1 \\ 3 & -7 \end{bmatrix}$

Solution:

detB=ad - bc = (2)(-7) - (-1)(3) = -14 + 3 = -11

c) C= $\begin{bmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

Solution:

Expanding along the last row...

detA= $(-1)^{3+1}(4)det\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} + (-1)^{3+2}(2)det\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$

=4(3+2) + (-2)(2+4)

=20-12

=8

d) D=$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & -1 \\ 3 & 4 & 0 \end{bmatrix}$

Solution:

Expand along the second column because it has the most zeros

det C=$(-1)^{3+2}(4)det\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

= -4(-1-2)

=12

e) E= $\begin{bmatrix} 3 & -1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 4 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

Solution: expand along the first column

det E= $(-1)^{1+1}(3)det\begin{bmatrix} 0 & 2 & 2 \\ 4 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

expand along the first column

det E=3 $\left[(-1)^{2+1}(4)det\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} + (-1)^{3+1}(1)det\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}\right]$

=3[(-4)(4)+1(-4)]

= 3(-16-4)

= - 60

f) F= $\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

Solution:

Expand along the first column

det G= $(-1)^{1+1}(2)det\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

expand along third column

det G= 2[$(-1)^{2+3}(2)det\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$]

=2[(-2)(1)]

= -4

g) $let\ A = \begin{bmatrix} 1 & 2 & -3 \\ a & b & c \\ d & e & f \end{bmatrix}$ and given

det $\begin{bmatrix} a & b \\ d & e \end{bmatrix} = 4$, det $\begin{bmatrix} a & c \\ d & f \end{bmatrix} = 2$, det $\begin{bmatrix} b & c \\ e & f \end{bmatrix} = 3$, Find det A.

Solution:

Find det A

$$detA = (-1)^{1+1}(1)\det\begin{bmatrix} b & c \\ e & f \end{bmatrix} + (-1)^{1+2}(2)\det\begin{bmatrix} a & c \\ d & f \end{bmatrix} +$$

$$(-1)^{1+3}(-3)\det\begin{bmatrix} a & b \\ d & e \end{bmatrix}$$

$$= (1)(3) + (-2)(2) + (-3)(4)$$

$$= 3 - 4 - 12$$

$$= -13$$

Now, back to minors and cofactors...

Example 2.7. Find the 3,2 minor and 3,2 cofactor of C= $\begin{bmatrix} 2 & 2 & 1 & 0 \\ -1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 1 & -1 \end{bmatrix}$.

Solution:

Delete the third row and second column

3,2 minor = det $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ expanding along top row

=$(-1)^{1+1}(2)$ det$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} + (-1)^{1+2}(1)det\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

=(2)(-3)+(-1)(0)= -6

3,2 cofactor= $(-1)^{3+2}(-6) = 6$

Example 2.8. Find the 2,2 minor and the 2,2 cofactor of D= $\begin{bmatrix} 3 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & 4 & 0 & 1 \\ 0 & -1 & 1 & 3 \end{bmatrix}$

Solution:

Delete the second row and second column

2,2 minor= det $\begin{bmatrix} 3 & 2 & 3 \\ -1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

expand along last row...

= 0 + $(-1)^{3+2}$(1) det $\begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix} + (-1)^{3+3}(3)det\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

= -1(3+3) +3(0+2)

=0

2,2 cofactor= $(-1)^{2+2}(0) = 0$

Example 2.9.

A is upper triangular, *B* is lower triangular and *C* is a diagonal matrix.

A= $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ det A = (1)(3)(5) = 15

B= $\begin{bmatrix} 6 & 0 & 0 \\ 5 & 2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ det B= (6)(2)(3)= 36

C= $\begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ det C = (6)(-2)(1)= - 12

Example 2.10. The determinant of $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ is (1)(3)(2)=6 because we can multiply along the main diagonal as this matrix is in upper triangular form.

Example 2.11. Find the determinant of each of the following by putting the matrix in upper or lower triangular form.

a) A= $\begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 3 \\ -3 & -1 & 4 \end{bmatrix}$

Solution:

R3 + R1→ $R3$

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

detA=(3)(1)(3)=9 multiply along main diagonal

b) B=$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 4 \\ -1 & -2 & 1 \end{bmatrix}$

Solution:

R3+R1 → $R3$ $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix}$ R3-R2 → $R3$ $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix}$ sign of the determinant)

detB=1(2)(-1)= -2 (multiply along the main diagonal)

c) C=$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 1 & 4 & 6 \end{bmatrix}$

Solution:

$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix}$ R3-2R2 → $R3$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & -6 \end{bmatrix}$$

detC= -6

d) D= $\begin{bmatrix} 0 & 2 & 2 & 2 \\ 2 & -2 & 1 & 1 \\ 0 & 4 & 5 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ switch rows 1 and 2 (switches sign of the determinant)

Solution:

det D= $-det \begin{bmatrix} 2 & -2 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 4 & 5 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ R3-2R2 $\rightarrow R3$

$= -det \begin{bmatrix} 2 & -2 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

det D= (- 1)2(2)(1)(1) = -4

Example 2.12. Find the determinant of A= $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 1 & 6 & 2 \end{bmatrix}$.

Solution: Using elementary row-operations, R3 – R1 → R2, we get: $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & 4 & 1 \end{bmatrix}$ Now, in order to finish, we would need to take: R3 -4/3 R2→R3!! YUCK!!!

So, switch to expanding along the first column.

detA= $(-1)^{1+1}(1)$ det $\begin{bmatrix} 3 & 3 \\ 4 & 1 \end{bmatrix} = (1)(1)(3-12) = -9$

Example 2.13. Find the determinant of each of the following matrices:

a) A= $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 1 & 4 & 1 \end{bmatrix}$

Solution:

det A=0 since there are two equal columns

b) B= $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 6 & 0 \\ 2 & 2 & 3 \end{bmatrix}$

Solution:

det B= (1)(6)(3)= 18 since it is in lower triangular form

c) C= $\begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & 0 \\ 3 & 1 & 4 \end{bmatrix}$

Solution:

detC=0 since there is a row of 0's

d) D= $\begin{bmatrix} 1 & 2 & 4 \\ 5 & -1 & 2 \\ -2 & -4 & -8 \end{bmatrix}$

Solution:

det D=0 since column row 1 times (-2) is equal to row 3

e) E= $\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 3 & 0 & 5 \end{bmatrix}$

Solution:

R3 - 2R2→R3 and R4 - 3R2→ $R4$

$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix}$ R4 - R3 → $R4$

$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

det E= (1)(-3)= -3

f) F= $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 0 & 0 & 3 \\ 3 & 1 & 1 & 2 \\ -1 & -1 & 0 & -1 \end{bmatrix}$

Solution:

R2-2R1 → $R2$ and R3-3R1 → $R3\ and\ R4 + R1 \rightarrow R4$

$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 1 & 4 & -4 \\ 0 & -1 & -1 & 1 \end{bmatrix}$ R4 + R3→ $R4$ $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 1 & 4 & -4 \\ 0 & 0 & 3 & -3 \end{bmatrix}$

you can switch to expanding along the first column, to avoid the fractions

det F= $(-1)^{1+1}(1)det\begin{bmatrix} 0 & 2 & -1 \\ 1 & 4 & -4 \\ 0 & 3 & -3 \end{bmatrix}$

expand along first column

det F= $(-1)^{2+1}(1)det\begin{bmatrix} 2 & -1 \\ 3 & -3 \end{bmatrix}$

= (-1) (-6+3)

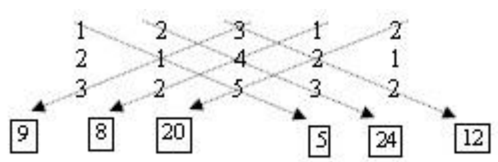
=3

<u>Example 2.14.</u> Use the basket-weave method to calculate the determinant of A=$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 2 & 5 \end{bmatrix}$

sum of right products= 5 + 24 + 12 = 41

sum of left products=9 + 8 + 20 = 37

det A= right products - left products = 41 - 37 = 4



Example 2.15. Use the basket-weave method to calculate the determinant of B=$\begin{bmatrix} 2 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & 2 & 5 \end{bmatrix}$

Solution:

R= 0 + 24 + 12 = 36 L= 0 + 16 - 20= -4

R - L = 36 - (-4) = 40

Example 2.16. Given det A=3, det B= 4 and A and B are both 3x3 matrices, use the properties above to answer each of the following:

Solution:

a) det(AB)= detA det B=(3)(4)=12

b) det (A^TB)= detA^T det B = (detA)(detB) =12

c) det(A^{-1})= 1/detA = 1/3

d) det(3A)= 3^3detA= 27(detA)=27(3) =81

e) det(2B)= 2^3 det B = 8(4) = 32

***Example 2.17**. If det $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 2, find$ det $\begin{bmatrix} 2a & 8b & 2c \\ d & 4e & f \\ g & 4h & i \end{bmatrix}$

Solution:

multiplied row 1 by 2 and column 2 by 4...multiplies determinant by 2 and by 4

new det = old det (2)(4) = 2(2)(4) = 16

* **Example 2.18**. Suppose A is a 3x3 matrix with det A = -5. Which one of the following is false? Circle all that apply.

A. det (A^T) = - 5	B. det (A^{-1}) = 1/5	C. det (A^2) =25	D. A is invertible	E. det (2A) = - 10

Solution:

A. det A^T = det A = - 5 true

B. det A^{-1} = 1/-5 = 1/5 false it should be -1/5

C. det(A^2) = det(AA)=det A (detA) = (-5)(-5)=25 true

D. true, since det A is NOT equal to 0

E. det (2A)= 2^3 det A = 8 (-5) = -40 false

The answer is b and e) are false.

* **Example 2.19**. Let A=$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and B=$\begin{bmatrix} -2a & -2b & -2c \\ d & e & f \\ g-2d & h-2e & i-2f \end{bmatrix}$. Find det B if you know that det A=36.

Solution:

det B = det A (-2)

det B = 36 (-2) = -72

Remember, g-2d back into g doesn't affect the determinant.

If it were g-2d back into d, it would multiply the determinant by -2.

* **Example 2.20**. If $A=\begin{bmatrix}4 & 3 & 0\\ k & 1 & k\\ 0 & 3 & 2\end{bmatrix}$, find the value of k for which det A=0.

Solution:

expand along bottom row (or you can basket weave)

det A = 0

$0 = (-1)^{3+2}(3)\det\begin{bmatrix}4 & 0\\ k & k\end{bmatrix} + (-1)^{3+3}(2)det\begin{bmatrix}4 & 3\\ k & 1\end{bmatrix}$

0 = -3 (4k - 0) + 2 (4- 3k)

0 = -12k+ 8 – 6k

18k = 8

k = 8/18 = 4/9

* **Example 2.21**. Let A be a 3x3 matrix with det A = 2. Find $\det(3A)\det(A^TA^{-1}A)$.

Solution:

$\det(3A)\det(A^TA^{-1}A)$

=det(3A)det (IA)

=det(3A)detA

$=3^3$ det A det A

=27(2)(2)

=108

* **Example 2.22**. If $\det\begin{bmatrix}2 & 0 & 0\\ 5 & k & 0\\ -1 & 6 & -3\end{bmatrix} = 72$, find the value of k.

Solution:

It is in triangular form, so multiply along the main diagonal to find the determinant

2(k) (-3) = 72

-6k = 72

k = - 12

* **Example 2.23.** Find $\det\begin{bmatrix} a & b & c \\ a-1 & b-1 & c-1 \\ a+5 & b+5 & c+5 \end{bmatrix} =$

Solution:

It is in triangular form, so multiply along the main diagonal to find the determinant

2(k) (-3) = 72

-6k = 72

k = - 12

Example 2.24. If A is a 4x4 matrix with $\det(2A^{-1}) = 5, find\ detA$.

A. 5/16	B. 16/5	C. 5/2	D. 2/5	E. none of the above

Solution:

$\det(2A^{-1}) = 5$

$2^4 detA^{-1} = 5$ since A is a 4x4 matrix

$16\left(\frac{1}{detA}\right) = 5$

$detA = \frac{16}{5}$

The answer is B).

Example 2.25. Solve using Cramer's Rule: 2x + y = 3

x + 3y = 15

$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ detA=6-1=5

A(1)=$\begin{bmatrix} 3 & 1 \\ 15 & 3 \end{bmatrix}$ detA(1)=9-15= -6

A(2)=$\begin{bmatrix} 2 & 3 \\ 1 & 15 \end{bmatrix}$ detA(2)=30-3= 27

$x = \frac{detA(1)}{detA} = \frac{-6}{5}$

$y = \frac{detA(2)}{detA} = \frac{27}{5}$

The solution is (-6/5, 27/5).

Example 2.26.

$A = \begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix}$…detA=-2 using basket weaving..R= -3+9+4 = 10 and left= 12+3-3= 12 and so

det A = right - left = 10 - 12 = - 2

Only use basket weaving for checking an answer on rough work paper!!

A(1)= $\begin{bmatrix} 10 & 3 & 4 \\ 2 & 1 & 1 \\ 4 & 1 & -1 \end{bmatrix}$ detA(1)=- 8 using basket weaving with R= -10 + 12 + 8 = 10 and

left= 16+ 10 -6 = 20 and det A = right - left = 10 – 20 = -10

x=detA(1)/detA= -10/-2= 5

Example 2.27. A= $\begin{bmatrix} 2 & 2 & a & -2 \\ 0 & 3 & b & -4 \\ 0 & 0 & c & 2 \\ 0 & 0 & d & 4 \end{bmatrix}$ det A=3

This involves determinants and finding a solution for a variable, so it is Cramer's Rule

To find x_3 we replace the third column of A with the b matrix (right hand side of the system of equations)

$2x_1+2x_2 + ax_3-2x_4=$ **-3**

$3x_2 + bx_3- 4x_4=$ **2**

$cx_3 + 2x_4=$ **-1**

$dx_3 + 4x_4=$ **0**

A(3)= $\begin{bmatrix} 2 & 2 & \mathbf{-3} & -2 \\ 0 & 3 & \mathbf{2} & -4 \\ 0 & 0 & \mathbf{-1} & 2 \\ 0 & 0 & \mathbf{0} & 4 \end{bmatrix}$

det A(3)= (2)(3)(-1)(4)= -24

$x_3= \frac{detA(3)}{\boldsymbol{det\ A}} = -\frac{24}{3} = -8$

Example 2.28. Find the 1,3 entry of the adjoint matrix:

a) delete the 3rd row and 1st column

1,3 entry of adjoint= 3,1 cofactor=$(-1)^{1+3}\det A_{31}$= (1) $\det\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$= 1

b) delete the 3rd row and 2nd column

2,3 entry of adjoint=3,2 cofactor= $(-1)^{2+3}\det A_{31}$= (-1) $\det\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$= -2

Example 2.29.

$C_{11}= (-1)^{1+1}\det A_{11}= -1$

$C_{12}= (-1)^{1+2}\det A_{21}= -3$

$C_{21}= (-1)^{2+1}\det A_{12}= -2$

$C_{22}= (-1)^{2+2}\det A_{22}= 2$

Adjoint matrix=$\begin{bmatrix} -1 & -3 \\ -2 & 2 \end{bmatrix}=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example 2.30. Given that det A=4 and Adj A=$\begin{bmatrix} 5 & 4 & 7 \\ 2 & 6 & 8 \\ 8 & -8 & 10 \end{bmatrix}$, find the (1,2) entry of A^{-1}.

Solution:

(1,2) of A^{-1}=$\frac{1}{detA}(1,2)\ of\ AdjA = \frac{1}{4}\begin{bmatrix} 5 & 4 & 7 \\ 2 & 6 & 8 \\ 8 & -8 & 10 \end{bmatrix}$ and the (1,2) entry is (1/4)(4)=1

For the (3,3) entry of A^{-1} we get: (3,3) of A^{-1}=$\frac{1}{detA}(3,3) of\ AdjA = \frac{1}{4}(10) = \frac{10}{4} = \frac{5}{2}$

Example 2.31. Given det A=4 and A is a 3x3 matrix, find det (Adj A).

Solution:

$\det(\text{Adj } A)=(\det A)^{n-1}$

$\det(\text{Adj } A)= (4)^{3-1} = 4^2 = 16$

Example 2.32. detA= -4

3,1 Adjoint=1,3 cofactor=delete 1st row and 3rd column= $(-1)^{3+1}$detA_{13}= (1) det$\begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}$=(1) (-2-2)= -4

$A^{-1} = \frac{1}{detA} AdjA = \frac{1}{-4}(-4) = 1$

Example 2.33. det(AdjA)= $(\text{detA})^{n-1}$

243 $=(3)^{n-1}$

$(3)^5=(3)^{n-1}$

5=n-1

n=6

So, it is a 6x6 matrix.

Example 2.34. Find det A if A=$\begin{bmatrix} a & b & c \\ 3 & -3 & 4 \\ d & e & f \end{bmatrix}$ $and\ Adj\ A = \begin{bmatrix} -22 & 8 & 17 \\ 8 & -7 & 2 \\ 17 & 2 & -7 \end{bmatrix}$

Solution:

det A = (ith row of A)· $(ith\ column\ of\ Adj\ A)$

$$= \begin{bmatrix} a & b & c \\ 3 & -3 & 4 \\ d & e & f \end{bmatrix} \begin{bmatrix} -22 & 8 & 17 \\ 8 & -7 & 2 \\ 17 & 2 & -7 \end{bmatrix}$$

(2nd row of A)· $(2nd\ column\ of\ Adj\ A)$

$$=\begin{bmatrix} 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ -7 \\ 2 \end{bmatrix} = (3)(8) + (-3)(-7) + (4)(2) = 24 + 21 + 8 = 53$$

2.9 Homework on Chapter 2

1. If A=$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then find detA.

multiply along the main diagonal since it is already in upper triangular

detA= (2)(2)(4)= 16

2. Find the 1,2 cofactor of the matrix A=$\begin{bmatrix} 2 & 0 & 3 \\ 2 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

delete the first row and second column

1,2 cofactor= $(-1)^{1+2} det \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = (-1)(2+3) = -5$

3. Find det $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 1 & 2 & 0 \\ 2 & 4 & 2 & 6 \end{bmatrix}$

The matrix is in lower triangular, so just multiply along the main diagonal to find the determinant

detA= (2)(3)(2)(6)= 72

4. Find det $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & -1 & -1 \end{bmatrix}$

=det $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & -1 & -1 \end{bmatrix}$ R2↔R3

$= -det \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix}$ R4 – R1→R4 and R3-2R2→ $R3$

$= -det \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -5 \\ 0 & -1 & 0 & -3 \end{bmatrix}$ R4+R2→R4

$= -det \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ R4+R3→R4

$= -det \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & -5 \end{bmatrix}$ = (-1)(1)(1)(-1)(-5)= -5

5. Find the 3,2 minor of B= $\begin{bmatrix} 5 & 1 & 2 \\ 4 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

delete the third row and second column

$= detA_{32} = det \begin{bmatrix} 5 & 2 \\ 4 & 1 \end{bmatrix} = (5 - 8) = -3$

6. Find det (A+B) where A= $\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and B= $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 7 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

****Find (A+B) first!!!**

note: you **cannot say det(A+B)=det A + det B...it is not true for any matrices A and B

A+B= $\begin{bmatrix} 2 & -1 & 6 \\ 0 & 8 & 3 \\ 1 & 1 & 5 \end{bmatrix}$ expand along the first column

det(A + B)= $(-1)^{1+1}(2)det\begin{bmatrix} 8 & 3 \\ 1 & 5 \end{bmatrix} + (-1)^{3+1}(1)det\begin{bmatrix} -1 & 6 \\ 8 & 3 \end{bmatrix}$

=(2)(40-3) + (1)(-3 -48)

=74 - 51

=23

7. If A and B are both 3x3 matrices, with det A=2 and det B=5, find det(A^TB^{-1}).

det(A^TB^{-1})= detA (1/detB)= 2 (1/5) = 2/5

8. If det $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$=4, find the determinant of $\begin{bmatrix} 2a & 2b & 2c \\ 3g & 3h & 3i \\ d & e & f \end{bmatrix}$

A. 6	B. -6	C. 24	D. -24	E. none of the above

switched rows x (-1) and then multiplied one row by 2, one row by 3 so det gets multiplied by 2 and 3

det=4(-1)(2)(3)= -24

The answer is d).

9. Find the 3,1 cofactor of B= $\begin{bmatrix} 4 & 1 & 2 \\ 4 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Delete the third row and first column

3,1 cofactor= $(-1)^{3+1}detB_{31} = (1)det\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = 1 + 2 = 3$

10. Find the 2,1 minor of the matrix A=$\begin{bmatrix} 3 & 3 & 2 \\ 0 & -1 & 1 \\ -1 & -2 & 3 \end{bmatrix}$

Delete the second row and first column

2,1 minor =$detA_{21} = det\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$= 9 + 4 = 13

11. Find det $\begin{bmatrix} 2 & 4 & 5 \\ 0 & \sqrt{2} & 2 \\ 0 & 0 & 4\sqrt{2} \end{bmatrix}$

matrix is in upper triangular form, so multiply along main diagonal to find the determinant

det= $2(\sqrt{2}(4\sqrt{2}) = 2(4)(2) = 16$ since a root times itself, is just the number under the root

12. Find det $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -6 & 6 & 0 & 0 \\ 6 & 6 & 1 & 0 \\ 2 & 2 & 3 & 2 \end{bmatrix}$

matrix is in lower triangular form, so multiply along main diagonal to find determinant

det=(1)(6)(1)(2)= 12

13. Find de$t$ $\begin{bmatrix} 0 & 0 & 1 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

switch row 1 and row 4, and multiply det by (-1)

= $-det\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ row reduce to get zero in row 4

R4-R3 → $R4$

=$-det\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

= -(1)(2)(1)(4)= -8

14. Find the determinant for each of the following:

a) $det\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

b) $det[-\sqrt{5}]$

c) $det\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -3 & 6 & -15 & 0 \\ 3 & 4 & 0 & 3 \end{bmatrix}$

a)det=ad-bc= 4 + 3 = 7

b) det=$-\sqrt{5}$

c) det=(2)(-1)(-15)(3)= 90

15. Find the determinant of $\begin{bmatrix} 2x & x & -2x \\ x & x+1 & -2x \\ 0 & 0 & 3x+1 \end{bmatrix}$

expand along third row

det=$(-1)^{3+3}(3x+1)det\begin{bmatrix} 2x & x \\ x & x+1 \end{bmatrix} = (3x+1)(2x^2+2x-x^2) = (3x+1)(x^2+2x)$

16. Using row and/or column operations, find det $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 4 \\ 2 & 2 & 4 \end{bmatrix}$.

$det\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 4 \\ 2 & 2 & 4 \end{bmatrix}$ using R3-2R1→R3 and R2-2R1→R2, then expand along the first column

$\begin{bmatrix} 1 & 3 & 0 \\ 0 & -5 & 4 \\ 0 & -4 & 4 \end{bmatrix}$

= $(-1)^{1+1}(1)det\begin{bmatrix} -5 & 4 \\ -4 & 4 \end{bmatrix} = -20 + 16 = -4$

17. The matrix $A=\begin{bmatrix}1 & 2 & 1\\ 1 & 1 & 0\\ 4 & 2 & 2\end{bmatrix}$ has det A= -4. Which of the following is ***false***?

A. $\det A^T = 4$	B. $\det A^{-1} = -1/4$	C. det (2A)= -8	D. $\det A^2 = 16$	E. A and C are false

A. is false since the transpose has the same det as the original matrix

B. is true since det (inverse)=1/detA

C. is false since $\det(2A)=2^3(\det A)=8(-4)= -32$

D. is true since $\det(A^2)=\det(AA)=\det A(\det A)= 16$

The answer is e).

18. If A and B are both nxn matrices, with det A=2 and det B=5, find $\det(A^TB^{-1})$.

A. 10	B. -10	C. 1/10	D. -2/5	E. none of the above

$\det(A^TB^{-1})= \det A\ (1/\det B)= 2/5$

The answer is e).

19. If A is a 3x3 matrix, with det A= 4, find $\det(2A)^T$.

A. 8	B. 1/8	C. 32	D. -32	E. none of the above

$\det(2A)^T=2^3(\det A)=8(4)=32$. The answer is c).

20. If det $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$=4, find the determinant of $\begin{bmatrix} 2a & 2d & 2g \\ b & e & h \\ c & f & i \end{bmatrix}$.

A. 4	B. -4	C. 8	D. -8	E. none of the above

The matrix is transposed, but this doesn't change the determinant

The first row is multiplied by 2, so the determinant gets multiplied by 2

det= 2(4)= 8

The answer is c).

21. If det $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$=5, find the determinant of $\begin{bmatrix} a-g & b-h & c-i \\ d & e & f \\ 2g & 2h & 2i \end{bmatrix}$

A. 5	B. -5	C. 10	D. -10	E. none of the above

The first elementary row-operation doesn't change the det, so just multiply it by 2 because the third row is multiplied by 2, so the new determinant is 5(2) =10.

The answer is c).

22. If A is a 4x4 matrix with det A= 2, which of the following statements are true?

I) $\det A^T=2$

II) det (-A)= -2

III) $\det (AA^{-1})=1$

IV) det(2A)=8

A. I) and IV)	B. I), II) and IV)	C. I), II) and III)	D. I) and III)	E. all of them

I) $\det A^T=2$ true

II) det (-A)= -2

$\det (-A)= (-1)^4(\det A)=2$ so II is false

III) $\det (AA^{-1})=1$

$\det (AA^{-1})=\det(I)=1$ is true

IV) $\det(2A)=2^4(\det A)=16(2)=32$ so IV) is false

The answer is d).

23. Find det (CD) where C=$\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$ and D= $\begin{bmatrix} 3 & -1 \\ 0 & 3 \end{bmatrix}$

A. 117	B. -63	C. 63	D. -117	E. none of the above

det(CD)=detCdetD= (3-10)(9-0)= (-7)(9)= -63. The answer is b).

24. Let A and B be nxn matrices with det A=7 and det B=6. Which of the following are false?

A. A and B are both invertible	B. Every linear system AX=b has a unique solution	C. Every linear system BX=b has no solution.	D. $\det(B^TA^T)=42$	E. The rank of A and B are both equal to n.

The answer is c). since if $\det B\neq 0$, there must be a unique solution

25. If A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and det A=8, which of the following matrices does not have a determinant equal to det A?

A.	B.	C.	D.
$\begin{bmatrix} 3a & \frac{1}{3}b \\ 3c & \frac{1}{3}d \end{bmatrix}$	$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$	$\begin{bmatrix} a & b \\ c+a & d+b \end{bmatrix}$	All of the following have a determinant equal to det A

for A, if you multiply one column by 3, you multiply det by 3, but then when you multiply another column by 1/3, you multiply det by 1/3 and get back where you started!

B, has same det, transpose doesn't change determinant

C, elementary row operations don't change det

The answer is d).

26. Determine the value of k so that matrix A=$\begin{bmatrix} 3 & 3 \\ k & 6 \end{bmatrix}$ does not have an inverse.

A. k=6	B. k= -6	C. $k \neq 6$	D. $k \neq -6$	E. None of the above

It won't have an inverse if detA=0

(3)(6) - 3k = 0

18 - 3k=0

k=6

The answer is a).

27. Determine the value of x so that matrix A= $\begin{bmatrix} 2 & 0 & 4 \\ x & 1 & 3 \\ 2 & 0 & x \end{bmatrix}$ has an inverse.

A. x=4	B. x= -5	C. $x \neq 4$	D. $x \neq -4$	E. none of the above

If det A=0, 2x-8 = 0 so x=4 and this would be the value for which there is NO inverse

The answer is c). Basket weave and get left = 8+0+0=8 and right=2x+0+0=2x and

right – left = 2x – 8 $\neq 0$,so we get x$\neq$ 4,since you know the determinant can't be 0 since it has an inverse. The answer is c).

28. Suppose A is an nxn matrix with det A≠0. Which of the following statements is true?

A. AX=b must have infinitely many solutions	B. There is at least one row of 0's in the row reduced echelon form of A	C. A is not invertible	D. AX=0 has no solution	E. rank A=n

The answer is e). because if detA=0, there can't be a unique solution or an inverse, or the identity when you row-reduce. Also, if det A =0, there is at least one row of 0's so the rank is less than n. The homogeneous system can't be a unique solution, so it must be infinitely many solutions.

29. Find $det\begin{bmatrix}1 & 0 & 5 & 3\\ 0 & -1 & 0 & 4\\ 0 & 0 & -3 & -2\\ 2 & 2 & 0 & 3\end{bmatrix}$

$R4 - 2R1 \rightarrow R4$

start row-reducing...

$\begin{bmatrix}1 & 0 & 5 & 3\\ 0 & -1 & 0 & 4\\ 0 & 0 & -3 & -2\\ 0 & 2 & -10 & -3\end{bmatrix}$ R4+2R2→ R4

$\begin{bmatrix}1 & 0 & 5 & 3\\ 0 & -1 & 0 & 4\\ 0 & 0 & -3 & -2\\ 0 & 0 & -10 & 5\end{bmatrix}$

expand along the first column...detA $(-1)^{1+1}(1)\det\begin{bmatrix}-1 & 0 & 4\\ 0 & -3 & -2\\ 0 & -10 & 5\end{bmatrix}$

expand along the first column of the 3x3...note: the numbers left in front of the 3x3 are just equal to 1

det A = $(-1)^{1+1}(-1)\det\begin{bmatrix}-3 & -2\\ -10 & 5\end{bmatrix} = -1(-15 - 20) = 35$

30. Find the (1,2) cofactor of A=$\begin{bmatrix}1 & 1 & 1\\ 3 & 3 & 3\\ 3 & 6 & 8\end{bmatrix}$

Delete the first row and second column

1,2 cofactor= $(-1)^{1+2}det\begin{bmatrix}3 & 3\\ 3 & 8\end{bmatrix} = (-1)(24 - 9) = -15$

31. If A= $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and det A = 4,

a) Find det $\begin{bmatrix} a & d & g \\ 4b & 4e & 4h \\ c & f & i \end{bmatrix}$

NEW det=4(4)=16

b) Find det $\begin{bmatrix} 3a & 3b & 3c \\ 4d & 4e & 4f \\ 2g-6a & 2h-6b & 2i-6c \end{bmatrix}$

NEW det=(3)(4)(2)(4)=96

c) Find det $\begin{bmatrix} a-3d & b-3e & c-3f \\ g & h & i \\ 3d & 3e & 3f \end{bmatrix}$

NEW det=(-1)(3)(4)= -12

d) Find det $\begin{bmatrix} 2a & 2d & 2g \\ b & e & h \\ c & f & i \end{bmatrix}$

NEW det=2(4)=8

32. Let A= $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. If det $\begin{bmatrix} 2a & 2d & 2g \\ b & e & h \\ c & f & i \end{bmatrix}$ = -40, find det A.

This one is backwards! You are finding det A, the original determinant and you know the final answer is -40.

2 det A = -40

Det A=1/2(-40)= -20

33. If B is a 3x3 matrix and the det (2B) =16 , then det B is:

A. 2	B. 1/2	C. 8	D. 8/3	E. none of the above

16=2^3 detB

det B=16/8 = 2

The answer is a).

34. Let A and B be 3×3 matrices with det A =5 and det B= 3. Find each of the following:

(i) det(AB)

(ii) $\det\left(A^T B^T A\right)$

(iii) $\det\left(A^{-1} B\right)$

(i) det (AB)= detA detB= 5(3)=15
(ii) det($A^T B^T A$)=(det A)(detB) (detA)=5(3)(5)=75
(iii) $\det\left(A^{-1} B\right)$
=det=(1/5)(3)=3/5

35. Find the det (A^{-1}B) if $A = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 3 & \sqrt[3]{5} \\ 0 & 0 & -2 \end{bmatrix}$ and B=$\begin{bmatrix} 8 & 0 & 0 \\ b & 3 & 0 \\ c & a & 1 \end{bmatrix}$

det(A^{-1}B)= (1/detA)(detB)=(1/-24)(24)=- 1 since they are both in triangular form so we can just multiply long main diagonal

36. Let A = $\begin{bmatrix} 2 & 3 & 2 \\ 0 & c & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Find c if det $A^{-1} = \frac{1}{8}$.

if det A^{-1}=1/8, then det A=8

detA=8= (2)(c)(2)...multiply along main diagonal since A is in triangular form

4c=8

c=2

37. Find all values of k for which A has an inverse if $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & k & 4 \\ 0 & 2k & k \end{bmatrix}$.

If A has an inverse, detA≠0...expand along first column

detA=$(-1)^{1+1}(2)det\begin{bmatrix} k & 4 \\ 2k & k \end{bmatrix} = (2)(k^2 - 8k) = 2k^2 - 16k = 2k(k-8)$=0

No inverse if k=0, 8

So, it will have an inverse as long as k≠0,8

38. For what value of k is the rank of A=$\begin{bmatrix} k & -10 \\ 1 & k+7 \end{bmatrix}$ equal to 1?

Given rank =1, (there is a row of 0's), so det A=0 since that means it is not invertible and not unique

k(k+7) +10 = 0

k^2 +7k +10 = 0

(k+2)(k+5)=0

k=-2, -5

39. Find det (-6I) where I is the identity matrix of order 70.

det (- 6I) = $(-6)^{70}$ det I = $(-6)^{70}$ (1) = $(-6)^{70}$

since the determinant of any identity matrix is 1

40. If C and D are 4x4 matrices where detC=2, detD=3, find det(2C) and det(3D).

det(2C)=2^4(2)= 32

det(3D)=3^4(3)=243

41. Find the determinant of $A=\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 2 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$

R3+R1 $\rightarrow R3$ *and* $R4 + R1 \rightarrow R4$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 3 & 3 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

then, expand along the first column

det A = $(-1)^{1+1}(1)$ det A_{11}

=(1)(1) det $\begin{bmatrix} 0 & 1 & 1 \\ 3 & 3 & -1 \\ 1 & 0 & -1 \end{bmatrix}$

Expand along the first column again

$=(-1)^{2+1}(3)\det\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + (-1)^{3+1}(1)\det\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ = (-3) (-1) + 1(-4) =3 – 4 = -1

42. Find k so that the matrix A has no inverse where:

$$A = \begin{bmatrix} k-3 & 2 & 3 \\ 0 & k+2 & 4 \\ 0 & 0 & k-4 \end{bmatrix}$$

$detA = (k-3)(k+2)(k-4) = 0$ since it is in upper triangular form

So, k=3, -2 and 4 so that A is not invertible.

43. If I is the 3x3 identity matrix, find $\det(3I^{-1} - 7I^T)$.

If I is the 3x3 identity matrix, find $det(3I^{-1} - 7I^T)$.

NOTE: the inverse and the transpose of I are the same matrix I, the identity matrix

$$det(3I^{-1} - 7I^T) = det(3I - 7I) = det(-4I) = (-4)^3 detI = -64(1) = -64$$

44. Find the determinant of matrix B= $\begin{bmatrix} 1 & 1 & -1 & 2 \\ -2 & 1 & 1 & 3 \\ 3 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 \end{bmatrix}$

R2 + 2R1 → $R2$

R3-3R1→ $R3$

R4 - 2R1→ $R4$

$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & -1 & 7 \\ 0 & -2 & 4 & -6 \\ 0 & -3 & 2 & -4 \end{bmatrix}$R4 + R2→ $R4$

$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & -1 & 7 \\ 0 & -2 & 4 & -6 \\ 0 & 0 & 1 & 3 \end{bmatrix}$...now expand along first column

det=$(-1)^{1+1}$ (1) det$\begin{bmatrix} 3 & -1 & 7 \\ -2 & 4 & -6 \\ 0 & 1 & 3 \end{bmatrix}$ = det$\begin{bmatrix} 3 & -1 & 7 \\ -2 & 4 & -6 \\ 0 & 1 & 3 \end{bmatrix}$...use any method to finish!

Determinant=right products - left products = 22 +12 = 34

45. If det $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = 4, find$ det $\begin{bmatrix} c & b & a \\ f & e & d \\ k-4c & h-4b & g-4a \end{bmatrix}$

New determinant = 4(-1) = -4 since they switched two columns

NOTE: k-4c is replacing k, so nothing is being done to the k, the spot being replaced, so this has no effect on the determinant

46. Let $a \in \mathbb{R}$ and let $A = \begin{bmatrix} 0 & 1 & -a & 0 \\ 2 & a & -1 & 0 \\ 1 & 1 & 1 & 0 \\ a & 4 & \pi & 1 \end{bmatrix}$. What value(s) of a is the matrix A invertible?

Let's find the determinant of the given matrix by expanding along the 4th column

$$\det A = \det \begin{bmatrix} 0 & 1 & -a & 0 \\ 2 & a & -1 & 0 \\ 1 & 1 & 1 & 0 \\ a & 4 & \pi & 1 \end{bmatrix} = 1\det \begin{bmatrix} 0 & 1 & -a \\ 2 & a & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Then, we can expand along the first row, which also has a zero, to get

$$\begin{aligned} \det A &= \det \begin{bmatrix} 0 & 1 & -a \\ 2 & a & -1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= -1\det \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} - a\det \begin{bmatrix} 2 & a \\ 1 & 1 \end{bmatrix} \\ &= -1\left[(2)(1) - (-1)(1)\right] - a\left[(2)(1) - a(1)\right] \\ &= -1(3) - a(2-a) \\ &= a^2 - 2a - 3 \\ &= (a-3)(a+1) \end{aligned}$$

Recall that a matrix is **invertible** when the determinant is **not equal to zero**.

Therefore, this matrix is invertible when $a \neq -1, 3$.

47. If det A = 3 and det B=2, and A and B are both 3x3 matrices, find each of the following:

a) det (AB) b) det (B^TA) c) det (A^{-1} B)

d) det (2A) e) det (-3B)

a) det (AB)= detAdetB= (3)(2)= 6

b) det (B^TA)= det B det A= (2)(3) =6

c) det (A^{-1} B)=$\frac{1}{detA}\,detB$=$\frac{1}{3}(2) = \frac{2}{3}$

d) det(2A)= 2^3 (3)= 24

e) det (-3B)= $(-3)^3(2) = -54$

48. In which one of the following matrices is the value of the determinant not equal to 12?

A. $\begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 7 & 5 & 6 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{bmatrix}$ C. $\begin{bmatrix} 0 & 0 & 6 \\ 0 & 2 & 3 \\ 1 & 5 & 6 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & -6 \end{bmatrix}$

C...det C is not equal to 12, while A, B and C are. (it is not the main diagonal, down and to the right)

49. If A=$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ and B=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$, find$ $\det(A + 2B)$.

A + 2B = $\begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$+2$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ = $\begin{bmatrix} 4 & 2 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

det (A + 2B)= 4(6)(-2)= - 48

50. If A=$\begin{bmatrix} -1 & 3 & -1 \\ 0 & 8 & k \\ 0 & 0 & 3 \end{bmatrix}$, find det ($A^{-1}$).

det A = (-1)(8)(3) = -24

det A^{-1} = -1/24

51. Find the value of k for which det $\begin{bmatrix} 4 & 2 \\ 1 & k \end{bmatrix} = det \begin{bmatrix} 12k & -2 \\ 1 & -3 \end{bmatrix}$.

4k - 2 = -36k+2

4k= -36k+2 +2

4k + 36k= 4

40k=4

k=1/10

52. If det $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 4$, find the determinant of $\begin{bmatrix} d & e & f \\ 3a & 3b & 3c \\ -2g & -2h & -2i \end{bmatrix}$.

switch row 1 and row 2 will multiply the det by -1...3 times row 2 will multiply the det by 3 and -2 times row 3 will multiply the original det by -2

So, new det= old det (-1)(2)(3)= 4(-1)(3)(-2) since the original det was 4...final answer = 24

53. 1,2 minor=detA_{12}= det$\begin{bmatrix} 0 & 2 & -1 \\ -1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

=$(-1)^{2+1}(-1)det\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + (-1)^{3+1}(2)det\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

=(-1)(-1)(7)+(1)(2)(2)=7+4=11

1,2 cofactor = $(-1)^{1+2}[1{,}2\ minor] = (-1)(11) = -11$

54.

a) $z = \frac{detA(3)}{detA} = \frac{9}{3} = 3$ The answer is C).\

b) $x = \frac{detA(1)}{detA} = -\frac{24}{3} = -8$ The answer is B).

c) $y = \frac{detA(2)}{detA} = -\frac{27}{3} = -9$ The answer is A).

55.det A= 6

A(1)=$\begin{bmatrix} 8 & 4 \\ 2 & -3 \end{bmatrix}$

det A(1)= - 24 - 8 = - 32

$$x = \frac{detA(1)}{detA} = -\frac{32}{6} = -\frac{16}{3}$$

56.det A = - 19

$$x3 = \frac{detA(3)}{detA}$$

$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ 4 & -3 & 2 \end{bmatrix}$ and b= $\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$

$A(3) = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ 4 & -3 & 2 \end{bmatrix}$ det A (3) = 4 by basket weaving...right products 2+48 – 36 = 14 and left products = 16 - 18 + 12= 10

and det A (3) = right - left = 14 – 10 = 4

So,

$$x3 = \frac{detA(3)}{detA} = \frac{4}{-19} = -\frac{4}{19}$$

Only use basket weaving for checking an answer on rough work paper!!

57. 2,3 Adj D = (3,2) Cofactor D= delete the 3rd row and 2nd column

(2,3) Adj D= (3,2) Cof D=$(-1)^{3+2}det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & -1 & -2 \end{bmatrix}$ =(-1)(right products – left products)

= -1 (-6 – (-5))= -1(-1) = 1

NOTE: I used basket-weaving, but you can find the determinant using any method

Right products= -4 + 0 + (-2) = -6

Left products = 0 -1 – 4 = -5

58. detA= 3 since det$A^{-1} = 1/3$

det(AdjA)= (detA)$^{n-1}$=(3)$^{4-1}$=3^3=27

59. $A^{-1} = \frac{1}{detA} AdjA$=$\frac{1}{-2}\begin{bmatrix} 2 & 0 & 1 \\ 2 & -1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$

2,1 entry is 2/-2= -1

60. det(AdjA)= (detA)$^{n-1}$

25 =(detA)$^{n-1}$

n=3 since it is a 3x3 matrix... 25=(det A)$^{3-1}$

25=(detA)^2

det A = ± 5

61. If A is a 3x3 invertible matrix, find det $\left[2A \frac{1}{detA}\ AdjA\right]$.

Recall, $A^{-1} = \frac{1}{detA} AdjA,$ so substitute this into the determinant above

det $\left[2A \frac{1}{detA}\ AdjA\right]$ move 1/detA to be next to the AdjA…it is a constant so this is allowed

$= \det\left[2A \frac{1}{detA}\ AdjA\ \right]$

$=$ det(2A)det[A^{-1}]

= 2^3 detA det [A^{-1}]

=8 detA (1/detA)

=8

62. det(AdjA)= $(\det A)^{n-1}=(3)^{3-1}=3^2=9$

63. Since $\det(A^{-1}) = \frac{1}{3}$, detA=3

det(AdjA)= $(\det A)^{n-1}=(3)^{4-1}=3^3=27$

The answer is d).

64. If $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 2 & 3 \\ 3 & 2 & 4 \end{bmatrix}$, find the (2,3)-entry of Adj A

2,3 Adjoint=3,2 cofactor= $(-1)^{2+3}\det A_{32}$...delete 3rd row and 2nd column

= (-1) det$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

=(-1)(3-8)

=5

65. If A is a 3x3 matrix with det A = -4 and Adj A = $\begin{bmatrix} 4 & 3 & 1 \\ 2 & 5 & 3 \\ \mathbf{6} & 7 & 4 \end{bmatrix}$, find the (3,1)-entry of A^{-1}.

(3,1) entry of $A^{-1} = \frac{1}{detA} AdjA = \frac{1}{-4}(6) = -3/2$

66.If A = $\begin{bmatrix} 3 & 6 & 2 \\ d & e & f \\ 1 & 2 & 2 \end{bmatrix}$ and Adj A = $\begin{bmatrix} 0 & -8 & 4 \\ -1 & 4 & -1 \\ 1 & 0 & -3 \end{bmatrix}$, find det A.

or use the formula

det A = (ith row of A)$\cdot$ (ith column of Adj A) and use either the 1st or 3rd rows/columns

det A = 1st row of A$\cdot$ 1st column of AdjA

=$(3,6,2) \cdot (0,-1,1) = 0 - 6 + 2 = -4$

67. This is Cramer's Rule

A(1)= $\begin{bmatrix} 4 & 4 \\ 3 & -3 \end{bmatrix}$ detA(1)= -12 – 12= -24

$$x = \frac{detA(1)}{detA} = -\frac{24}{48} = -\frac{1}{2}$$

68. *det A = (ith row of A)·* $(ith\ column\ of\ Adj\ A)$

=(2nd row of A)· $(2nd\ column\ of\ Adj\ A)$

detA= $\begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} -9 \\ 4 \\ 7 \end{bmatrix} = (0)(-9) + (1)(4) + (3)(7) = 0 + 4 + 21 = 25$

69. det(AdjB)= $(detB)^{4-1}$

125= $(detB)^3$

detB=5

70. $x = \frac{detA(1)}{detA} = -\frac{6}{6} = -1$

$y = \frac{detA(2)}{detA} = \frac{36}{6} = 6$ The solution is (-1,6)

The answer is b). (Cramer's Rule)

71. You can see that the matrix on top is A(1) since the second column in the original system has been replaced by the constant terms on the right of the system.

$$x = \frac{detA(1)}{detA}$$

Using Cramer's rule, the answer is a).

72. $det(A^{-1}adjA) = det(A^{-1})\ det\ (AdjA) = (1/detA)(detA)^{n-1} = (1/-3)(-3)^2 = -3$

The answer is e).

73.

det A = 16 (multiply along main diagonal since it is in triangular form)

$$x = \frac{detA(1)}{detA}$$

$$A(1) = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ 1 & 0 & 2 \end{bmatrix} \text{ and b=} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

det A(1)= (8+3+0) – (8+0+12)=11 – 20 = -9 by basket weaving

$$x = \frac{detA(1)}{detA} = \frac{-9}{16}$$

74. a) is true

b) is false

c) is true, det $(AB)^T$=(det (B^T A^T)=detB^T detA^T = detB detA=detA detB

d) is false, many determinants are 2 but the matrices don't have to be equal for both matrices to have a determinant of 2

e) is false

f) is true det $(AB)^{-1}$= det ($B^{-1}A^{-1}$)= det B^{-1} det A^{-1}=$\frac{1}{detB}\left(\frac{1}{detA}\right) = \frac{1}{detAdetB}$

So, a, c and f) are all true.

75.For the system:

x+2y+3z=6

-x+2y+4z=8

2x+4y+5z=2

Find the value of y in the solution using Cramer's Rule

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

det A = -4 from basket weaving:

down and right = 10 + 16 – 12=14

down and left = 12 +16 -10= 18

det A = R – L = 14 – 18 = - 4

Only use basket weaving for checking an answer on rough work paper!!

$$A(2) = \begin{bmatrix} 1 & 6 & 3 \\ -1 & 8 & 4 \\ 2 & 2 & 5 \end{bmatrix}$$

det A(2) = -40 from basket weaving:

down and right = 40+48 -6 = 82

down and left = 48 + 8 – 30 = 26

det A = R – L = 82 – 26 = 56

The solution for y is: $y = \frac{detA(2)}{detA} = \frac{56}{-4} = -14$

Find the value of z in the solution using Cramer's Rule.

$$A(3) = \begin{bmatrix} 1 & 2 & 6 \\ -1 & 2 & 8 \\ 2 & 4 & 2 \end{bmatrix}$$

det A(3) = -40 from basket weaving:

down and right = 4+32 – 24 = 12

down and left = 24+32 – 4 = 52

det A(3) = R – L = 12 – 52 = -40

$$z = \frac{detA(3)}{detA} = \frac{-40}{-4} = 10$$

Only use basket weaving for checking an answer on rough work paper!!

76. If A is a 3x3 invertible matrix, find det $\left[A\frac{1}{detA}3AA^{-1}\ (AdjA)\right]$.

Recall, $A^{-1} = \frac{1}{detA}\ (AdjA)$ **we will use this near the last step**

det $\left[A\frac{1}{detA}3AA^{-1}\ (AdjA)\right]$ = det $\left[A\frac{1}{detA}3I\ (AdjA)\right]$=$det\left[3A\frac{1}{detA}\ I(AdjA)\right]$

=$det\left[3A\frac{1}{detA}\ (AdjA)\right]$=det$(3AA^{-1}) = 3^3 detA\ detA^{-1} = 27 \det A\left(\frac{1}{detA}\right) =$
$27(1) = 27$

77. det(AdjA)= $(\text{detA})^{n-1}$

64 =$(2)^{n-1}$

$(2)^6(2)^{n-1}$

6=n-1

n=7

So, it is a 7x7 matrix

3. Eigenvalues and Eigenvectors

Example 3.1. Find the characteristic polynomial and the eigenvalues for the matrix

$A = \begin{bmatrix} 4 & 3 \\ -2 & -3 \end{bmatrix}.$

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{bmatrix} 4 & 3 \\ -2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4-\lambda & 3 \\ -2 & -3-\lambda \end{bmatrix}$$

$\det(A - \lambda I) = ad - bc = (4-\lambda)(-3-\lambda) - (3)(-2) = 0$

$-12 - 4\lambda + 3\lambda + \lambda^2 + 6 = 0$

Therefore, the characteristic polynomial is $\lambda^2 - \lambda - 6 = 0$

Solving $\lambda^2 - \lambda - 6 = 0$, we get $(\lambda - 3)(\lambda + 2) = 0$

(Factor the trinomial, by finding two numbers that add to -1 and multiply to -6)

Therefore, $\lambda = 3, -2$ are the eigenvalues.

Find the eigenvectors:

Once you have the eigenvalues, let vector v=(v1,v2,v3,...vn) corresponding to an eigenvalue λ, and solve the system of linear equations given by:

$(A - \lambda I)v = 0$ or $A\vec{v} = \lambda\vec{v}$

If $\lambda = -2$, let v=(v1,v2) then $(A - \lambda I)\vec{x} = \vec{0}$ $becomes$ $(A + 2I)v = 0$

$\lambda = -2$ $\quad \left(\begin{bmatrix} 4 & 3 \\ -2 & -3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 6 & 3 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{matrix} 6 & 3 \\ -2 & -1 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$ R1 ÷6 → R1 $\left[\begin{matrix} 1 & 1/2 \\ -2 & -1 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$ R2+2R1→ R2

$\left[\begin{matrix} 1 & 1/2 \\ \mathbf{0} & \mathbf{0} \end{matrix}\middle|\begin{matrix} 0 \\ \mathbf{0} \end{matrix}\right]$

$\therefore\ x_2 = t$

**** When row-reducing to find eigenvectors, you ALWAYS get at least 1 row of 0's**

$x_1 + \frac{1}{2}t = 0$

$x_1 = -\frac{1}{2}t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

The eigenspace is span $\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$

So, all eigenvectors corresponding to $\lambda = -2$ are multiples of the vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

$\lambda = 3 \qquad (A - \lambda I)\vec{x} = \vec{0}$

$\left(\begin{bmatrix} 4 & 3 \\ -2 & -3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ -2 & -6 & 0 \end{array}\right]$ R2 + 2R1 → R2

$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = t$

$x_1 + 3t = 0$

$x_1 = -3t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

∴ the eigenspace is span $\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$.

So, all eigenvectors corresponding to $\lambda = 3$ are multiples of the vector $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Example 3.2. Find the characteristic polynomial and the eigenvalues for the matrix

$A = \begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix}$.

Solution:

$\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ -2 & -1-\lambda \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$\therefore (2-\lambda)(-1-\lambda) - (3)(-2) = 0$

$-2 - 2\lambda + \lambda + \lambda^2 + 6$

$\lambda^2 - \lambda + 4 = 0$

$a = 1 \quad b = -1 \quad c = 4$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4(1)(4)}}{2(1)} = \frac{1 \pm \sqrt{-15}}{2} = \frac{1 \pm \sqrt{15}i}{2}$$

The eigenvalues are $\frac{1 \pm \sqrt{15}i}{2}$.

Example 3.3. Given matrix A below, which has eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, what is the corresponding eigenvalue? A= $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$

A. $\lambda = 2$	B. $\lambda = 3$	C. $\lambda = -1$	D. $\lambda = 1$

Solution:

$A\vec{v} = \lambda\vec{v}$

$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -2+0 \\ -1+3 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \therefore \lambda = 2$

The answer is A.

Example 3.4. a) Find the eigenvalues and eigenvectors for A=$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$.

Solution:

Find the eigenvalues:

$\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{bmatrix} = 0$$

$ad - bc = 0$

$(2-\lambda)(3-\lambda) = 0 \qquad \lambda = 2, 3$ eigenvalues

Find the eigenvectors:

Only use a short-cut method for checking an answer on rough work paper!!

Short-cut: $\lambda = 2 \quad eigenvector\ multiple\ of$

$$\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 0 \\ 2-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \therefore must\ use\ the\ other\ form$$

$$\begin{bmatrix} \lambda - d \\ c \end{bmatrix} = \begin{bmatrix} 2-3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \therefore non-zero\ multiples\ of\ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{or} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 3 \qquad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 0 \\ 3-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \therefore non-zero\ multiples\ of\ \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Long Method:

$\lambda = 2$ Find the eigenvector

$\begin{bmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{bmatrix}$ substitute $\lambda = 2$

$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ row reducing: $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Let $v_2 = t$

$v_1 + t = 0$

$\therefore \ v_1 = -t$

$\therefore$ eigenvector is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 3$

$\begin{bmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{bmatrix}$ substitute $\lambda = 3$

$\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$ R1 x -1 → R1

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$ R2-R1 → R2

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = t$

$x_1 = 0$

$\therefore$ eigenvector is $t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

b) Find $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}^{12}\begin{bmatrix} -2 \\ 6 \end{bmatrix}$

<u>Solution: From Theorem 4.19</u>

$\vec{\boldsymbol{x}} = \boldsymbol{c_1}\vec{\boldsymbol{v}}_{\boldsymbol{1}} + \boldsymbol{c_2}\boldsymbol{v}\vec{\boldsymbol{v}}_{\boldsymbol{2}} + \cdots + \boldsymbol{c_m}\vec{\boldsymbol{v}}_{\boldsymbol{m}}$ then, for any integer k,

$\boldsymbol{A^k}\vec{\boldsymbol{x}} = \boldsymbol{c_1}\boldsymbol{\lambda_1^k}\vec{\boldsymbol{v}}_{\boldsymbol{1}} + \boldsymbol{c_2}\boldsymbol{\lambda_2^k}\vec{\boldsymbol{v}}_{\boldsymbol{2}} + \cdots + \boldsymbol{c_m}\boldsymbol{\lambda_m^k}\vec{\boldsymbol{v}}_{\boldsymbol{m}}$

Let $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$. From part a) we know:

$\lambda_1 = 2 \quad \lambda_2 = 3$

$\vec{V}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \vec{V}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x} = C_1\overrightarrow{v_1} + C_2\overrightarrow{v_2}$

$\begin{bmatrix} -2 \\ 6 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$-2 = C_1$

$\therefore 6 = -(-2) + C_2$

$6 = 2 + C_2$

$C_2 = 4$

$\therefore\ \vec{x} = -2\overrightarrow{v_1} + 4\overrightarrow{v_2}$

$\therefore A^{12}\vec{x} = -2(2)^{12}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 4(3)^{12}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A^{12}\vec{x} = \begin{bmatrix} (-2)(2^{12}) \\ (2^{13}) + 4(3)^{12} \end{bmatrix}$

<u>Example 3.5.</u> a) Find the eigenvalues and eigenvectors for A= $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$.

<u>Solution:</u>

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 2-\lambda & -1 \\ 3 & 2-\lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(2-\lambda)(2-\lambda) + 3 = 0$

$4 - 4\lambda + \lambda^2 + 3 = 0$

$\lambda^2 - 4\lambda + 7 = 0$ characteristic polynomial

Quadratic formula: $\lambda^2 - 4\lambda + 7 = 0$

$a = 1,\quad b = -4,\quad c = 7 \qquad \lambda = \frac{4 \pm \sqrt{16-4(7)}}{2} \quad \therefore \lambda = \frac{4 \pm \sqrt{-12}}{2}$

$\lambda = \frac{4 \pm \sqrt{4}\sqrt{-3}}{2} = \frac{4 \pm 2\sqrt{3}i}{2}$

$\lambda = 2 + \sqrt{3}i,\ 2 - \sqrt{3}i$

Short-Cut: **<u>For rough work to check ONLY!!</u>**

eigenvector $\lambda = 2 + \sqrt{3}i \qquad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -1 \\ 2 + \sqrt{3}i - 2 \end{bmatrix} = \begin{bmatrix} -1 \\ \sqrt{3}i \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -\sqrt{3}i \end{bmatrix}$

$\lambda = 2 - \sqrt{3}i \quad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} -1 \\ 2 - \sqrt{3}i - 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -\sqrt{3}i \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3}i \end{bmatrix}$

Long-Method:

$\lambda = 2 + \sqrt{3}i$

$A\vec{v} = \lambda\vec{v}$

$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left(2 + \sqrt{3}i\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$2v_1 - v_2 = (2 + \sqrt{3}i)v_1$

$3v_1 - 2v_2 = (2 + \sqrt{3}i)v_2$

$2v_1 - v_2 = 2v_1 + \sqrt{3}i\, v_1$

$2v_1 - 2v_1 - \sqrt{3}iv_1 = v_2$

$v_2 = -\sqrt{3}iv_1$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -\sqrt{3}iv_1 \end{bmatrix}$

Let $v_1 = 1$

$v_2 = -\sqrt{3}i$

$\therefore \begin{bmatrix} 1 \\ -\sqrt{3}i \end{bmatrix}$ is the eigenvector for $\lambda = 2 + \sqrt{3}i$.

Vector $\vec{v} = \begin{bmatrix} -1 \\ \sqrt{3}i \end{bmatrix}$ is an eigenvector of matrix A, with eigenvalue $\lambda = 2 + \sqrt{3}i$. If $\vec{w} = \begin{bmatrix} i \\ \sqrt{3} \end{bmatrix}$, then what is the value of $A\vec{w}$?

$\begin{bmatrix} i \\ \sqrt{3} \end{bmatrix} \times i = \begin{bmatrix} i^2 \\ \sqrt{3}i \end{bmatrix} = \begin{bmatrix} -1 \\ \sqrt{3}i \end{bmatrix} = \vec{v}$

$\therefore\ \vec{v}\ and\ \vec{w}$ are equal

$A\vec{W} = A\vec{v} = \lambda\vec{v} = \left[2 + \sqrt{3}i\right]\begin{bmatrix} i \\ \sqrt{3} \end{bmatrix}$

$= \begin{bmatrix} 2i + \sqrt{3}i^2 \\ 2\sqrt{3} + 3i \end{bmatrix} = \begin{bmatrix} 2i - \sqrt{3} \\ 2\sqrt{3} + 3i \end{bmatrix}$

Example 3.6. Given A=$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, find:

a) the characteristic polynomial and the eigenvalues.

Solution:

$$\det(A - \lambda I) = 0$$

$$det\left(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0$$

$$det\begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix} = 0$$

$$\det\begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix} = 0$$

$$(-1)^{1+1}(2-\lambda)\det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} + (-1)^{2+1}(1)\det\begin{bmatrix} 1 & 0 \\ 1 & 2-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)[(2-\lambda)(2-\lambda)-1] + (-1)(2-\lambda-0) = 0$$

$$(2-\lambda)(4-2\lambda-2\lambda+\lambda^2-1)-2+\lambda = 0$$

$$(2-\lambda)(\lambda^2-4\lambda+3)-2+\lambda = 0$$

$$2\lambda^2-8\lambda+6-\lambda^3+4\lambda^2-3\lambda-2+\lambda = 0$$

$\therefore$ *the characteristic polynomial is*

$$-\lambda^3+6\lambda^2-10\lambda+4 = 0$$

Or $\lambda^3-6\lambda^2+10\lambda-4 = 0$

This is a cubic $\therefore$ use The Factor Theorem

$$f(\lambda) = \lambda^3-6\lambda^2+10\lambda-4$$

Try factors of the constant term $\pm1, \pm2, \pm4$

$$f(1) = 1-6+10-4 \neq 0$$

$$f(-1) = -1-6-10-4 \neq 0$$

$f(2) = 8 - 6(2)^2 + 10(2) - 4 = 8 - 24 + 20 - 4 = 0$

$\therefore$ $(\lambda - 2)$ is a factor

$$
\begin{array}{r|l}
 & \lambda^2 - 4\lambda + 2 \\
\hline
\lambda - 2 & \lambda^3 - 6\lambda^2 + 10\lambda - 4 \\
 & \underline{\lambda^3 - 2\lambda^2} \\
 & \quad -4\lambda^2 + 10\lambda \\
 & \quad \underline{-4\lambda^2 + 8\lambda} \\
 & \qquad 2\lambda - 4 \\
 & \qquad \underline{2\lambda - 4}
\end{array}
$$

OR

$\therefore$ $\lambda^3 - 6\lambda^2 + 10\lambda - 4 = 0$

$(\lambda - 2)(\lambda^2 - 4\lambda + 2) = 0 \qquad a = 1, b = -4, c = 2$

$\lambda = 2 \quad \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\lambda = \dfrac{4 \pm \sqrt{16 - 4(1)(2)}}{2(1)}$

$\lambda = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm \sqrt{4}\sqrt{2}}{2}$

$\lambda = \dfrac{4 \pm 2\sqrt{2}}{2}$

$\lambda = 2 \pm \sqrt{2}$

$\therefore$ the eigenvalues are $2, 2 + \sqrt{2}, 2 - \sqrt{2}$

b) Find the eigenvectors for $\lambda = 2 + \sqrt{2}\ and\ 2$.

Find the eigenvector for $\lambda = 2$

$$(A - \lambda I)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \text{R1} \leftrightarrow \text{R2} \left[\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \text{R3-R2} \rightarrow \text{R3}$$

t

$$\left[\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$$

Let $v_3 = t$

$v_2 = 0 \quad v_1 + t = 0 \ \therefore v_1 = -t$

$\therefore$ eigenvector is $t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Eigenvector for $\lambda = 2 + \sqrt{2}$

$$\left(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} - (2 + \sqrt{2})\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - 2 - \sqrt{2} & 1 & 0 \\ 1 & 2 - 2 - \sqrt{2} & 1 \\ 0 & 1 & 2 - 2 - \sqrt{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{matrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{matrix}\middle|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \text{R1} \leftrightarrow \text{R2}$$

$$\left[\begin{array}{ccc|c} 1 & -\sqrt{2} & 1 & 0 \\ -\sqrt{2} & 1 & 0 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \end{array}\right] \text{R2+} \sqrt{2}\ \text{R1} \rightarrow \text{R2} \quad 1+\sqrt{2}\left(-\sqrt{2}\right) = 1-2 = -1$$

$$\left[\begin{array}{ccc|c} 1 & -\sqrt{2} & 1 & 0 \\ 0 & -1 & \sqrt{2} & 0 \\ 0 & 1 & -\sqrt{2} & 0 \end{array}\right] \text{R2 x -1} \rightarrow \text{R2} \quad \text{R3+R2} \rightarrow \text{R3}$$

$$\left[\begin{array}{ccc|c} 1 & -\sqrt{2} & 1 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{R1 + } \sqrt{2}\ \text{R2} \rightarrow \text{R1} \quad 1+\sqrt{2}\left(-\sqrt{2}\right) = 1-2 = -1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 - t = 0$

$v_1 = t$

Let $v_3 = t$

$v_2 - \sqrt{2}t = 0$

$v_2 = \sqrt{2}t$

∴ eigenvector is $\begin{bmatrix} t \\ \sqrt{2}t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

Example 3.7.

Determine whether the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is independent, where $\vec{v}_1 = (1,2,0,8)$, $\vec{v}_2 = (1,0,8,8)$, $\vec{v}_3 = (1,2,1,-8)$ and $\vec{v}_4 = (-2,-4,0,-16)$.

Solution:

We'll use the test for linear independence.

Step 1: Write the vectors in the set as the columns of a matrix.

First, we'll construct a matrix whose first column is the vector $\vec{v}_1 = (1,2,0,8)$, whose second column is the vector $\vec{v}_2 = (1,0,8,8)$, whose third column is the vector $\vec{v}_3 = (1,2,1,-8)$, and whose fourth column is the vector $\vec{v}_4 = (-2,-4,0,-16)$.

This matrix looks like: $\begin{bmatrix} 1 & 1 & 1 & -2 \\ 2 & 0 & 2 & -4 \\ 0 & 8 & 1 & 0 \\ 8 & 8 & -8 & -16 \end{bmatrix}$

Step 2:

Row-reduce the matrix.

Now we row-reduce this matrix to find that it reduces to:

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The determinant of the resulting matrix is 0 (because there is a zero row). Therefore, the set is *dependent*. This should come as no surprise as it can be seen that $\vec{v}_4 = -2\vec{v}_1$ (but it won't always be this obvious!)

Example 3.8. Let's say we have $\lambda = -1$ and we have: $A - \lambda I = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$ *and we substitute and get:*

$$(A - \lambda I)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and row-reducing we get: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$ R2 – R1→ R2 R3 – R1→ R3

$$\begin{matrix} & s & t & \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = s \quad x_3 = t$

$x_1 + s + t = 0 \quad x_1 = -s - t$

$$\therefore \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -s - t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$E_{-1} = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$. So, the eigenspace of eigenvalue -1 has 2 eigenvectors!

3.9 Homework on Chapter 3

1. If $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$, find the characteristic polynomial, eigenvalues and the eigenvectors.

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 6-\lambda & 16 \\ -1 & -4-\lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(6-\lambda)(-4-\lambda) + 16 = 0$

$-24 - 6\lambda + 4\lambda + \lambda^2 + 16 = 0$

$\lambda^2 - 2\lambda - 8 = 0$

$(\lambda - 4)(\lambda + 2) = 0$

$\lambda = 4, -2$

$\lambda = 4$

$(A - \lambda I)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$\left(\begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 16 \\ -1 & -8 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 2 & 16 & 0 \\ -1 & -8 & 0 \end{array}\right]$ R1 ÷2→ R1 $\left[\begin{array}{cc|c} 1 & 8 & 0 \\ -1 & -8 & 0 \end{array}\right]$

R2+R1 → R2

$\left[\begin{array}{cc|c} 1 & 8 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = t \quad x_1 + 8t = 0 \; x_1 = -8t$

$\therefore t\begin{bmatrix} -8 \\ 1 \end{bmatrix}$

$\lambda = -2$

$(A - \lambda I)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left(\begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 8 & 16 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{matrix} 8 & 16 \\ -1 & -2 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$ R1 ÷8→ R1

$\left[\begin{matrix} 1 & 2 \\ -1 & -2 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$ R2+R1 → R2 $\left[\begin{matrix} 1 & 2 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$

$x_2 = t \quad x_1 + 2t = 0 \; x_1 = -2t$

$\therefore t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

2. If $A = \begin{bmatrix} -4 & 2 \\ 3 & -5 \end{bmatrix}$, find the characteristic polynomial, eigenvalues and the eigenvector corresponding to the largest eigenvalue.

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} -4 & 2 \\ 3 & -5 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} -4-\lambda & 2 \\ 3 & -5-\lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(-4-\lambda)(-5-\lambda) - 6 = 0$

$20 + 4\lambda + 5\lambda + \lambda^2 - 6 = 0$

$\lambda^2 + 9\lambda + 14 = 0$

$(\lambda + 7)(\lambda + 2) = 0$

$\lambda = -7, -2$

Largest is $\lambda = -2$

$$(A - \lambda I)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\left(\begin{bmatrix} -4 & 2 \\ 3 & -5 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{matrix} -2 & 2 \\ 3 & -3 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$ R1÷2→ R1

$\left[\begin{matrix} 1 & -1 \\ 3 & -3 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$ R2 - 3R1 → R2

$\left[\begin{matrix} 1 & -1 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$

$x_2 = t \quad x_1 - t = 0 \; x_1 = t$

$\therefore t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, find the eigenvector associated with each eigenvalue of A and the eigenspace.

$\det(A - \lambda I) = 0 \quad \therefore \det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} = 0$ R3 – R2 → R3

$\det\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda - 1 \end{bmatrix} = 0$

$(-1)^{1+1}(-\lambda)\det\begin{bmatrix} -\lambda & 1 \\ 1+\lambda & -\lambda - 1 \end{bmatrix} + (-1)^{2+1}(1)\det\begin{bmatrix} 1 & 1 \\ 1+\lambda & -\lambda - 1 \end{bmatrix} = 0$

$-\lambda\,[(\lambda^2 + \lambda) - (1 + \lambda)] + (-1)[1(-\lambda - 1) - 1(1 + \lambda)] = 0$

$-\lambda\,(\lambda^2 + \lambda - 1 - \lambda) + (-1)(-\lambda - 1 - 1 - \lambda) = 0$

$-\lambda\,(\lambda^2 - 1) - 1(-2\lambda - 2) = 0$

$-\lambda^3 + \lambda + 2\lambda + 2 = 0$

$-\lambda^3 + 3\lambda + 2 = 0$

$\lambda^3 - 3\lambda - 2 = 0$

<u>Factor Theorem</u>

$f(\lambda) = \lambda^3 + 3\lambda + 2$ try factors of -2 $(\pm 1, \pm 2)$

$f(1) = 1 - 3 - 2 \neq 0$

$f(-1) = -1 + 3 - 2 = 0$

$\therefore$ $(\lambda + 1)$ is a factor

Using Long division, we get:

$$
\begin{array}{r|l}
 & \lambda^2 - \lambda - 2 \\
\lambda + 1 & \lambda^3 + 0\lambda^2 - 3\lambda - 2 \\
 & \lambda^3 + \lambda^2 \\
\hline
 & -\lambda^2 - 3\lambda \\
 & -\lambda^2 - \lambda \\
\hline
 & -2\lambda - 2 \\
 & -2\lambda - 2 \\
\hline
 & 0
\end{array}
$$

$\lambda^3 - 3\lambda - 2 = 0$

$(\lambda + 1)(\lambda^2 - \lambda - 2) = 0$

$(\lambda + 1)(\lambda - 2)(\lambda + 1) = 0$

$\lambda = -1$ (algebraic multiplicity 2 and $\lambda = 2$.

<u>**Synthetic Division:**</u>

$let\ f(\lambda) = \lambda^3 - 3\lambda - 2$ $\quad f(-1) = -1 + 3 - 2 = 0$

$(\lambda + 1) is\ a\ factor$ $\quad -1$

1	0	−3	−2	
↓	−1	1	2	
1	-1	-2	0	R

$\therefore (\lambda + 1)(\lambda^2 - \lambda - 2) = 0$

$(\lambda + 1)(\lambda - 2)(\lambda + 1) = 0 \quad \lambda = -1, -1, 2$

Find Eigenvectors

$\lambda = 2$

$$(A - \lambda I)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$ R1 ←→ R2

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$ R3-R1 → R3 R2+2R1→ R2

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$ R2÷-3→ R2 R3+R2→ R3

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ R1 +2R2 → R1

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t \quad x_1 - t = 0 \; x_1 = t$

$x_2 - t = 0 \quad x_2 = t$

$\therefore$ eigenvector is $\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$

$\lambda = -1$

$$\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\1 & 1 & 1 & 0\\1 & 1 & 1 & 0\end{array}\right]$$ R2 – R1→ R2 R3 – R1→ R3

$$\begin{array}{ccc} & s & t\end{array}$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

$x_2 = s \quad x_3 = t$

$x_1 + s + t = 0 \quad x_1 = -s - t$

$$\therefore \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}-s-t\\s\\t\end{bmatrix} = s\begin{bmatrix}-1\\1\\0\end{bmatrix} + t\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$E_{-1} = \text{span}\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right)$$

4. If $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$, find the eigenvector associated with the largest eigenvalue of A and the eigenspace.

$$A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$$

Only use a short-cut method for checking an answer on rough work paper!!

Using the short-cut method:

$$tr(A) = 7 + 3 = 10$$

$$detA = ad - bc$$

$$= 7(3) - (-1)(4)$$

$$= 21 + 4 \quad = 25$$

$$\lambda^2 - tr(A)\lambda + detA = 0$$

$$\lambda^2 - 10\lambda + 25 = 0$$

$$(\lambda - 5)(\lambda - 5) = 0 \qquad \lambda = 5,\ 5$$

$Find\ the\ eigenvector(s)$:

$$\left(\begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 4 & -2 & 0 \end{array}\right]$ R1÷2→ R1

$\left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 4 & -2 & 0 \end{array}\right]$ R2 - 4R1 → R2

$\left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_2 = t \quad x_1 - \frac{1}{2}t = 0\ x_1 = \frac{1}{2}t$

$$\therefore t\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

5. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$. Find:

(a) the characteristic polynomials of *A*.

(b) the eigenvalues of *A*.

(c) the corresponding eigenvectors and the eigenspace for each

a) $(-1)^{1+1}(1-\lambda)\det\begin{bmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} + (-1)^{2+1}(-1)\det\begin{bmatrix} -1 & 0 \\ -1 & 1-\lambda \end{bmatrix} = 0$

$(1-\lambda)[(2-\lambda)(1-\lambda)-1] + 1\,(-1+\lambda-0) = 0$

$(1-\lambda)(2-3\lambda+\lambda^2-1) - 1 + \lambda = 0$

$(1-\lambda)(1-3\lambda+\lambda^2) - 1 + \lambda = 0$

$1 - 3\lambda + \lambda^2 - \lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda = 0$

$-\lambda^3 + 4\lambda^2 - 3\lambda = 0$

Or $\lambda^3 - 4\lambda^2 + 3\lambda = 0$ (Characteristic polynomial).

$\lambda\,(\lambda^2 - 4\lambda + 3) = 0$

$\lambda\,(\lambda-1)(\lambda-3) = 0$

b) $\lambda = 0, 1, 3$ (eigenvalues)

c) $\lambda = 0$

$(A - \lambda I)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left(\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - 0\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$ R2+R1 → R2

$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$ R3 + R2 → R1

$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ R1 + R2 → R1

$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_3 = t$

$x_1 - t = 0 \quad x_1 = t$

$x_2 - t = 0 \quad x_2 = t$

$\therefore t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$E_0 = \text{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 1$

$\left(\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$ R1 ←→ R2

$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$ R1 x -1 → R1 R3 – R2 → R3

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ R2 x -1 → R2

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ R1 + R2 → R1 $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_3 = t$

$x_1 + t = 0 \;\; \therefore \;\; x_1 = -t$

$x_2 = 0$

$$\begin{bmatrix}-t\\0\\t\end{bmatrix}=t\begin{bmatrix}-1\\0\\1\end{bmatrix} \qquad E_1=\text{span}\left(\begin{bmatrix}-1\\0\\1\end{bmatrix}\right)$$

$\lambda = 3$

$$\left(\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}-3\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-2&-1&0\\-1&-1&-1\\0&-1&-2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}-2&-1&0&0\\-1&-1&-1&0\\0&-1&-2&0\end{array}\right]$$ R1 ←→ R2

$$\left[\begin{array}{ccc|c}-1&-1&-1&0\\-2&-1&0&0\\0&-1&-2&0\end{array}\right]$$ R1 x -1 → R1

$$\left[\begin{array}{ccc|c}1&1&1&0\\-2&-1&0&0\\0&-1&-2&0\end{array}\right]$$ R2 + 2R1 → R2

$$\left[\begin{array}{ccc|c}1&1&1&0\\0&1&2&0\\0&-1&-2&0\end{array}\right]$$ R3 + R2 → R3

$$\left[\begin{array}{ccc|c}1&1&1&0\\0&1&2&0\\0&0&0&0\end{array}\right]$$ R1 – R2 → R1

$$\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&2&0\\0&0&0&0\end{array}\right]$$

$x_3 = t$

$x_1 - t = 0 \quad \therefore \quad x_1 = t$

$x_2 + 2t = 0 \quad \therefore \; x_2 = -2t$

$$\begin{bmatrix}t\\-2t\\t\end{bmatrix}=t\begin{bmatrix}1\\-2\\1\end{bmatrix} \qquad E_3=\text{span}\left(\begin{bmatrix}1\\-2\\1\end{bmatrix}\right)$$

6. Given the matrix, find:

$$A = \begin{bmatrix} 1 & -2 & -2 \\ 4 & -5 & -2 \\ 8 & -4 & 5 \end{bmatrix}$$

(a) the characteristic polynomials of A.

(b) the eigenvalues of A.

(c) the corresponding eigenvector for the largest eigenvalue.

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 1 & -2 & -2 \\ 4 & -5 & -2 \\ 8 & -4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & -2 & -2 \\ 4 & -5-\lambda & -2 \\ 8 & -4 & 5-\lambda \end{bmatrix} = 0$$

$$(-1)^{1+1}(1-\lambda)\det\begin{bmatrix} -5-\lambda & -2 \\ -4 & 5-\lambda \end{bmatrix} + (-1)^{2+1}(4)\det\begin{bmatrix} -2 & -2 \\ -4 & 5-\lambda \end{bmatrix} +$$
$$(-1)^{3+1}(8)\det\begin{bmatrix} -2 & -2 \\ -5-\lambda & -2 \end{bmatrix} = 0$$

$(1-\lambda)[(-5-\lambda)(5-\lambda) - 8] + (-4)[-2(5-\lambda) - 8] + 8\,[4 + 2(-5-\lambda)] = 0$

$(1-\lambda)\,(-25 - 5\lambda + 5\lambda + \lambda^2 - 8) + (-4)(-10 + 2\lambda - 8) + 8(4 - 10 - 2\lambda) = 0$

$(1-\lambda)(-25 + \lambda^2 - 8) + 40 - 8\lambda + 32 + 32 - 80 - 16\lambda = 0$

$-25 + \lambda^2 - 8 + 25\lambda - \lambda^3 + 8\lambda + 40 - 8\lambda + 32 + 32 - 80 - 16\lambda = 0$

$-\lambda^3 + \lambda^2 + 9\lambda - 9 = 0$

$\lambda^3 - \lambda^2 - 9\lambda + 9 = 0$ factor by grouping

$\lambda^2(\lambda - 1) - 9\,(\lambda - 1) = 0$

$(\lambda^2 - 9)(\lambda - 1) = 0$

$(\lambda - 3)(\lambda + 3)(\lambda - 1) = 0$

$\lambda = 3, -3, 1$

$\lambda = 3$

$$(A - \lambda I)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix}1 & -2 & -2\\4 & -5 & -2\\8 & -4 & 5\end{bmatrix} - 3\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\left[\begin{array}{ccc|c}-2 & -2 & -2 & 0\\4 & -8 & -2 & 0\\8 & -4 & 2 & 0\end{array}\right]$ R1÷2 → R2 R2+2R1 → R2

$\left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\0 & -12 & -6 & 0\\8 & -4 & 2 & 0\end{array}\right]$ R2 ÷ -12 → R2

$\left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\0 & 1 & 1/2 & 0\\8 & -4 & 2 & 0\end{array}\right]$ R3 – 8R1 → R3

$\left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\0 & 1 & 1/2 & 0\\0 & -12 & -6 & 0\end{array}\right]$ R3 + 12 R2 → R3

$\left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\0 & 1 & 1/2 & 0\\0 & 0 & 0 & 0\end{array}\right]$ R1 – R2 → R1

$\left[\begin{array}{ccc|c}1 & 0 & 1/2 & 0\\0 & 1 & 1/2 & 0\\0 & 0 & 0 & 0\end{array}\right]$

$x_3 = t$

$x_1 + \frac{1}{2}t = 0 \quad \therefore \; x_1 = -\frac{1}{2}t$

$x_2 + \frac{1}{2}t = 0 \quad \therefore \; x_2 = -\frac{1}{2}t$

$\begin{bmatrix}-\frac{1}{2}t\\-\frac{1}{2}t\\t\end{bmatrix} = t\begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}$ Or $\left(\begin{bmatrix}1\\1\\-2\end{bmatrix}\right)$

It didn't ask for the other eigenvectors, but here they are solved algebraically. You get the same answer if you use matrices!

$\lambda = 1$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 1 & -2 & -2 \\ 4 & -5 & -2 \\ 8 & -4 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 1\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$v_1 - 2v_2 - 2v_3 = v_1$ $\boxed{1}$ $\quad -2v_2 = 2v_3 \quad v_2 = -v_3$

$4v_1 - 5v_2 - 2v_3 = v_2$ $\boxed{2}$

$8v_1 - 4v_2 + 5v_3 = v_3$ $\boxed{3}$

$8v_1 - 4v_2 + 4v_3 = 0$

$8v_1 - 4(-v_3) + 4v_3 = 0$

$8v_1 + 4v_3 + v_3 = 0$

$8v_1 + 8v_3 = 0$

$8v_1 = -8v_3 \qquad v_1 = -v_3$

$\therefore let\ v_3 = 1 \quad \rightarrow v_2 = -1 \quad v_1 = -1$

$vector \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} for\ \lambda = 1$

$check \quad A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 1 & -2 & -2 \\ 4 & -5 & -2 \\ 8 & -4 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad and\ \lambda\vec{v} = 1\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$let\ \lambda = -3$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 1 & -2 & -2 \\ 4 & -5 & -2 \\ 8 & -4 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = -3\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$v_1 - 2v_2 - 2v_3 = -3v_1$ $\boxed{1}$

$4v_1 - 5v_2 - 2v_3 = -3v_2$ $\boxed{2}$

$8v_1 - 4v_2 + 5v_3 = -3v_3$ $\boxed{3}$

$from \boxed{1} \quad -2v_2 - 2v_3 = -4v_1 \qquad v_2 + v_3 = 2v_1$

$from \boxed{2} \quad 4v_1 - 2v_3 = 2v_2 \qquad 2v_1 - v_3 = v_2 \quad same\ as \boxed{1}$

$from \boxed{3} \quad 8v_1 - 4v_2 = -8v_3 \qquad 2v_1 - v_2 = -2v_3$

$$2v_1 = -2v_3 + v_2 \quad sub\ into \boxed{1}$$

$\therefore v_2 + v_3 = -2v_3 + v_2$

$3v_3 = 0 \qquad v_3 = 0$

$$sub\ v_3 = 0 \quad into \boxed{2} \quad 2v_1 - 0 = v_2$$

$$v_2 = 2v_1$$

$$v = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

7. An eigenvalue of the matrix $\begin{bmatrix} 2 & 1 & -2 \\ -3 & 0 & 4 \\ -2 & -1 & 4 \end{bmatrix}$ is 2. What is the eigenvector that corresponds to this eigenvalue?

A. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	B. $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$	C. $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$	D. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$	E. none of these are correct

$$\left(\begin{bmatrix} 2 & 1 & -2 \\ -3 & 0 & 4 \\ -2 & -1 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -2 \\ -3 & -2 & 4 \\ -2 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\left[\begin{array}{ccc|c} 0 & 1 & -2 & 0 \\ -3 & -2 & 4 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right]$ R1←→ R3

$\left[\begin{array}{ccc|c} -2 & -1 & 2 & 0 \\ -3 & -2 & 4 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$ R1 ÷-2 → R1

$\left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 0 \\ -3 & -2 & 4 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$ R2 + 3R1 → R2 $\quad -2+\frac{3}{2} \quad -\frac{4}{2}+\frac{3}{2} = -\frac{1}{2}$

$\left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 0 \\ 0 & -1/2 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$ R1 x -2 → R1

$\left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$ R3 – R2 → R3

$\left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ R1 – ½ R2 → R1

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 0$

$x_3 = t$

$x_2 - 2t = 0 \quad \therefore\ x_2 = 2t$

$\therefore \begin{bmatrix} 0 \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$\therefore B)\ is\ the\ solution$

NOTE: [0 0 0] is NEVER an eigenvector

8. The matrix A=$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ has which of the following as an eigenvalue?

A. 3	B. 0	C. -1	D. i	E. both A and B

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 2-\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(2-\lambda)(1-\lambda) - 2 = 0$

$2 - 2\lambda - \lambda + \lambda^2 - 2 = 0$

$\lambda^2 - 3\lambda = 0$

$\lambda(\lambda - 3) = 0$

$\lambda = 0, 3$

The answer is E).

9. a) Given A=$\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$, *which of the following are the eigenvalues for A.*

A. 1	B. -1	C. -3	D. 3	E. both A and D

It is in lower triangular form, so 1 and 3 are the eigenvalues.

$\therefore$ *E is the solution*

b) Find $\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}^{10}\begin{bmatrix}2\\ 1\end{bmatrix}$

$A = \begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$ $\vec{x} = \begin{bmatrix}3\\ 1\end{bmatrix}$

$\lambda = 1, 3$

$\lambda = 1$ Find the eigenvector

$\left(\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix} - 1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$\left[\begin{matrix}0 & 0\\ 2 & 2\end{matrix}\middle|\begin{matrix}0\\ 0\end{matrix}\right]$ R1 ←→ R2

$\left[\begin{matrix}2 & 2\\ 0 & 0\end{matrix}\middle|\begin{matrix}0\\ 0\end{matrix}\right]$ R1 ÷ R2 → R1

$\left[\begin{matrix}1 & 1\\ 0 & 0\end{matrix}\middle|\begin{matrix}0\\ 0\end{matrix}\right]$

$x_2 = t$

$x_1 + t = 0 \qquad x_1 = -t$

$\therefore\ \lambda = 1$ has eigenvector $\overrightarrow{x_1} = \begin{bmatrix}1\\ -1\end{bmatrix}$

$\lambda = 3$ Find the eigenvector

$\left(\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix} - 3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$\left[\begin{matrix}-2 & 0\\ 2 & 0\end{matrix}\middle|\begin{matrix}0\\ 0\end{matrix}\right]$ R1 ÷ -2 → R1

$\left[\begin{matrix}1 & 0\\ 2 & 0\end{matrix}\middle|\begin{matrix}0\\ 0\end{matrix}\right]$ R2 -2R1 → R2

$\left[\begin{matrix}1 & 0\\ 0 & 0\end{matrix}\middle|\begin{matrix}0\\ 0\end{matrix}\right]$

$x_2 = t \qquad x_1 = 0$

$\therefore\ \lambda = 3$ has eigenvector $\overrightarrow{x_2} = \begin{bmatrix}0\\ 1\end{bmatrix}$

$A\overrightarrow{v_1} = 1\overrightarrow{v_1}\ and\ A\overrightarrow{v_2} = 3\overrightarrow{v_2}$

Since $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ forms a basis for R^2, we can write $\vec{x}$ as a linear combination of $\overrightarrow{v_1}\ and\ \overrightarrow{v_2}$

$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x} = 2\overrightarrow{v_1} + 3\overrightarrow{v_2}\ \therefore\ C_1 = 2, C_2 = 3$

$\therefore A^k\vec{x} = C_1\lambda_1^k\overrightarrow{v_1} + C_2\lambda_2^k\overrightarrow{v_2}$

$\therefore A^{10}\vec{x} = 2(1)^{10}\overrightarrow{v_1} + 3(3)^{10}\overrightarrow{v_2}$

$= 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3^{11}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -2 + 3^{11} \end{bmatrix}$

10. Given matrices A, B and C, match the eigenvectors associated with eigenvalue 2 to each matrix.

$A = \begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix}$ B=$\begin{bmatrix} 6 & 3 \\ 0 & 2 \end{bmatrix}$ and C=$\begin{bmatrix} 4 & 2 \\ -2 & 0 \end{bmatrix}$

Here, I will use the short cut method for finding the eigenvectors!!

Only use a short-cut method for checking an answer on rough work paper!!

Vectors V1=$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ V2=$\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ V3=$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and V4=$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

A. $\det\left(\begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 2-\lambda & 3 \\ 0 & 6-\lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(2-\lambda)(6-\lambda) - 0 = 0$

$12 - 8\lambda + \lambda^2 = 0$

$\lambda^2 - 8\lambda + 12 = 0$

$(\lambda - 2)(\lambda - 6) = 0$

$\lambda = 2,6$

$eigenvector \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 3 \\ 2-2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ or any multiple

$\therefore v_1$

B. $\det\left(\begin{bmatrix} 6 & 3 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 6-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(6-\lambda)(2-\lambda) = 0$

$12 - 6\lambda - 2\lambda + \lambda^2 = 0$

$\lambda^2 - 8\lambda + 12 = 0$

$(\lambda - 2)(\lambda - 6) = 0$

$\lambda = 2,6\ eigenvector \quad \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 3 \\ 2-6 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$\therefore v_2$

$C.\ \det\left(\begin{bmatrix} 4 & 2 \\ -2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 4-\lambda & 2 \\ -2 & -\lambda \end{bmatrix} = 0$

$(4-\lambda)(-\lambda) + 4 = 0$

$-4\lambda + \lambda^2 + 4 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)(\lambda - 2) = 0$

$\lambda = 2,2$

$\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ 2-4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ *or any multiple*

$\therefore v_3$

Only use a short-cut method for checking an answer on rough work paper!!

11. The eigenvalues of A= $\begin{bmatrix} 0 & -1 \\ 0 & 4 \end{bmatrix}$ are:

A. 4, 0	B. 0, -4	C. -1, 4	D.-4, 1	E. undefined, since detA=0

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 0 & -1 \\ 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} -\lambda & -1 \\ 0 & 4-\lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(-\lambda)(4-\lambda) + 0 = 0$

$-4\lambda + \lambda^2 = 0$

$\lambda^2 - 4\lambda = 0$

$\lambda(\lambda - 4) = 0$

$\lambda = 0{,}4$

$\therefore A)\ is\ the\ answer$

NOTE: It is in upper triangular form, so the eigenvalues are just the numbers along the main diagonal.

12. a) Find the characteristic polynomial of A and the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$.

b) Find the eigenvector for each eigenvalue found in part a)

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 1-\lambda & 2 \\ -1 & 1-\lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(1-\lambda)(1-\lambda) + 2 = 0$

$1 - 2\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 2\lambda + 3 = 0$

Won't factor, so we use the quadratic formula:

$\lambda^2 - 2\lambda + 3 = 0 \qquad a = 1 \quad b = -2 \quad c = 3$

$$\lambda = -\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} = \frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm \sqrt{4}\sqrt{2}i}{2}$$

$\lambda = \frac{2+2\sqrt{2}i}{2}, \frac{2-2\sqrt{2}i}{2} \quad or \quad 1 + \sqrt{2}\,i, 1 - \sqrt{2}i$

$\lambda = 1 + \sqrt{2}i$

$(A - \lambda I)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left(\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} - (1 + \sqrt{2}i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 - (1 + \sqrt{2}i) & 2 \\ -1 & 1 - (1 + \sqrt{2}i) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -\sqrt{2}i & 2 \\ -1 & -\sqrt{2}i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} -\sqrt{2}i & 2 & 0 \\ -1 & -\sqrt{2}i & 0 \end{array}\right]$ R1 ←→ R2

$\left[\begin{array}{cc|c} -1 & -\sqrt{2}i & 0 \\ -\sqrt{2}i & 2 & 0 \end{array}\right]$ R1 x -1 → R1

$\left[\begin{array}{cc|c} 1 & \sqrt{2}i & 0 \\ -\sqrt{2}i & 2 & 0 \end{array}\right]$ R2 + $(-\sqrt{2}i)$R1 → R2

$2 + (-\sqrt{2}i)(-\sqrt{2}i)$

$= 1 + 1i^2$

$= 2 + 2\,(-1) = 0$

$\left[\begin{array}{cc|c} 1 & \sqrt{2}i & 0 \\ 0 & 2 & 0 \end{array}\right]$

$x_2 = t$

$x_1 + (\sqrt{2}i)t = 0$

$x_1 = -\sqrt{2}i\,t$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}i\,t \\ t \end{bmatrix} = t\begin{bmatrix} -\sqrt{2}i \\ 1 \end{bmatrix}$

$\lambda = 1 - \sqrt{2}i$

$\left(\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} - (1-\sqrt{2}i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1-(1-\sqrt{2}i) & 2 \\ -1 & 1-(1-\sqrt{2}i) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{matrix} \sqrt{2}i & 2 \\ -1 & \sqrt{2}i \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$ R1 ←→ R2

$\left[\begin{matrix} -1 & \sqrt{2}i \\ \sqrt{2}i & 2 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$ R1 x -1 → R1

$\left[\begin{matrix} 1 & -\sqrt{2}i \\ \sqrt{2}i & 2 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$ R2 $-(\sqrt{2}i)$R1 → R2

$\sqrt{2}i - \sqrt{2}i = 0$

$2 - (\sqrt{2}i)(-\sqrt{2}i)$

$= 2 + 2i^2$

$= 2 + 2\,(-1)$

$= 0$

$\left[\begin{matrix} 1 & -\sqrt{2}i \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$

$x_2 = t$

$x_1 - \sqrt{2}i\,t = 0 \; x_1 = \sqrt{2}i\,t$

$\begin{bmatrix} \sqrt{2}i\,t \\ t \end{bmatrix} = t\begin{bmatrix} \sqrt{2}i \\ 1 \end{bmatrix}$

b) **Only use a short-cut method for checking an answer on rough work paper!!**

Using the short-cut method:

Eigenvector $\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ 1+\sqrt{2}i-1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2}i \end{bmatrix}$

$\lambda = 1 - \sqrt{2}i$

Eigenvector$=\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \begin{bmatrix} 2 \\ 1-\sqrt{2}i-1 \end{bmatrix} = \begin{bmatrix} 2 \\ -\sqrt{2}i \end{bmatrix}$

13.a) This matrix is in upper triangular form. As a result, the numbers on the main diagonal are always the eigenvalues. (for any matrix in triangular form)

$$\therefore \lambda = 3,6,2$$

b) Find the eigenvector for the smallest eigenvalue.

$\lambda = 2$

$$\left(\begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 4 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{R2} \div 4 \rightarrow \text{R2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 5/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{R1} - 2\text{R2} \rightarrow \text{R1} \qquad 3 - 2\left(\frac{5}{2}\right) = 3 - 5 = -2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 5/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$

$x_1 - 2t = 0 \;\therefore\; x_1 = 2t$

$x_2 + \frac{5}{2}t = 0 \qquad x_2 = -\frac{5}{2}t$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ -\frac{5}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -5/2 \\ 1 \end{bmatrix} \times 2 \text{ or } t\begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$$

14. Show $\vec{v} = \begin{bmatrix} 16 \\ -2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$

$A\vec{v} = \lambda\vec{v}$

$A\vec{v} = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 16 \\ -2 \end{bmatrix} = \begin{bmatrix} 96-32 \\ -16+8 \end{bmatrix} = \begin{bmatrix} 64 \\ -8 \end{bmatrix}$ this is a multiple of $\vec{v}$

$\begin{bmatrix} 64 \\ -8 \end{bmatrix} = \lambda\begin{bmatrix} 16 \\ -2 \end{bmatrix}$

$\therefore \lambda = 4$

∴ the eigenvalue is 4

15.

$$\det\left(\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} \overbrace{-2-\lambda}^{1,1} & 0 & 0 & 0 \\ 0 & 4-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 3 & 4-\lambda \end{bmatrix} = 0$$

$$(-1)^{1+1}(-2-\lambda)\det\begin{bmatrix} \overbrace{4-\lambda}^{1,1} & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 3 & 4-\lambda \end{bmatrix} = 0$$

$$(-2-\lambda)((-1)^{1+1}(4-\lambda)\det\begin{bmatrix} 3-\lambda & 0 \\ 3 & 4-\lambda \end{bmatrix} = 0$$

$$(-2-\lambda)(4-\lambda)[(3-\lambda)(4-\lambda)] = 0$$

$$\therefore \lambda = -2, 4, 3, 4$$

Find e-vector for $\lambda = 4$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 4\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$\boxed{1} -2v_1 = 4v_1 \quad \therefore 6v_1 = 0 \;\therefore v_1 = 0$$

$$\boxed{2}\, 4v_2 = 4v_2 \leftarrow \text{doesn't mean } v_2 = 0$$

$$\boxed{3}\, 3v_3 = 3v_3 \quad \therefore v_3 = 0$$

$$\boxed{4}\, 3v_3 + 4v_4 = 4v_4 \quad \therefore v_3 = 0$$

$$\therefore \text{eigenvector} = \begin{bmatrix} 0 \\ v_2 \\ 0 \\ v_4 \end{bmatrix} = v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

↑ ↑

2 e-vectors

Find e-vector for $\lambda = -2$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = -2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$\boxed{1} -2v_1 = -2v_1 \quad \leftarrow \text{doesn't mean } v_1 = 0$$

$$\boxed{2}\, 4v_2 = -2v_2 \;\therefore 6v_2 = 0 \;\therefore v_2 = 0$$

$$\boxed{3}\, 3v_3 = -2v_3 \quad \therefore 5v_3 = 0 \;\therefore v_3 = 0$$

$$\boxed{4}\, 3v_3 + 4v_4 = -2v_4 \quad \therefore 3v_3 = -6v_4$$

$$v_3 = -2v_4$$

But $v_3 = 0 \;\therefore v_4 = 0$

$$\therefore \begin{bmatrix} v_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

↑

e-vector for $\lambda = -2$

16. A= $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$,

Only use a short-cut method for checking an answer on rough work paper!!

Short-cut Method: $tr(A) = 8$

$\det A = ad - bc = 5(3) - (-2)(1)$

$= 17$

$\lambda^2 - tr(A)\lambda + \det A = 0$

$\lambda^2 - 8\lambda + 17 = 0$

Doesn't factor

$\lambda^2 - 8\lambda + 16 = -17 + 16$

$(\lambda - 4)^2 = -1$

$\lambda - 4 = \pm\sqrt{-1}$

$\lambda - 4 = i, \quad \lambda - 4 = -i$

$\lambda = 4 + i, 4 - i$

Long Method: Find the eigenvector:

$\lambda = 4 + i$

$A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (4 + i)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$5v_1 - 2v_2 = 4v_1 + iv_1 \quad \boxed{1}$

$v_1 - iv_1 = 2v_2$

$v_1(1 - i) = 2v_2$

$$v_2 = \frac{v_1(1-i)}{2}$$

Let $v_1 = 2 \;\; v_2 = 1 - i$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$$

Note: it could also be

$$\begin{bmatrix} 2 \\ 1-i \end{bmatrix} \begin{matrix} \times 1+i \\ \times 1+i \end{matrix}$$

$$\begin{bmatrix} 2+2i \\ 1+i-i-i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2i \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2+2i \\ 2 \end{bmatrix}$$

$$= 2\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

$$(A - \lambda I)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda = 4 + i$$

$$\begin{bmatrix} 5-(4+i) & -2 \\ 1 & 3-(4+i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-i)v_1 - 2v_2 = 0$$

$$(1-i)v_1 = 2v_2$$

$$v_2 = \frac{(1-i)}{2}v_1$$

Let $v_1 = 2$

$$v_2 = 1 - i$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$$

Note: it could also be

$$\begin{bmatrix} 2 \\ 1-i \end{bmatrix} \begin{matrix} \times 1+i \\ \times 1+i \end{matrix}$$

$$\begin{bmatrix} 2+2i \\ 1+i-i-i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2i \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2+2i \\ 2 \end{bmatrix}$$

$$= 2\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

17. $A = \begin{bmatrix} 5 & -4 \\ 1 & 3 \end{bmatrix}$ Find eigenvalues and the e-vector for $a - bi$

<u>Long method</u>

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 5 & -4 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 5-\lambda & -4 \\ 1 & 3-\lambda \end{bmatrix} = 0$$

$ad - bc = 0$

$(5-\lambda)(3-\lambda) + 4 = 0$

$15 - 5\lambda - 3\lambda + \lambda^2 + 4 = 0$

$\lambda^2 - 8\lambda + 19 = 0$

$\lambda^2 - 8\lambda = -19$

$\lambda^2 - 8\lambda + 16 = -19 + 16$

$(\lambda - 4)^2 = -3$

$\lambda - 4 = \pm\sqrt{-3}$

$\lambda = 4 \pm \sqrt{-3}$

$(A - \lambda I)\vec{v} = 0$

$$\begin{bmatrix} 5-\lambda & -4 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = 4 + \sqrt{-3}$

$$\begin{bmatrix} 5-\left(4+\sqrt{-3}\right) & -4 \\ 1 & 3-\left(4+\sqrt{-3}\right) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\sqrt{-3}i & -4 \\ 1 & -1-\sqrt{-3}i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_1 + \left(-1 - \sqrt{3}i\right)v_2 = 0$

$v_1 = -\left(-1 - \sqrt{3}i\right)v_2$

$v_1 = \left(1 + \sqrt{3}i\right)v_2$

Let $v_2 = 1$

$\therefore \begin{bmatrix} 1+\sqrt{3}i \\ 1 \end{bmatrix}$ is the eigenvector

18. $A = \begin{bmatrix} \overbrace{2}^{a} & \overbrace{-4}^{b} \\ \underbrace{5}_{c} & \underbrace{5}_{d} \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 2 & -4 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} 2-\lambda & -4 \\ 5 & 5-\lambda \end{bmatrix} = 0$

$ad - bc = 0$

$(2-\lambda)(5-\lambda) + 20 = 0$

$10 - 7\lambda + \lambda^2 + 20 = 0$

$\lambda^2 - 7\lambda + 30 = 0$

$\lambda = \dfrac{7 \pm \sqrt{49 - 4(1)(30)}}{2(1)}$

$\lambda = \dfrac{7 \pm \sqrt{71}i}{2}$

$\lambda = \dfrac{7 + \sqrt{71}i}{2}, \lambda = \dfrac{7 - \sqrt{71}i}{2}$

$(A - \lambda I)\vec{v} = \vec{0}$

$\left(\begin{bmatrix} 2 & -4 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 - \lambda & -4 \\ 5 & 5 - \lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = \dfrac{7 + \sqrt{71}i}{2}$

$$\begin{bmatrix} 2 - \left(\dfrac{7 + \sqrt{71}i}{2}\right) & -4 \\ 5 & 5 - \left(\dfrac{7 + \sqrt{71}i}{2}\right) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{-3 - \sqrt{71}i}{2} & -4 \\ 5 & \dfrac{3 - \sqrt{71}i}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$5v_1 + \dfrac{3 - \sqrt{71}i}{2}v_2 = 0$

$5v_1 = -\dfrac{\left(3 - \sqrt{71}i\right)}{2}v_2$

$v_1 = \dfrac{-3 + \sqrt{71}i}{2}v_2$

Let $v_2 = 2, v_1 = -3 + \sqrt{71}i$

$\therefore$ the eigenvector is $\begin{bmatrix} -3 + \sqrt{71}i \\ 2 \end{bmatrix}$

4. Similarity and Diagonalization

Example 4.1. Are matrices $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ similar?

Since $\det A = 1 - 9 = -8$

$\det B = 9 - 1 = 8$

$\det A \neq \det B$

∴ they are not similar

Example 4.2. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a diagonal matrix

∴ eigenvalues are 1, 2, 5

A^{-1} would have eigenvalues

$\frac{1}{1}, \frac{1}{2}, \frac{1}{5}$

and $\det A = 1(2)(5) = 10$

and $tr(A) = 1 + 2 + 5 = 8$

Example 4.3. If possible, find matrix P that diagonalizes $\boldsymbol{A} = \begin{bmatrix} \mathbf{7} & \mathbf{-1} \\ \mathbf{4} & \mathbf{3} \end{bmatrix}$.

From a homework question in chapter 2, we know that the eigenvalues are 5,5 and we can only obtain one eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. As a result we can't get 2 linearly independent eigenvectors, so A is not diagonalizable.

Example. 4.4. If possible, find matrix P that diagonalizes $\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{1} & \mathbf{3} \end{bmatrix}$.

We can easily check that the eigenvalues of this matrix are 2 and 3 and the corresponding eigenvectors are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *respectively*. These vectors are linearly independent.

The matrix P is a matrix with eigenvectors are columns, so P=$\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ and D=$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

D is a matrix with the eigenvalues as the diagonals.

We know that P is invertible and $P^{-1}AP = D$.

Example. 4.5. Given the matrix $A=\begin{bmatrix}1 & 3 & -3\\ -1 & -3 & 1\\ 1 & 1 & -3\end{bmatrix}$, find an invertible matrix *P* and a diagonal matrix *D* such that $P^{-1}AP = D$.

Solution:
First, we will find the eigenvalues of *A*.

To do this, we need to find the characteristic equation by solving $\det\left(A-\lambda I_3\right)=0$.

$$\begin{aligned}A-\lambda I_3 &= \begin{bmatrix}1 & 3 & -3\\ -1 & -3 & 1\\ 1 & 1 & -3\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}1 & 3 & -3\\ -1 & -3 & 1\\ 1 & 1 & -3\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}\\ &= \begin{bmatrix}1-\lambda & 3 & -3\\ -1 & -3-\lambda & 1\\ 1 & 1 & -3-\lambda\end{bmatrix}\end{aligned}$$

We can find the determinant of this matrix by performing cofactor expansion and expanding along the first row.

$$\begin{aligned}\det\left(A-\lambda I_3\right) &= \left(1-\lambda\right)\det\begin{bmatrix}-3-\lambda & 1\\ 1 & -3-\lambda\end{bmatrix}-3\det\begin{bmatrix}-1 & 1\\ 1 & -3-\lambda\end{bmatrix}-3\det\begin{bmatrix}-1 & -3-\lambda\\ 1 & 1\end{bmatrix}\\ &= \left(1-\lambda\right)\left[\left(-3-\lambda\right)\left(-3-\lambda\right)-\left(1\right)\left(1\right)\right]-3\left[\left(-1\right)\left(-3-\lambda\right)-\left(1\right)\left(1\right)\right]-3\left[\left(-1\right)\left(1\right)-\left(-3-\lambda\right)\left(1\right)\right]\\ &= \left(1-\lambda\right)\left(9+6\lambda+\lambda^2-1\right)-3\left(3+\lambda-1\right)-3\left(-1+3+\lambda\right)\\ &= \left(1-\lambda\right)\left(\lambda^2+6\lambda+8\right)-3\left(\lambda+2\right)-3\left(\lambda+2\right)\end{aligned}$$

At this point, we could continue expanding the characteristic equation, and then factor the cubic polynomial. *But*… notice that the quadratic term on the left can be factored right now… let's work with that term.

$$= (1-\lambda)(\lambda+2)(\lambda+4) - 6(\lambda+2)$$
$$= (\lambda+2)\left[(1-\lambda)(\lambda+4) - 6\right]$$
$$= (\lambda+2)\left(\lambda+4-\lambda^2-4\lambda-6\right)$$
$$= (\lambda+2)\left(-\lambda^2-3\lambda-2\right)$$
$$= -(\lambda+2)\left(\lambda^2+3\lambda+2\right)$$
$$= -(\lambda+2)(\lambda+1)(\lambda+2)$$

Solving for $\det(A-\lambda I)=0$ gives us two eigenvalues: $\lambda=-1$ (algebraic multiplicity of 1) and $\lambda=-2$ (algebraic multiplicity of 2).

Next, let's find the corresponding eigenvectors for each eigenvalue.

$\boxed{\lambda=-1}$

$$null\ space\ (A+I_3) = \left[\begin{array}{ccc|c} 2 & 3 & -3 & 0 \\ -1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -1 & -2 & 1 & 0 \\ 2 & 3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow[R_3-2R_1 \to R_3]{R_2+R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3+R_2 \to R_3]{R_1-R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the above matrix, we can see that there is a free variable in column 3, and fixed variables in columns 1 and 2.

The solution is therefore

$x = 3t$

$y = -t$

$z = t, t \in \mathbb{R}$

In vector form, we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.

Therefore, a basis for $null\ space\ (A + I_3)$ is $\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\}$.

An eigenvector for $\lambda = -1$ is any $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$. A basis for E_{-1} is therefore $\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\}$ (geometric multiplicity is 1).

$\boxed{\lambda = -2}$

$$null\ space(A + 2I_3) = \left[\begin{array}{ccc|c} 3 & 3 & -3 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 3 & 3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 - 3R_1 \to R_3]{R_2 + R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the above matrix, we can see that there are free variables in columns 2 and 3, and a fixed variable in column 1.

The solution is therefore

$x = -s + t$

$y = s$

$z = t$

$s, t \in \mathbb{R}$

In vector form, we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Therefore, a basis for $null\ space\ (A + 2I_3)$ is $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

An eigenvector for $\lambda = -2$ is any $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. A basis for E_{-2} is therefore $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ (geometric multiplicity is 2).

Summary:

Eigenvalue, λ	Algebraic Multiplicity, $almu(\lambda)$	Eigenvector(s)	Geometric Multiplicity, $gemu(\lambda)$
-1	1	$\left\{\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}\right\}$	1
-2	2	$\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$	2

Now we can construct the desired matrices *P* and *D*.

P is constructed by "gluing" the eigenvectors together:

$$P = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

D is constructed by putting the eigenvalues on the diagonal of a blank matrix.

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Note: Make sure that the order in which you put the eigenvalues is the same order in which you arrange the eigenvectors. For example, if you put the eigenvalue $\lambda = -1$ in the first column of *D*, then make sure that you also put the eigenvector for $\lambda = -1$ in the first column of *P*.

Example. 4.6.

Given the matrix $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix}$, find an invertible matrix *P* and a diagonal matrix *D* such that $P^{-1}AP = D$.

Solution:

First, we will find the eigenvalues of *A*.

To do this, we need to find the characteristic equation by solving $\det(A - \lambda I_4) = 0$.

$$A - \lambda I_4 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 1 & 0 & 0 \\ 2 & 1-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 3 & -\lambda \end{bmatrix}$$

We can find the determinant of this matrix by performing cofactor expansion and expanding along the first row (every row/column has two zeroes).

$$\det(A-\lambda I_4)=(2-\lambda)\det\begin{bmatrix}1-\lambda & 0 & 0\\ 0 & -\lambda & 3\\ 0 & 3 & -\lambda\end{bmatrix}-1\det\begin{bmatrix}2 & 0 & 0\\ 0 & -\lambda & 3\\ 0 & 3 & -\lambda\end{bmatrix}$$
$$=(2-\lambda)\left\{(1-\lambda)\det\begin{bmatrix}-\lambda & 3\\ 3 & -\lambda\end{bmatrix}\right\}-1\left\{2\det\begin{bmatrix}-\lambda & 3\\ 3 & -\lambda\end{bmatrix}\right\}$$
$$=(2-\lambda)(1-\lambda)\left[(-\lambda)^2-(3)^2\right]-1(2)\left[(-\lambda)^2-(3)^2\right]$$
$$=(2-\lambda)(1-\lambda)(\lambda^2-9)-2(\lambda^2-9)$$

At this point, we could continue expanding the characteristic equation, and then factor the cubic polynomial. *But*… notice that the left term and the right term both contain (λ^2-9)… let's work with that!

$$=(\lambda^2-9)\left[(2-\lambda)(1-\lambda)-2\right]$$
$$=(\lambda^2-9)(2-3\lambda+\lambda^2-2)$$
$$=(\lambda^2-9)(\lambda^2-3\lambda)$$
$$=\lambda(\lambda+3)(\lambda-3)^2$$

Solving for $\det(A-\lambda I_4)=0$ gives us three eigenvalues: $\lambda=-3$ (algebraic multiplicity of 1), $\lambda=0$ (algebraic multiplicity of 1) and $\lambda=3$ (algebraic multiplicity of 2).

Next, let's find the corresponding eigenvectors for each eigenvalue.

$\boxed{\lambda = -3}$

$$null\ space\ (A + 3I_4) = \left[\begin{array}{cccc|c} 5 & 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right] \xrightarrow{R_1 \div 5 \to R_1} \left[\begin{array}{cccc|c} 1 & 1/5 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cccc|c} 1 & 1/5 & 0 & 0 & 0 \\ 0 & 18/5 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2/(18/5) \to R_2} \left[\begin{array}{cccc|c} 1 & 1/5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - \left(\frac{1}{5}\right)R_2 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \div 3 \to R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 - 3R_3 \to R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From the above matrix, we can see that there is a free variable in column 4, and fixed variables in columns 1, 2, and 3.

The solution is therefore:

$x = 0$

$y = 0$

$z = -t$

$w = t, t \in \mathbb{R}$

In vector form, we get $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

Therefore, a basis for $null\ space(A + 3I_4)$ is $\left\{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\}$.

An eigenvector for $\lambda = -3$ is any $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$. A basis for E_{-3} is therefore $\left\{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\}$ (geometric multiplicity is 1).

$\boxed{\lambda = 0}$ $null\ space(A - 0I_4) = \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \div 2 \to R_1} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right]$

$$\xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cccc|c} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 \div 3 \to R_4]{R_3 \div 3 \to R_3} \left[\begin{array}{cccc|c} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

From the above matrix, we can see that there is a free variable in column 2, and fixed variables in columns 1, 3, and 4.

The solution is therefore

$x = -\frac{1}{2}t$

$y = t, t \in \mathbb{R}$

$z = 0$

$w = 0$

In vector form, we get $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t\begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Therefore, a basis for $null\ space\ (A - 0I_4)$ is $\left\{\begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$.

An eigenvector for $\lambda = 0$ is any $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t\begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. A basis for E_0 is therefore $\left\{\begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$ (geometric multiplicity is 1).

$\boxed{\lambda = 3}$

$$null\ space(A - 3I_4) = \left[\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 3 & 0 \\ 0 & 0 & 3 & -3 & 0 \end{array}\right] \xrightarrow{-R_1 \to R_1} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 3 & 0 \\ 0 & 0 & 3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 3 & 0 \\ 0 & 0 & 3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \div 3 \to R_3} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 - 3R_3 \to R_4} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From the above matrix, we can see that there are free variables in columns 2 and 4, and fixed variables in columns 1 and 3.

The solution is therefore

$x = s$

$y = s$

$z = t$

$w = t$

$s, t \in \mathbb{R}$

In vector form, we get $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Therefore, a basis for $null\ space(A - 3I_4)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

An eigenvector for $\lambda = 3$ is any $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. A basis for E_3 is therefore $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ (geometric multiplicity is 2).

Summary:

Eigenvalue, λ	Algebraic Multiplicity, $almu(\lambda)$	Eigenvector(s)	Geometric Multiplicity, $gemu(\lambda)$
-3	1	$\left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$	1
0	1	$\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$	1
3	2	$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$	2

Now we can construct the desired matrices *P* and *D*.

P is constructed by "gluing" the eigenvectors together:

$$P = \begin{bmatrix} 0 & -1/2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

D is constructed by putting the eigenvalues on the diagonal of a blank matrix.

$$D = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Example. 4.7.

Given the matrix $A=\begin{bmatrix}2&0&0\\1&1&2\\1&2&1\end{bmatrix}$, compute A^3 indirectly (i.e., without directly computing $A\times A\times A$). Show all of your work.

Solution:

To compute A^3 indirectly, we need to diagonalizable the matrix *A*.

That means… finding matrices *P* and *D*. How do we do that? Eigenvalues and eigenvectors!

To begin, we need to find the characteristic equation by solving $\det(A-\lambda I_3)=0$.

$$\begin{aligned}A-\lambda I_3&=\begin{bmatrix}2&0&0\\1&1&2\\1&2&1\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\\&=\begin{bmatrix}2&0&0\\1&1&2\\1&2&1\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}\\&=\begin{bmatrix}2-\lambda&0&0\\1&1-\lambda&2\\1&2&1-\lambda\end{bmatrix}\end{aligned}$$

We can find the determinant of this matrix by performing cofactor expansion and expanding along the first row.

$$\begin{aligned}\det(A-\lambda I_3)&=(2-\lambda)\det\begin{bmatrix}1-\lambda&2\\2&1-\lambda\end{bmatrix}\\&=(2-\lambda)\left[(1-\lambda)(1-\lambda)-(2)^2\right]\\&=(2-\lambda)(1-2\lambda+\lambda^2-4)\\&=(2-\lambda)(\lambda^2-2\lambda-3)\\&=(2-\lambda)(\lambda-3)(\lambda+1)\end{aligned}$$

Solving for $\det(A-\lambda I_3)=0$ gives us three eigenvalues, each with an algebraic multiplicity of 1: $\lambda=-1$, $\lambda=2$, and $\lambda=3$.

Next, let's find the corresponding eigenvectors for each eigenvalue.

$\boxed{\lambda=-1}$ $null\ space(A+I_3)=\left[\begin{array}{ccc|c}3&0&0&0\\1&2&2&0\\1&2&2&0\end{array}\right]\xrightarrow{R_3\div 3\to R_3}\left[\begin{array}{ccc|c}1&0&0&0\\1&2&2&0\\1&2&2&0\end{array}\right]$

$$\xrightarrow[R_3-R_1\to R_3]{R_2-R_1\to R_2}\left[\begin{array}{ccc|c}1&0&0&0\\0&2&2&0\\0&2&2&0\end{array}\right]$$

$$\xrightarrow{R_2\div 2\to R_2}\left[\begin{array}{ccc|c}1&0&0&0\\0&1&1&0\\0&2&2&0\end{array}\right]$$

$$\xrightarrow{R_3-2R_2\to R_3}\left[\begin{array}{ccc|c}1&0&0&0\\0&1&1&0\\0&0&0&0\end{array}\right]$$

From the above matrix, we can see that there is a free variable in column 3, and fixed variables in columns 1 and 2.

The solution is therefore

$x=0$

$y=-t$

$z=t, t\ \in\mathbb{R}$

In vector form, we get $\begin{bmatrix}x\\y\\z\end{bmatrix}=t\begin{bmatrix}0\\-1\\0\end{bmatrix}$.

Therefore, a basis for $null\ space\ (A+I_3)$ is $\left\{\begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$.

An eigenvector for $\lambda = -1$ is any $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$. A basis for E_{-1} is therefore $\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$ (geometric multiplicity is 1).

$\boxed{\lambda = 2}$

$$\text{null space } (A - 2I_3) = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \div -3 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the above matrix, we can see that there is a free variable in column 3, and fixed variables in columns 1 and 2.

The solution is therefore

$x = -t$

$y = t$

$z = t, t \in \mathbb{R}$

In vector form, we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

Therefore, a basis for *null space* $(A - 2I_3)$ is $\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}$.

An eigenvector for $\lambda = 2$ is any $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. A basis for E_2 is therefore $\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}$ (geometric multiplicity is 1).

$\boxed{\lambda = 3}$

$$null\ space(A - 3I_3) = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right] \xrightarrow{-R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \div -2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the above matrix, we can see that there is a free variable in column 3, and fixed variables in columns 1 and 2.

The solution is therefore

$x = 0$

$y = t$

$z = t, t \in \mathbb{R}$

In vector form, we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Therefore, a basis for $null\ space(A - 3I_3)$ is $\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$.

An eigenvector for $\lambda = 3$ is any $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. A basis for E_3 is therefore $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ (geometric multiplicity is 1).

Summary:

Eigenvalue, λ	Algebraic Multiplicity, $almu(\lambda)$	Eigenvector(s)	Geometric Multiplicity, $gemu(\lambda)$
-1	1	$\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$	1
2	1	$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$	1
3	1	$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$	1

Now we can construct the desired matrices *P* and *D*.

P is constructed by "gluing" the eigenvectors together:

$$P = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

D is constructed by putting the eigenvalues on the diagonal of a blank matrix.

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Now we can indirectly calculate A^3 using the formula $A^k = PD^kP^{-1}$.

Let's find P^{-1} first.

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 + R_1 \to R_2} \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2 \to R_2} \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - 2R_2 \to R_3]{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 1 & 0 & 1 \\ 0 & \boxed{1} & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \div 2 \to R_3} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 1 & 0 & 1 \\ 0 & \boxed{1} & 0 & -1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \boxed{1} & 0 & -1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Therefore, $P^{-1} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ -1 & 0 & 0 \\ 1 & 1/2 & 1/2 \end{bmatrix}$.

Next, let's find D^3. Since D is a diagonal matrix, this isn't so bad!

$$D^3 = \begin{bmatrix} (-1)^3 & 0 & 0 \\ 0 & 2^3 & 0 \\ 0 & 0 & 3^3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

Now we can compute A^3 using the formula by matrix multiplication: $A^3 = PD^3P^{-1}$.

$$PD^3 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix} = \begin{bmatrix} 0 & -8 & 0 \\ 1 & 8 & 27 \\ -1 & 8 & 27 \end{bmatrix}$$

$$PD^3P^{-1} = \begin{bmatrix} 0 & -8 & 0 \\ 1 & 8 & 27 \\ -1 & 8 & 27 \end{bmatrix}\begin{bmatrix} 0 & -1/2 & 1/2 \\ -1 & 0 & 0 \\ 1 & 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 19 & 13 & 14 \\ 19 & 14 & 13 \end{bmatrix}$$

Therefore, $A^3 = \begin{bmatrix} 8 & 0 & 0 \\ 19 & 13 & 14 \\ 19 & 14 & 13 \end{bmatrix}$.

4.5 Homework on Chapter 4

1. Determine which of the matrices below are diagonalizable. You must give your reasons.

(a) $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

(b) $B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(a) Characteristic polynomial:

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-2 & 0 & 0 \\ -1 & \lambda-1 & -2 \\ -1 & -2 & \lambda-1 \end{bmatrix}$$

$$= (\lambda-2)\cdot[(\lambda-1)(\lambda-2)-(-2)(-2)]$$

$$= (\lambda-2)\cdot[\lambda^2-2\lambda+1-4]$$

$$= (\lambda-2)\cdot[\lambda^2-2\lambda-3]$$

$$= (\lambda-2)(\lambda-3)(\lambda+1)$$

There are three distinct eigenvalues.

Therefore, the matrix is diagonalizable.

(b) Characteristic polynomial:

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-2 & 0 & 0 \\ -1 & \lambda-1 & 0 \\ -1 & 0 & \lambda-1 \end{bmatrix}$$

$$= (\lambda-2)\cdot[(\lambda-1)(\lambda-1)]$$

The eigenvalues are $\lambda = 2,1$. We must now ensure that the dimension of the eigenspace for eigenvalue $\lambda = 1$ has dimension 2.

To find the dimension of the eigenspace: $B\bar{x} = \lambda\bar{x} \Rightarrow B\bar{x} - \lambda\bar{x} = 0 \Rightarrow (B - \lambda I)\bar{x} = \bar{0}$.

Thus reduce $B - 1 \cdot I$ as follows: $B - 1 \cdot I \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The null space of this matrix has dimension 2, thus *B* is diagonalizable.

2. The symmetric matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ has 3 distinct eigenvalues, namely $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 2$.

(a) Which eigenvalues do the eigenvectors $\bar{x}_1 = (1,0,0)$ and $\bar{x}_2 = (0,1,1)$ correspond to?

(b) Find an eigenvector that is orthogonal to both $\bar{x}_1$ and $\bar{x}_2$.

(c) Find an <u>orthogonal</u> matrix *P* such that $P^{-1}AP$ is a diagonal matrix. Also write down this diagonal matrix.

(a)

$$A\bar{x}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus, the eigenvector $\bar{x}_1$ corresponds to the eigenvalue 2.

$$A\bar{x}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Thus, the eigenvector $\bar{x}_2$ corresponds to the eigenvalue 1.

(b) The matrix A is symmetric. Its eigenvectors are therefore orthogonal. So, to find the eigenvector that corresponds to λ = -1, we simply find the cross product of the other two eigen vectors.

$$\vec{x}_3 = \vec{x}_1 \times \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Thus, the other eigenvector is given by $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

(c) $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

3. Suppose the (n,n)-matrix A has the eigenvector x with eigenvalue $\lambda = -5$. Show in complete detail that A^3 has the eigenvector x with the eigenvalue $\kappa = -125$.

$$Ax = -5x \Rightarrow A^3 x = A^2 \cdot Ax = A^2(-5x) = (-5)A^2 x = (-5)A \cdot Ax = (-5)A(-5x) \Rightarrow$$

$$A^3 x = (-5)(-5)Ax = (-5)(-5)(-5)x = -125x \Rightarrow \kappa = -125$$ is an eigenvalue of A^3.

4. Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$. Its eigenvalues are $\lambda = 2$ and $\lambda = 3$. You are given the eigenvectors $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP=D$.

Since the eigenvalues for this matrix are (1,1) and (1,2). The desired matrix P is $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, since the two eigenvalues of these matrix are different, and therefore the eigenvector we have are l.i, the diagonal matrix $D = P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

5. (a) Find all eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}$.

(b) Find a matrix *P* and a diagonal matrix *D* such that $P^{-1}AP=D$.

(a) $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 8 \\ 2 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 16 \Rightarrow \lambda(\lambda - 8) = 0 \Rightarrow \lambda = 0,8$

$\lambda = 0$:

$\begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 4x + 8y = 0 \\ 2x + 4y = 0 \end{matrix} \Rightarrow x = -2y$ and the eigenspace is $y(-2,1)$ with $y \in \Re$.

$\lambda = 8$:

$\begin{pmatrix} -4 & 8 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -4x + 8y = 0 \\ 2x - 4y = 0 \end{matrix} \Rightarrow x = 2y$ so the eigenspace is $y(2,1)$ with $y \in \Re$.

(b) $P = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$ and therefore $D = \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix}$.

6. Find the eigenvalues and the eigenvectors for the matrix $A=\begin{pmatrix}5 & 0 & 0\\ 1 & 0 & 3\\ 0 & 0 & -2\end{pmatrix}$

Can A be diagonalized? If so, find a matrix P and diagonal matrix D such that $P^{-1}AP=D$.

$$\det(A-\lambda I)=\begin{vmatrix}5-\lambda & 0 & 0\\ 1 & -\lambda & 3\\ 0 & 0 & -2-\lambda\end{vmatrix}=(5-\lambda)[-\lambda(-2-\lambda)]=\lambda(5-\lambda)(\lambda+2)$$

So, $\det(A-\lambda I)=0 \Leftrightarrow \lambda=0,5,-2.$

For $\lambda=0$:

$\begin{pmatrix}5 & 0 & 0\\ 1 & 0 & 3\\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow \begin{matrix}5x=0\\ x+3z=0\\ -2z=0\end{matrix}\Rightarrow \begin{matrix}x=0\\ z=0, y\in\Re\\ z=0\end{matrix}$, and the eigenspace for this eigenvalue is $(0,y,0)=y(0,1,0)$ with $y\in\Re$.

For $\lambda=5$:

$\begin{pmatrix}0 & 0 & 0\\ 1 & -5 & 3\\ 0 & 0 & -7\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow \begin{matrix}0=0, x\in\Re\\ x-5y+3z=0\\ -7z=0\end{matrix}\Rightarrow \begin{matrix}x\in\Re\\ y=\frac{x}{5}\\ z=0\end{matrix}$, and therefore, the eigenspace for this eigenvalue is $\left(x,\frac{x}{5},0\right)=\frac{x}{5}(5,1,0)$ with $x\in\Re$.

For $\lambda=-2$:

$\begin{pmatrix}7 & 0 & 0\\ 1 & 2 & 3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow \begin{matrix}7x=0\\ x+2y+3z=0\\ 0\cdot z=0\end{matrix}\Rightarrow \begin{matrix}x=0\\ y=-\frac{3}{2}z\\ z\in\Re\end{matrix}$, and therefore, the eigenspace for this eigenvalue is $\left(0,-\frac{3}{2}z,z\right)=\frac{z}{2}(0,-3,2)$ with $z\in\Re$.

Finally, the vectors (0,1,0), (5,1,0), and (0,-3,2) are three eigenvectors and therefore the matrix P that diagonalizes A is the matrix: $P=\begin{pmatrix}0 & 5 & 0\\ 1 & 1 & -3\\ 0 & 0 & 2\end{pmatrix}$.

7. Let $A = \begin{bmatrix} 5 & 3 & -6 \\ -4 & -2 & 4 \\ 3 & 3 & -4 \end{bmatrix}$.

Find the characteristic polynomial and the eigenvalues of *A*. Without actually finding *P*, explain how you can be certain that *A* is diagonalizable.

We find the characteristic polynomial as follows:

$$\det(\lambda I - A) = \begin{bmatrix} \lambda-5 & -3 & 6 \\ 4 & \lambda+2 & -4 \\ -3 & -3 & \lambda+4 \end{bmatrix}$$

$$= (\lambda-5)\cdot[(\lambda+2)\cdot(\lambda+4)-12] + 3\cdot[4(\lambda+4)-12] + 6\cdot[-12+3\cdot(\lambda+2)]$$

$$= \lambda^3 + \lambda^2 - 4\lambda - 4$$

In order to find the eigenvalues, we set the characteristic polynomial equal to zero and we solve for λ.

We have: $\lambda^3 + \lambda^2 - 4\lambda - 4 = 0$.

Note that –1 is a factor of the characteristic polynomial, i.e. $(-1)^3 + (-1)^2 - 4\cdot(-1) - 4 = 0$.
Therefore $x+1$ is a root.

$$\begin{array}{r|l} & \lambda^2 - 4 \\ \lambda+1 & \lambda^3 + \lambda^2 - 4\lambda - 4 \\ & \underline{\lambda^3 + \lambda^2} \\ & \underline{-4\lambda - 4} \\ & 0 \end{array}$$

Thus: $\lambda^3 + \lambda^2 - 4\lambda - 4 = (\lambda+1)\cdot(\lambda^2 - 4) = (\lambda+1)\cdot(\lambda+2)\cdot(\lambda-2)$

$$(\lambda+1)\cdot(\lambda+2)\cdot(\lambda-2) = 0 \Rightarrow \lambda = -1, -2, 2$$

Therefore, the eigenvalues are 2, –2, and –1.

Since this 3×3 matrix A has three distinct eigenvalues, we know that the matrix, we know that it is diagonalizable to $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

8. (a) Find all eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}$.

(b) Find a matrix P and a diagonal matrix D such that $P^{-1}AP=D$.

(a) $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 8 \\ 2 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 16 \Rightarrow \lambda(\lambda - 8) = 0 \Rightarrow \lambda = 0,8$

$\lambda = 0$:

$\begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 4x + 8y = 0 \\ 2x + 4y = 0 \end{matrix} \Rightarrow x = -2y$ and the eigenspace is $y(-2,1)$ with $y \in \Re$.

$\lambda = 8$:

$\begin{pmatrix} -4 & 8 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -4x + 8y = 0 \\ 2x - 4y = 0 \end{matrix} \Rightarrow x = 2y$ so the eigenspace is $y(2,1)$ with $y \in \Re$.

(b) $P = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$ and therefore $D = \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix}$.

9. Find the eigenvalues and the eigenvectors for the matrix $A=\begin{pmatrix}5 & 0 & 0\\ 1 & 0 & 3\\ 0 & 0 & -2\end{pmatrix}$

Can A be diagonalized? If so, find a matrix P and diagonal matrix D such that $P^{-1}AP=D$.

$$\det(A-\lambda I)=\begin{vmatrix}5-\lambda & 0 & 0\\ 1 & -\lambda & 3\\ 0 & 0 & -2-\lambda\end{vmatrix}=(5-\lambda)[-\lambda(-2-\lambda)]=\lambda(5-\lambda)(\lambda+2)$$

So, $\det(A-\lambda I)=0 \Leftrightarrow \lambda=0,5,-2.$

For $\lambda=0$:

$$\begin{pmatrix}5 & 0 & 0\\ 1 & 0 & 3\\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow \begin{matrix}5x=0\\ x+3z=0\\ -2z=0\end{matrix}\Rightarrow \begin{matrix}x=0\\ z=0,\ y\in\Re\\ z=0\end{matrix}$$

, and the eigenspace for this eigenvalue is $(0,y,0)=y(0,1,0)$ with $y\in\Re$.

For $\lambda=5$:

$$\begin{pmatrix}0 & 0 & 0\\ 1 & -5 & 3\\ 0 & 0 & -7\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow \begin{matrix}0=0,\ x\in\Re\\ x-5y+3z=0\\ -7z=0\end{matrix}\Rightarrow \begin{matrix}x\in\Re\\ y=\frac{x}{5}\\ z=0\end{matrix}$$

, and therefore, the eigenspace for this eigenvalue is $\left(x,\frac{x}{5},0\right)=\frac{x}{5}(5,1,0)$ with $x\in\Re$.

For $\lambda=-2$:

$$\begin{pmatrix}7 & 0 & 0\\ 1 & 2 & 3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow \begin{matrix}7x=0\\ x+2y+3z=0\\ 0\cdot z=0\end{matrix}\Rightarrow \begin{matrix}x=0\\ y=-\frac{3}{2}z\\ z\in\Re\end{matrix}$$

, and therefore, the eigenspace for this eigenvalue is $\left(0,-\frac{3}{2}z,z\right)=\frac{z}{2}(0,-3,2)$ with $z\in\Re$.

Finally, the vectors (0,1,0), (5,1,0), and (0,-3,2) are three l.i. eigenvectors and therefore the matrix P that diagonalizes A is the matrix: $P=\begin{pmatrix}0 & 5 & 0\\ 1 & 1 & -3\\ 0 & 0 & 2\end{pmatrix}$.

10. Let $A=\begin{bmatrix}5 & 3 & -6\\ -4 & -2 & 4\\ 3 & 3 & -4\end{bmatrix}$.

Find the characteristic polynomial and the eigenvalues of *A*. Without actually finding *P*, explain how you can be certain that *A* is diagonalizable.

We find the characteristic polynomial as follows:

$$\det(\lambda I - A)=\begin{bmatrix}\lambda-5 & -3 & 6\\ 4 & \lambda+2 & -4\\ -3 & -3 & \lambda+4\end{bmatrix}$$

$$=(\lambda-5)\cdot[(\lambda+2)\cdot(\lambda+4)-12]+3\cdot[4(\lambda+4)-12]+6\cdot[-12+3\cdot(\lambda+2)]$$

$$=\lambda^3+\lambda^2-4\lambda-4$$

In order to find the eigenvalues, we set the characteristic polynomial equal to zero and we solve for λ.

We have: $\lambda^3+\lambda^2-4\lambda-4=0$.

Note that –1 is a factor of the characteristic polynomial, i.e. $(-1)^3+(-1)^2-4\cdot(-1)-4=0$. Therefore $x+1$ is a root.

$$\begin{array}{r|l} & \lambda^2-4 \\ \hline \lambda+1 & \lambda^3+\lambda^2-4\lambda-4 \\ & \underline{\lambda^3+\lambda^2} \\ & \qquad\quad \underline{-4\lambda-4} \\ & \qquad\qquad 0 \end{array}$$

Thus: $\lambda^3+\lambda^2-4\lambda-4=(\lambda+1)\cdot(\lambda^2-4)=(\lambda+1)\cdot(\lambda+2)\cdot(\lambda-2)$

$$(\lambda+1)\cdot(\lambda+2)\cdot(\lambda-2)=0\Rightarrow\lambda=-1,-2,2$$

Therefore, the eigenvalues are 2, –2, and –1.

Since this 3×3 matrix A has three distinct eigenvalues, we know that the matrix, we know that it is diagonalizable to $D=\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -1\end{bmatrix}$.

11. The eigenvalues of the matrix $B = \begin{bmatrix} 4 & 4 & -4 \\ 4 & -2 & 8 \\ -4 & 8 & -2 \end{bmatrix}$ are $\lambda_1 = 6$ and $\lambda_2 = -12$.

i) Determine a basis for each eigenspace.
ii) Is B diagonalizable? Explain why or why not.

a) Eigenspace of $\lambda_1 = 6$:

basis for solution space of $(\lambda I - B)\mathbf{x} = \mathbf{0}$ is also basis for eigenspace of λ

First, sub all the known values into the equation

$$(\lambda I - B)\mathbf{x} = \mathbf{0}$$
$$(6I - B)\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 2 & -4 & 4 \\ -4 & 8 & -8 \\ 4 & -8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This has augmented matrix

$$\left[\begin{array}{ccc|c} 2 & -4 & 4 & 0 \\ -4 & 8 & -8 & 0 \\ 4 & -8 & 8 & 0 \end{array}\right]$$

which reduces to

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This means that the solution space is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2s - 2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},\ s, t \in \mathbf{R}$$

which has $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ as a basis

Therefore, a basis of the eigenspace of $\lambda_1 = 6$ is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Eigenspace of $\lambda_2 = -12$:

basis for solution space of $(\lambda I - B)\mathbf{x} = \mathbf{0}$ is also basis for eigenspace of λ

First, sub all the known values into the equation

$$(\lambda I - B)\mathbf{x} = \mathbf{0}$$
$$(-12I - B)\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} -16 & -4 & 4 \\ -4 & -10 & -8 \\ 4 & -8 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This has augmented matrix

$$\left[\begin{array}{ccc|c} -16 & -4 & 4 & 0 \\ -4 & -10 & -8 & 0 \\ 4 & -8 & -10 & 0 \end{array}\right]$$

which reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & -\tfrac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This means that the solution space is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} \tfrac{1}{2} \\ -1 \\ 1 \end{bmatrix}, \ t \in \mathbf{R}$$

Therefore, a basis of the eigenspace of $\lambda_2 = -12$ is $\left\{ \begin{bmatrix} \tfrac{1}{2} \\ -1 \\ 1 \end{bmatrix} \right\}$.

b) When we count the number of basis vectors in the eigenspace of *B*, we get a total of 3 (2 from the eigenspace of 6 and 1 from the eigenspace of –12). Since *B* is *3 x 3*, the dimension of the eigenspace of *B* = *n* = 3. Therefore, *B* is diagonalizable.

More detailed justification: $P = \begin{bmatrix} 2 & -2 & \frac{1}{2} \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ and find P^{-1} to prove that it exists.

12. Given $P^{-1}AP = D$, list the eigenvalues of D and bases for the corresponding eigenvectors.

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 & -3 \\ -1 & -3 & 1 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ ∴ eigenvalues are $-1, -2, -2$.

Since $P = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ we know the eigenvectors are

$\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = -1$ $\lambda = -2$

∴ $\lambda = -1$ has eigenspace span $\left(\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}\right)$ and $\lambda = -2$ has eigenspace span $\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

5. Orthogonality

Example 5.1.

Is the set B below an orthonormal basis for $\mathbf{R}^3$? Explain why or why not.

$$B = \left\{ \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} \right\}$$

Solution:

According to the definition for orthonormal basis, there are three conditions that a set must meet in order to be orthonormal:

- ✓ the set must be orthogonal (each pair of vectors must be orthogonal)
- ✓ each vector in the set must have magnitude of 1
- ✓ the set must be a basis for R^3

We'll disprove that it's an orthonormal basis by finding two vectors that are not orthogonal as follows:

Since $\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \bullet \begin{bmatrix} \frac{-2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{1}{3}\left(\frac{-2}{3}\right) + \frac{2}{3}\left(\frac{2}{3}\right) + \frac{2}{3}\left(\frac{1}{3}\right) = \frac{4}{9} \neq 0$, condition 1 is not satisfied.

Therefore, since the first condition of orthonormal bases doesn't hold, the set *B* cannot be an orthonormal basis.

This set is **not** even an orthogonal basis for R^3, let alone an orthonormal basis.

Example 5.2.

Find an orthogonal basis for the subspace w of $\mathbb{R}^3$ given by:

$$w = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 3z = 0 \right\}$$

Solution:

Solve for x: $x = y - 3z$

$$\therefore \begin{bmatrix} y-3z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \vec{v}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \text{are of basis for } w,$$

So, we need to make them orthogonal.

Since they are not orthogonal, we add another non zero vector in w that is orthogonal to either one of $\vec{u}$ or $\vec{v}$.

Let $\vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a vector in w that is orthogonal to $\vec{u}$.

Then $x - y + 3z = 0$, since $\vec{w}$ is in the plane w. Since $\vec{u} \cdot \vec{w} = 0$,

we know $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$\therefore x + y = 0$

Solve the system:

$x - y + 3z = 0$

$x + y = 0$

$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \rightarrow$ RREF $\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \end{array}\right]$

$z = t$

$x = -\frac{3}{2}t$

$y = \frac{3}{2}t$

$\therefore \vec{w} = \begin{bmatrix} -\frac{3}{2}t \\ \frac{3}{2}t \\ t \end{bmatrix}$ or let t=2 and we get $\vec{w} = \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix}$

$\therefore \{\vec{u}, \vec{w}\}$ is an orthogonal basis for w, since $\dim w = 2$

$\left(\text{check } \vec{u} \cdot \vec{w} = 0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix} = -3 + 3 + 0 = 0\right)$

Example 5.3.

Find the coordinates of $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ with respect to an orthogonal basis $\mathcal{B} = \{v_1, v_2, v_3\}$

where $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \overrightarrow{v_2} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, and $\overrightarrow{v_3} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

Solution:

$$C_1 = \frac{\vec{w} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} = \frac{(1,2,4) \cdot (1,4,1)}{(1,4,1) \cdot (1,4,1)} = \frac{13}{18}$$

$$C_2 = \frac{\vec{w} \cdot \overrightarrow{v_2}}{\overrightarrow{v_2} \cdot \overrightarrow{v_2}} = \frac{(1,2,4) \cdot (2,0,-2)}{(2,0,-2) \cdot (2,0,-2)} = \frac{-6}{8} = -\frac{3}{4}$$

$$C_3 = \frac{\vec{w} \cdot \overrightarrow{v_3}}{\overrightarrow{v_3} \cdot \overrightarrow{v_3}} = \frac{(1,2,4) \cdot (2,-1,2)}{(2,-1,2) \cdot (2,-1,2)} = \frac{8}{9}$$

$\therefore \vec{w} = C_1\overrightarrow{v_1} + C_2\overrightarrow{v_2} + C_3\overrightarrow{v_3}$

$$\vec{w} = \frac{13}{18}\overrightarrow{v_1} - \frac{3}{4}\overrightarrow{v_2} + \frac{8}{9}\overrightarrow{v_3}$$

$$\therefore [\vec{w}]_B = \begin{bmatrix} \frac{13}{18} \\ -\frac{3}{4} \\ \frac{8}{9} \end{bmatrix}$$

Example 5.4. Determine whether the matrix is orthogonal. If it is, find its inverse.

a) $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Solution:

$$Q^T Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ It is orthogonal and $Q^T = Q^{-1}$

$$\therefore Q^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

b) $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$Q^TQ = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore Q$ is orthogonal and $Q^{-1} = Q^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

<u>Example 5.5.</u> Show that the given vectors form an orthogonal basis for R^2 and then express $\vec{w}$ as a linear combination of these basis vectors and find the coordinate vector $[\vec{w}]_B$ with respect to your basis.

$$\vec{v_1} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

<u>Solution:</u>

$$\vec{v_1} \cdot \vec{v_2} = 5 - 5 = 0$$

Since the vectors are orthogonal, they're linearly independent. Since there are 2 vectors, they form an orthogonal basis for R^2.

$C_1 = \dfrac{\vec{w} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} = \dfrac{(2,3) \cdot (5,-1)}{(5,-1) \cdot (5,-1)} = \dfrac{7}{26}$

$C_2 = \dfrac{\vec{w} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} = \dfrac{(2,3) \cdot (1,5)}{(1,5) \cdot (1,5)} = \dfrac{17}{26}$

$\therefore \vec{w} = C_1\vec{v_1} + C_2\vec{v_2}$

$\vec{w} = \dfrac{7}{26}\begin{bmatrix} 5 \\ -1 \end{bmatrix} + \dfrac{17}{26}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$\therefore [\vec{w}]_B = \begin{bmatrix} \frac{7}{26} \\ \frac{17}{26} \end{bmatrix}$

Example 5.6.

Find the orthogonal complement and $W^{\perp}$ of W and give a basis for $W^{\perp}$.

a) $W = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : 3x - y = 0\right\}$

Solution:

$y = 3x$ from the equation, so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3x \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix} = span\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$

W is in the column space of $A = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

so that $W^{\perp}$ is the null space of $A^T = [1 \quad 3]$

The augmented matrix is

$\qquad x \quad y$

$[A^T|0]$ → $[1 \quad 3 \quad |0]$ and is already row reduced Let y=t be a free variable and we get:

x + 3t = 0 and $x = -3t$

null (A^T) = span $\begin{bmatrix} -3t \\ t \end{bmatrix}$ = t $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

So, a basis for the null space, and thus for $W^{\perp}$ is given by: $\left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}$.

b) $W = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + z = 0\right\}$ $\quad z = -x + y$

Solution:

W consists of $\begin{bmatrix} x \\ y \\ -x+y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, so that $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for W

Then W is the column space of $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$ so that $W^{\perp}$ is the null space of A^T.

The augmented matrix is

$$\begin{matrix} & x & y & z & \end{matrix}$$

$[A^T|0]$ → $\begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix}$ and is already row reduced.

$$Let\ z = t\ be\ a\ free\ variable$$

x – t = 0 and x=t and from the second row, $y + t = 0\ and\ y = -t$

$(\text{row}\ (A))^{\perp} = \text{null}(A)$

$\therefore \text{null}(A^T) = \text{span}\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \text{span}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and therefore, a basis for $W^{\perp}$ is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Example 5.7. Find the orthogonal projection of $\vec{v}$ onto the subspace W spanned by the vector $\overrightarrow{u_i}$. (Assume $\overrightarrow{u_i}$ are orthogonal vectors).

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \overrightarrow{u_1} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \quad \overrightarrow{u_2} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$proj_W\vec{v} = \left(\frac{\overrightarrow{u_1} \cdot \vec{v}}{\overrightarrow{u_1} \cdot \overrightarrow{u_1}}\right) \cdot \overrightarrow{u_1} + \left(\frac{\overrightarrow{u_2} \cdot \vec{v}}{\overrightarrow{u_2} \cdot \overrightarrow{u_2}}\right) \cdot \overrightarrow{u_2}$$

$$= \frac{(-2,2,1) \cdot (1,-1,2)}{(-2,2,1) \cdot (-2,2,1)} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + \frac{(1,-1,4) \cdot (1,-1,2)}{(1,-1,4) \cdot (1,-1,4)} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$= \frac{-2}{9} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + \frac{10}{18} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \frac{-2}{9} \overrightarrow{u_1} + \frac{10}{18} \overrightarrow{u_2} = \begin{bmatrix} 4/9 \\ -4/9 \\ -2/9 \end{bmatrix} + \begin{bmatrix} 10/18 \\ -10/18 \\ 4/18 \end{bmatrix}$$

$$= \begin{bmatrix} 4/9 \\ -4/9 \\ -2/9 \end{bmatrix} + \begin{bmatrix} 5/9 \\ -5/9 \\ 2/9 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

NOTE: For the dot product on the bottom, we can do the magnitude of the vectors squared as that is easier for most!

Example 5.8. Find the orthogonal decomposition of $\vec{v}$ with respect to W.

$\vec{v} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} \quad w = \text{span} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Since w is spanned by vector $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ we know

$$proj_w\vec{v} = \left(\frac{\overrightarrow{u_1} \cdot \vec{v}}{\overrightarrow{u_1} \cdot \overrightarrow{u_1}}\right)$$

$$= \frac{(1,4)\cdot(3,-3)}{(1,4)\cdot(1,4)} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{-9}{7}\overrightarrow{u_1}$$

$$= \frac{-9}{17}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -9/17 \\ -36/17 \end{bmatrix}$$

∴ the component of $\vec{v}$ orthogonal to w is $proj_{w^{\perp}}(\vec{v}) = \vec{v} - proj_w\vec{v}$

$$= \begin{bmatrix} 3 \\ -3 \end{bmatrix} - \begin{bmatrix} -9/17 \\ -36/17 \end{bmatrix}$$

$$= \begin{bmatrix} 51/17 \\ -51/17 \end{bmatrix} - \begin{bmatrix} -9/17 \\ -36/17 \end{bmatrix}$$

$$= \begin{bmatrix} 60/17 \\ -15/17 \end{bmatrix}$$

Example 5.9. If A=$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 1 \\ 3 & 10 & 13 \\ 0 & 8 & 2 \end{bmatrix}$, *find the bases for the fundamental subspaces of A.*

From earlier, in example 8.7. we found the following:

a) Find bases for the row space, column space, and null space of *A*.

b) What is *dim row A, dim col A,* and *dim null space of A*?

Solution:

$$\begin{bmatrix} \boxed{1} & 2 & 4 \\ 0 & 4 & 1 \\ 3 & 10 & 13 \\ 0 & 8 & 2 \end{bmatrix} \xrightarrow{R3-3R1\to R3} \begin{bmatrix} \boxed{1} & 2 & 4 \\ 0 & 4 & 1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{bmatrix}$$

$$\xrightarrow{R2\div 4\to R2} \begin{bmatrix} \boxed{1} & 2 & 4 \\ 0 & \boxed{1} & 1/4 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{bmatrix}$$

$$\xrightarrow[R3-4R2\to R3]{R1-2R2\to R1 \atop R4-8R2\to R4} \begin{bmatrix} \boxed{1} & 0 & 7/2 \\ 0 & \boxed{1} & 1/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$null\ A = \left[\begin{array}{ccc|c} 1 & 0 & 7/2 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x+7/2z=0$

$y+1/4z=0$

$z=t$

Row (A)=$\{[1,0,7/2],[0,1,1/4]\}$ is a basis for the row space of *A*.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -7/2 \\ -1/4 \\ 1 \end{bmatrix}$$

Col (A)= $\left\{\begin{bmatrix}1\\0\\3\\4\end{bmatrix},\begin{bmatrix}2\\4\\10\\8\end{bmatrix}\right\}$ is a basis for the column space of A.

Null (A)= $\left\{\begin{bmatrix}-7/2\\-1/4\\1\end{bmatrix}\right\}$ is a basis for the null space. This is the solution of the system in RREF.

dim row A=2

dim col A=2

dim null space = 1

So, now we just need to find the bases for the null space of A^T.

$$\left[\begin{array}{cccc|c}1 & 0 & 3 & 0 & 0\\2 & 4 & 10 & 8 & 0\\4 & 1 & 13 & 2 & 0\end{array}\right] \dots \text{RREF}$$

$x_1 \; x_2 \; x_3 \; x_4$

$$\left[\begin{array}{cccc|c}1 & 0 & 3 & 0 & 0\\0 & 1 & 1 & 2 & 0\\0 & 0 & 0 & 0 & 0\end{array}\right]$$

$x_3 = s, x_4 = t$

$x_1 + 3s = 0 \qquad x_1 = -3s$

$x_2 + s + 2t = 0 \quad x_2 = -s - 2t$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-3s\\-s-2t\\s\\t\end{bmatrix} = s\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix} + t\begin{bmatrix}0\\-2\\0\\1\end{bmatrix}$$

$$\therefore null\ (A^T) = span\ \left\{\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix},\begin{bmatrix}0\\-2\\0\\1\end{bmatrix}\right\}$$

Example 5.10.

The following vectors form a basis for R^2 or R^3. Apply the Gram-Schmidt process to obtain an orthogonal basis. Then, find an orthonormal basis.

a) $\overrightarrow{x_1} = \begin{bmatrix}1\\1\end{bmatrix} \quad \overrightarrow{x_2} = \begin{bmatrix}1\\3\end{bmatrix}$

Solution:

$\overrightarrow{v_1} = \overrightarrow{x_1} = \begin{bmatrix}1\\1\end{bmatrix}$

$\overrightarrow{v_2} = \overrightarrow{x_2} - \text{proj}_{\overrightarrow{v_1}}(\overrightarrow{x_2}) \cdot \overrightarrow{v_1} = \overrightarrow{x_2} - \left(\frac{\overrightarrow{v_1} \cdot \overrightarrow{x_2}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}}\right)\overrightarrow{v_1}$

$= \begin{bmatrix}1\\3\end{bmatrix} - \frac{(1,1)\cdot(1,3)}{(1,1)\cdot(1,1)}\begin{bmatrix}1\\1\end{bmatrix}$

$= \begin{bmatrix}1\\3\end{bmatrix} - \frac{4}{2}\begin{bmatrix}1\\1\end{bmatrix}$

$= \begin{bmatrix}1\\3\end{bmatrix} - \begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$

$\therefore$ an orthogonal basis is $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$

To find an orthonormal basis we need to normalize each vector.

$\overrightarrow{q_1} = \frac{\overrightarrow{v_1}}{\|\overrightarrow{v_1}\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

$\overrightarrow{q_2} = \frac{\overrightarrow{v_2}}{\|\overrightarrow{v_2}\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

b) $\vec{x_1} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ $\vec{x_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Solution:

$\vec{v_1} = \vec{x_1} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$\vec{v_2} = \vec{x_2} - \text{proj}_{\vec{v_1}}(\vec{x_2}) \cdot \vec{v_1} = \vec{x_2} - \left(\frac{\vec{v_1} \cdot \vec{x_2}}{\vec{v_1} \cdot \vec{v_1}}\right)\vec{v_1}$

$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{(2,-2)\cdot(2,1)}{(2,-2)\cdot(2,-2)}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{2}{8}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$

$\therefore$ an orthogonal basis is $\left\{\begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}\right\}$

To find an orthonormal basis we need to normalize each vector.

$\vec{q_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{1}{\sqrt{2^2+(-2)^2}}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$\therefore \vec{q_1} = \frac{1}{\sqrt{4}\sqrt{2}}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$\vec{q_1} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$\vec{q_2} = \frac{\vec{v_2}}{\|\vec{v_2}\|} = \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2}}\begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$

Note: $\sqrt{\frac{18}{4}} = \frac{\sqrt{9}\sqrt{2}}{\sqrt{4}} = \frac{3\sqrt{2}}{2}$

$\vec{q_2} = \frac{1}{\frac{3\sqrt{2}}{2}}\begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} = \frac{2}{3\sqrt{2}}\begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Example 5.11. Fill in the missing elements of Q to make it an orthogonal matrix.

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & x \\ 1/\sqrt{2} & -1/\sqrt{3} & x \\ 0 & 1/\sqrt{3} & x \end{bmatrix}$$

Solution:

Let $\overrightarrow{q_1} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ and $\overrightarrow{q_2} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

Let the third column be $\overrightarrow{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\det Q = (-1)^{1+1}\left(\frac{1}{\sqrt{2}}\right)\det\begin{bmatrix} -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 1 \end{bmatrix} + (-1)^{2+1}\left(\frac{1}{\sqrt{2}}\right)\det\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 1 \end{bmatrix}$$

$$\det Q = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{3}}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{-2}{\sqrt{6}} \neq 0$$

∴ the matrix Q is invertible and the 3 column vectors form a basis for R^3.

The first 2 vectors are orthogonal and unit vectors, so we orthogonalize the 3^{rd} vector:

$$\overrightarrow{v_3} = \overrightarrow{x_3} - \left(\frac{\overrightarrow{q_1}\cdot\overrightarrow{x_3}}{\overrightarrow{q_1}\cdot\overrightarrow{q_1}}\right)\overrightarrow{q_1} - \left(\frac{\overrightarrow{q_2}\cdot\overrightarrow{x_3}}{\overrightarrow{q_2}\cdot\overrightarrow{q_2}}\right)\overrightarrow{q_2}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{0}{\overrightarrow{q_1}\cdot\overrightarrow{q_1}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} - \frac{\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\cdot(0,0,1)}{\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\cdot\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}\begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \frac{\frac{1}{\sqrt{3}}}{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{3}} \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$\therefore \overrightarrow{v_3} = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

Normalize $\overrightarrow{v_3}$:

$$\overrightarrow{q_3} = \frac{\overrightarrow{v_3}}{\|\overrightarrow{v_3}\|}$$

$$= \frac{\left(-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)}{\sqrt{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}}}$$

$$= \frac{\left(-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)}{\sqrt{\frac{6}{9}}}$$

$$= \left(-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)\left(\frac{3}{\sqrt{6}}\right) = \left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

$$\therefore Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & \frac{-1}{\sqrt{6}} \\ 1/\sqrt{2} & -1/\sqrt{3} & \frac{1}{\sqrt{6}} \\ 0 & 1/\sqrt{3} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

5.4 Homework on Chapter 5

1. Determine whether the given set of orthogonal vectors is orthonormal. If it isn't, normalize the vectors to form an orthonormal set

a)

$$\overrightarrow{v_1} = \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix} \qquad \overrightarrow{v_2} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

$\overrightarrow{v_1} \cdot \overrightarrow{v_2} = 0$ ∴ orthogonal

$$||\overrightarrow{v_1}|| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(-\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1$$

$$||\overrightarrow{v_2}|| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$$

∴ the set is orthonormal.

b) $s = \{(1,0,1,0), (1,0,-1,0)(0,1,0,1)\}$

$\overrightarrow{v_1} \cdot \overrightarrow{v_2} = 0, \overrightarrow{v_2} \cdot \overrightarrow{v_3} = 0$ and $\overrightarrow{v_1} \cdot \overrightarrow{v_3} = 0$

(orthogonal)

$$||\overrightarrow{v_1}|| = \sqrt{1^2 + 0^2 + 1^2 + 0^2} = \sqrt{2} \neq 1$$

$$||\overrightarrow{v_2}|| = \sqrt{2} \neq 1$$

$$||\overrightarrow{v_3}|| = \sqrt{2} \neq 1$$

∴ normalize the vectors

$$\frac{\overrightarrow{v_1}}{||\overrightarrow{v_1}||} = \frac{(1,0,1,0)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$$

$$\frac{\overrightarrow{v_2}}{||\overrightarrow{v_2}||} = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, 0\right)$$

$$\frac{\overrightarrow{v_3}}{||\overrightarrow{v_3}||} = \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

These are an orthonormal set.

2. Show that the given vectors form an orthogonal basis for R^2 and then express $\vec{w}$ as a linear combination of these basis vectors and find the coordinate vector $[\vec{w}]_B$ with respect to your basis.

$\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\overrightarrow{v_2} = \begin{bmatrix} -8 \\ 2 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\overrightarrow{v_1} \cdot \overrightarrow{v_2} = 5 - 5 = 0$

$$C_1 = \frac{\vec{w} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} = \frac{(1,3) \cdot (1,4)}{(1,4) \cdot (1,4)} = \frac{13}{17}$$

$$C_2 = \frac{\vec{w} \cdot \overrightarrow{v_2}}{\overrightarrow{v_2} \cdot \overrightarrow{v_2}} = \frac{(1,3) \cdot (-8,2)}{(-8,2) \cdot (-8,2)} = \frac{-2}{68} = -\frac{1}{34}$$

$\therefore \vec{w} = C_1\overrightarrow{v_1} + C_2\overrightarrow{v_2}$

$$\vec{w} = \frac{13}{17}\overrightarrow{v_1} - \frac{1}{34}\overrightarrow{v_2}$$

$$\therefore [\vec{w}]_B = \begin{bmatrix} \frac{13}{17} \\ \frac{-1}{34} \end{bmatrix}$$

3. Construct an orthonormal basis for R^3 from vectors in question 2. NOTE: The orthogonal vectors were: $\begin{bmatrix}1\\4\\1\end{bmatrix}$, $\begin{bmatrix}2\\0\\-2\end{bmatrix}$, and $\begin{bmatrix}2\\-1\\2\end{bmatrix}$.

$\overrightarrow{q_1} = \frac{1}{\|\overrightarrow{v_1}\|}\overrightarrow{v_1}$

$\overrightarrow{q_2} = \frac{1}{\|\overrightarrow{v_2}\|}\overrightarrow{v_2}$

$\overrightarrow{q_3} = \frac{1}{\|\overrightarrow{v_3}\|}\overrightarrow{v_3}$

$$\overrightarrow{q_1} = \frac{1}{\sqrt{18}}\begin{bmatrix}1\\4\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{18}}\\\frac{4}{\sqrt{18}}\\\frac{1}{\sqrt{18}}\end{bmatrix} = \begin{bmatrix}1/3\sqrt{2}\\4/3\sqrt{2}\\1/3\sqrt{2}\end{bmatrix}$$

$$\overrightarrow{q_2} = \frac{1}{\sqrt{8}}\begin{bmatrix}2\\0\\-2\end{bmatrix} = \frac{1}{\sqrt{4\sqrt{2}}}\begin{bmatrix}2\\0\\-2\end{bmatrix}$$

$$\therefore \overrightarrow{q_2} = \begin{bmatrix}2/2\sqrt{2}\\0\\-2/2\sqrt{2}\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}$$

$$\overrightarrow{q_3} = \frac{1}{3}\begin{bmatrix}2\\-1\\2\end{bmatrix} = \begin{bmatrix}2/3\\-1/3\\2/3\end{bmatrix}$$

4. Find the orthogonal complement and $W^{\perp}$ of W and give a basis for $W^{\perp}$.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x + y - z = 0 \right\} \qquad \therefore z = 2x + y$$

W consists of $\begin{bmatrix} x \\ y \\ 2x+y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, so that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for W.

Then W is the column space of $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$ so that $W^{\perp}$ is the null space of A^T.

The augmented matrix is

$$\begin{matrix} x & y & z & \\ \end{matrix}$$

$$\left[\begin{matrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right]$$ and is already row reduced.

$x = -2z$

$y = -z$

$$\therefore \text{null}(A^T) = \text{span} \begin{bmatrix} -2z \\ -z \\ z \end{bmatrix} = \text{span} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

and therefore, a basis for $W^{\perp}$ is $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

5. Let w be the subspace spanned by the given vectors. Find a basis for $W^{\perp}$.

a) $\overrightarrow{w_1} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \overrightarrow{w_2} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

Let $A = [\overrightarrow{w_1} \quad |\overrightarrow{w_2}\,] = \begin{bmatrix} 2 & 4 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}$

Then the subspace w spanned by $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ is the column space of A

and $W^{\perp} = \left(\text{col(A)}\right)^{\perp} = \text{null}(\text{A}^{\text{T}})$

$[A^T|0] \rightarrow \begin{bmatrix} 2 & 1 & -1 & |0 \\ 4 & 0 & 2 & |0 \end{bmatrix} \rightarrow$ RREF… $\begin{bmatrix} 1 & 0 & 1/2 & |0 \\ 0 & 1 & -2 & |0 \end{bmatrix}$

∴ null(A^T) is the set of solutions of this augmented matrix which is:

$z = t$

$x = -\frac{1}{2}t$

$y = 2t$

∴ $\begin{bmatrix} -\frac{1}{2}t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2} \\ 2 \\ 1 \end{bmatrix} = \text{span}\begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$ so this vector is a basis for $W^{\perp}$.

b) $\overrightarrow{w_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \overrightarrow{w_2} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$

Let $A = [\overrightarrow{w_1} \quad |\overrightarrow{w_2}\,] = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}$

Then the subspace w spanned by $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ is the column space of A

and $w^{\perp} = \left(\text{col(A)}\right)^{\perp} = \text{null}(\text{A}^{\text{T}})$

$[A^T|0] \rightarrow \begin{bmatrix} 1 & 2 & 1 & |0 \\ 3 & 0 & -1 & |0 \end{bmatrix} \rightarrow$ RREF… $\begin{bmatrix} 1 & 0 & -1/3 & |0 \\ 0 & 1 & 2/3 & |0 \end{bmatrix}$

∴ null(A^T) is the set of solutions of this augmented matrix which is:

$z = t$

$x = \frac{1}{3}t$

$y = -\frac{2}{3}t$

$\therefore \begin{bmatrix} \frac{1}{3}t \\ -\frac{2}{3}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix} = \text{span}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ so this vector is a basis for $W^{\perp}$.

c) $\overrightarrow{w_1} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}$ $\overrightarrow{w_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ $\overrightarrow{w_3} = \begin{bmatrix} 2 \\ 4 \\ 3 \\ -1 \end{bmatrix}$

Let $A = [\overrightarrow{w_1} \mid \overrightarrow{w_2} \mid \overrightarrow{w_3}] = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -1 \end{bmatrix}$

Then the subspace W spanned by $\overrightarrow{w_1}$, $\overrightarrow{w_2}$ $\overrightarrow{w_3}$ is the column space of A

and $W^{\perp} = \left(\text{col(A)}\right)^{\perp} = \text{null}(\text{A}^{\text{T}})$

$[A^T|0] \rightarrow \left[\begin{matrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 0 & 1 \\ 2 & 4 & 3 & -1 \end{matrix}\middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \rightarrow \text{RREF} = \left[\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{matrix}\middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$

let $w_3 = t$

$w_1 = -t$

$w_2 = 0$

$w_3 = t$

The null(A^T) is the set of solutions to the augmented matrix:

$\begin{bmatrix} -t \\ 0 \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \text{span}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ $\therefore$ this vector forms a basis for $W^{\perp}$.

6. Find the orthogonal projection of $\vec{v}$ onto the subspace W spanned by the vector $\vec{u_i}$. (Assume $\vec{u_i}$ are orthogonal vectors).

a) $\vec{v} = \begin{bmatrix} 6 \\ -4 \end{bmatrix} \quad \vec{u_i} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$proj_W\vec{v} = \left(\frac{\vec{u_i} \cdot \vec{v}}{\vec{u_i} \cdot \vec{u_i}}\right) \cdot \vec{u_i}$$

$$= \frac{(1,1) \cdot (6,-4)}{(1,1) \cdot (1,1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

b) $\vec{v} = \begin{bmatrix} 10 \\ -4 \end{bmatrix} \quad \vec{u_i} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$proj_W\vec{v} = \left(\frac{\vec{u_i} \cdot \vec{v}}{\vec{u_i} \cdot \vec{u_i}}\right) \cdot \vec{u_i}$$

$$= \frac{(1,2) \cdot (10,-4)}{(1,2) \cdot (1,2)} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{2}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2/5 \\ 4/5 \end{bmatrix}$$

c) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $\quad \overrightarrow{u_1} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$ $\quad \overrightarrow{u_2} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}$

$$proj_w\vec{v} = \left(\frac{\overrightarrow{u_1} \cdot \vec{v}}{\overrightarrow{u_1} \cdot \overrightarrow{u_1}}\right) \cdot \overrightarrow{u_1} + \left(\frac{\overrightarrow{u_2} \cdot \vec{v}}{\overrightarrow{u_2} \cdot \overrightarrow{u_2}}\right) \cdot \overrightarrow{u_2}$$

$$= \frac{(2,-2,-1) \cdot (1,2,4)}{(2,-2,-1) \cdot (2,-2,-1)} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} + \frac{(4,2,4) \cdot (1,2,4)}{(4,2,4) \cdot (4,2,4)} \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}$$

$$= \frac{-6}{9} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} + \frac{24}{36} \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}$$

$$= \frac{-2}{3} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -4/3 \\ 4/3 \\ 2/3 \end{bmatrix} + \begin{bmatrix} 8/3 \\ 4/3 \\ 8/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 8/3 \\ 10/3 \end{bmatrix}$$

7. Find the orthogonal decomposition of $\vec{v}$ with respect to W.

a) $\vec{v} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ $\quad W = \text{span} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Since W is spanned by vector $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ we know

$$proj_w\vec{v} = \left(\frac{\overrightarrow{u_1} \cdot \vec{v}}{\overrightarrow{u_1} \cdot \overrightarrow{u_1}}\right)$$

$$= \frac{(1,2) \cdot (5,-2)}{(1,2) \cdot (1,2)} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5}\overrightarrow{u_1}$$

$$= \frac{1}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}$$

∴ the component of $\vec{v}$ orthogonal to w is $proj_{w^\perp}(\vec{v}) = \vec{v} - proj_w\vec{v}$

$$= \begin{bmatrix} 5 \\ -2 \end{bmatrix} - \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}$$

$$= \begin{bmatrix} 25/5 \\ -10/5 \end{bmatrix} - \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 24/5 \\ -12/5 \end{bmatrix}$$

b) $\vec{v} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ W= span $\left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\right)$

Since W is spanned by vector $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ and $\overrightarrow{u_2} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ we have:

$$proj_w\vec{v} = \left(\frac{\overrightarrow{u_1} \cdot \vec{v}}{\overrightarrow{u_1} \cdot \overrightarrow{u_1}}\right) \cdot \overrightarrow{u_1} + \left(\frac{\overrightarrow{u_2} \cdot \vec{v}}{\overrightarrow{u_2} \cdot \overrightarrow{u_2}}\right) \cdot \overrightarrow{u_2}$$

$$= \frac{(1,4,1) \cdot (4,2,-1)}{(1,4,1) \cdot (1,4,1)} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + \frac{(2,-1,2) \cdot (4,2,-1)}{(2,-1,2) \cdot (2,-1,2)} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$= \frac{11}{18} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + \frac{4}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 11/18 \\ 4/18 \\ 11/18 \end{bmatrix} + \begin{bmatrix} 8/9 \\ -4/9 \\ 8/9 \end{bmatrix}$$

$$= \begin{bmatrix} 11/18 \\ 4/18 \\ 11/18 \end{bmatrix} + \begin{bmatrix} 16/18 \\ -8/18 \\ 16/18 \end{bmatrix} = \begin{bmatrix} 27/18 \\ -4/18 \\ 27/18 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 \\ -2/9 \\ 3/2 \end{bmatrix}$$

The component of $\vec{v}$ orthogonal to W is $proj_{w^{\perp}}(\vec{v}) = \vec{v} - proj_w\vec{v}$

$$= \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 3/2 \\ -2/9 \\ 3/2 \end{bmatrix}$$

$$= \begin{bmatrix} 8/2 \\ 18/9 \\ -2/2 \end{bmatrix} - \begin{bmatrix} 3/2 \\ -2/9 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 20/9 \\ -5/2 \end{bmatrix}$$

8. The following vectors form a basis for R^2 or R^3. Apply the Gram-Schmidt process to obtain an orthogonal basis. Then, find an orthonormal basis.

a) $\vec{x_1} = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \vec{x_2} = \begin{bmatrix}1\\1\\0\end{bmatrix} \quad \vec{x_3} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$$\vec{v_1} = \vec{x_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\vec{v_2} = \vec{x_2} - \text{proj}_{\vec{v_1}}(\vec{x_2}) \cdot \vec{v_1} = \vec{x_2} - \left(\frac{\vec{v_1} \cdot \vec{x_2}}{\vec{v_1} \cdot \vec{v_1}}\right)\vec{v_1}$$

$$\vec{v_2} = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{(1,0,0)\cdot(1,1,0)}{(1,0,0)\cdot(1,0,0)}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\therefore \ \vec{v_2} = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{1}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\vec{v_3} = \vec{x_3} - \text{proj}_{\vec{v_1}}(\vec{x_3}) \cdot \vec{v_1} - \text{proj}_{\vec{v_2}}(\vec{x_3}) \cdot \vec{v_2} = \vec{x_3} - \left(\frac{\vec{v_1} \cdot \vec{x_3}}{\vec{v_1} \cdot \vec{v_1}}\right)\vec{v_1} - \left(\frac{\vec{v_2} \cdot \vec{x_3}}{\vec{v_2} \cdot \vec{v_2}}\right)\vec{v_2}$$

$$\vec{v_3} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{(1,0,0)\cdot(1,1,1)}{(1,0,0)\cdot(1,0,0)}\begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{(0,1,0)\cdot(1,1,1)}{(0,1,0)\cdot(0,1,0)}\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\vec{v_3} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{1}\begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{1}\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\vec{v_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$\therefore$ an orthogonal basis is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

Note: this is already orthonormal since each vector has magnitude of 1.

b) $\overrightarrow{x_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $\overrightarrow{x_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ $\overrightarrow{x_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\overrightarrow{v_1} = \overrightarrow{x_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\overrightarrow{v_2} = \overrightarrow{x_2} - \text{proj}_{\overrightarrow{v_1}}(\overrightarrow{x_2}) \cdot \overrightarrow{v_1} = \overrightarrow{x_2} - \left(\frac{\overrightarrow{v_1} \cdot \overrightarrow{x_2}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}}\right)\overrightarrow{v_1}$

$\overrightarrow{v_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{(2,1,0)\cdot(2,0,1)}{(2,1,0)\cdot(2,1,0)}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\therefore \overrightarrow{v_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\overrightarrow{v_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 8/5 \\ 4/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/5 \\ -4/5 \\ 1 \end{bmatrix}$

$\overrightarrow{v_3} = \overrightarrow{x_3} - \text{proj}_{\overrightarrow{v_1}}(\overrightarrow{x_3}) \cdot \overrightarrow{v_1}$

$\overrightarrow{v_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{(2,1,0)\cdot(0,1,0)}{(2,1,0)\cdot(2,1,0)}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\overrightarrow{v_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\overrightarrow{v_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2/5 \\ -1/5 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 4/5 \\ 0 \end{bmatrix}$

$\therefore$ an orthogonal basis is $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2/5 \\ -4/5 \\ 1 \end{bmatrix}, \begin{bmatrix} -2/5 \\ 4/5 \\ 0 \end{bmatrix}\right\}$

An orthonormal basis is found by normalizing each vector

$\overrightarrow{q_1} = \frac{\overrightarrow{v_1}}{\|\overrightarrow{v_1}\|} = \frac{(2,1,0)}{\sqrt{2^2+1^2+0^2}} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$

$\overrightarrow{q_2} = \frac{\overrightarrow{v_2}}{\|\overrightarrow{v_2}\|} = \frac{(\frac{2}{5}, -\frac{4}{5}, 1)}{\sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{-4}{5}\right)^2 + 1^2}}$

Note: $\sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{-4}{5}\right)^2 + 1^2} = \sqrt{\frac{4}{25} + \frac{16}{25} + \frac{25}{25}} = \sqrt{\frac{24}{25}} = \frac{\sqrt{9}\sqrt{5}}{5} = \frac{3\sqrt{5}}{5}$

$\therefore \overrightarrow{q_2} = \frac{\left(\frac{2}{5}, -\frac{4}{5}, 1\right)}{\frac{3\sqrt{5}}{5}}$

$\overrightarrow{q_2} = \frac{5}{3\sqrt{5}} \left(\frac{2}{5}, \frac{-4}{5}, 1\right)$

$\overrightarrow{q_2} = \begin{bmatrix} 2/3\sqrt{5} \\ -4/3\sqrt{5} \\ 5/3\sqrt{5} \end{bmatrix}$

$\overrightarrow{q_2} = \frac{\overrightarrow{v_3}}{\|\overrightarrow{v_3}\|} = \frac{(\frac{-2}{5}, \frac{4}{5}, 0)}{\sqrt{\left(\frac{-2}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + 0^2}}$

$= \frac{(\frac{-2}{5}, \frac{4}{5}, 0)}{\sqrt{\frac{20}{25}}} = \frac{(\frac{-2}{5}, \frac{4}{5}, 0)}{\frac{\sqrt{4}\sqrt{5}}{5}}$

$= \frac{(\frac{-2}{5}, \frac{4}{5}, 0)}{\frac{2\sqrt{5}}{5}}$

$= \frac{2\sqrt{5}}{5} \left(\frac{-2}{5}, \frac{4}{5}, 0\right)$

$= \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$

$\overrightarrow{q_3} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$

$\therefore$ an orthogonal basis is $\left\{ \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 2/3\sqrt{5} \\ -4/3\sqrt{5} \\ 5/3\sqrt{5} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \right\}$

9. The following vectors form a basis for a subspace W of R^3. Find the orthogonal basis for W using the Gram-Schmidt process.

$$\vec{x_1} = \begin{bmatrix}1\\0\\4\end{bmatrix} \quad \vec{x_2} = \begin{bmatrix}3\\3\\1\end{bmatrix}$$

$$\vec{v_1} = \vec{x_1} = \begin{bmatrix}1\\0\\4\end{bmatrix}$$

$$\vec{v_2} = \vec{x_2} - \text{proj}_{\vec{v_1}}(\vec{x_2}) \cdot \vec{v_1} = \vec{x_2} - \left(\frac{\vec{v_1} \cdot \vec{x_2}}{\vec{v_1} \cdot \vec{v_1}}\right)\vec{v_1}$$

$$\vec{v_2} = \begin{bmatrix}3\\3\\1\end{bmatrix} - \frac{(1,0,4)\cdot(3,3,1)}{(1,0,4)\cdot(1,0,4)}\begin{bmatrix}1\\0\\4\end{bmatrix}$$

$$\therefore \ \vec{v_2} = \begin{bmatrix}3\\3\\1\end{bmatrix} - \frac{7}{17}\begin{bmatrix}1\\0\\4\end{bmatrix}$$

$$\vec{v_2} = \begin{bmatrix}51/17\\51/17\\17/17\end{bmatrix} - \begin{bmatrix}7/17\\0\\28/17\end{bmatrix} = \begin{bmatrix}44/17\\51/17\\-11/17\end{bmatrix}$$

$$\vec{v_2} = \begin{bmatrix}44/17\\51/17\\-11/17\end{bmatrix}$$

The orthonormal basis consists of vectors v1, v2 and v3.

10. Fill in the missing element of Q to make it an orthogonal matrix.

$$Q = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & a \\ 0 & 1/\sqrt{3} & b \\ 1/\sqrt{2} & 1/\sqrt{3} & c \end{bmatrix}$$

Let $\overrightarrow{q_1} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ and $\overrightarrow{q_2} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ These are already unit vectors.

Let the third column be $\overrightarrow{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\det Q = (-1)^{1+1}\left(-\frac{1}{\sqrt{2}}\right)\det\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 1 \end{bmatrix} + 0 + (-1)^{3+1}\left(\frac{1}{\sqrt{2}}\right)\det\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix}$$

$$\det Q = \frac{-1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}} - 0\right) + 0$$

$$= \frac{-1}{\sqrt{6}} \neq 0$$

$\therefore Q$ is invertible and the 3 column vectors form a basis for R^3.

The first two vectors are orthogonal (and unit vectors), so we need to orthogonalize the third vector.

$$\overrightarrow{v_3} = \overrightarrow{x_3} - \left(\frac{\overrightarrow{q_1}\cdot\overrightarrow{x_3}}{\overrightarrow{q_1}\cdot\overrightarrow{q_1}}\right)\overrightarrow{q_1} - \left(\frac{\overrightarrow{q_2}\cdot\overrightarrow{x_3}}{\overrightarrow{q_2}\cdot\overrightarrow{q_2}}\right)\overrightarrow{q_2}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)\cdot(0,0,1)}{\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)\cdot\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)}\begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \frac{\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\cdot(0,0,1)}{\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\cdot\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)}\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\frac{1}{\sqrt{2}}}{1} \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \frac{\frac{1}{\sqrt{3}}}{1} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

$$\overrightarrow{v_3} = \begin{bmatrix} 1/6 \\ -1/3 \\ 1/6 \end{bmatrix}$$

Normalize $\overrightarrow{v_3}$: we get

$$\overrightarrow{q_3} = \frac{\overrightarrow{v_3}}{\|\overrightarrow{v_3}\|} = \frac{\left(\frac{1}{6}, \frac{-1}{3}, \frac{1}{6}\right)}{\sqrt{\frac{1}{36} + \frac{1}{9} + \frac{4}{36}}}$$

$$\overrightarrow{q_3} = \frac{\left(\frac{1}{6}, \frac{-1}{3}, \frac{1}{6}\right)}{\sqrt{\frac{1+4+1}{36}}} = \frac{\left(\frac{1}{6}, \frac{-1}{3}, \frac{1}{6}\right)}{\sqrt{\frac{5}{36}}}$$

$$= \left(\frac{1}{6}, \frac{-1}{3}, \frac{1}{6}\right)\left(\frac{6}{\sqrt{5}}\right)$$

$$\therefore \overrightarrow{q_3} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$\therefore Q = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{5} \\ 0 & 1/\sqrt{3} & -2/\sqrt{5} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{5} \end{bmatrix}$$

11. Find an orthogonal basis for R^3 that contains the vector $\overrightarrow{x_1} = \begin{bmatrix}2\\1\\1\end{bmatrix}$.

If we add $\overrightarrow{x_2} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\overrightarrow{x_3} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ we have a basis for $R^3(\det A \neq 0)$

Now, orthogonalize that basis:

$$\overrightarrow{v_1} = \overrightarrow{x_1} = \begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$\overrightarrow{v_2} = \overrightarrow{x_2} - \text{proj}_{\overrightarrow{v_1}}(\overrightarrow{x_2}) \cdot \overrightarrow{v_1} = \overrightarrow{x_2} - \left(\frac{\overrightarrow{v_1} \cdot \overrightarrow{x_2}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}}\right)\overrightarrow{v_1}$$

Note: $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$

$$\overrightarrow{v_2} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{(1,0,0)\cdot(2,1,1)}{2^2+1^2+1^2}\begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{2}{6}\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}2/3\\1/3\\1/3\end{bmatrix} = \begin{bmatrix}1/3\\-1/3\\-1/3\end{bmatrix}$$

$$\overrightarrow{v_3} = \overrightarrow{x_3} - \text{proj}_{\overrightarrow{v_1}}(\overrightarrow{x_3}) \cdot \overrightarrow{v_1} - \text{proj}_{\overrightarrow{v_2}}(\overrightarrow{x_3}) \cdot \overrightarrow{v_2}$$

$$\overrightarrow{v_3} = \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{(0,1,0)\cdot(2,1,1)}{2^2+1^2+1^2}\begin{bmatrix}2\\1\\1\end{bmatrix} - \frac{(0,1,0)\cdot\left(\frac{1}{3},-\frac{1}{3},-\frac{1}{3}\right)}{\left(\frac{1}{3}\right)^2+\left(-\frac{1}{3}\right)^2+\left(-\frac{1}{3}\right)^2}\begin{bmatrix}1/3\\-1/3\\-1/3\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{1}{6}\begin{bmatrix}2\\1\\1\end{bmatrix} - \frac{-\frac{1}{3}}{\frac{1}{3}}\begin{bmatrix}1/3\\-1/3\\-1/3\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}1/3\\1/6\\1/6\end{bmatrix} + \begin{bmatrix}1/3\\-1/3\\-1/3\end{bmatrix}$$

$$\overrightarrow{v_3} = \begin{bmatrix}0\\1/2\\-1/2\end{bmatrix}$$

The orthonormal basis consists of vectors v1, v2 and v3.

12. Find an orthogonal basis for the subspace W in $\mathbb{R}^4$ spanned by

$$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \{(1,1,3,2), (1,-2,0,-1), (0,2,1,2)\}$$

We are given a spanning set. Let's use the Gram-Schmidt algorithm to make the vectors in the spanning set orthogonal. Orthogonal vectors are naturally linearly independent! Thus, we will create an **orthogonal basis** by applying the Gram-Schmidt algorithm.

Note that it is **easier** to perform the algorithm by starting with a vector with 0's in it.

$$\vec{X}_1 = \vec{w}_3 = (0,2,1,2)$$

$$\begin{aligned}
\vec{X}_2 &= \vec{w}_2 - proj_{\vec{X}_1}\vec{w}_2 \\
&= (1,-2,0,-1) - \frac{(1,-2,0,-1)\cdot(0,2,1,2)}{(0,2,1,2)\cdot(0,2,1,2)}(0,2,1,2) \\
&= (1,-2,0,-1) - \left(-\frac{2}{3}\right)(0,2,1,2) \\
&= (1,-2,0,-1) + \left(0,\frac{4}{3},\frac{2}{3},\frac{4}{3}\right) \\
&= \left(1,-\frac{2}{3},\frac{2}{3},\frac{1}{3}\right) \\
&\quad OR \\
&= (3,-2,2,1)
\end{aligned}$$

$$\begin{aligned}
\vec{X}_3 &= \vec{w}_1 - proj_{\vec{X}_1}\vec{w}_1 - proj_{\vec{X}_2}\vec{w}_1 \\
&= (1,1,3,2) - \frac{(1,1,3,2)\cdot(0,2,1,2)}{(0,2,1,2)\cdot(0,2,1,2)}(0,2,1,2) - \frac{(1,1,3,2)\cdot(3,-2,2,1)}{(3,-2,2,1)\cdot(3,-2,2,1)}(3,-2,2,1) \\
&= (1,1,3,2) - \left(\frac{9}{9}\right)(0,2,1,2) - \left(\frac{9}{18}\right)(3,-2,2,1) \\
&= (1,1,3,2) - (0,2,1,2) - \left(\frac{3}{2},-1,1,\frac{1}{2}\right) \\
&= \left(-\frac{1}{2},0,1,-\frac{1}{2}\right) \\
&\quad OR \\
&= (-1,0,2,-1)
\end{aligned}$$

Therefore, an orthogonal basis for W is $\{\vec{X}_1, \vec{X}_2, \vec{X}_3\} = \{(0,2,1,2), (3,-2,2,1), (-1,0,2,-1)\}$.

13. Let $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^3$. Find an orthonormal basis for $\mathbb{R}^3$.

First, we shall Gram-Schmidt!

Starting with the given basis and using the vector with the most zeroes to make our life easier, we get…

Note: To take up less room on the page, I am writing the column vectors as row vectors until the final answer.

$$\vec{X}_1 = (-1,0,1)^T$$

$$\begin{aligned}
\vec{X}_2 &= (1,-1,1)^T - proj_{\vec{X}_1}(1,-1,1)^T \\
&= (1,-1,1)^T - \frac{(-1,0,1)^T \cdot (1,-1,1)^T}{(-1,0,1)^T \cdot (-1,0,1)^T}(-1,0,1)^T \\
&= (1,-1,1)^T - \frac{0}{2}(-1,0,1)^T \\
&= (1,-1,1)^T
\end{aligned}$$

$$\begin{aligned}
\vec{X}_3 &= (1,1,2)^T - proj_{\vec{X}_1}(1,1,2)^T - proj_{\vec{X}_2}(1,1,2)^T \\
&= (1,1,2)^T - \frac{(-1,0,1)^T \cdot (1,1,2)^T}{(-1,0,1)^T \cdot (-1,0,1)^T}(-1,0,1)^T - \frac{(1,-1,1)^T \cdot (1,1,2)^T}{(1,-1,1)^T \cdot (1,-1,1)^T}(1,-1,1)^T \\
&= (1,1,2)^T - \frac{1}{2}(-1,0,1)^T - \frac{2}{3}(1,-1,1)^T \\
&= \left(\frac{5}{6}, \frac{5}{3}, \frac{5}{6}\right)^T \\
&\quad OR \\
&= (5,10,5)^T
\end{aligned}$$

An orthogonal basis is $\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}5\\10\\5\end{bmatrix}\right\}$.

Now we need to turn every vector in the orthogonal basis into a unit vector.

Let’s find the magnitude of each vector.

$$\|\vec{X}_1\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$$
$$\|\vec{X}_2\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$
$$\|\vec{X}_3\| = \sqrt{5^2 + 10^2 + 5^2} = \sqrt{150} = \sqrt{25}\sqrt{6} = 5\sqrt{6}$$

Therefore, an orthonormal basis is $\left\{\begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix},\begin{bmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix},\begin{bmatrix}5/\left(5\sqrt{6}\right)\\10/\left(5\sqrt{6}\right)\\5/\left(5\sqrt{6}\right)\end{bmatrix}\right\}$ or (more preferably)

$\left\{\begin{bmatrix}-\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix},\begin{bmatrix}\sqrt{3}/3\\-\sqrt{3}/3\\\sqrt{3}/3\end{bmatrix},\begin{bmatrix}\sqrt{6}/6\\\sqrt{6}/3\\\sqrt{6}/6\end{bmatrix}\right\}$.

This is the original vector!

14. Find an orthogonal basis for the null space of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

First, let's find the null space:

Null space $A = \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

From the above matrix, we can see that there are free variables in columns 2 and 3, and fixed variables in column 1.

Therefore, the general solution is

$x = -s - t$

$y = s$

$z = t$

$s, t \in \mathbb{R}$

In vector form, we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Therefore, a basis for the null space is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Now let's turn this into an orthogonal basis using the Gram-Schmidt algorithm.

BUT WAIT. We've seen this basis before!! From the previous question – so we can "borrow" from that answer.

Therefore, an orthogonal basis for the null space is $\left\{ \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{6}/6 \\ -\sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix} \right\}$ (note that I'm using the orthonormal basis, which is orthogonal, by definition).

15. Find an orthogonal basis for the subspace of $\mathbf{R}^4$ spanned by

$\{\mathbf{w}_1 = (1, 1, 3, 2), \mathbf{w}_2 = (1, -2, 0, -1), \mathbf{w}_3 = (0, 2, 1, 2)\}$.

Gram-Schmidt process to find orthogonal basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$

$\mathbf{v}_1 = \mathbf{w}_1 = (1, 1, 3, 2)$

$$\mathbf{v}_2 = \mathbf{w}_2 - proj_{\mathbf{v}_1}\mathbf{w}_2 = \mathbf{w}_2 - \frac{\mathbf{w}_2 \bullet \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \left(\frac{6}{5}, -\frac{9}{5}, \frac{3}{5}, -\frac{3}{5}\right)$$

$$\mathbf{v}_3 = \mathbf{w}_3 - proj_{span\{\mathbf{v}_1, \mathbf{v}_2\}}\mathbf{w}_3 = \mathbf{w}_3 - \frac{\mathbf{w}_3 \bullet \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{w}_3 \bullet \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \left(\frac{1}{3}, 0, -\frac{1}{3}, \frac{1}{3}\right)$$

Therefore, an orthogonal basis is $\left\{(1, 1, 3, 2), \left(\frac{6}{5}, -\frac{9}{5}, \frac{3}{5}, -\frac{3}{5}\right), \left(\frac{1}{3}, 0, -\frac{1}{3}, \frac{1}{3}\right)\right\}$.

Note: Recall that a projection is $proj_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$.

16. Let $S=\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}0\\2\\2\end{bmatrix}\right\}$. Find $S^{\perp}$.

To find the orthogonal complement, we need to find the null space of the column space of S^T.

(Don't forget to write the given vectors as **row vectors**).

null space $S^T = \begin{bmatrix}1 & 0 & -1 & 0\\0 & 2 & 2 & 0\end{bmatrix}$ R2 ÷ 2 → $\begin{bmatrix}1 & 0 & -1 & 0\\0 & 1 & 1 & 0\end{bmatrix}$

From the above matrix, we can see that there are fixed variables in columns 1 and 2, and a free variable in column 3.

The solution is therefore

$x=t$

$y=-t$

$z=t, t\in\mathbb{R}$

In vector form, we get $\begin{bmatrix}x\\y\\z\end{bmatrix}=t\begin{bmatrix}1\\-1\\1\end{bmatrix}$.

Therefore, a basis for $\ker\left(S^T\right)$ is $\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$.

Therefore, $S^{\perp}=span\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$.

17. Find an orthogonal basis for the subspace W in $\mathbb{R}^4$ spanned by

$$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \{(1,1,3,2), (1,-2,0,-1), (0,2,1,2)\}$$

We are given a spanning set. Let's use the Gram-Schmidt algorithm to make the vectors in the spanning set orthogonal. Orthogonal vectors are naturally linearly independent! Thus, we will create an **orthogonal basis** by applying the Gram-Schmidt algorithm.

Note that it is **easier** to perform the algorithm by starting with a vector with 0's in it.

$$\vec{X}_1 = \vec{w}_3 = (0,2,1,2)$$

$$\begin{aligned}
\vec{X}_2 &= \vec{w}_2 - proj_{\vec{X}_1}\vec{w}_2 \\
&= (1,-2,0,-1) - \frac{(1,-2,0,-1)\cdot(0,2,1,2)}{(0,2,1,2)\cdot(0,2,1,2)}(0,2,1,2) \\
&= (1,-2,0,-1) - \left(-\frac{2}{3}\right)(0,2,1,2) \\
&= (1,-2,0,-1) + \left(0,\frac{4}{3},\frac{2}{3},\frac{4}{3}\right) \\
&= \left(1,-\frac{2}{3},\frac{2}{3},\frac{1}{3}\right) \\
&\quad OR \\
&= (3,-2,2,1)
\end{aligned}$$

$$\begin{aligned}
\vec{X}_3 &= \vec{w}_1 - proj_{\vec{X}_1}\vec{w}_1 - proj_{\vec{X}_2}\vec{w}_1 \\
&= (1,1,3,2) - \frac{(1,1,3,2)\cdot(0,2,1,2)}{(0,2,1,2)\cdot(0,2,1,2)}(0,2,1,2) - \frac{(1,1,3,2)\cdot(3,-2,2,1)}{(3,-2,2,1)\cdot(3,-2,2,1)}(3,-2,2,1) \\
&= (1,1,3,2) - \left(\frac{9}{9}\right)(0,2,1,2) - \left(\frac{9}{18}\right)(3,-2,2,1) \\
&= (1,1,3,2) - (0,2,1,2) - \left(\frac{3}{2},-1,1,\frac{1}{2}\right) \\
&= \left(-\frac{1}{2},0,1,-\frac{1}{2}\right) \\
&\quad OR \\
&= (-1,0,2,-1)
\end{aligned}$$

Therefore, an orthogonal basis for W is $\{\vec{X}_1, \vec{X}_2, \vec{X}_3\} = \{(0,2,1,2), (3,-2,2,1), (-1,0,2,-1)\}$.

Best of luck on the exam!!!