MATH 1229 Final Exam Booklet Solutions (2024)

TEST ONE MATERIAL 2

A. Distance or Length (1.1) 2
B. Dot Product (1.2) 4
C. The Cross Product (1.2) 5
D. Area of a Triangle and Parallelogram (1.2) 6
E. Volume of a Parallelepiped (1.2) 8
F. The Most Challenging Exam Questions on Vectors (1.1, 1.2) 9
G. Practice Exam Questions on Vector Operations (1.1, 1.2) 13
H. Point-Normal Form (1.3) 20
I. Standard Form of the Equation (1.3) 21
J. Point-Parallel Form Equation For a Line (1.3) 23
K. Two-Point Form Equation for a Line (1.3) 24
L. Distance from a Point to a Line (1.3) 25
M. Distance from a Point to a Plane (1.3) 26
N. Determining if Planes are Parallel, Perpendicular, Identical or None of These (1.3) 27
O. Vectors Parallel to Lines (1.3) 28
P. Vectors Perpendicular to Lines (1.3) 30
Q. Point of Intersection of a Line and a Plane (1.3) 31
R. Intersection of Lines (1.3) 32
S. The Most Challenging Exam Questions on Lines and Planes (1.3) 33
T. Practice Exam Questions on Equations of Lines and Planes (1.3) 38
U. Vectors in R^M (2.1) 51

TEST TWO MATERIAL 55

A. Systems of Equations (2.2) 55
B. Elimination Method (2.2) 56
C. Row Reduction of Matrices to Solve Linear Systems (2.3) 57
D. Extra Practice Exam Questions on Systems of Equations and Row Reduction (2.3) 71
E. Rank 87
F. Operations on Matrices (3.1) 88
G. Matrix Equations and Inverses (3.2) 97

POST TEST TWO MATERIAL 112

A. Theory of Linear Systems (3.3) 112
B. Final Exam Questions on RREF, Matrices, Inverses and Theory of Linear Systems 118
C. Minor and Co-Factors (4.1) 122
D. Determinants (4.1) 123
E. Triangular Form Method (4.1) 127
F. Basket-Weave Method (4.1) 130
G. Properties of Determinants (4.2) 131
H. Practice Exam Questions on Determinants (4.1, 4.2) 134
I. Cramer's Rule (4.3) 150
J. Adjoint of a Matrix (4.3) 153
K. Finding the Inverse of a Matrix using the Determinant and the Adjoint (4.3) 155
L. An Application to Find the Determinant (4.3) 157
M. Practice Exam Questions on Determinant, Adjoint and Cramer's Rule (Ch. 4) 158

Test One Material

A. Distance or Length (1.1)

A1. b) $= \sqrt{(5+2)^2 + (-1+1)^2 + (-2-1)^2}$
$= \sqrt{49+0+9}$
$= \sqrt{58}$

c) $= \sqrt{(2+1)^2 + (-1+2)^2 + (-2-3)^2}$
$= \sqrt{9+1+25}$
$= \sqrt{35}$

d) $= \sqrt{(2+2)^2 + (-1+1)^2 + (-2-1)^2}$
$= \sqrt{16+0+9}$
$= \sqrt{25}$
$= 5$

A2. b) $= \sqrt{(-2)^2 + (-1)^2 + 1^2}$
$= \sqrt{6}$

c) $= \sqrt{5^2 + (-1)^2 + (-2)^2}$
$= \sqrt{25+1+4}$
$= \sqrt{30}$

d) $= \sqrt{2^2 + (-1)^2 + (-2)^2}$
$= \sqrt{4+1+4} \quad = \sqrt{9} \quad = 3$

Example. a) $= \sqrt{25}\sqrt{3} = 5\sqrt{3}$
b) $= \sqrt{36}\sqrt{3} = 6\sqrt{3}$
c) $= \sqrt{100}\sqrt{7} = 10\sqrt{7}$

Scalar Multiplication

A3. b) $= 5(-2,-1,1) = (-10,-5,5)$
$c) = -2(-1,-2,3) = (2,4,-6)$
d) $= \|-2(5,-1,-2)\|$
$= \|(-10,2,4)\| = \sqrt{(-10)^2 + 2^2 + 4^2}$
$= \sqrt{100+4+16} \quad = \sqrt{120}$

A4. $\vec{u} = -1(-2,-4,6)$

$\vec{u} = (2,4,-6)$

$\|\vec{u}\| = \sqrt{2^2 + 4^2 + (-6)^2} = \sqrt{4 + 16 + 36} = \sqrt{56}$

$\therefore \frac{\vec{u}}{\|\vec{u}\|} = \left(\frac{2}{\sqrt{56}}, \frac{4}{\sqrt{56}}, \frac{-6}{\sqrt{56}}\right)$

A5. $\sqrt{(2c)^2 + (2c)^2 + (3c)^2} = 1$

$4c^2 + 4c^2 + 9c^2 = 1$

$17c^2 = 1$

$c^2 = \frac{1}{17}$

$c = \pm\sqrt{\frac{1}{17}} = \pm\frac{1}{\sqrt{17}}$

A6. b) $= (-2,-1,1) + (5,-1,-2) = (3,-2,-1)$

c) $= (2,-1,-2) + (-1,-2,3) = (1,-3,1)$

A7. a) $= 3(-1,-2,3) - 5(-2,-1,1)$

$= (-3,-6,9) + (10,5,-5) = (7,-1,4)$

b) $= -(-2,-1,1) + 3(5,-1,-2)$

$= (2,1,-1) + (15,-3,-6) = (17,-2,-7)$

c) $= 3(-1,-2,3) - (-2,-1,1) + 2(5,-1,-2)$

$= (-3,-6,9) + (2,1,-1) + (10,-2,-4)$

$= (9,-7,4)$

B. Dot Product (1.2)

B1. b) $= (-2,-1,1)\cdot(5,-1,-2)$
$= -10 + 1 + (-2) = -11$
c) $= (-1,-2,3)\cdot(2,-1,-2)$
$= -2 + 2 - 6 = -6$

**Dots Product is ALWAYS a Number or Scalar

Angles between Vectors

B2b) $\vec{v}\cdot\vec{w} = (-2,-1,1)\cdot(5,-1,-2)$
$= -10 + 1 - 2 \quad = -11$

$\|\vec{v}\| = \sqrt{(-2)^2 + (-1)^2 + 1^2} = \sqrt{6}$

$\|\vec{w}\| = \sqrt{5^2 + (-1)^2 + (-2)^2} = \sqrt{25+1+4} = \sqrt{30}$

$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \frac{-11}{\sqrt{6}\sqrt{30}} \quad or \quad \frac{-11}{\sqrt{180}}$

c) $\cos\theta = \frac{\vec{r}\cdot\vec{u}}{\|\vec{r}\|\ \|\vec{u}\|}$

$\vec{r}\cdot\vec{u} = (2,-1,-2)\cdot(-1,-2,3) = -2 + 2 - 6 = -6$

$\|\vec{r}\| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$

$\|\vec{u}\| = \sqrt{(-1)^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$

$\cos\theta = \frac{-6}{3\sqrt{14}} = -\frac{2}{\sqrt{14}}$

B3. $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|} = 0$

$\frac{(1,-4)\cdot(k,3)}{\sqrt{1^2+(-4)^2}\ \sqrt{k^2+3^2}} = 0$

$k - 12 = 0$

$k = 12$

Orthogonal Vectors

B4. $(-4,2,0)\cdot(-1,-2,6) = 4 - 4 + 0$

$= 0 \quad \therefore Yes, orthonal$

B5. $(-3,3,-1)\cdot(-1,-1,2) = 3 - 3 - 2$

$= -2 \quad \therefore No, not\ orthogonal$

C. The Cross Product (1.2)

**Cross Product is ALWAYS a Vector.

Example

$$\begin{matrix} 1 & 2 & 1 & 1 & 2 & 1 \\ -1 & -2 & 4 & -1 & -2 & 4 \end{matrix}$$

$$(8-(-2),\ -4-4,\ -2+2)$$

$$\vec{u}\times\vec{v}=(10,-5,0)$$

C1. $\begin{matrix} 3 & -3 & -1 & 3 & -3 & -1 \\ -1 & -1 & 2 & -1 & -1 & 2 \end{matrix}$

$$\vec{u}\times\vec{v}=(-6-1,\ 1-6,-3-3)$$

$$=(-7,-5,-6)$$

C2. $\vec{v}\times\vec{u}$ $\begin{matrix} -1 & 2 & 3 & -1 & 2 & 3 \\ 5 & 2 & -1 & 5 & 2 & -1 \end{matrix}$

$$=(-2-6,\ \ 15-1,\ -2-10)$$

$$=(-8,14,-12)$$

Orthogonal Example p.24

$$\begin{matrix} \frac{1}{4} & \frac{2}{5} & \frac{-6}{5} & \frac{1}{4} & \frac{2}{5} & \frac{-6}{5} \end{matrix}$$

$$\times \quad \times \quad \times$$

$$\begin{matrix} \frac{-1}{3} & \frac{-3}{1} & \frac{2}{3} & \frac{-1}{3} & \frac{-3}{1} & \frac{2}{3} \end{matrix}$$

$$\left(\frac{4}{15}-\frac{18}{5},\ \ \frac{6}{15}-\frac{2}{12},\ \ \frac{-3}{4}+\frac{2}{15}\right)$$

$$=\left(\frac{4}{15}-\frac{54}{15},\ \ \frac{24}{60}-\frac{10}{60},\ \ \frac{-45}{60}+\frac{8}{60}\right)=\left(\frac{-50}{15},\frac{14}{60},\frac{-37}{60}\right)=\left(\frac{-10}{3},\frac{7}{30},\frac{-37}{60}\right)$$

Unit Vectors p.25

It would be $(5,-3,2)$

D. Area of a Triangle and Parallelogram (1.2)

D1. S D1. Step 1 $\vec{u} = q - p = (1,-1,3) - (3,-1,4) = (-2,0,-1)$

$\vec{v} = r - p = (4,-3,2) - (3,-1,4) = (1,-2,-2)$

Step 2 $\begin{matrix} -2 & 0 & -1 & -2 & 0 & -1 \\ 1 & -2 & -2 & 1 & -2 & -2 \end{matrix}$

$\vec{u} \times \vec{v} = (0-2, -1-4,\ 4-0) = (-2,-5,4)$

Step 3 $A = \frac{1}{2}\ \|(-2,-5,4)\|$

$= \frac{1}{2}\sqrt{(-2)^2 + (-5)^2 + 4^2}$

$= \frac{1}{2}\sqrt{4+25+16}$

$= \frac{1}{2}\sqrt{45}$

D2. $\vec{u} = b - a = (-1,\ 1,-1)$

$\vec{v} = c - a = (0,2,-1)$

$\begin{matrix} -1 & 1 & -1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 0 & 2 & -1 \end{matrix}$

$\vec{u} \times \vec{v} = (-1+2, 0-1, -2-0)$

$= (1,-1,-2)$

$A = \frac{1}{2}\ \|(1,-1,-2)\|$

$A = \frac{1}{2}\sqrt{1^2 + (-1)^2 + (-2)^2}$

$= \frac{1}{2}\sqrt{6}\ \ or\ \ \frac{\sqrt{6}}{2}$

D3. $\vec{u} \times \vec{v}$
$$\begin{matrix} 1 & 3 & 1 & 1 & 3 & 1 \\ -2 & 4 & -1 & -2 & 4 & -1 \end{matrix}$$

$$= (-3 - 4, -2 + 1,\ 4 + 6)$$

$$= (-7, -1, 10)$$

$$A = \|(-7, -1, 10)\| = \sqrt{49 + 1 + 100} = \sqrt{150}$$

D4.

$$\vec{u} = q - p = (3, -1, 0)$$

$$\vec{v} = q - r = (-2, 2, 0)$$

$\vec{u} \times \vec{v}$
$$\begin{matrix} 3 & -1 & 0 & 3 & -1 & 0 \\ -2 & 2 & 0 & -2 & 2 & 0 \end{matrix}$$

$$= (0,0,4)$$

$$A = \|(0,0,4)\|$$

$$= \sqrt{4^2} = \sqrt{16} = 4$$

E. Volume of a Parallelepiped (1.2)

E1. $V = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$

$$\vec{u} \times \vec{v} = \begin{matrix} 4 & -1 & 0 & 4 & -1 & 0 \\ 4 & 2 & -1 & 4 & 2 & -1 \end{matrix}$$

$$\vec{u} \times \vec{v} = (1-0,\ 0+4,\ 8+4) = (1,4,12)$$

$$V = |(1,4,12) \cdot (-1,2,4)|$$

$$= |-1+8+48|$$

$$= 55\, units^3$$

E2. $\vec{u} \times \vec{v} \quad \begin{matrix} 1 & 3 & 4 & 1 & 3 & 4 \\ 1 & -2 & 0 & 1 & -2 & 0 \end{matrix}$

$$= (0+8,\ \ 4-0, -2-3) = (8,4,-5)$$

$$V = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$$

$$= \left|(8,4,-5) \cdot \left(\frac{-2}{5}, \frac{3}{4}, \frac{1}{3}\right)\right|$$

$$= \left|\frac{-16}{5} + \frac{3}{1} - \frac{5}{3}\right| = \left|\frac{-48}{15} + \frac{45}{15} + \frac{(-25)}{15}\right| = \left|\frac{-28}{15}\right| = \frac{28}{15}$$

F. The Most Challenging Exam Questions on Vectors (1.1, 1.2)

F1. $\sqrt{k^2 + 2^2} = \sqrt{10}$

$k^2 + 4 = 10$

$k^2 = 6$

$k = \pm\sqrt{6}$

F2. $(14, k, k) \cdot (4, k, 15) = 0$

$56 + k^2 + 15k = 0$

$k^2 + 15k + 56 = 0$

$(k + 7)(k + 8) = 0$

$k = -7, -8$

F3. a) $(2,8)\ collinear\ to\ \ (6, k)$

$\nwarrow \times 3 \nearrow$

$2 \times 3 = 6 \quad \therefore 8 \times 3 = k$

$k = 24$

b) to get from 2 to 3, we multiply by 3/2.

So, to get from 8 to c, we multiply by the same factor, 3/2.

So, $8 \times \frac{3}{2} = \frac{24}{2} = 12 = c$

c) (h+2,4h-1) collinear to (1,3), write it as (1,3) collinear to (h+2, 4h-1)

1 (h+2) = h+2, so the same multiple works for the second component:

3(h+2) = 4h - 1

Cross multiply, and get: 3h + 6 = 4h -1 and solve h = 7

F4. $\vec{u} = 3i + 3j - 6k + 4j - 4k$

$= 3i + 7j - 10k$

$= (3,7,-10)$

$\|\vec{u}\| = \sqrt{3^2 + 7^2 + (-10)^2} = \sqrt{9 + 49 + 100} = \sqrt{158}$

F5. $\sqrt{c^2 + (\frac{c}{3})^2 + (\frac{c}{2})^2 = 1}$

$c^2 + \frac{c^2}{9} + \frac{c^2}{4} = 1$

$\frac{36c^2}{36} + \frac{4c^2}{36} + \frac{9c^2}{36} = 1$

$\frac{49c^2}{36} = 1$

$49c^2 = 36$

$c^2 = \frac{36}{49}$

$c = \pm\sqrt{\frac{36}{49}} = \pm\frac{6}{7}$

F6. $\overrightarrow{PQ} = q - p = (2,3,2)$

$\overrightarrow{RS} = s - r = (-2,3,1)$

$(2,3,2) \cdot (-2,3,1) = -4 + 9 + 2 \neq 0 \quad \therefore \text{not perpendicular}$

$\overrightarrow{PQ}$ *is not a multiple of* $\overrightarrow{RS} \quad \therefore A$ *is false*

Therefore, the answer is D).

F7. $I)$ *is false*

$II)$ *is true*

$III)$ *is true*

Therefore, the answer is E).

F8. The directed line segment QP= P- Q = (3,5) – (-4,-3) = (7, 8). Always do second minus first!

F9. Which of the following is NOT defined for vectors $\vec{u}, \vec{v}$ *and* $\vec{w}$ in R^3.

A. $\vec{u} \cdot (\vec{v} \times \vec{w})$	B. $\vec{u} \cdot (\overrightarrow{2v} - 3\vec{w})$	C. $\vec{u} \cdot (\vec{v} \cdot \vec{w})$	D. $5\vec{u} + 9\vec{v}$	E. $\vec{u} \times (\vec{v} + \vec{w})$

You can only dot or cross or add/subtract a vector with another vector.

You can't do $\vec{u} + 5$ or $\vec{v} - 9$ or $\vec{u} \times 5$ or $\vec{w} \cdot 7$

C) is the answer since once you do the bracket you get a scalar (or number) from the dot product and you can't do the dot product of a number with a vector

F10.

$\vec{u} = q - p = (4,2,2)$

$\vec{v} = r - p = (3,3,-1)$

$\vec{w} = s - p = (5,5,1)$

$$\vec{u} \times \vec{v} = \begin{matrix} 4 & 2 & 2 & 4 & 2 & 2 \\ 3 & 3 & -1 & 3 & 3 & -1 \end{matrix}$$

$$= (-2 - 6, 6 + 4, 12 - 6) = (-8,\ 10, 6)$$

$$V = |(\vec{u} \times \vec{v}) \cdot \vec{w}| = (-8, 10,6) \cdot (5,5,1)$$

$$= |-40 + 50 + 6|$$

$$= 16$$

F11. $\|\vec{v}\| = 4$, and $\vec{u} \cdot \vec{v} = 0$, so using the property in section B Dot Product,

we know $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

$(4\vec{u} - \vec{v}) \cdot \vec{v} = 4\vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{v} = 0 - \|\vec{v}\|^2 = 0 - 16 = -16$

F12.

$4\vec{u} \times (2\vec{u} + 4\vec{v})$
$= 4\vec{u} \times 2\vec{u} + 4\vec{u} \times 4\vec{v}$
$= 8\vec{u} \times \vec{u} + 16\,(\vec{u} \times \vec{v})$
$= 8\,(0, 0, 0) + 16\,(-1, -3, -1)$
$= (0, 0, 0) + (-16, -48, -16)$
$= (-16, -48, -16)$

Note: $\vec{u} \times \vec{u} = \vec{0}$ for any vector $\vec{u}$ and cross product has property $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

G. Practice Exam Questions on Vector Operations (1.1, 1.2)

G1. $\vec{u} \times \vec{v} = \begin{matrix} 4 & 1 & 2 & 4 & 1 & 2 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{matrix}$

$= (-1-2,\ 0+4,\ 4-0) = (-3,\ 4,\ 4)$

$V = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$

$= |(-3,4,4) \cdot (2,-1,4)|$

$= |-6-4+16|$

$= 6$

G2. $d\ u, v = \sqrt{(-1-1)^2 + (4-3)^2 + (5-2)^2}$

$= \sqrt{4+1+9} = \sqrt{14}$

G3. $3(1,2,1) - (1,3,-1)$

$= (3,6,3) - (1,3,-1) = (2,3,4)$

Therefore, the answer is C).

G4. $2(3,6,1) - 3(0,-1,3)$

$= (6,12,2) - (0,-3,9) = (6,15,-7)$

Therefore, the answer is B).

G5. $\|\vec{u}\| = \sqrt{3^2 + 1^2 + 5^2}$

$= \sqrt{9+1+25} = \sqrt{35}$

Therefore, the answer is C).

G6. $= \|(1,3,1) + (2,5,1)\|$

$= \|(3,8,2)\|$

$= \sqrt{3^2 + 8^2 + 2^2}$

$= \sqrt{9+64+4} = \sqrt{77}$

Therefore, the answer is D).

G7. $\|\vec{w}\| = \sqrt{2^2 + 1^2 + 3^2}$

$$= \sqrt{4 + 1 + 9} = \sqrt{14}$$

Therefore, the answer is D).

G8. $\|\vec{v}\| = \sqrt{1^2 + 1^2 + 2^2}$

$= \sqrt{1 + 1 + 4} = \sqrt{6} \quad \therefore \frac{-\vec{v}}{\|\vec{v}\|}$ since it is in the opposite direction

Therefore, the answer is A).

G9. $\vec{u} \cdot \vec{v} = -1 - 6 - 16 = -23$

Therefore, the answer is B).

G10. $\begin{matrix} 1 & 2 & -3 & 1 & 2 & -3 \\ 1 & 3 & 2 & 1 & 3 & 2 \end{matrix}$

$\vec{u} \times \vec{v} = (4 + 9, -3 - 2, 3 - 2) = (13, -5, 1)$

Therefore, the answer is C).

G11. $\vec{u} \times \vec{v} = \begin{matrix} 5 & 1 & 2 & 5 & 1 & 2 \\ -1 & 1 & 0 & -1 & 1 & 0 \end{matrix}$

$$= (0 - 2, -2 - 0, 5 + 1) = (-2, -2, 6)$$

$$\|\vec{u}\| = \sqrt{(-2)^2 + (-2)^2 + (6)^2} = \sqrt{4 + 4 + 36} = \sqrt{44}$$

$\therefore \frac{\vec{u}}{\|\vec{u}\|} = \left(\frac{-2}{\sqrt{44}}, \frac{-2}{\sqrt{44}}, \frac{6}{\sqrt{44}}\right)$ is a unit vector.

Therefore, the answer is E.

G12. C is undefined as once you dot $\vec{v} \cdot \vec{w}$ you get a number and you can't do $\vec{u}$ $\cdot$ (some number)

Therefore, the answer is C).

G13. $\vec{u} \times \vec{v} = \begin{matrix} 1 & 2 & 1 & 1 & 2 & 1 \\ 2 & 1 & 4 & 2 & 1 & 4 \end{matrix}$

$$= (8 - 1, 2 - 4, 1 - 4) = (7, -2, -3)$$

$$A = \|\vec{u} \times \vec{v}\| = \sqrt{7^2 + (-2)^2 + (-3)^2} = \sqrt{49 + 4 + 9}$$

$$= \sqrt{62}$$

Therefore, the answer is E).

G14. $\vec{v} \times \vec{w}$ $\begin{matrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 1 & 4 & 1 & 1 & 4 \end{matrix}$

$$= (8-3, 3-4, 1-2) = (5,-1,-1)$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (-2,3,1) \cdot (5,-1,-1)$$

$$= -10 - 3 - 1 = -14$$

Therefore, the answer is A).

G15. $\vec{u} \cdot \vec{v} = 8(0) + 1(1) + 6(2)$

$$= 0 + 1 + 12 = 13$$

Therefore, the answer is D).

G16. (I) $A = \|\vec{u} \times \vec{v}\| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6} \quad \therefore false$

(II) $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{6}{5\cdot 3} = \frac{6}{15} = \frac{2}{5} \quad \therefore false$

Therefore, the answer is D).

G17. $\overrightarrow{PQ} = q - p = (-1,-6,-2) - (1,3,5)$

$$= (-2,-9,-7)$$

Therefore, the answer is A).

G18. $2\vec{u} = 2(1,3,6) = (2,6,12)$

Therefore, the answer is B).

G19. $\vec{u} = b - a = (-2, 4, -1)$

$\vec{v} = c - a = (-2, 3, 1)$

$\vec{u} \times \vec{v}$ $\begin{matrix} -2 & 4 & -1 & -2 & 4 & -1 \\ -2 & 3 & 1 & -2 & 3 & 1 \end{matrix}$

$= (4+3, 2+2, -6+8) = (7,4,2) \qquad A = \frac{1}{2}\|(7,4,2)\| = \frac{1}{2}\sqrt{7^2 + 4^2 + 2^2}$

$$= \frac{1}{2}\sqrt{49 + 16 + 4} = \frac{1}{2}\sqrt{69}$$

G20. $\vec{u}\cdot\vec{v} = (1,2)\cdot(-1,1) = -1+2 = 1$

$\|\vec{u}\| = \sqrt{1^2+2^2} = \sqrt{5}$

$\|\vec{v}\| = \sqrt{(-1)^2+1^2} = \sqrt{2}$

$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

$= \frac{1}{\sqrt{5}\sqrt{2}} = \frac{1}{\sqrt{10}}$

G21. $(2,k,3k)\cdot(2,-5,1) = 0$

$4-5k+3k = 0$

$-2k = -4$

$k = 2$

Therefore, the answer is C).

G22. $\|(2,3,-1)+2(1,2,-1)\|$

$= \|(2,3,-1)+(2,4,-2)\|$

$= \|(4,7,-3)\| = \sqrt{16+49+9} = \sqrt{74}$

Therefore, the answer is C).

G23. $\overrightarrow{QP} = p-q = (-2,-3,4,6)-(1,2,3,4)$

$= (-3,-5,1,2)$

Therefore, the answer is B).

G24. $(5,4k,3,6)\cdot(-2,2,-3,k) = 0$

$-10+8k-9+6k = 0$

$14k-19 = 0$

$14k = 19$

$k = \frac{19}{14}$

Therefore, the answer is B).

G25. $\vec{u} \times \vec{v} = \begin{matrix} 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 \end{matrix}$

$= (0+1, 1-0, -1-2) = (1,1,-3)$

$V = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$

$= |(1,1,-3) \cdot (-3,1,1)|$

$= |-3+1-3| = |-5| \;\; = 5$

G26. $(8,9,4) \;\; collinear \quad (16,12,6)$

$\times 2$

$8 \times 2 = 16 \qquad a \times 2 = 12 \qquad 4 \times 2 = b$

$a = 6 \qquad b = 8$

Therefore, the answer is A).

G27. $\vec{u} = (6,-3,0)$

$\|\vec{u}\| = \sqrt{6^2 + (-3)^2 + 0^2} = \sqrt{36+9} = \sqrt{45}$

G28. $\sqrt{(2c)^2 + (3c)^2 + c^2 + 0^2} = 1$

$4c^2 + 9c^2 + c^2 = 1$

$14c^2 = 1$

$c^2 = \frac{1}{14} \qquad c = \pm\sqrt{\frac{1}{14}} = \pm\frac{1}{\sqrt{14}}$

G29. $\overrightarrow{PQ} = q - p = (4,4,4,4)$

$\overrightarrow{RS} = s - r = (2,2,2,2)$

$\therefore \overrightarrow{PQ} = 2\overrightarrow{RS}$

$\therefore parallel$

G30. $(2,k)$ $collinear$ $(7,9)$

$$\times\frac{7}{2}$$

$$2\times\frac{7}{2}=7 \qquad \therefore k\times\frac{7}{2}=9$$

$$\frac{7k}{2}=9$$

$$7k=18$$

$$k=\frac{18}{7}$$

Therefore, the answer is B).

G31. $i)$ $true$ ii) false $iii)$ $i\times i=j\times j=\vec{0}$ true

$iv)$ $\|(1,0,0)+(0,1,0)+(0,0,1)\|=\|(1,1,1)\|$ true

$$=\sqrt{1^2+1^2+1^2}=\sqrt{3}$$

Therefore, the answer is C).

G32. The answer is D).

G33. $A(1,3,1)$ $B(0,1,5)$ $C(0,0,0)$

$$\vec{u}=b-a=(-1,-2,4)$$

$$\vec{v}=c-a=(-1,-3,-1)$$

$$\vec{u}\times\vec{v}=\begin{matrix}-1 & -2 & 4 & -1 & -2 & 4\\ -1 & -3 & -1 & -1 & -3 & -1\end{matrix}$$

$$=(2+12,-4-1,3-2)$$

$$=(14,-5,1)$$

$$A=\frac{1}{2}\|(14,-5,1)\|$$

$$=\frac{1}{2}\sqrt{14^2+(-5)^2+1^2}$$

$$=\frac{1}{2}\sqrt{196+25+1}=\quad\frac{1}{2}\sqrt{222}$$

G34. $\|\vec{u}\| = \sqrt{3^2 + 3^2 + 3^2 + 3^2}$

$= \sqrt{9+9+9+9} = \sqrt{36} = 6$

$\vec{v} = k(3,3,3,3) = \left(\frac{-3}{6}, \frac{-3}{6}, \frac{-3}{6}, \frac{-3}{6}\right)$ ←unit vector in opposite direction

$\therefore k = \frac{-1}{6}$

Therefore, the answer is E).

G35. i) true ii) true iii) false iv) false v) true

Therefore, the answer is D).

G36. $\sqrt{c^2 + 4^2} = 7$

$c^2 + 16 = 49$

$c^2 = 33$

$c = \pm\sqrt{33}$

Therefore, the answer is A).

G37. $\vec{u} \times \vec{v}\ is\ only\ defined\ in\ R^3$

i) defined ii) defined iii) undefined iv) undefined v) undefined

Therefore, the answer is D). How would your answer differ if it were in R^3? iii) and iv) would be undefined since when you do the brackets first, you get a number since it is dot product and then you can’t dot or cross a vector with a number, so you can’t do it!

G38. $\vec{u} \cdot \vec{v} = (2,-1,6) \cdot (2,2,2)$

$= 4 - 2 + 12 = 14$

$\|\vec{u}\| = \sqrt{2^2 + (-1)^2 + 6^2} = \sqrt{4+1+36} = \sqrt{41}$

$\|\vec{v}\| = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12}$

$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{14}{\sqrt{41}\sqrt{12}}$

H. Point-Normal Form (1.3)

H1. a) $\vec{n} \cdot (x - p) = 0$

$$(1,2,3) \cdot \big(x - (7,8,9)\big) = 0$$

b) $(5,3,64) \cdot \big(x - (0,-1,2)\big) = 0$

c) $(2,5,-2) \cdot \big(x - (1,1,-1)\big) = 0$

I. Standard Form of the Equation (1.3)

I1. b) $a = 5, \quad b = 3, \quad c = 4$

$$5x + 3y + 4z = (5,3,4) \cdot (0,-1,2)$$

$$5x + 3y + 4z = 0 - 3 + 8$$

$$5x + 3y + 4z = 5$$

c) $a = 1, \quad b = 3, \quad c = -2$

$$1x + 3y - 2z = (1,3,-2) \cdot (2,2,-1)$$

$$x + 3y - 2z = 2 + 6 + 2$$

$$x + 3y - 2z = 10$$

I2. $\vec{n}(3,-1,2)$

Find a point P. $Let\ x = z = 0$

$$3(0) - y + 2(0) = 9$$

$$-y = 9$$

$$y = -9 \quad \therefore P(0,-9,0)$$

$$\vec{n} \cdot (x - p) = 0$$

$$(3,-1,2) \cdot \big(x - (0,-9,0)\big) = 0$$

I3. $\vec{u} = q - p = (1,2,1)$

$$\vec{v} = r - p = (3,4,3)$$

$$\vec{n} = \vec{u} \times \vec{v} \quad \begin{matrix} 1 & 2 & 1 & 1 & 2 & 1 \\ 3 & 4 & 3 & 3 & 4 & 3 \end{matrix}$$

$$\vec{u} \times \vec{v} = (6 - 4, 3 - 3, 4 - 6) = (2,0,-2)$$

$$\vec{n} \cdot (x - p) = 0$$

$$(2,0,-2) \cdot \big(x - (1,-1,3)\big) = 0$$

$\uparrow$ *any point $P, Q,$ or R*

I4. $ax + by + cz = \vec{n} \cdot p$

$\vec{u} = q - p = (2,1,-2)$

$\vec{v} = r - p = (2,0,-3)$

$$\vec{n} = \vec{u} \times \vec{v} \quad \begin{matrix} 2 & 1 & -2 & 2 & 1 & -2 \\ 2 & 0 & -3 & 2 & 0 & -3 \end{matrix}$$

$$\vec{n} = (-3 + 0, -4 + 6, 0 - 2) = (-3, 2, -2)$$

$$-3x + 2y - 2z = (-3,2,-2) \cdot (1,2,4)$$

$$-3x + 2y - 2z = -3 + 4 - 8$$

$$-3x + 2y - 2z = -7$$

I5. $\vec{u} = q - p = (1,2,-5)$

$\vec{v} = r - p = (-5,0,-2)$

$$\vec{n} = \vec{u} \times \vec{v} \quad \begin{matrix} 1 & 2 & -5 & 1 & 2 & -5 \\ -5 & 0 & -2 & -5 & 0 & -2 \end{matrix}$$

$$\vec{n} = (-4 + 0, 25 + 2, 0 + 10) = (-4, 27, 10)$$

$$\vec{n} \cdot (x - p) = 0$$

$$(-4,27,10) \cdot \big(x - (4,-1,3)\big) = 0$$

$\uparrow P, Q \; or \; R$

J. Point-Parallel Form Equation For a Line (1.3)

J1. $\vec{x} = p + t\vec{v}$

$(x, y, z) = (-1,4,3) + t(5,0,-1)$

$x = -1 + 5t$

$y = 4$

$z = 3 - t$

J2. $(x, y, z) = (2,-3,4) + t(4,1,-2)$

$x = 2 + 4t$

$y = -3 + t$

$z = 4 - 2t$

J3. $\vec{v} = q - p = (2,0,1)$

$\vec{x} = p + t\vec{v}$

$(x, y, z) = (1,4,6) + t(2,0,1)$

$x = 1 + 2t \qquad y = 4 \qquad z = 6 + t$

K. Two-Point Form Equation for a Line (1.3)

Example 2. This is a 2-point form, so $\vec{v} = p - q$

$$= (4,5,9) - (1,3,4)$$

$$= (3,2,5)$$

Example 3. $\vec{v} = (4,5,9)$ since it is in point parallel form

K1. $(x, y, z) = (1 - t)(-1,3,-5) + t(-2,5,7)$

$$x = -1(1 - t) - 2(t) = -1 + t - 2t = -1 - t$$

$$y = 3(1 - t) + 5(t) = 3 - 3t + 5t = 3 + 2t$$

$$z = -5(1 - t) + t(7) = -5 + 5t + 7t = -5 + 12t$$

K2. $(x, y, z) = (1 - t)(1,3,4) + t(7,5,2)$

$$x = 1(1 - t) + t(7) = 1 - t + 7t = 1 + 6t$$

$$y = 3(1 - t) + t(5) = 3 - 3t + 5t = 3 + 2t$$

$$z = 4(1 - t) + t(2) = 4 - 4t + 2t = 4 - 2t$$

L. Distance from a Point to a Line (1.3)

L1. $\vec{n} \cdot p = -12$

$\vec{n} = (3, -4)$

$\|\vec{n}\| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

$\vec{n} \cdot q = (3, -4) \cdot (4, -1) = 12 + 4 = 16$

$d = \frac{|\vec{n} \cdot q - \vec{n} \cdot p|}{\|\vec{n}\|} = \frac{|16 - (-12)|}{5} = \frac{28}{5}$

L2. $\vec{v} = (-1, -4)$

To convert from $\vec{v}\ to\ \vec{n}\ or\ \overline{\vec{n}}\ to\ \bar{\vec{v}}\ in\ R^2$

1. Switch the sign of y
2. Switch $x\ and\ y$

$\therefore \vec{n} = (4, -1)$

$\|\vec{n}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$

$\vec{n} \cdot p = (4, -1) \cdot (2,5) = 8 - 5 = 3$

$\vec{n} \cdot q = (4, -1) \cdot (7, -3) = 28 + 3 = 31$

$d = \frac{|31 - 3|}{\sqrt{17}} = \frac{28}{\sqrt{17}}$

M. Distance from a Point to a Plane (1.3)

M1. $\vec{n} \cdot p = 8$ $\quad\quad$ $\vec{n} = (2, -1,3)$

$$\|\vec{n}\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

$$\vec{n} \cdot q = (2, -1,3) \cdot (1,2,1) = 2 - 2 + 3 = 3$$

$$d = \frac{|3-8|}{\sqrt{14}} = \frac{5}{\sqrt{14}}$$

M2. $\vec{n} \cdot p = 10$

$$\vec{n} = (2,1,3)$$

$$\|\vec{n}\| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$

$$\vec{n} \cdot q = (2,1,3) \cdot (3, -2,1) = 6 - 2 + 3 = 7$$

$$d = \frac{|7-10|}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

N. Determining if Planes are Parallel, Perpendicular, Identical or None of These (1.3)

N1. $\vec{n}_1 = (4,-2,6)$ $\vec{n}_2 = (6,-3,9)$

$$\vec{n}_1 \times \frac{3}{2} = \vec{n}_2 \quad and \quad 3 \times \frac{3}{2} = 4.5$$

∴The constant terms on the right follow the same multiple as the normals ∴They're identical planes

N2. $(4,-2,2) \cdot (5,-3,-15) = 20 + 6 - 30 \neq 0$

∴not perpendicular

$\vec{n}_1$ is not a multiple of $\vec{n}_2$ ∴not identical or parallel

Therefore, the answer is none of these.

O. Vectors Parallel to Lines (1.3)

O1. Standard form $\therefore \vec{n} = (3, -5)$

$\therefore \vec{v} = vector\ parallel = (5, 3)$. Since this isn't there, we multiply it by -3 and

therefore, the answer is D).

O2. Point normal form $\therefore \vec{n} = (-2,5)$

$\vec{v} = parallel\ vector = (-5, -2)$

$x - 2$

$= (10,4)$

Therefore, the answer is C).

O3. Point parallel form $\therefore \vec{v} = (10, -7)$

Therefore, the answer is B).

O4. Parallel to $3x - 5y = 6$

$\overrightarrow{n_1} = (3, -5) = \overrightarrow{n_2}$ since they are parallel

$$standard\ form\ \ ax + by = \vec{n} \cdot p$$

$$through\ P(1,2)$$

$\therefore 3x - 5y = (3, -5) \cdot (1,2)$

$3x - 5y = 3 - 10$

$3x - 5y = -7$

O5. Parallel to $4x + 7y = 4$

$\overrightarrow{n_1} = (4,7)$

$\overrightarrow{v_1} = (-7,4)$

$\therefore (x, y) = p + t\vec{v}$ Pt. Parallel Form

$(x, y) = (-1,3) + t(-7,4)$

Therefore, the answer is E).

O6. Between A and C It has to have the same normal since it is a parallel line $\therefore$ $A\ is\ parallel$

Between B and D $\overrightarrow{n_1}(2,3)$ $\therefore \overrightarrow{v_1}(-3,2)$

$\therefore B\ is\ parallel\ since\ it\ has\ the\ same\ directing\ vector\ \vec{v}$

$\therefore A\ and\ B\ are\ correct$

P. Vectors Perpendicular to Lines (1.3)

P1. Point normal form $\vec{n} = (-3,5,6)$ or any multiple. Multiply by -2 and you get C).

P2. Standard form $\vec{n} = (6,-7,8)$

P3. Standard form $\vec{n} = (5,-7)$ or any multiple. Here, we multiply the components by -3. Therefore, the answer is D).

P4. Point parallel form $\vec{v} = (2,-1)$ $\therefore \vec{n} = (1,2)$ or any multiple. Switch the signs and you get (-1,-2).Therefore, the answer is D).

P5. Perpendicular to $5x - 4y = 2$ $\overrightarrow{n_1} = (5,-4)$ $\overrightarrow{v_1} = (4,5)$

Standard form $ax + by = \vec{n} \cdot p$

$\overrightarrow{n_2} = \overrightarrow{v_1} = (4,5)$ since perpendicular

$\therefore 4x + 5y = (4,5) \cdot (2,3)$

$4x + 5y = 8 + 15$

$4x + 5y = 23$

P6. $\overrightarrow{n_1} = (-2,6)$ $\overrightarrow{v_1} = (-6,-2) or\ (6,2)$

Standard $\overrightarrow{n_2} = \overrightarrow{v_1}$ since perpendicular

$\overrightarrow{n_2} = (6,2)$

$6x + 2y = (6,2) \cdot (1,2)$

$6x + 2y = 10$ ∴A, B are false as point normal is always equal to 0. So we don't need to check it. C is false as point normal is always equal to 0, so we didn't need to check it.

D and E Point parallel form $\overrightarrow{v_2} = \overrightarrow{n_1} = (-2,6)$

$\vec{x} = (1,2) + t(-2,6)$ $\therefore D\ is\ true$

Note: F is 2 point and we don't know 2 points, so it can't be correct

Therefore, the answer is D).

P7. $\overrightarrow{v_2} = \overrightarrow{n_1} = (4,-2,7)$ $since\ perpendicular$

$\therefore \vec{x} = p + t\vec{v}$

$\vec{x} = (1,3,2) + t(4,-2,7)$

Q. Point of Intersection of a Line and a Plane (1.3)

Q1. $x = 5 + t,\ \ y = -1 + 2t,\ \ z = -2 + 3t$

$3x + 2y + z = 21$

$3(5 + t) + 2(-1 + 2t) + (-2 + 3t) = 21$

$15 + 3t - 2 + 4t - 2 + 3t = 21$

$10t + 11 = 21$

$10t = 21 - 11$

$10t = 10$

$t = 1$

$\therefore x = 5 + 1 = 6$

$y = -1 + 2(1) = 1$

$z = -2 + 3(1) = 1 \qquad \therefore POI\ (6,1,1)$

Q2. $x = 1 + t \qquad y = -1 + 2t \qquad z = t$

3x – y + 2z = 10

$3(1 + t) - (-1 + 2t) + 2(t) = 10$

$3 + 3t + 1 - 2t + 2t = 10$

$3t + 4 = 10$

$3t = 6$

$t = \frac{6}{3} = 2$

$b = y = -1 + 2(2) = 3$

R. Intersection of Lines (1.3)

Example 1. $x = 1 + t \quad y = 3 - 4t$

$x = 2 + 4s \quad y = 3 - 2s$

$x = x \qquad y = y$

$2 + 4s = 1 + t \qquad 3 - 2s = 3 - 4t$

$4s - t = -1 \; \boxed{1} \qquad -2s + 4t = 0 \; \boxed{2}$

$4s - t = -1$

$-2s + 4t = 0 \quad (\times 2)$

$4s - t = -1$

$-4s + 8t = 0$

Add $7t = -1$

$t = -\frac{1}{7}$

$x = 1 - \frac{1}{7} = \frac{7}{7} - \frac{1}{7} = \frac{6}{7}$

$y = 3 - 4\left(\frac{-1}{7}\right) = \frac{21}{7} + \frac{4}{7} = \frac{25}{7} \quad \therefore POI \left(\frac{6}{7}. \frac{25}{7}\right)$

Example 2. $x = -1 + 3t \qquad x = 1 + 2s$

$y = 3 - 2t \qquad y = 4 - s$

$x = x \qquad y = y$

$1 + 2s = -1 + 3t \qquad 3 - s = 3 - 2t$

$2s - 3t = -2 \qquad -s + 2t = -1$

$2s - 3t = -2$

$-s + 2t = -1 \quad \times 2$

$2s - 3t = -2$

$-2s + 4t = -2$

$Add \quad t = -4$

$a = x = -1 + 3(-4) = -13$

S. The Most Challenging Exam Questions on Lines and Planes (1.3)

S1. $\vec{n} = \vec{u} \times \vec{v}$

$$\begin{matrix} 3 & -1 & -4 & 3 & -1 & -4 \\ -1 & -2 & 3 & -1 & -2 & 3 \end{matrix}$$

$\vec{n} = (-3-8, 4-9, -6-1) = (-11, -5, -7)$

S2. $LS = 2(1) + 3(3) = 11$ $RS = 6$

$\therefore LS \neq RS$ $\therefore No, the\ point\ is\ not\ on\ the\ line.$

S3. $t = 1\ gives\ (2,3) + 1(4,-2) = (6,1) \therefore B\ is\ on\ line$

$t = 2\ gives\ (2,3) + 2(4,-2) = (10,-1) \therefore C\ is\ on\ line$

CHECK D

$(4,3) = (2,3) + t(4,-2)$

x	y
$4 = 2 + 4t$	$3 = 3 - 2t$
$2 = 4t$	$0 = -2t$
$t = \frac{1}{2}$	$t = 0$

$\therefore Point\ is\ not\ on\ line$ (different values of t)

Therefore, the answer is D).

S4. $\overrightarrow{n_2} = \overrightarrow{v_1}$ since perpendicular

Therefore, the answer is A).

S5.

$\pi 1: (x, y, z) = (4, -2, -4) + t(1,2,3)$

$\overrightarrow{v_1} = (1,2,3)$

$\pi 2: \overrightarrow{v_2} = q - p = (2,4,6)$

So, we notice that $2\overrightarrow{v_1} = \overrightarrow{v_2}$, so the lines are parallel

Now, are they identical? Check if the point (3,2,4) on line 2 is also on line 1?

From line 1, we have parametric equations below and we check to see if we get the same value of t from each equation:

(3,2,4)=(4,-2,-4) + t(1,2,3)

For x

3=4+t

t=-1

For y

2=-2+2t

t=2 so it is not the same t value, so they are just parallel and not identical.

The answer is B).

S6. $(3, -2,1) \times 3 = (9, -6,3)$

$\therefore$ *they are parallel and C isn't involved* (parallel only involves the left hand side) Therefore, the answer is D).

S7. $5 \times 3 = c$

$c = 15$

Therefore, the answer is C).

S8. $(3,-2c,-1)\cdot(9,-6,3)=0$

$27+12c-3=0$

$12c+24=0$

$12c=-24$

$c=-2$

Therefore, the answer is C).

S9. $\vec{n}=\vec{u}\times\vec{v}$

$$\begin{matrix} 1 & 0 & 2 & 1 & 0 & 2 \\ 2 & -2 & 1 & 2 & -2 & 1 \end{matrix}$$

$\vec{n}=(0+4,4-1,-2-0)=(4,3,-2)$

$ax+by+cz=\vec{n}\cdot p$

$4x+3y-2z=(4,3,-2)\cdot(2,3,4)$

$4x+3y-2z=8+9-8$

$4x+3y-2z=9$

b) 4(1-t)+3(2+3t)-2(4+t)=9

4 – 4t + 6 + 9t -8 – 2t = 9

3t +2 = 9

3t = 7

t=7/3

x= 1- t = 1 – 7/3 = 3/3 – 7/3 = -4/3

S10. $x = x$ $\quad\quad$ $y = y$

$3 + s = 2 + t$ $\quad\quad$ $3 - 2s = 3 - 4t$

$s - t = -1$ $\quad\quad$ $-2s + 4t = 0$

$s - t = -1 \quad \times 2$

$-2s + 4t = 0$

$2s - 2t = -2$

$-2s + 4t = 0$

$Add \quad 2t = -2$

$t = -1$

$s - t = -1$

$s - (-1) = -1$

$s + 1 = -1$

$s = -2$

$x = 3 + s = 3 + 2 = 1$

$y = 3 - 2s = 3 - 2(-2) = 7$

$z = s = -2$

$x = 2 + t = 2 - 1 = 1$

$y = 3 - 4t = 3 - 4(-1) = 7$

$z = t - 1 = -1 - 1 = -2$

$\therefore POI\ (1,7,-2) \quad \therefore y = b = 7$

Therefore, the answer is C).

S11. If they're perpendicular, $\overrightarrow{v_1} \cdot \overrightarrow{v_2} = 0$

$(2,-5,1) \cdot (1,1,-3) = 2 - 5 - 3 \neq 0$

$(2,-5,1) \cdot (1,3,1) = 2 - 15 + 1 \neq 0$

$(2,-5,1) \cdot (-1,2,3) = -2 - 10 + 3 \neq 0$

$(2,-5,1) \cdot (2,-1,-7) = 4 + 5 - 7 \neq 0$

Therefore, the answer is E).

S12. Standard form $\overrightarrow{n_1} = (3, -1)$ $\overrightarrow{v_1} = (1,3)$

$\overrightarrow{n_2} = \overrightarrow{v_1} = (1,3)$ since perpendicular

$\therefore A \; and \; B \; are \; false$

Point parallel $\overrightarrow{v_2} = \overrightarrow{n_1} = (3, -1)$

Therefore, the answer is C).

S13. $\vec{n} \cdot \vec{v} = 0$ (they are always perpendicular to each other)

$(3, -4,2) \cdot (1,1, -3) = 3 - 4 - 6 \neq 0$

$(3, -4,2) \cdot (2,0, -3) = 6 + 0 - 6 = 0$

$(3, -4,2) \cdot (1, -1,2) = 3 + 4 + 4 \neq 0$

Therefore, the answer is B).

S14. Find a two-point equation parallel to x=(5,0,2) + t(6,8,2) and through the point (1,4,2).

From the point parallel form given:

$\vec{v}$=(6,8,2) so since they are parallel, they have the same v

So, since $\vec{v}$=q-p we get:

(6,8,2) = q - (1,4,2)

Solving for point Q we get: (6,8,2) + (1,4,2) = (7,12,4)

The two-point equation is $\vec{x} = (1 - t)p + tq$

$\vec{x} = (1 - t)(1,4,2) + t\,(7,12,4)$

T. Practice Exam Questions on Equations of Lines and Planes (1.3)

T1. $\vec{n}(1,-3,4)$ $\vec{n}\cdot p = 0$ (right side in standard form)

$$\|\vec{n}\| = \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{1+9+16} = \sqrt{26}$$

$$\vec{n}\cdot q = (1,-3,4)\cdot(1,2,-1) = 1 - 6 - 4 = -9$$

$$d = \frac{|\vec{n}\cdot q - \vec{n}\cdot p|}{\|\vec{n}\|} = \frac{|-9-0|}{\sqrt{26}} = \frac{9}{\sqrt{26}}$$

T2. $\vec{n}\cdot p = 12$ (right side in standard form)

$$\vec{n}(4,-7) \qquad \|\vec{n}\| = \sqrt{4^2 + (-7)^2}$$

$$= \sqrt{16+49}$$

$$= \sqrt{65}$$

$$\vec{n}\cdot q = (4,-7)\cdot(2,5) = 8 - 35 = -27$$

$$d = \frac{|\vec{n}\cdot q - \vec{n}\cdot p|}{\|\vec{n}\|} = \frac{|-27-12|}{\sqrt{65}} = \frac{39}{\sqrt{65}}$$

T3. $x = 2 + 5t$ $x = 1 + 4s$

$y = 2 + 3t$ $y = 3 + 2s$

$x = x$ $y = y$

$1 + 4s = 2 + 5t$ $3 + 2s = 2 + 3t$

$4s - 5t = 1$ $2s - 3t = -1$

$$4s - 5t = 1$$

$$2s - 3t = -1 \quad \times(-2)$$

$$4s - 5t = 1$$

$$-4s + 6t = 2$$

$$Add \qquad t = 3$$

$$x = 2 + 5(3) = 17$$

$$y = 2 + 3(3) = 11$$

$POI\ (17, 11)$

T4. Therefore, the answer is A).

T5. Perpendicular to $4x - 3y = 7$

$\overrightarrow{n_1} = (4, -3)$

$\overrightarrow{v_1} = (3,4)$

Standard $ax + by = \vec{n} \cdot p$

$through\ P(1, -2)$

$\overrightarrow{n_2} = \overrightarrow{v_1} = (3,4)$

$$3x + 4y = (3,4) \cdot (1, -2)$$

$$3x + 4y = 3 - 8$$

$$3x + 4y = -5$$

Therefore, the answer is E).

T6. $\overrightarrow{n_2} = \overrightarrow{v_1} = (3,4)$

Therefore, the answer is C).

T7. $-x + 3y + 4z = (-1,3,4) \cdot (1,2,1)$

$$-x + 3y + 4z = -1 + 6 + 4$$

$$-x + 3y + 4z = 9$$

Therefore, the answer is B).

T8. $\vec{v} = q - p = (4,2)$

$\vec{n} = (-2,4)$

Therefore, the answer is B).

T9. Therefore, the answer is A). since it has the same $\vec{v}$ and goes through the point (1,4,5)

T10. $ax + by + cz = \vec{n} \cdot p$

$$3x + 1y + 2z = (3,1,2) \cdot (1,2,3)$$

$$3x + y + 2z = 11$$

Therefore, the answer is E).

T11. $\vec{v} = q - p = (1, -2, -1, -1)$

$\vec{x} = p + t\vec{v}$

$(x_1, x_2, x_3, x_4) = (1,3,4,5) + t(1, -2, -1, -1)$

$x_1 = 1 + t$

$x_2 = 3 - 2t$

$x_3 = 4 - t$

$x_4 = 5 - t$

T12. $-5x + 4y + 2z = (-5,4,2) \cdot (4,1,2)$

$-5x + 4y + 2z = -20 + 4 + 4$

$-5x + 4y + 2z = -12$

Therefore, the answer is B).

T13. $x = 4 + t$

$y = -3 + 3t$

$z = 1 + 2t$

$2x + 3y + z = 13$

$2(4 + t) + 3(-3 + 3t) + (1 + 2t) = 13$

$8 + 2t - 9 + 9t + 1 + 2t = 13$

$13t = 13$

$t = 1$

$b = y = -3 + 3(1) = 0$

Therefore, the answer is C).

T14. Since they are perpendicular, n2= v1.

Therefore, the answer is B).

T15. $\vec{v} = q - p = (2, -1, -8)$

$\vec{x} = (1,2,3) + t(2, -1, -8)$

$x = 1 + 2t$

$y = 2 - t$

$z = 3 - 8t$

Therefore, the answer is D).

T16. See which point gives $LS = RS$ Check (2,0,2)

$LS = 4(2) - 0 + 2 = 10$

$RS = 10 \qquad \therefore LS = RS$

Therefore, the answer is B).

T17. E has the same $\vec{n}$ so it is parallel

The rest are in Point Parallel form $\vec{x} = p + t\vec{v}$

$5x - 7y = 10 \qquad \overrightarrow{n_1} = (5, -7) = \overrightarrow{v_2}$ since perpendicular

$\therefore \vec{x} = (1,4) + t(5, -7)$

Therefore, the answer is A).

T18. Check i) $\pi_1 \times 2 = \pi_2 \quad \therefore true$

Check ii) $(3, -2,1) \cdot (5,1, -13) = 15 - 2 - 13 = 0 \quad \therefore true$

Check iii) false – the normal's are not multiples

Therefore, the answer is C).

T19. $(2,1,2) = (2, -3,4) + t(0, -8,4)$

x	y	z
$2 = 2 + 0$	$1 = -3 - 8t$	$2 = 4 + 4t$
	$4 = -8t$	$-2 = 4t$
	$t = \frac{-1}{2}$	$t = \frac{-1}{2}$

Therefore, the answer is E).

T20. $x = 3 + t$ $y = -2$ $z = 1 + 3t$

$4x - 3y + 2z = 40$

$4(3 + t) - 3(-2) + 2(1 + 3t) = 40$

$12 + 4t + 6 + 2 + 6t = 40$

$10t + 20 = 40$

$10t = 20$

$t = 2$

$b = y = -2$

Therefore, the answer is B).

T21. $\vec{n}(5, -2, -1)$ $\vec{n} \cdot p = 3$

$\|\vec{n}\| = \sqrt{5^2 + (-2)^2 + (-1)^2} = \sqrt{25 + 4 + 1} = \sqrt{30}$

$\vec{n} \cdot q = (5, -2, -1) \cdot (1, -1, 2) = 5 + 2 - 2 = 5$

$d = \frac{|5-3|}{\sqrt{30}} = \frac{2}{\sqrt{30}}$

T22. $\overrightarrow{n_1} = (2, -3, 5)$ $\overrightarrow{n_2} = (3, \frac{-9}{2}, \frac{15}{2})$

$\overrightarrow{n_1} \times \frac{3}{2} = \overrightarrow{n_2}$ ∴they are parallel for any value of c since c is on the right side and not on the left side (parallel only means the left sides are multiples, ie. the normals are multiples)

Therefore, the answer is E).

T23. $(2, -4, c)$ $parallel$ $(4, -8, 12)$

$2 \times 2 = 4$ $-4 \times 2 = -8$ $c \times 2 = 12$

$c = 6$

Therefore, the answer is D).

T24. $x = x$ $\qquad y = y$

$$5 - 2s = 8 + t \qquad -4 + 3s = -1 + 8t$$

$$-2s - t = 3 \quad \boxed{1} \qquad 3s - 8t = 3 \quad \boxed{2}$$

$$-2s - t = 3 \quad \times(-8)$$

$$3s - 8t = 3$$

$$16s + 8t = -24$$

$$3s - 8t = 3$$

$$Add \quad 19s = -21$$

$$s = \frac{-21}{19}$$

$$x = 5 - 2s$$

$$= 5 - 2\left(\frac{-21}{19}\right) = \frac{95+42}{19} = \frac{137}{19} \qquad \therefore a = \frac{137}{19}$$

T25. $\vec{v} = q - p = (4,4,4) \textit{or any multiple}$

Therefore, the answer is D).

T26. $t = 1$ $(2,-4) + (1,3) = (3,-1)$ So, A is on the line

$t = 2$ $(2,-4) + 2(1,3) = (4,2)$ So, B is on the line

$t = -1$ $(2,-4) - 1(1,3) = (1,-7)$ So, C is on the line

$t = 3$ $(2,-4) + 3(1,3) = (5,5)$ So, D is on the line

Therefore, the answer is E).

Or substitute each point in on the left of the equation and see if the parametric equations give the same value for t in each equation. If t is the same, the point is on the line.

For A, we get: (3,-1) =(2,-4) + t(1,3)

x gives us 3=2 + t

and t=1

y gives us -1= -4 + 3t

3=3t

t=1, so the point is on the line.

For B, we get (4,2)=(2,-4) + t(1,3)

For x, we get: 4= 2+t

t=2 and for y, we get: 2= -4 + 3t

6=3t

t=2, so the t is the same and the point is on the line. Etc.

T27. i) $x + y = (1,1) \cdot (1,7)$

$x + y = 8 \quad true$

ii) $x + y = (1,1,0) \cdot (1,7,0)$

$x + y = 8 \quad true$

iii) False – it should be called a hyperplane

Therefore, the answer is E).

T28. $\overrightarrow{n_2} = \overrightarrow{v_1}$ since perpendicular

Therefore, the answer is B).

T29. $x = 3 + t \quad y = 1 + 3t \quad z = 5 + 2t$

$$2x - y + z = 28$$

$$2(3 + t) - (1 + 3t) + 5 + 2t = 28$$

$$6 + 2t - 1 - 3t + 5 + 2t = 28$$

$$t + 10 = 28$$

$$t = 18$$

$$a = x = 3 + t = 3 + 18 = 21$$

Therefore, the answer is A).

T30. Check i) $(2,1,-3) \cdot (-4,5,-1) = -8 + 5 + 3 = 0 \quad \therefore true$

Check ii) No, they're not multiples on the left

Check iii) yes, $\pi_1 \times 5 = \pi_2 \quad \therefore true$

Therefore, the answer is D).

T31. $\overrightarrow{n_1} \times 4 = \overrightarrow{n_2}$ ∴They're parallel for any value of c

Therefore, the answer is A).

T32. $(2, -c, 1)\ parallel\ to\ (6, -12,3)$

ie. they are multiples on the left $2 \times 3 = 6$

$$\therefore -c \times 3 = -12$$

$$-3c = -12$$

$$c = 4$$

Therefore, the answer is C).

T33. $x = x$ $y = y$

$3 + 4s = 3 - t$ $5 - 3s = 2 + 2t$

$4s + t = 0$ $-3s - 2t = -3$

$4s + t = 0 \quad \times 2$

$-3s - 2t = -3$

$8s + 2t = 0$

$-3s - 2t = -3$

$Add \quad 5s = -3$

$s = \frac{-3}{5}$

$x = a = 3 + 4\left(\frac{-3}{5}\right) = \frac{15}{5} - \frac{12}{5} = \frac{3}{5}$

T34. From the original line (x,y,z)=(1,4,4) + t(3,-3,-7)

$\vec{v} = (3, -3, -7)$ (the numbers beside the t's in the parametric equations)

New line is in point parallel form, or x=p+ tv

(x,y,z)= (2,-1,-5) +t(3,-3,-7)

Therefore, the answer is B).

T35. $(3, k, 5) = (1,2,1) + t(4,6,8)$

x y z

$3 = 1 + 4t$ $k = 2 + 6t$ 5=1+8t

$2 = 4t$ $k = 2 + 6(\frac{1}{2})$ 4=8t and t=1/2 (same as from x)

$t = \frac{1}{2}$ $k = 2 + 3 = 5$ $\therefore k = 5$

T36. $\vec{u} = \overrightarrow{PQ} = q - p = (1,2)$

$\vec{v} = \overrightarrow{RS} = s - r = (-2, -4)$

Therefore, the answer is B). It's true since $-\overrightarrow{2u} = \vec{v}$

T37. $\overrightarrow{v_1} = \overrightarrow{n_2} = (4, -5, 6)$ since perpendicular

$ax + by + cz = \vec{n} \cdot p$

$4x - 5y + 6z = (4, -5, 6) \cdot (1, 4, 1)$

$4x - 5y + 6z = 4 - 20 + 6$

$4x - 5y + 6z = -10$

T38. Perpendicular to $3x - 2y = 6$

$\overrightarrow{n_1} = (3, -2)$

$\overrightarrow{v_1} = (2, 3)$

Standard Form

$ax + by = \vec{n} \cdot p$

$\overrightarrow{n_2} = \overrightarrow{v_1}$ since perpendicular

$\overrightarrow{n_2} = (\begin{smallmatrix} 2, 3 \\ a, \ b \end{smallmatrix})$

$2x + 3y = (2, 3) \cdot (1, 2)$

$2x + 3y = 8$

T39. A vector parallel to the line is the direction vector v. The normal is (8,-5), so the v is (5,8). Therefore, the answer is C).

T40. A vector perpendicular to the line is the normal and we have standard form and the normal
is (8, -5). Therefore, the answer is A).

T41. A vector parallel to the line is the direction vector v and this line is in point-parallel form, so the v is (-5,1). Therefore, the answer is B).

T42. The vector perpendicular to the line is the normal and the v is (-5,1), so the normal is $(-1, -5)$ *or* (1,5). Therefore, the answer is C).

T43. The vector parallel to the line means to find the v and we have point-normal form with a normal

(-3,6), so the direction vector v would be $(-6,-3)$ or $(6,3)$ or $(12,6)$. (multiply by -2). Therefore, the answer is D).

T44. We want a vector perpendicular to the line, so we want the normal and this is in point-normal form and the normal is (-3,6). Therefore, the answer is A).

T45. (x,y,z)=(2,0,2)+t(5,-1,1) The numbers beside the t's are the direction vector v.

$\overrightarrow{v_1} = (5,-1,1)$

$\overrightarrow{n_2} = \overrightarrow{v_1}$ since perpendicular

$\therefore \overrightarrow{n_2} = (5,-1,1)$

$5x - 1y + 1z = (5,-1,1) \cdot (4,0,1)$

$5x - y + z = 20 + 0 + 1$

$5x - y + z = 21$

T46. The answer is A) since a line perpendicular to the plane would have a direction vector or parallel vector equal to the normal of the plane.

T47. $\vec{v} = (-1,4)$

$\vec{n} = (-4,-1)$ $or\ any\ multiple$

$x - 3$

$= (12,3)$

Therefore, the answer is C).

T48. $\vec{v}$ is parallel

$\vec{n} \cdot \vec{v} = 0$

$\therefore (5,-4,3) \cdot (2,1,-2) = 10 - 4 - 6 = 0$

Therefore, the answer is C).

T49. $(c, 4) = (3,8) + t(-1, 2)$

x	y
$c = 3 - t$	$4 = 8 + 2t$
$c = 3 - (-2) = 5$	$-4 = 2t$
	$t = -2$

T50. $\vec{v} = (-1,7) \quad \therefore \vec{n} = (-7,-1) \ \ or \ \ (7,1)$

$$7x + 1y = (7,1) \cdot (3,5)$$

$$7x + y = 21 + 5$$

$$7x + y = 26$$

Therefore, the answer is B).

T51. $\vec{n} = \vec{u} \times \vec{v}$

$$\begin{matrix} 2 & 1 & 3 & 2 & 1 & 3 \\ 1 & 2 & 1 & 1 & 2 & 1 \end{matrix}$$

$$\vec{n} = (1 - 6, 3 - 2, 4 - 1) = (-5, 1, 3)$$

$$\vec{n} \cdot (x - p) = 0$$

$\therefore (-5,1,3) \cdot \big(x - (1,4,5)\big) = 0 \qquad or\ (5,6,-1)$ can be used as Pt P

Therefore, the answer is A). since the normal is a multiple

T52. $\vec{v} = q - p = (3,3,3)$

$\therefore Point\ parallel\ is\ \vec{x} = p + t\vec{v}$

$$(x, y, z) = (1,2,3) + t(3,3,3)$$

$$or \quad (x, y, z) = (4,5,6) + t(3,3,3)$$

Therefore, the answer is D).

T53. $\overrightarrow{n_1} = (3, -8)$ $\overrightarrow{v_1} = (8,3)$

Since the lines are parallel, $\overrightarrow{v_2} = \overrightarrow{v_1} = (8,3)$

$P = (5, -3)$

$\therefore \vec{x} = p + t\vec{v}$

$\vec{x} = (5, -3) + t(8,3)$

Therefore, the answer is C).

*T54. The direction vector is the numbers in front of s, so $\vec{v} = (-3,4,2)$ and point P is (1,2,4)

So, the equation is $\therefore \vec{x} = p + t\vec{v}$

$\vec{x} = (1,2,4) + t(-3,4,2)$

*T55. Find the distance from point Q(4, -3) to the line $\vec{x} = (2,6) + t(1, -4)$.

$\vec{v} = (1, -4)$

To convert from $\vec{v}$ to $\vec{n}$ or $\overrightarrow{n}$ to $\bar{\bar{v}}$ in R^2

1. Switch the sign of y
2. Switch x and y

$\therefore \vec{n} = (4,1)$

$\|\vec{n}\| = \sqrt{4^2 + (1)^2} = \sqrt{17}$

$\vec{n} \cdot p = (4,1) \cdot (2,6) = 8 + 6 = 14$

$\vec{n} \cdot q = (4,1) \cdot (4, -3) = 16 - 3 = 13$

$d = \frac{|13-14|}{\sqrt{17}} = \frac{1}{\sqrt{17}}$

T56. x=(1-t)(1,2,3) + t(4,5,6)

U. Vectors in R^m (2.1)

U1. a) $= (1,2,3,4) + (3,1,-2,4) = (4,3,1,8)$

b) $= (1,2,3,4) - (3,1,-2,4) = (-2,1,5,0)$

c) $= 3(1,2,3,4) - 2(3,1,-2,4) + 4(1,-1,2,4)$

$= (3,6,9,12) + (-6,-2,4,-8) + (4,-4,8,16)$

$= (1,0,21,20)$

d) $= \sqrt{(3-2)^2 + (1-2)^2 + (-2-3)^2 + (4-4)^2}$

$= \sqrt{4+1+25+0} = \sqrt{30}$

e) $= \sqrt{1^2 + 2^2 + 3^2 + 4^2}$

$= \sqrt{1+4+9+16} = \sqrt{30}$

f) $= (1,2,3,4) \cdot (3,1,-2,4) = 3 + 2 - 6 + 16 = 15$

g) $= \|2(3,1,-2,4)\| \quad = \|(6,2,-4,8)\|$

$= \sqrt{6^2 + 2^2 + (-4)^2 + 8^2} = \sqrt{36+4+16+64} = \sqrt{120}$

U2. a) $\vec{v} = (3,1,-2,4) \qquad -\vec{u} = (-1,-2,-3,-4)$

$d = \sqrt{(-1-3)^2 + (-2-1)^2 + (-3+2)^2 + (-4-4)^2}$

$= \sqrt{16+9+1+64} = \sqrt{90}$

b) undefined since $\vec{u} \cdot \vec{w} = a$ number and we can't dot product a constant with a vector.

c) $= (3,6,9,12) \cdot (-2,2,-4,-8)$

$= -6 + 12 - 36 - 96 \quad = -126$

d) $= \|2[(12,3,4) - (3,,1,-2,4)]\|$

$= \|2(-2,1,5,0)\|$

$= \|(-4,2,10,0)\|$

$= \sqrt{(-4)^2 + 2^2 + 10^2 + 0^2}$

$= \sqrt{16+4+100} \quad = \sqrt{120}$

U3. a) $\vec{v} \cdot \vec{w} = -2 - 4 + 6 + 0 = 0$

Yes, it's orthogonal

b) $\vec{u} \cdot \vec{v} = -3 - 2 + 3 + 8 \neq 0 \quad \therefore \textit{not orthogonal}$

U4. a) $\|\vec{u}\| = \sqrt{3^2 + 1^2 + 1^2 + 4^2}$

$= \sqrt{9 + 1 + 1 + 16} = \sqrt{27}$

$\frac{\vec{u}}{\|\vec{u}\|} = (\frac{3}{\sqrt{27}}, \frac{1}{\sqrt{27}}, \frac{1}{\sqrt{27}}, \frac{4}{\sqrt{27}})$

b) $\|\vec{v}\| = \sqrt{(-1)^2 + (-2)^2 + 3^2 + 2^2}$

$= \sqrt{1 + 4 + 9 + 4} = \sqrt{18}$

$\frac{\vec{u}}{\|\vec{u}\|} = (\frac{-1}{\sqrt{18}}, \frac{-2}{\sqrt{18}}, \frac{3}{\sqrt{18}}, \frac{2}{\sqrt{18}})$

U5. $\|\vec{v}\| = \sqrt{(-1)^2 + (-2)^2 + 3^2 + 2^2}$

$= \sqrt{1 + 4 + 9 + 4} = \sqrt{18}$

$\frac{-u}{\|\vec{u}\|} = (\frac{1}{\sqrt{18}}, \frac{2}{\sqrt{18}}, \frac{-3}{\sqrt{18}}, \frac{-2}{\sqrt{18}})$

Switch all signs since it says opposite direction

U6. $\vec{x} = p + t\vec{v}$

$(x_1, x_2, x_3, x_4) = (2,4,5,-1) + t(1,2,1,3)$

$x_1 = 2 + t$

$x_2 = 4 + 2t$

$x_3 = 5 + t$

$x_4 = -1 + 3t$

U7. $\vec{n} \cdot (x - p) = 0$ point-normal form

$(2,1,-1,0,1) \cdot (x - (2,4,6,1,-1) = 0$

Standard is $2x_1 + 1x_2 - 1x_3 + 0x_4 + 1x_5 = (2,1,-1,0,1) \cdot (2,4,6,1,-1)$

$2x_1 + x_2 - x_3 + x_5 = 4 + 4 - 6 + 0 - 1$

$2x_1 + x_2 - x_3 + x_5 = 1$

U8. $(1,0,-2,1)\cdot\big(x-(5,3,4,-1)\big)=0$

$1x_1+0x_2-2x_3+1x_4=(1,0,-2,1)\cdot(5,,3,4,-1)$

$x_1-2x_3+x_4=5+0-8-1$

$x_1-2x_3+x_4=-4$

U9. 2-point

$(x_1,x_2,x_3,x_4,x_5)=(1-t)(3,4,0,-1,0)+t(1,2,3,-1,1)$

$\vec{v}=q-p=(-2,-2,3,0,1)$

Point parallel

$\vec{x}=(3,4,0,-1,0)+t(-2,-2,3,0,1)$

$x_1=3-2t$

$x_2=4-2t$

$x_3=3t$

$x_4=-1$

$x_5=t$

U10. 2-point

$\vec{x}=(1-t)(1,3,3,2,1)+t(1,2,1,3,0)$

$\vec{v}=q-p=(0,-1,-2,1,-1)$

Point parallel

$\vec{x}=p+t\vec{v}$

$\vec{x}=(1,3,3,2,1)+t(0,-1,-2,1,-1)$

$x_1=1$

$x_2=3-t$

$x_3=3-2t$

$x_4=2+t$

$x_5=1-t$

*U11. Find a two-point equation parallel to x=(5,0,2,3,4) + t(6,8,2,-1,2) and through the point (1,4,2,3,5).

From the point parallel form given:

$\vec{v}$=(6,8,2,-1,2) and we have a point P on the new line P=(1,4,2,3,5)

So, since $\vec{v}$=q-p we get:

(6,8,2,-1,2) = q - (1,4,2,3,5)

Solving for point Q we get: (6,8,2,-1,2) + (1,4,2,3,5)=(7, 12, 4, 2, 7)

The two-point equation is $\vec{x} = (1-t)p + tq$

$$\vec{x} = (1-t)(1,4,2,3,5) + t\,(7, 12, 4, 2, 7)$$

Test Two Material

A. Systems of Equations (2.2)

Example 1.i) and iii) are linear...answer is D

Example 2. i) and iii)...answer D

Example 3.
i) yes ii) no iii) no...answer is A

Example 4. i) linear ii) non-linear iii) non-linear answer is E.

Example 5.
i) linear
ii) non linear
iii)linear
iv) non linear
v) linear
The answer is B.

A1. A) and D) are linear

A2. i), iii) and iv) are linear. The answer is C.

A3. All are non-linear. The answer is E.

A4. i) no ii) yes iii) no iv) yes...the answer is B

A5. The answer is E, none of them.

B. Elimination Method (2.2)

Example 1. 2, plane

Example 2. 1, line

Example 3. 0, point

C. Row Reduction of Matrices to Solve Linear Systems (2.3)

Example 1.

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 2 \\ 1 & -3 & 2 & 4 \\ 2 & -1 & 4 & 7 \end{array}\right]$$

Example 2.

Make sure the equations are in the proper form first, with all of the variables on the left and the constant terms on the right.

x - 2y + 3z = 4
x - 3y - 2z = -8
y + 2z = 3

The augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 1 & -3 & -2 & -8 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

Example 3.

$$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 1 & 0 & 3 \\ 1 & -1 & 3 \end{array}\right]$$

p.107 #3 let y=t and get x=1-2t...solution is (1-2t, t)

p.109 let x_2=s, x_4=t and get x_1+2s=1 and x_1= 1-2s
Second row gives x_3+t=2 and x_3=2-t...solution is (1-2s, s, 2-t, t)

Example 4.

a) yes, it is the identity matrix, so it is a unique solution POI (2,3,4)
b) no, the row of 0's must be in the bottom
c) yes...infinitely many solutions
the parameter is $x_4 = t$ (not a leading 1)

From the first row, we get $x_1 = -2t$
From the second row, we get $x_2 = 3$
and from the last row, we get $x_3 + t = 0 \; or \; x_3 = -t$.

The solution is (-2t, 3 -t, t) a line of intersection (one parameter)

d) yes, it has infinitely many solutions...let y=t be the parameter
The first row gives x+2t = 2 and we get x= 2 - 2t and the second gives z=5.
The solution is (2-2t, t, 5)

e) yes, and since the last row is 0=2, there is no solution

f) yes, and the last row indicates no solution

g) no, as the third leading 1 (last row) has a 1 above it and we can only have 0's above and below leading 1's.

h) no, the third leading 1 has a 2 above it, so it is not in RREF

i) yes, and it has no solution even though row 4 says infinitely many because row 3 says no solution, so there is no solution to the whole system

j) yes, and there are 3 leading 1's, so 6 -3 = 3 parameters

Let $x_2 = r$ and $x_4 = s$ and $x_5 = t$
from the first row, we get $x_1 + 2r + 3s + t = 1$ and $x_1 = 1 - 2r - 3s - t$
from the second row we get $x_3 + 2s + 2t = 3$ and $x_3 = 3 - 2s - 2t$
and from the last row we get $x_6 = 1$.

k) it is a unique solution because it has the 2x2 identity. The solution is (0,2).

Example 5.

x+y = 8
y=6

$\left[\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 6 \end{array}\right]$ $R_1 - R_2 \to R_2$

$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 6 \end{array}\right]$

Solution is x=2, y=6 or (2,6)

Example 6.

3x+3y=15
2x+3y = 13

$\left[\begin{array}{cc|c} 3 & 3 & 15 \\ 2 & 3 & 13 \end{array}\right]$ divide R1 by 3

$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 3 & 13 \end{array}\right]$ $R_2 - 2R_1 \rightarrow R_2$

$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \end{array}\right]$ $R_1 - R_2 \rightarrow R_1$

$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$

The POI is (2,3)

Example 7.

$\left[\begin{array}{cc|c} 12 & -3 & 6 \\ -16 & 4 & -8 \end{array}\right]$ Divide the second row by 4 first:

$\left[\begin{array}{cc|c} 12 & -3 & 6 \\ -4 & 1 & -2 \end{array}\right]$ $R_1 \div 12 \rightarrow R_1$

$\left[\begin{array}{cc|c} 1 & \frac{-1}{4} & \frac{1}{2} \\ -4 & 1 & -2 \end{array}\right]$ $R_2 + 4R_1 \rightarrow R_2$

$\left[\begin{array}{cc|c} 1 & \frac{-1}{4} & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$

Infinitely many solutions
Let y=t

$$x - \frac{1}{4}t = \frac{1}{2}$$

$$x = \frac{1}{2} + \frac{1}{4}t$$

The solution is $(\frac{1}{2} + \frac{1}{4}t, t)$

Example 8.

$$\left[\begin{array}{cc|c} 4 & -8 & 6 \\ 1 & -2 & 5 \end{array}\right]$$ $R_1 \leftrightarrow R_2$

$$\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 4 & -8 & 6 \end{array}\right]$$ $R_2 - 4R_1 \rightarrow R_2$

$$\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 0 & -12 \end{array}\right]$$ $R_2 - 2R_1 \rightarrow R_2$

The last line says 0 0 / # so there is no solution

Example 9.

$$\left[\begin{array}{ccc|c} 5 & 4 & -1 & 0 \\ 0 & 20 & -6 & 22 \\ 0 & 0 & 2 & 6 \end{array}\right]$$ $R_1 \div 5 \rightarrow R_1$

$R_2 \div 20 \rightarrow R_2$ and $R_3 \div 2 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 4/5 & -1/5 & 0 \\ 0 & 1 & -3/10 & 11/10 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$R_2 + 3/10\, R_3 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 4/5 & -1/5 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$R_1 - 4/5\, R_2 \rightarrow R1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/5 & -8/5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$R_1 + 1/5\, R_3 \rightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The point of intersection is (-1,2,3)...exactly one solution

Example 10.

$$\left[\begin{array}{ccc|c} 4 & -2 & 2 & 2 \\ 3 & 2 & -4 & 4 \\ -6 & 3 & -3 & 2 \end{array}\right]$$ $R_1 \div 4 \to R1$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 1/2 \\ 3 & 2 & -4 & 4 \\ -6 & 3 & -3 & 2 \end{array}\right]$$ $R_2 - 3R_1 \to R_2$, $R_3 + 6R_1 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 7/2 & -11/2 & 5/2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

There is no solution since 0=5 is not true.

Example 11.

$$\left[\begin{array}{cccc|c} 2 & -4 & 6 & 2 & -6 \\ 2 & -1 & 3 & -1 & 0 \end{array}\right]$$ $R_1 \div 2 \to R1$

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & -3 \\ 2 & -1 & 3 & -1 & 0 \end{array}\right]$$ $R_2 - 2R_1 \to R2$

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & -3 \\ 0 & 3 & -3 & -3 & 6 \end{array}\right]$$ $R_2 \div 3 \to R2$

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & -3 \\ 0 & 1 & -1 & -1 & 2 \end{array}\right]$$ $R_1 + 2R_2 \to R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

Let x_3=s and x_4=t

$x_1 = 1 - s + t$

$x_2 = 2 + s + t$

The solution is (1 -s + t, 2+s+t, s, t)...two parameters, therefore it is a plane of intersection

Example 12.

$$\left[\begin{array}{cc|c} 4 & -2 & 6 \\ 12 & 2k & 4 \end{array}\right]$$

$R_2 - 3R1 \to R_2$

$$\left[\begin{array}{cc|c} 4 & -2 & 6 \\ 0 & 2k+6 & -14 \end{array}\right]$$

If k= -3, we get 0 0 /-14 which means no solution

If k≠ -3, we can row-reduce the matrix to get the identity matrix, so there is exactly one solution.

Since all k values except k=-3 give the identity, it is impossible to get a row of zeros...So, there is "no value of k" that will result in infinitely many solutions ie. 0 0/ 0

Example 13.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 2k \\ 0 & 5 & k & 4 \end{array}\right] R3 - 5R2 \to R3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 2\quad k \\ 0 & 0 & k-5 & 4-10k \end{array}\right]$$

If k=5, we get 0 0 0/ -46 which means no solution
If k≠ 5, we get a unique solution
There is no value of k that will make the bottom row a full row of 0’s, so infinitely many solutions is impossible.

Example 14.

$\left[\begin{array}{cc|c} 2 & -1 & 2 \\ 3 & k & 5 \end{array}\right]$ $R_1 \div 2 \rightarrow R1$

$\left[\begin{array}{cc|c} 1 & -1/2 & 1 \\ 3 & k & 3 \end{array}\right]$ $R_2 - 3R_1 \rightarrow R_2$

$\left[\begin{array}{cc|c} 1 & -1/2 & 1 \\ 0 & k+\frac{3}{2} & 0 \end{array}\right]$

a) there is no value of k that will result in no solution
b) if k=-3/2 or -1.5 there will be a row of zeros...infinitely many solutions
Any other value of k can be row-reduced to the identity matrix...exactly one solution
c) If k≠-3/2 there is exactly one solution

Example 15.

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & k^2-16 & 4+k \end{array}\right]$

For what value of k does the matrix above have:

a) infinitely many solutions?

If k= -4 we get a row of zeros

b) no solution?

If k=4, we get 0 0 0/ 8...no solution

c) exactly one solution?

If k≠ 4, -4 we get a unique solution

Example 16. $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 3 & 4 & 0 \\ 0 & 1 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & c^2-36 & c+6 \end{array}\right]$

For what value of c does the matrix above have:

a) a 3-parameter family of solutions?

If c= -6, we get 0 0 0 0 0 /0 which is infinitely many solutions...this will have two leading 1's and therefore, 5-2=3 parameters

b) a 2-parameter family of solutions?

If c≠6, -6, we can get 3 leading 1's and therefore 5-3=2 parameters

c) no solution?
If c=6, we get 0 0 0 0 0 /6 which means no solution
d) exactly one solution?
no value of c...we can't get a 5x5 identity since there are only three rows

Example 17.

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & k-5 \\ 0 & 1 & 9 & 3 \\ 1 & 0 & 2 & 2k \end{array}\right] R_3 - R_1 \rightarrow R_3$

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & k-5 \\ 0 & 1 & 9 & 3 \\ 0 & 0 & 1 & k+5 \end{array}\right]$ R1-R3→ $R1$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & 9 & 3 \\ 0 & 0 & 1 & k+5 \end{array}\right]$ R2-9R3→ R_3

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -9k-42 \\ 0 & 0 & 1 & k+5 \end{array}\right]$ This matrix is always the identity no matter what k is
a) no value of k
b) no value of k
c) all k∈R

Your answer would be different in this case as if you do R3 - R1 into R3 you will get a row of 0's on the left and it will be 0 0 0/ 2k- (k-5) which is 0 0 0 / k+5 and this would mean
a) k= -5 gives infinitely many solutions
b) k$\neq -5$ gives no solution and
c) no value of k gives the identity or a unique solution since the last row is all 0's so it can't be row-reduced to be the identity matrix.

Example 18.
A unique solution means we get the identity and with 4 columns we would need at least 4 rows to get a 4x4 identity. So, a unique solution is impossible. ie. D. is the solution, no value of k.

Example 19.
It is the same matrix as the last example, so we know that a unique solution is impossible. For no solution we would need all 0's before the line and then the 6 after, ie. 0 0 0 0/ 6 but there is no value of k that will make both k+1 and k-5 equal to zero at the same time, so no solution is also impossible. Therefore, the system ALWAYS has infinitely many solutions and the answer is E.

It is the same matrix as the last example, so we know that a unique solution is impossible. For no solution we would need all 0's before the line and then the 6 after, ie. 0 0 0 0/ 6 but there is no value of k that will make both k+1 and k-5 equal to zero at the same time, so no solution is also impossible. Therefore, the system ALWAYS has infinitely many solutions and the answer is E.

NOTE:

If k=-1, $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & -6 & 6 \end{array}\right]$ and we could divide the last row by -6 to get another leading 1, and we have # parameters= n – r = 4 – 3 = 1

If k=5, $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 6 & 0 & 6 \end{array}\right]$ and we could divide the last row by 6 to get another leading 1, and we have # parameters= n – r = 4 – 3 = 1

Example 20.
This matrix has the identity on the left, so it is always a unique solution, so a) is no value of k and b) is no value of k and c) is all values of k.

Example 21.

$$\left[\begin{array}{cc|c} 1 & -3 & k \\ k & 1 & 4 \end{array}\right]$$

$R_2 - kR_1 \rightarrow R_2$

$$\left[\begin{array}{cc|c} 1 & -3 & k \\ 0 & 1+3k & 4-k^2 \end{array}\right]$$

infinitely many is impossible since k= -1/3 makes the left side 0 and
k=2,-2 makes the right side 0
no solution k= -1/3
unique k$\neq -1/3$

Example 22. Which of the following is a solution for: $2x + 3y - z = 4$
$3x + 2y - 3z = 2$

Check A $(1,1,-1)$ $2x + 3y - z = 4$

$LS = 2(1) + 3(1) + 1$ $RS = 4$

$= 6 \neq 4$

Therefore, (1,1,-1) is not a solution to equation (1) so it isn't a solution to this system.

Check B $(1,1,1)$ $2x + 3y - z = 4$

$LS = 2(1) + 3(1) - 1$ $RS = 4$

$= 4$ LS=RS

$3x + 2y - 3z = 2$

$LS = 3(1) + 2(1) - 3(1)$ $RS = 2$

$= 2$ LS=RS

So, (1,1,1) is a solution to the system above since it is a solution to ALL equations in this system.

Example 23.

$$\#\,parameter = n - r = number\ of\ columns - rank$$

$$\#\,parameter = n - r = 4\ columns - 2\ leading\ 1's$$

$$= 2$$

$\therefore\ this\ means\ the\ last\ row\ must\ be\ all\ 0's$ (since we already have two leading 1's)

$\therefore k^2 - 9 = 0 \quad and \quad k - 3 = 0 \qquad \therefore k = 3$

Example 24.

$$\begin{bmatrix}1 & 2 & 3 & 4\\5 & 7 & 6 & 8\\9 & 8 & 7 & 6\end{bmatrix} \quad R1\times 2 \rightarrow R1$$

$$\begin{bmatrix}2 & 4 & 6 & 8\\5 & 7 & 6 & 8\\9 & 8 & 7 & 6\end{bmatrix} \quad R2 - R1 \rightarrow R2$$

$$\begin{bmatrix}2 & 4 & 6 & 8\\3 & 3 & 0 & 0\\9 & 8 & 7 & 6\end{bmatrix} \quad R2 \leftrightarrow R3 \quad \begin{bmatrix}2 & 4 & 6 & 8\\9 & 8 & 7 & 6\\3 & 3 & 0 & 0\end{bmatrix}$$

Practice Exam Questions on Row-Reducing

C1.

$$\left[\begin{array}{cc|c}4 & 2 & 18\\3 & -1 & 16\end{array}\right] R_1 \div 4 \rightarrow R_1 \qquad \left[\begin{array}{cc|c}1 & \frac{1}{2} & \frac{9}{2}\\3 & -1 & 16\end{array}\right] R_2 - 3R_1 \rightarrow R_2$$

$$\left[\begin{array}{cc|c}1 & \frac{1}{2} & \frac{9}{2}\\0 & \frac{-5}{2} & \frac{5}{2}\end{array}\right] R_2 \times (-\tfrac{2}{5}) \rightarrow R_2 \qquad \left[\begin{array}{cc|c}1 & \frac{1}{2} & \frac{9}{2}\\0 & 1 & -1\end{array}\right] R_1 - \tfrac{1}{2}R_3 \rightarrow R_1$$

$\left[\begin{array}{cc|c}1 & 0 & 5\\0 & 1 & -1\end{array}\right]$ The solution is (5,-1).

C2.

$\left[\begin{array}{cc|c}2 & -6 & 12\\3 & -2 & 4\end{array}\right]$ R1÷ 2 $\rightarrow R_1$ $\left[\begin{array}{cc|c}1 & -3 & 6\\3 & -2 & 4\end{array}\right]$

$R_2 - 3R_1 \rightarrow R_2$

$$\left[\begin{array}{cc|c}1 & -3 & 6\\0 & 7 & -14\end{array}\right] R_2 \div 7 \rightarrow R_2$$

$$\left[\begin{array}{cc|c}1 & -3 & 6\\0 & 1 & -2\end{array}\right] R_1 + 3R_2 \rightarrow R_1$$

$$\left[\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & -2\end{array}\right]$$

The solution is x=0 and y= -2

C3. $\left[\begin{array}{ccc|c} 2 & -6 & 6 & -8 \\ 2 & 3 & -1 & 15 \\ 4 & -3 & -1 & 19 \end{array}\right]$ Divide row 2 by 2

$\left[\begin{array}{ccc|c} 1 & -3 & 3 & -4 \\ 2 & 3 & -1 & 15 \\ 4 & -3 & -1 & 19 \end{array}\right]$
R2-2R1 $\rightarrow R2$
R3-4R1 $\rightarrow R3$

$\left[\begin{array}{ccc|c} 1 & -3 & 3 & -4 \\ 0 & 9 & -7 & 23 \\ 0 & 9 & -13 & 35 \end{array}\right]$
R2$\div$ 9 $\rightarrow R2$
R3-R2 $\rightarrow R3$

$\left[\begin{array}{ccc|c} 1 & -3 & 3 & -4 \\ 0 & 1 & -7/9 & 23/9 \\ 0 & 0 & -6 & 12 \end{array}\right]$
R3$\div -6 \rightarrow R3$

$\left[\begin{array}{ccc|c} 1 & -3 & 3 & -4 \\ 0 & 1 & -7/9 & 23/9 \\ 0 & 0 & 1 & -2 \end{array}\right]$
R2+7/9R3 $\rightarrow R2$

$\left[\begin{array}{ccc|c} 1 & -3 & 3 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$
R1-3R3 $\rightarrow R1$

$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$
R1+3R2 $\rightarrow R1$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$ The solution is (5, 1, -2).

C4. $\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 6 & 3 & -8 & 12 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 6 & 3 & -8 & 12 \end{array}\right]$
R2-2R1 $\rightarrow R2$
R3-6R1 $\rightarrow R3$

$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 0 & -1 & 5 & -6 \\ 0 & -3 & 10 & -12 \end{array}\right]$
-R2$\rightarrow R2$
R3-3R2 $\rightarrow R3$

$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$
1-R2$\rightarrow R1$

$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

There is a row of zeros in the matrix. Therefore, there are infinitely many solutions.

We need to introduce a parameter.

Let z=t.
From the second row, we get: y - 5z = 6
y - 5t = 6
y=6+5t
From the first row, we get: x +2t=-2
x=-2-2t

The solution is (-2 - 2t, 6 + 5t, t).

C5. $\left[\begin{matrix}1 & -1\\4 & -4\end{matrix}\middle|\begin{matrix}3\\2k\end{matrix}\right]$ R_2 - $4R_1 \rightarrow R_2$

$\left[\begin{matrix}1 & -1\\0 & 0\end{matrix}\middle|\begin{matrix}3\\2k-12\end{matrix}\right]$ 2k-12 = 0, so k=6

If k=6, we get a row of 0's...infinitely many solutions

If k≠6, we get no solution

There is no value of k that would give the identity or exactly one solution.

C6. $\left[\begin{matrix}1 & k\\2-k & 1\end{matrix}\middle|\begin{matrix}2\\4\end{matrix}\right]$ R_2 + (k-2) $R_1 \rightarrow R_2$

$\left[\begin{matrix}1 & k\\0 & k^2-2k+1\end{matrix}\middle|\begin{matrix}2\\2k\end{matrix}\right]$

$\left[\begin{matrix}1 & k\\0 & (k-1)(k-1)\end{matrix}\middle|\begin{matrix}2\\2k\end{matrix}\right]$

If k=1, we get 0 0 /2 which means no solution

C7. $\left[\begin{matrix}3 & -2\\12 & 2k\end{matrix}\middle|\begin{matrix}3\\8\end{matrix}\right]$

$\left[\begin{matrix}3 & -2\\12 & 2k\end{matrix}\middle|\begin{matrix}3\\8\end{matrix}\right]$ $R_1 \div 3 \rightarrow R1$

$\left[\begin{matrix}1 & -2/3\\12 & 2k\end{matrix}\middle|\begin{matrix}1\\8\end{matrix}\right]$ R_2 -$12R_1 \rightarrow R_2$

$\left[\begin{matrix}1 & -2/3\\0 & 2k+8\end{matrix}\middle|\begin{matrix}1\\-4\end{matrix}\right]$

If k= -4, we get 0 0/ -4 which means no solution.
Note: in this example if k≠ -4 we could turn it into the identity and get exactly one solution...there is no value of k that would give us a whole row of zeros...so infinitely many solutions is impossible.

C8. $\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 6 & 4 & -8 & 12 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 6 & 4 & -8 & 12 \end{array}\right]$ $R_2 - 2R_1 \rightarrow R_2$, $R_3 - 6R_1 \rightarrow R_3$

$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 0 & -1 & 5 & -6 \\ 0 & -2 & 10 & -12 \end{array}\right]$ R_2 x(-1) $\rightarrow R2$

$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 0 & 1 & -5 & 6 \\ 0 & -2 & 10 & -12 \end{array}\right]$ $R_3 + 2R_2 \rightarrow R_3$

$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $R_1 - R_2 \rightarrow R_1$

$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

A complete row of 0's... infinitely many solutions
Let z=t since x and y are leading 1's
x + 2t = -2
x=-2 - 2t
y-5t = 6
y=6+5t

The solution is (-2-2t, 6+5t, t)....intersection is a line since there is one parameter.

D. Extra Practice Exam Questions on Systems of Equations and Row Reduction (2.3)

D1. The answer is c).

D2. Since there is a row of zeros, there are infinitely many solutions.
From the second row, y=1. Let z=t
Substitute y=1 and z=t into the first row (equation).

x + z = 5
x + t = 5
x= 5 - t
Therefore, the solution is (5- t, 1, t).

D3. $\begin{bmatrix} 2 & 2 & 4 \\ -2 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}$ Divide row 1 by 2.

$\begin{bmatrix} 1 & 1 & 2 \\ -2 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}$ $R_3 - 3R_1 \to R_3$ and $R_2 + 2R_1 \to R_2$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 5 \\ 0 & -1 & -6 \end{bmatrix}$ $R_3 + \frac{1}{3}R_2 \to R_3$

$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & \frac{-13}{3} \end{bmatrix}$ $\frac{-3}{13}R_3 \to R_3$ (same as dividing by -13/3) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ $\frac{1}{3}R_2 \to R_2$
$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \end{bmatrix}$

$R2 - \frac{5}{3}R_3 \to R_2$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ R1-R2 $\to R1$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ R1-2R3$\to R1$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D4. Row reduce: $\left[\begin{array}{ccc|c} 1 & 5 & 0 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ divide row 2 by 2: $\left[\begin{array}{ccc|c} 1 & 5 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$
From the last row, we see a row of zeros, and therefore there are infinitely many solutions.
From row #2, z=1 so let y=t (don't pick leading 1's as a parameter)

x + 5t =2
x=2 - 5t
The solution is (2-5t, t, 1)

D5.
i) is linear
ii) is not linear...we can't divide by variables
iii) is not linear...we can't take square root of a variable

Therefore, ii) and iii) are NOT linear
The answer is c).

D6. The augmented matrix is b).

D7.
There is a row of zeros, so there are infinitely many solutions
From row #2, z=4, so let y=t
From row #1, we get x + 5t = 2 so x=2 - 5t

The solution is (2 - 5t, t, 4)
The answer is d).

D8. If x=0 and y=1, this matrix will be in RREF.
The answer is b).

D9. Which matrix is NOT in row reduced echelon form?

ii) and iii) are not in RREF, so the answer is C).

D10. Write the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 0 & 3 & 3 & 3 \\ 0 & 4 & 4 & 8 \end{array}\right] \text{R1} \div 2 \rightarrow R1 \text{ and R3} \div 4 \rightarrow R3 \text{ and R2} \div 3 \rightarrow R3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right] \text{R3 – R2} \rightarrow \text{R3} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

...we don't need to finish!
From the last row, we can see that there is no solution. So, we don't need to row-reduce any further.

D11. The augmented matrix of a system of linear equations has row-reduced echelon

form $\left[\begin{array}{ccc|c} 1 & 6 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ The solution is:

There are infinitely many solutions. We need # parameters = 3 unknowns - 2 rank= 1

From row 2, z=4 so let y=t

x + 6t = 1
x=1-6t
The solution is (1-6t, t, 4)

D12. The augmented matrix of a system of linear equations has row-reduced echelon form
$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$
The solution is

The third row is a row of zeros, so we know there are infinitely many solutions.
Since there are 5 columns, there are 5 variables.

The # of parameters= # variables - rank= 5 - 2 = 3

Don't pick leading 1's for parameters. Let x2, x3 and x5 be parameters.
x2=r, x3=s and x5=t
From the second row, x4 +2t =3 so x4=3 – 2t

From the first row,

$x_1 + 3r + t = 0$
$x_1 = -3r - t$

The solution is (-3r - t, r,s, 3 -2t, t).

D13.
Row-reduce the matrix first

$R_3 - 4R_2 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & k \\ 0 & 4 & k & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & k \\ 0 & 0 & k-4 & 3-4k \end{array}\right]$$

Now, for there to be no solution, the last line of the matrix must read:

0 0 0 / # where the number to the right of the line is NOT a "0".

For k – 4= 0 we get k=4.

The answer is a).

D14.
Row-reducing we get: $R_3 - 5R_2 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & k-5 & -5 \end{array}\right]$$

If k=5 we get no solution

If there is a unique solution, then k≠5...since we can get the identity by row-reducing.

The answer is c).
NOTE: Infinitely many solutions is impossible, ie. no value of k

D15.

If k=5, the last line of the matrix reads: 0 0 0 / 10
and this means no solution

So, i) is true.

If k=-5, the last line of the matrix reads: 0 0 0 / 0
and this means infinitely many solutions

So, iii) is true.

The only case left is a unique solution, which occurs for k≠-5,5 so ii) is false.

The answer is b).

D16. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & k^2-16 & 2k-8 \end{array}\right]$

If we have no solution, we get 0 0 0/ # where the # is not zero.

If k= -4, we get 0 0 0/ -16 which gives no solution.

The answer is b).

D17.
If k=4, we get 0 = 0 which means infinitely many solutions. We already saw in #16 that we get no solution for k= -4. So, if k≠4, -4 we will get a unique solution.
The answer is e).

D18.
We get a row of zeros in our matrix when k=4. ie 0 0 0/ 0
The answer is a).

D19. Row-reduce and write the solution to each system:

$\left[\begin{array}{ccccc|c} 2 & 4 & 2 & 0 & 2 & 2 \\ 1 & 1 & -1 & 1 & 2 & 2 \end{array}\right]$ Divide row 1 by 2:

$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 1 & 2 & 2 \end{array}\right]$ R2 - R1→R2

$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & -1 & -2 & 1 & 1 & 1 \end{array}\right]$ -R2→R2

$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 & -1 & -1 \end{array}\right]$ R1 - 2R2→R1

$\left[\begin{array}{ccccc|c} 1 & 0 & -3 & 2 & 3 & 3 \\ 0 & 1 & 2 & -1 & -1 & -1 \end{array}\right]$

So, there are leading 1's in the x_1 and x_2 columns and so there must be 3 parameters
Let x_3=r, x_4=s and x_5=t

From the second row, we get $1x_2+2r - 1s - t = -1$ and solving for x_2:

$x_2=-1 - 2r + s + t$

From the first row, we get: $x_1 - 3r + 2s + 3t = 3$ and solving for $x_1$we get:

$x_1=3+3r - 2s - 3t$

(3+3r-2s-3t, -1-2r+s+t, r,s,t)

D20. $\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 4 & 0 & 2 \\ 1 & 1 & -1 & 1 & 2 & 2 \end{array}\right]$ Divide row 2 by 2

$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 1 & 2 & 2 \end{array}\right]$ R3 - R1→R3

$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & -3 & 0 & 1 & 2 \end{array}\right]$ R1 - R2→R1

$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & -3 & 0 & 1 & 2 \end{array}\right]$ R3÷-3→R3

$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1/3 & -2/3 \end{array}\right]$ R2 - R3→R2 $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 2 & 1/3 & 5/3 \\ 0 & 0 & 1 & 0 & -1/3 & -2/3 \end{array}\right]$ R1-R3→R1

$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 4/3 & -1/3 \\ 0 & 1 & 0 & 2 & 1/3 & 5/3 \\ 0 & 0 & 1 & 0 & -1/3 & -2/3 \end{array}\right]$

The first three columns have leading 1's...so, there are two parameters representing the 4th and 5th columns

Let x_4=s and x_5=t

The last row gives us....x_3 - 1/3t= -2/3 or x_3= -2/3 + 1/3 t

The second row gives us...x_2 + 2s + 1/3 t = 5/3 or x_2= 5/3 - 2s - 1/3 t

The first row gives us...x_1 -s + 4/3 t = -1/3 or x_1= -1/3 + s - 4/3 t
The solution is a plane since there are two parameters and it is:
(-1/3 + s - 4/3 t, 5/3 - 2s - 1/3t, -2/3 + 1/3t, s, t)

*D21. Describe the region that is the intersection of the following planes:
x+y+2z=-2
3x -y + 14z = 6
x + 2y = -5

$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 3 & -1 & 14 & 6 \\ 2 & 4 & 0 & -10 \end{array}\right]$ $R_2 - 3R_1 \rightarrow R_2$ $\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & -4 & 8 & 12 \\ 2 & 4 & 0 & -10 \end{array}\right]$ R2$\div -4 \rightarrow R2$ and $R3 - 2$R1$\rightarrow R3$

$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 2 & -4 & -6 \end{array}\right]$ R3- 2R2 $\rightarrow R3$

$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ R1-R2 $\rightarrow R1$

$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

infinitely many solutions

parameters= n – r = 3 – 2 = 1
Since there is only 1 parameter, it is a line of intersection
z=t

The solution (not required) is:
from the first-row x+4t=1 and x=1-4t
from the second-row y-2t=-3 and y=-3 +2t

(1-4t, -3 +2t, t)

*D22. Describe the region that is the intersection of the following planes:
x - y - 3z = 1
6x - 6y - 18z = 6
-4x + 4y + 12z = -4

$$\left[\begin{array}{ccc|c} 2 & -2 & -6 & 2 \\ 6 & -6 & -18 & 6 \\ -4 & 4 & 12 & -4 \end{array}\right]$$ R1÷2→R1

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 1 \\ 6 & -6 & -18 & 6 \\ -4 & 4 & 12 & -4 \end{array}\right]$$ R2 - 6R1→R2 and R3 + 4R1→R3

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ This is in row-reduced form

There is one leading one (column) and two other columns representing parameters.
Let y=s and z=t

The intersection is a plane, since we have 2 parameters.

The solution (not asked for) is:
From the first row, we get x - y - 3z = 1
and with the parameters, we get x -s - 3t =1 or x=1+s+3t

The solution is (1+s+3t, s, t)

*D23. The answer is D.

*D24. Choose h and k such that the system below has (a) no solution, (b) a unique solution, and (c) many solutions.

$\left[\begin{array}{cc|c} 2 & 6 & 4 \\ 3 & h & k \end{array}\right]$ Divide row 1 by 2:

$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right]$ R2 - 3R1→R2 $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{array}\right]$

(a) for no solution, we need 0 0 / # where the # is not 0

So, h=9 and k≠6 because k=6 would make it a whole row of zeros

(c) is easier to answer first...h=9 and k=6 would make the last row 0 0/ 0 and there would be infinitely many solutions

(b) for a unique solution, we can't have h=9, but any other value could reduce the matrix to the identity....so h≠9, k€R

*D25. Find the value of "k" so that the following matrix has infinitely many solutions.

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 6 \\ 0 & 3 & k & 9 \end{array}\right]$

R3 - 3R2→R3

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & k-9 & -9 \end{array}\right]$

Now, for there to be infinitely many solutions, we need a row of zeros. If k=9, we get 0 0 0/ -9 and infinitely many solutions is impossible. So, the value of k is "no value of k".

*D26. From D25, if k=9 we get 0 0 0/-9 which would mean no solution. Any other value of "k" would result in the identity matrix if we finished row-reducing and therefore a unique solution. So, it would have exactly one solution if k≠9.

*D27. See D25. The answer is no solution if k=9.

D28.

Row-reducing, we get: R2- R1→R2 and divide row 1 by 2:

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ and this has ***infinitely many solutions*** because of the row of zeros. There is only one leading "1" so there are two parameters in the solution.

D29. Find the value(s) of k for which the following system is consistent:

$\left[\begin{array}{ccc|c} 3 & 3 & -3 & 6 \\ 2 & -2 & 2 & 4 \\ -1 & -1 & k & 4 \end{array}\right]$ Divide row 1 by 3:

$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & -2 & 2 & 4 \\ -1 & -1 & k & 4 \end{array}\right]$ ... $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & -1+k & 6 \end{array}\right]$...consistent means there is a solution (either unique

R2 - 2R1→R2
R3 + R1→R3

or infinitely many) ...inconsistent means it has no solution.

So, looking at the third row, if k=1, we get 0 0 0/6 which means no solution
So, the system is consistent and has some solution as long as k≠1

D30. Find the value of k so that the system below has infinitely many solutions.

$\left[\begin{array}{ccc|c} 2 & 2 & 6 & 0 \\ 4 & 3 & k & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]$ Divide row 1 by 2

...RREF $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 4 & 3 & k & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]$... $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & k-12 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right]$ ... $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 12-k & 0 \\ 0 & 0 & 8-k & 0 \end{array}\right]$

R3-2R1→R3 -R2→R2
R2-4R1→R2 R3-R2→R3

and there are infinitely many solutions (a full row of zeros) if k=8

D31.

$$\left[\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 2 & 2k & 2 & 2 \\ 3 & 3 & 3k & 3 \end{array}\right]$$ R1$\leftrightarrow$ $R3$ and R2$\div$ 2 $\rightarrow$ $R2$ and divide row 3 by 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{array}\right]$$ R2-R1 $\rightarrow$ $R2$ and R3 -kR1 $\rightarrow$ $R3$

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & -1+k & -k+1 & 0 \\ 0 & -k+1 & -k^2+1 & -k+1 \end{array}\right]$$ R3+R2 $\rightarrow$ $R3$

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & -1+k & -k+1 & 0 \\ 0 & 0 & -k^2-k+2 & -k+1 \end{array}\right]$$

factor $-k^2 - k + 2 = 0$

$-(k^2 + k - 2) = 0$

$-(k+2)(k-1) = 0$

k=-2, 1

If k=1...

the last row becomes 0 0 0 / 0 infinitely many solutions

If k= -2...

the last row becomes 0 0 0/3 no solution

If k$\neq 1, -2$ there is a unique (exactly one) solution

D32. $\left[\begin{array}{ccc|c} 2 & 4 & -6 & 8 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right]$ Divide row 1 by 2:

$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right]$ R2-3R1 → $R2$ and R3-4R1 → $R3$

$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & -7 & a^2-2 & a-14 \end{array}\right]$ R3+7R2 → $R3$

$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$

If a= -4, we get 0 = -8 which means no solution

If a=4, we get 0 = 0 which means infinitely many solutions

Therefore, there is a unique solution if a≠4, -4

D33. yes or no for row-reduced...solutions below

a) no, because the leading 1 in row 2 must have only 0's above and below it
b) yes
c) no, the row of 0's must be at the bottom
d) yes
e) yes
f) yes
g) yes
h) no, the 2 can't be above the leading one in last row
i) yes
j) yes
k) yes
l) yes

Solutions:
b) x=6, y= 7, z= 8
POI is (6,7,8)
d) 0 = 8 is the last row of the matrix and this means no solution

e) a row of zeros means infinitely many solutions, so we need to decide which variable to let the parameter be.

Don't pick leading 1's to be parameters, so let x=t
The solution is (t,3,3)

f) z=5 so let y=t
x+ t =3
x=3 -t
The solution is (3-t, t, 5).

g) # parameters= 5-3=2

x_1=1-3s-3t
x_2=s
x_3=4
x_4=t
x_5=0
The solution is (1-3s-3t,s,4,t,0)

i) no solution since it is 0 0 0 /1

j) x_2=s and x_4=t
x_1+5s-3t=2
x_1+=2-5s+3t

x_3+t=3
x_3=3-t
(2-5s+3t, s, 3-t, t) a plane (s and t= two parameters)

k) x_1=1, x_2=2, x_3=3 and x_4=4...POI is (1,2,3,4)

l) x_2 =r, x_4=s, x_5=t
$x_1 = 1 - 2x_2 - 3x_4 - x_5$
x_1=1-2r-3s-t

x_3=3-2x_4-2x_5
x_3=3-2s-2t

x_6=4
The solution is (1-2r-3s-t, r, 3-2s-2t, s, t,4)

D34.

$$\left[\begin{matrix}1 & -2\\ k & 4\end{matrix}\middle|\begin{matrix}k\\ 4\end{matrix}\right]$$

$R_2 - kR_1 \rightarrow R_2$

$$\left[\begin{matrix}1 & -2\\ 0 & 4+2k\end{matrix}\middle|\begin{matrix}k\\ 4-k^2\end{matrix}\right]$$

a) infinitely many is k= -2 since it makes 0 0/ 0
b) no solution is impossible as only k = -2 makes the left side 0 and it won't make the right side equal to a non-zero number
c) unique k$\neq -2$

D35. Find the value of k for which the matrix has no solution, infinitely many solutions and a unique solution.

$$\left[\begin{matrix}4 & -12\\ k & 6\end{matrix}\middle|\begin{matrix}4\\ 1\end{matrix}\right]$$

R1$\div$ $4 \rightarrow R1$ $\left[\begin{matrix}1 & -3\\ k & 6\end{matrix}\middle|\begin{matrix}1\\ 1\end{matrix}\right]$

R2$-$kR1 $\rightarrow R2$

$$\left[\begin{matrix}1 & -3\\ 0 & 6+3k\end{matrix}\middle|\begin{matrix}1\\ 1-k\end{matrix}\right]$$

No solution if k=-2
Infinitely many solutions is impossible, ie no value of k (can't get 00/0 with the same k value)
Unique solution if $k \neq -2$

*D36.

$$\left[\begin{matrix}1 & 0 & 0\\ 0 & 1 & 3\\ 0 & k & 2k+1\end{matrix}\middle|\begin{matrix}2\\ 4\\ 2k-8\end{matrix}\right] \text{R3}-\text{kR2} \rightarrow R3$$

$$\left[\begin{matrix}1 & 0 & 0\\ 0 & 1 & 3\\ 0 & 0 & -k+1\end{matrix}\middle|\begin{matrix}2\\ 4\\ -2k-8\end{matrix}\right]$$

If k=1, we get no solution
If k$\neq$ 1, we get a unique solution
Infinitely many solutions is impossible as we can't get 0 0 0/ 0

D37. $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -2 & 4 \\ 2 & -2 & 4 \end{array}\right]$

$R3 - 2R1 \rightarrow R3$

$R2 - 3R1 \rightarrow R2$

$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -5 & -5 \\ 0 & -4 & -2 \end{array}\right]$ $R2 \div (-5) \rightarrow R2$

$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & -4 & -2 \end{array}\right]$ $R3 + 4R2 \rightarrow R3$

$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{array}\right]$ $R1 - R2 \rightarrow R1$

$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{array}\right]$ $\therefore$ *no solution*

D38. $\left[\begin{array}{cc|c} 5 & -1 & 2 \\ 6 & k & 6 \end{array}\right]$ R_2-$R_1 \rightarrow R2$...if you want to avoid fractions, you must do this in several steps.

$\left[\begin{array}{cc|c} 5 & -1 & 2 \\ 1 & k+1 & 4 \end{array}\right]$ $R_1 \leftrightarrow R_2$

$\left[\begin{array}{cc|c} 1 & k+1 & 4 \\ 5 & -1 & 2 \end{array}\right]$ R_2-$5R_1 \rightarrow R2$

$\left[\begin{array}{cc|c} 1 & k+1 & 4 \\ 0 & -5k-6 & -18 \end{array}\right]$

a) if -5k-6=0 then -5k=6 and k= -6/5 and this will result in no solution

b) there is no value of k that will result in 0 0/ 0

c) Any other value of k can be row-reduced to the identity matrix...exactly one solution
If k≠ -6/5 there is exactly one solution

E. Rank

Example 2.

a) Not homogeneous

b) YES, it is homogeneous

Trivial solution...the first matrix says x=0, y=0 and z=0 ie a point of intersection (0,0,0)

The second matrix indicates infinitely many solutions..ie z=t and x=0, y=0...solution (0,0,t) where t is any real number

F. Operations on Matrices (3.1)

Example 1.
b) dim is 2x3
c) dim is 1x3
d) dim is 3x2
e) dim is 4x3

Example 4. $\begin{bmatrix} 3 & 5 \\ 2 & 2 \end{bmatrix}$

Example 5. $\begin{bmatrix} 0 & 3 & 0 \\ 2 & 8 & 2 \\ 7 & 7 & 12 \end{bmatrix}$

Example 6.
-4A= $-4\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -8 \\ -12 & -16 \end{bmatrix}$
The (2,1) entry would be the second row, 1st column spot= -12

Example 7.
a)
$$2\begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & -1 \\ 3 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 & -2 \\ 2 & 3 & 2 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -11 & 10 \\ -2 & -5 & -8 \\ 3 & 5 & -5 \end{bmatrix}$$

The (1,2) entry is the number in the first row and second column= -11

b) 2A - 3B - 4I = 2C

$$2C = \begin{bmatrix} -4 & -11 & 10 \\ -2 & -5 & -8 \\ 3 & 5 & -5 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$2C = \begin{bmatrix} -8 & -11 & 10 \\ -2 & -9 & -8 \\ 3 & 5 & -9 \end{bmatrix}$$

$$C = \begin{bmatrix} -4 & -\frac{11}{2} & 5 \\ -1 & -9/2 & -4 \\ 3/2 & 5/2 & -9/2 \end{bmatrix}$$

Example 8.

b) 2x2 multiplied by 2x2
Possible and answer is 2x2

$$\begin{bmatrix} -1+10 & 5+6 \\ 1+10 & -5+6 \end{bmatrix} = \begin{bmatrix} 9 & 11 \\ 11 & 1 \end{bmatrix}$$

c) 3x3 multiplied by 3x3
Possible and answer is 3x3

$$\begin{bmatrix} 2-1+6 & 2-1+9 & -1-2+3 \\ 6+2+4 & 6+2+6 & -3+4+2 \\ 4+1+2 & 4+1+3 & -2+2+1 \end{bmatrix} = \begin{bmatrix} 7 & 10 & 0 \\ 12 & 14 & 3 \\ 7 & 8 & 1 \end{bmatrix}$$

d) 3x2 multiplied by 2x2

Possible and answer is 3x2

$$\begin{bmatrix} 0+9 & 0+6 \\ 2+3 & 1+2 \\ 6-3 & 3-2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 5 & 3 \\ 3 & 1 \end{bmatrix}$$

e) multiplied by 3x1

Possible and answer is 1x1

$$[-4+10+12] = [18]$$

f) 2x4 multiplied by 3x3

Since 4≠3, this product is NOT possible.

g) 3x1 multiplied by a 1x3, gives a 3x3

$$\begin{bmatrix} -1 & -3 & -4 \\ 2 & 6 & 8 \\ 1 & 3 & 4 \end{bmatrix}$$

Example 9.

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 4\\5 & 6 & 0\end{bmatrix}\begin{bmatrix}a & b & 5\\20 & c & -4\\-5 & 4 & 1\end{bmatrix}=\begin{bmatrix}1 & \mathbf{0} & 0\\0 & 1 & 0\\0 & \mathbf{0} & 1\end{bmatrix}$$

Find a row and column that can be multiplied to find "b"...

1,2 entry= first row of M multiplied by the second column of N

$\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}b\\c\\4\end{bmatrix}$=0 since the 1,2 entry of matrix I, the 3x3 identity is 0

$b + 2c + 12 = 0$
b= -12 – 2c

3,2 entry= third row of M multiplied by the second row of N

$\begin{bmatrix}5 & 6 & 0\end{bmatrix}\begin{bmatrix}b\\c\\4\end{bmatrix} =$0 since the 3,2 entry of matrix I, is 0

this gives us 5b+6c + 0 = 0 substitue b= -12 – 2c
and get:
5(-12 – 2c)+6c = 0
-60 – 10c + 6c = 0
-4c=60
C= -15
b= -12 – 2c = -12 -2(-15) = -12 + 30 = 18

Example 10.

$A^2 = \begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 9\end{bmatrix}$...the 2,2 entry is 3^2=9

$A^3 = AA^2 = \begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 9\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 27\end{bmatrix}$...the 2,2 entry is 3^3=27

$A^{45} = \begin{bmatrix}1 & 0\\0 & 3^{45}\end{bmatrix}$

Example 11.

$$\left(\begin{bmatrix}1 & -1 & 0\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} - 3\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)^2$$

$$= \begin{bmatrix}-2 & -1 & 0\\0 & -2 & 2\\0 & 0 & -2\end{bmatrix}^2$$

$$= \begin{bmatrix}-2 & -1 & 0\\0 & -2 & 2\\0 & 0 & -2\end{bmatrix}\begin{bmatrix}-2 & -1 & 0\\0 & -2 & 2\\0 & 0 & -2\end{bmatrix}$$

$$= \begin{bmatrix}4 & 4 & -2\\0 & 4 & -8\\0 & 0 & 4\end{bmatrix}$$

Example 12.

a) $A^2 = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}$

$A^3 = \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & -3\\0 & 1\end{bmatrix}$

$A^n = \begin{bmatrix}1 & -n\\0 & 1\end{bmatrix}$

Example 13.

a) $A^T = \begin{bmatrix}1 & 4\\2 & 5\\3 & 6\end{bmatrix}$

b) $B^T = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$

c) $C^T = \begin{bmatrix}2\\3\\4\end{bmatrix}$

Example 14.

A^T is a 4x2
A^TC is a 4x2 2x3 which gives a 4x3

so the final answer is a 4x3 + 3x4 which is not defined and the answer is A.

Example 15.

a) $AB^T = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 3 & 4 \end{bmatrix}$
c_{12}= the number in the first row, second column after you do the product of AB^T which is 3.

b) $A^TA - AA^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$
$= \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix} - \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & -5 \end{bmatrix}$

F1.

$a)\ AB = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -1 \\ 9 & 2 & 3 \\ 2 & 1 & -2 \end{bmatrix}$

b) $A^TB^T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 3 & -1 \\ 5 & 10 & 2 \end{bmatrix}$

c) $B^TA^T = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 2 \\ 0 & 2 & 1 \\ -1 & 3 & -2 \end{bmatrix} = (AB)^T$

d) $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 3 & 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix} = A$

f) $A^2 = A \times A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 6 \\ 3 & 7 & 15 \\ 0 & 3 & 1 \end{bmatrix}$

F2.
$A^2 = \begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$

The answer is d).

F3.
$\begin{bmatrix} 5 & 13 \\ 7 & 17 \end{bmatrix}$

The answer for the (2,1) entry is 7.
The answer is B).

F4.
$\begin{bmatrix} 2 & 0 & 1 \\ 5 & 3 & 2 \end{bmatrix}$

The (1,3) entry is 1.

The answer is C).

F5.
$A^T = \begin{bmatrix} 3 & 5 & 7 \\ 2 & 6 & 6 \\ 1 & 8 & 5 \end{bmatrix}$

The (1,2) entry of the transpose matrix is 5.

The answer is b).

F6. $B^2 = B \times B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$B^3 = B^2 \times B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$B^{35} = \begin{bmatrix} 1 & 35 \\ 0 & 1 \end{bmatrix}$
The (1,2) entry is 35.

The answer is c).

F7. $C = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 5 & 8 \end{bmatrix}$

The (2,2) entry of C is 8.

The answer is d).

F8.
$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 0 & 4 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 6 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 14 \\ 0 & 8 \end{bmatrix}$

The (1,2) entry is 14.

The answer is d).

F9.
A^2 is only defined for square matrices. So, it is undefined.
The answer is e).

F10. $A^T B^T = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 13 \\ 8 & 23 \\ -1 & -4 \end{bmatrix}$
The (1,2) entry is 13.

The answer is A).

F11.
3x3 by 3x2 by 2x2

=3x2 by 2x2

=3x2
The answer is a).

F12.
i) AC^T=3x4 by 4x3 = 3x3=defined

ii) A +4C^TB= 3x4 + (4x3) by (3x3)
= 3x4 + 4x3
=undefined

iii) 3x4 (4x3) by (3x3)
=3x3 by 3x3
= 3x3

The answer is E). ii only is undefined

F13. $2A - B = 2\begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ -2 & 7 \end{bmatrix}$
The answer is $\begin{bmatrix} 1 & 8 \\ -2 & 7 \end{bmatrix}$.

F14.
$C^2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$

$C^3 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix}$...look at the pattern in the matrices

$C^6 = \begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix}$

The (1,2) entry is -12.

The answer is d).

F15. Let A=$\begin{bmatrix}1\\2\\3\end{bmatrix}$ and B=$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$. Which of the following is defined?

A). B+2A ...no, matrices must be the same dimension to add them
B) A^T +5B ...no
C) $2AB^T$ +4A= 3x1 = no
D) A^2 +7A= you can't square matrices that aren't square, so not defined
E) B^TA+5A=3x3 3x1 + 3x1= 3x1 + 3x1 Yes this is defined

Therefore, E) is defined.

*F16. $\begin{bmatrix}1 & 0\\4 & k\end{bmatrix}\begin{bmatrix}1 & 0\\4 & k\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$$\begin{bmatrix}1 & 0\\4+4k & k^2\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

The only value that works for both the (2,1) and (2,2 positions is k= -1. NOTE: k=1 doesn't give the 0 in the (2,1) entry for the identity

The answer is B).

F17. If I is the 3x3 identity matrix, find $\det(2I^{-1}-7I^T)$.
NOTE: the inverse and the transpose of I are the same matrix I, the identity matrix

$$2I^{-1}-7I^T=2I-7I=-5I=\begin{bmatrix}-5 & 0 & 0\\0 & -5 & 0\\0 & 0 & -5\end{bmatrix}$$

G. Matrix Equations and Inverses (3.2)

Example 1.

a)

$$\begin{bmatrix} 5 & -3 & 2 \\ 2 & -3 & -4 \\ 0 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ 0 \end{bmatrix}$$

b) $$\begin{bmatrix} 5 & -3 & 4 \\ -1 & -2 & -1 \\ 2 & 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix}$$

Example 3. a)

$$A^{-1} = \frac{1}{(1)(5)-(4)(2)}\begin{bmatrix} 5 & -4 \\ -2 & 1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 5 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & \frac{4}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

b) $$B^{-1} = \frac{1}{(1)(-1)-(3)(-1)}\begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Example 5.

a)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 2 & 5 & -1 & 0 & 0 & 1 \end{array}\right]$$ R3 -2R1→ $R3$ and R2(-1) → $R2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right]$$ R3 – R2 → $R3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right]$$ R1 + R3→ $R1$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right]$$ R1 – 2R2 → $R1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & 3 & 1 \\ 0 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

b) $\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ R3 +R1→ $R3$ and R2 (-1) → $R2$

$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 1 \end{array}\right]$ R3 + R2 → $R3$

$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$ R2 + 2R3→ $R2$

$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -3 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$ R1 – 2R3→ $R1$

$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 2 & -2 \\ 0 & 1 & 0 & 2 & -3 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$ R1 + R2→ $R1$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2 & -3 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

The (3,2) entry of the inverse matrix is the number in the third row and second column= -1

Example 6.

no inverse if det A = 0
ad - bc = 0
(c)(c) - (-5)(5) = 0
$c^2 + 25 = 0$

c^2= -25 and so c has no real value since we can't take the square root of a negative number

Example 7.
inverse if detA≠0
ad - bc ≠ 0

k(k+6) – (-3)(3) ≠ 0
k^2+6k+9≠ 0
(k+3)(k+3) ≠ 0
k≠ − 3

Invertible if k≠ − 3
and no inverse if k=-3

Example 8.
a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \\ 10 \end{bmatrix}$
So, x=7 and y=4

b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2-2+1 \\ 0-2+0 \\ 2+6-1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}$

The solution is x= -3, y= -2 and z=7.

Example 9.
The formula is $X = A^{-1}b$, so we need to find the inverse matrix first.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2 & -1 & 0 \\ 1 & -3 & -2 \\ 1/2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2-1+0 \\ 3-3-4 \\ 3/2-1-2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -4 \\ -3/2 \end{bmatrix}$$

POI is (1/2, -4, -3/2)

Example 10.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & -c & 0 & 0 & 1 \end{array}\right] \ldots R3 + R1 \rightarrow R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 7-c & 1 & 0 & 1 \end{array}\right] R3 + R2 \rightarrow R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 6-c & 1 & 1 & 1 \end{array}\right]$$

If c=6...we can't get the identity matrix because we get a row of 0's. As long as $c \neq 6$, we get the identity and the matrix is invertible.

Example 11.

$$b = \begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix}$$

$$X = A^{-1}b$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 4 \\ -1 & 6 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix}$$

$\therefore x_2 = 1(7) + 1(3) + 4(2)$

$x_2 = 18$

Example 12.

Multiply on the left of both sides by B and get

$BA = BB^{-1}CB$

$BA = ICB$

$BA = CB$

Or multiply on the right by B^{-1} and get another equivalent expression

$A = B^{-1}CB$

$AB^{-1} = B^{-1}CBB^{-1}$

$AB^{-1} = B^{-1}CI$

$AB^{-1} = B^{-1}C$

Example 13.

$$\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = A^{-1}b$$

$$\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} k \\ 4 \\ 8 \end{bmatrix}$$

$x1 = 1(k) + (-1)(4) + 0(8)$ substitute x1=10
10=$k - 4 + 0$
$k = 14$

***Example 14.**

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 \\ c & d \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$
So, if we want the first column of A^{-1}, we dot the first row of A and the first column of A^{-1} and set it equal to the (1,1) entry of A^{-1}, *which is* 1.

Check A $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \mathbf{-1} & \square \\ \mathbf{2} & \square \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
(2,1) ·(-1,2)= 2(-1)+1(2)=0 $\neq 1$ $\therefore$ *not* $\boxed{A}$

Check B $\begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix}$

$$\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \mathbf{1/5} & \square \\ \mathbf{3/5} & \square \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$(2,1) \cdot \left(\frac{1}{5}, \frac{3}{5}\right) = 2\left(\frac{1}{5}\right) + 1\left(\frac{3}{5}\right) = \frac{2}{5} + \frac{3}{5} = 1$
EQUAL $\therefore$ $\boxed{B}$ *is the answer*

Example 15. $2AA^{-1}+3A^{-1}A$=

AA^{-1}=I and $A^{-1}A=I$, so we have 2I + 3I=5I and I is a 2x2 identity since matrix A is 2x2.

So, the answer is $5\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}$

G1.

$$\mathrm{A}^{-1}=\frac{1}{(1)(-1)-5(3)}\begin{bmatrix}-1 & -5\\-3 & 1\end{bmatrix}=-\frac{1}{16}\begin{bmatrix}-1 & -5\\-3 & 1\end{bmatrix}$$

The (1,2) entry is 5/16.
The answer is b).

G2.
b is in the (1,2) entry of the inverse.

$$A^{-1}=\frac{1}{-4+5}\begin{bmatrix}-1 & 5\\-1 & 4\end{bmatrix}=1\begin{bmatrix}-1 & 5\\-1 & 4\end{bmatrix}$$

The (1,2) entry is 5.
The answer is b).

G3.

$$\left[\begin{array}{ccc|ccc}2 & 0 & 2 & 1 & 0 & 0\\0 & -1 & 0 & 0 & 1 & 0\\2 & 2 & 2 & 0 & 0 & 1\end{array}\right]$$ R3-R1 → $R3$ and R1 divided by 2, R2 x (-1) → $R2$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1/2 & 0 & 0\\0 & 1 & 0 & 0 & -1 & 0\\0 & 2 & 0 & -1 & 0 & 1\end{array}\right]$$ R3-2R2 → $R3$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1/2 & 0 & 0\\0 & 1 & 0 & 0 & 1 & 0\\0 & 0 & 0 & -1 & 2 & 1\end{array}\right]$$

We get a row of 0's on the left, so the inverse doesn't exist.
The answer is e).

G4. There is no inverse if detA=0 or if ad-bc=0.

(3)(6) - (k)(-2)=0
2k= -18
Solving for k, we get: x= -9
The answer is b).

G5. There is no inverse if detA=0 or if ad-bc=0.
(k)(k)-(10)(10)=0
$k^2 = 100$
k=10, -10
The answer is e).

G6. $[A|I] \rightarrow [I|A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & x & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ -1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & x & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & 3 & x+1 & 1 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & x & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & 0 & x-3 & 1 & -1 & 1 \end{array}\right]$$

R3+R1 $\rightarrow R3$ R3 - R2$\rightarrow R3$

If the matrix is invertible, we get the identity matrix I when we row-reduce.
If it is not invertible, we get a row of 0's and can't get the identity matrix...
From the last row, x - 3=0, and x=3.

The answer is a).

*G7. Determine the first row of the 2x2 matrix X that satisfies the equation

$$(X + 2I)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

First, take the inverse of both sides of the equation...

$[(X + 2I)^{-1}]^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$ ...we find the inverse of the 2x2 matrix on the right side

$X+2I = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$X+2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{(1)(1)-(1)(0)}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$X+\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$X = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$

The first row is [-1 -1]

*G8. For what value of k is the following linear system consistent?

3x + 3y - 3z = 6
2x - 2y + 2z =4
-x - y + k z = 2 Divide the first equation by 3 and the second equation by 2:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 2 \\ -1 & -1 & k & 2 \end{array}\right] \ldots \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -1+k & 4 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1+k & 4 \end{array}\right]$$

R2 - R1→R2 R2÷-2→R2
R3 + R1→R3

The system is inconsistent when k = 1, and is consistent when k≠ 1.
If a linear system is consistent, then it may have a unique solution or may have infinitely may solutions. Remember, inconsistent means there is no solution.

*G9. Find the entry in the 1st row and 2nd column of the inverse of A =

$[A|I] \rightarrow [I|A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \ldots \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 2 & 0 & -2 & 0 & 1 \end{array}\right] \ldots \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right]$$

R3 - 2R1→R1 R3 - 2R2→R3 R3x (-1) → $R3$
R2÷2→ $R2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right] \ldots \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1/2 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right] \ldots \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & \mathbf{-1} & 1 \\ 0 & 1 & 0 & -1 & 0 & 1/2 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

R2 - 1/2 R3→R2 R1 - R3→R1

The a_{12} position is = -1

*G10. The matrix has an inverse as long as detA≠0 ie. ad - bc ≠0
There is no inverse if ad-bc=0
$-9-k^2=0$
$-k^2=9$
$k^2= -9$
no solution in the real numbers (no inverse)
So, it has an inverse for all $k\epsilon R$

The answer is e).

*G11. The formula for AX=b is $X=A^{-1}b$ where b is the matrix of constant terms

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -2 & 4 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{4} \\ \mathbf{3} \\ \mathbf{-3} \end{bmatrix}$$

z=(1)(4) + (0)(3) + (-1)(-3)= 7...dot product of third row and the column to find z

The answer is b).

*G12.
$A^5=I$
$AA^4=I$
$A^4=A^{-1}$

A matrix and its inverse have a product equal to the identity matrix, I.
The answer is d).

*G13. Let A be a 3x3 matrix such that $A^2=I$ where I is the 3x3 identity matrix. Then, which of the following must always be true?
If two matrices multiply to give the identity matrix, then they are inverse matrices of one another...So, since $A^2=I$, then $AA=I$ and so $A=A^{-1}$. Now, if that answer were here, we would be done.
Looking at the other solutions, we can find out what A^3 equals...$A^3= AA^2=A^{-1}AA=IA=A$ so $A^3=A$...so b) is true

$A^4=A^2\ A^2=I(I)=I$ and c) is true.
The answer is d).

*G14. Let B, C and D be nxn matrixes. If it is also true that $D=C^{-1}AC$, which of the following must always be true?
To solve these questions, always multiply by various matrices until you get the identity...for example, if we multiply on the left side of both sides of this equation by C, we get:
$D=C^{-1}AC$
$CD=\mathit{C\ C^{-1}}\ AC$ ... $C\ C^{-1}=I$ and remember multiplying by the identity, doesn't change the matrix.

∴$CD=AC$...and that isn't there. We can't say D=A because there is no such thing as dividing matrices.
What if you hadn't tried that first...say we multiplied on the right side by C^{-1}:

$D=C^{-1}BC$
$DC^{-1}=C^{-1}A\ \mathit{CC^{-1}}$ ... $C\ C^{-1}=I$ and remember multiplying by the identity, doesn't change the matrix.

∴ $DC^{-1}=C^{-1}A$ and the order of answer c) is correct, so the answer is c).

*G15. The formula for AX=b is X=A^{-1}b where b is the matrix of constant terms

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

Since we want z, we take the last row, times the column 5, 1, -1 (dot product)

z=$\begin{bmatrix} 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$= (2)(5) + (1)(1) + (5)(-1)= 6

Therefore, z=6.

G16. $\left[\begin{array}{ccc|ccc} 6 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -6 & 0 & -2 & 0 & 0 & 1 \end{array}\right]$ $R3 + R1 \rightarrow R3$

$$\left[\begin{array}{ccc|ccc} 6 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \quad R1 \div (6) \rightarrow R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1/6 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \quad R1 - \frac{1}{2}R3 \rightarrow R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 0 & -1/2 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

The (1,2) entry is 0.

The answer is a).

*G17. $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ R3↔R2

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$ R2 + R1 →R2

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$ R2 – 2R3→R2

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$ R1 – R3 →R1

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$ R1 – R2 →R1

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$

$$A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

G18.

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2(1) - (-2)(-3)}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

1,2 entry=b= -1/4(2) = -2/4 = -1/2

G19.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}b = \begin{bmatrix} -2 & 4 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

x= -2(1) + 4(4) + (-1)(2) = -2 + 16 – 2 =12

G20. $(AB)^T = B^T A^T$ is a known property you have to memorize

So, $((AB)C^T)^T = C(AB)^T$ and the transpose of the transpose is just the original matrix
$= C(B^T A^T) or\ (CB^T)A^T$
The answer is e).

G21. $BB^{-1}\ is\ just\ the\ identity\ matrix\ I, so\ we\ just\ do\ B + I\ \ and\ square\ it$

$(\begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})^2$
$= \begin{bmatrix} 3 & 5 \\ 3 & 3 \end{bmatrix}^2$
$= \begin{bmatrix} 3 & 5 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 24 & 30 \\ 18 & 24 \end{bmatrix}$

G22. $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

From the 1,1 entry we get 2a=1 or a=1/2...not there…A is false
from the 1,2 entry we get b+d=0 or b=-d or d=-b...not there…B is false
from the 1,3 entry we get c+g=0 so c=-g or g=-c...D is true
from the 3,3 entry i + i = 1, so 2i=1 and i=1/2, so C is true. The answer is E).

G23. $\begin{bmatrix} a & d & 1 \\ b & e & 3 \\ c & f & -2 \end{bmatrix}\begin{bmatrix} 4 & 0 & x \\ 2 & -2 & y \\ 0 & 3 & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ since they are inverses of each other. We need to find y which is in the third column, but it is too difficult since x and z are also in the third column and there are too many unknowns. But, if AB=I, then BA=I as well, so switch the order of the matrices.

$$\begin{bmatrix} 4 & 0 & x \\ 2 & -2 & y \\ 0 & 3 & z \end{bmatrix}\begin{bmatrix} a & d & 1 \\ b & e & 3 \\ c & f & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, y is in the second row of the first matrix and we can dot product it with the third column of the second matrix and set it equal to the (2,3) entry of I which is a 0.

2(1) + (-2)(3) + (y)(-2) = 0
2 -6 – 2y = 0
-4 =2y
y= -2

G24. B times B inverse is just the identity matrix I and the second row of I is [0 1].
The answer is E).

G25. i). true
ii) false, if it has a row of 0's it is not invertible
iii) true
iv) false
The answer is A).

G26.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{5(0)-1(3)}\begin{bmatrix} 0 & -1 \\ -3 & 5 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 0 & -1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 \\ 1 & -5/3 \end{bmatrix}$$

the first row is [0   1/3].

*G27. R3+R1→R3

B I I B^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 12 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -c & 0 & 0 & 1 \end{array}\right] \ldots \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 12-c & 1 & 0 & 1 \end{array}\right]$$

If c=12...we can't get the identity matrix because we get a row of 0's. As long as c≠12, we get the identity and the matrix is invertible.

*G28. Find the value of b_1 given the matrix below given (10, -14, -7) is the unique solution to the system below.

$$A^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ 5 & -3 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$ and

$ax_1+bx_2+cx_3=b_1$
$dx_1+ex_2+fx_3=b_2$
$gx_1+hx_2+jx_3=b_3$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}b = \begin{bmatrix} 5 & 0 & 0 \\ 5 & -3 & -2 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} \mathrm{b1} \\ \mathrm{b2} \\ \mathrm{b3} \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ -14 \\ -7 \end{bmatrix} = A^{-1}b = \begin{bmatrix} 5 & 0 & 0 \\ 5 & -3 & -2 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} \mathrm{b1} \\ \mathrm{b2} \\ \mathrm{b3} \end{bmatrix}$$

Look at the top row on the left and dot the top row with the column on the right
$10=5\mathrm{b1} + 0\ \mathrm{b2} + 0\ \mathrm{b3}$
$\mathrm{b1} = 2$

Post Test Two Material

A. Theory of Linear Systems (3.3)

Example 1. det A=0 means A is not invertible...means rank A<n and AX=b does not have a unique solution, and AX=0 has infinitely many solutions...the answer is C since AX=b is not unique.

Example 2. Homogeneous + More unknowns than equations = infinitely many solutions Answer is C.

Example 3. If rank A< rankA/b....there is NO solution. The answer is A.

Example 4. If A is invertible, it has rank=n and it has a unique solution...the homogeneous would have unique solution x=y=z=0 (trivial)...row reduced form is the identity matrix...det A=0 is false since if determinant of A is 0, it would NOT have an inverse...answer is E.

Example 5. The number of parameters = n - r where n=# unknowns and r=rank

parameters = 15 - 5 = 10...the answer is A.

Example 6.

Again, look at the list we made...see p.187. if A is an invertible 7x7 matrix, then the rank A=7, the rank A/b=7, AX=b has a unique solution, AX=0 has a unique solution, ie. the trivial solution. The one that is false is A. since there can't be a row of 0's or the rank would NOT be 7, and it would not be invertible.

Example 7. # parameters= n-r = 6 -4 =2...the answer is D.

Example 8. Since rank A= rankA/b=n=6, there is exactly one solution, so E is the answer.

Example 9. Since there are only 3 rows, we can get at most 3 leading 1's. So, C is the answer.

Example 10. The answer is A.

Example 11. The answer is E. If it is homogeneous, it cannot be “no solution”.

Example 12. The answer is B. The rank can be at most 4, so A and E are false. Since it is not a square matrix, it cannot be invertible, so C is false. The homogeneous system is NEVER "no solution" and since the matrix has no inverse, it must be infinitely many solutions.

Example 13.

i) True
ii) False, it must be a unique solution since rank A=9
iii) True
iv) True, it is a unique solution and since it is homogenous it must be the trivial solution

Example 14. The system must have infinitely many solutions

i) The system must no solution
ii) The system must have only the trivial solution (0,0,0,0)

A. i) only
B. ii) only
C. iii) only
D. i) and iii) only
E. All of i) ii) and iii)

If det A=0, the system can't have a unique solution. If it is homogeneous, we also can't get "no solution", so it must be infinitely many solutions and A is the answer. NOTE: The trivial solution is a unique solution, so iii) is false as well.

A1.

R2-2R1 $\rightarrow R2$ and R3-3R1 $\rightarrow R3$

RREF… $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ R2$\times -1 \rightarrow R2$ $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ R1-R2$\rightarrow R1$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank=2 non-zero rows

The answer is c).

A2.

RREF… $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

R2-R1 → $R2$ R3+R2 → $R3$ R2÷ -2 → $R2$ R4+R2 → $R4$
R1+R3→ $R3$
R4-2R1 → $R4$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ R4 -1/2 R3 → $R4$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ R1-R2→ $R1$

$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ R1 -1/2 R3 → $R1$ and R2 +1/2 R3 → $R2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

There are three non-zero rows.
The rank is 3.
The answer is d).

A3.
Rank=2 means last row is all zeros

X= -2…0 0 0 / 0

The answer is b).

A4. You need to know rank to conclude exactly how many parameters. You can say there are always at least (n-m) parameters.
The answer is e).

A5. If rank=3, # parameters= n - rank (r)= 5 - 3 = 2
The answer is c).

A6. # parameters= n - r = 5 - 3 = 2
The answer is b). Remember, it isn't homogeneous, so it could be no solution as well.

A7.
parameters= n - r = 6 - 5 = 1
The answer is b).

A8. There can't be a unique solution, because there aren't enough rows to row-reduce and get the identity matrix. There could be no solution, or infinitely many. If there are infinitely many solutions, the number of parameters is n-r, so it depends on whether there is a row of zeros. It could be 2 or more parameters. You need to know the rank to conclude exactly how many parameters you have.
The answer is d).

A9.

A) is true because if it is homogeneous, the b vector is 0
B) is true because all homogeneous systems have a solution, either a unique solution or infinitely many solutions
C) is false ... a homogeneous system always has a solution
D) True
E) True
The answer is C).

A10. False. If the rank A=rank [A/b] = n, the number of unknowns, then it has a unique solution. It would equal the number of equations only if it is a square matrix.

A11.

A) is false, you wouldn't know unless you row-reduced
B) False, the number of parameters is n - r and we don't know the rank
C) False, If the rank is 4, the number of parameters is n-r=5-4=1 parameter, not (trivial) unique solution.
D) is TRUE. It is a 9x5, so it can be any of the three possibilities. If it were a 5x9, unique would be impossible.
The answer is d).

A12. The answer is d). (n-r=7-3=4)

A13.
parameters= 18 - 8 = 10
The answer is b).

A14.
n-r=10 – 8 =2
The answer is d). We can't say it always has a solution (consistent) or it never has a solution (inconsistent) as we don't know what the matrix numbers are. We can only tell the number of parameters if we know the rank.

A15.
You don't have as many rows as you do columns, so...
It is impossible to get the identity matrix, so there can't be a unique solution. It is homogeneous, so it can't be no solution either.
The answer is c).

A16. C. is false because if matrix A is not invertible, there could be no solution or infinitely many solutions.

A17. If rankA=4 < rankA/b then there is no solution, which is called inconsistent. The answer is d).

A18. Rank A < n if there is no solution where n= number of unknowns. The answer is c). Since the matrix is square, we know that m=n ie. number of equations is the same as the number of unknowns.

A19. The answer is d). Homogeneous systems can either be unique or infinitely many solutions...

A20.The answer is e). If det A=0, you do not get the identity or a unique solution when you row-reduce.

A21. rank A= rankA/b =n= 6...the answer is b).

A22.
i) if a system is homogeneous, no solution is impossible...so false
ii) is false since if detA=0 then the system does not have a unique solution
iii) If det A = 0, then there is no unique solution so since it is also homogeneous, yes it MUST have infinitely many solutions...true
iv) false...det A = 0 so non-invertible
v) true
The ones that are true are iii) and v). The answer is e).

A23. 8 - 7 = 1 parameter...answer is a).

A24. 3x5
i) false, it can't be unique, there are not enough rows
ii) true, a 2x5 cannot be unique
iii) you can't tell the number of parameters unless you know the rank, so false
iv) true, there are at least 5 - 2 = 3 parameters if there is a solution
v) if rank A=rankA/b = 2, then there is exactly n - r = 5 -2 = 3 parameters...true
The answer is ii, iv and v) are true. The answer is e).

A25.6x6...we don't know if it has a solution or not.
c) is true, if it is homogeneous, it always has the unique, trivial solution. If the ranks both equal n, it is a unique solution, so d) is true as well. The answer is e).

A26. The answer is d), since we know any homogeneous system with more unknowns than equations must have infinitely many solutions. If it isn't homogeneous, it could be no solution as well! If n=m+2, then n is greater than m, so there are more unknowns than equations, and therefore infinitely many solutions.

A27. The answer is a). Draw a matrix and notice that if you have 6 columns and 8 rows, you can't have 8 leading 1's or some of them would be on top of each other. So, you can only get up to 6 leading 1's at the most.

B. Final Exam Questions on RREF, Matrices, Inverses and Theory of Linear Systems

B1. b) and d) are NOT row-reduced. a) and c) are row-reduced...Answer is a and c). answer C)

B2.
b) is not linear because you can't take the square root of x, and c) is not linear because you can't multiply variables...a) and d) are both linear...answer B)

B3. Find the value(s) of k for which the matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & k \\ 0 & 2 & k & 2 \end{array}\right]$ has no solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & k \\ 0 & 0 & k-6 & 2-2k \end{array}\right]$$

k=6 gives 0 0 0 / -10 no solution

B4.

$$A+B=\begin{bmatrix} 8 & 4 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 5 \end{bmatrix}$$

$$(A+B)^T=\begin{bmatrix} 8 & 0 & -1 \\ 4 & 2 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

C_{32}=1 (row 3, column 2)

B5.

$$2\begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=2\begin{bmatrix} 10 & 3 \\ 2 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 20 & 6 \\ 4 & 14 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 17 & 6 \\ 4 & 11 \end{bmatrix}$$

B6.
$X=A^{-1}b$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

x=8+0+2=10
The answer is e).

B7.
homogeneous + more unknowns than equations= infinitely many solutions

The answer is c).

B8.
If rank A= 3< rank A/b=4 , there is no solution

The answer is a).

B9.
Non-invertible matrices have rankA <n and do not have unique solutions. In order to get the identity matrix, we need an invertible matrix and therefore a unique solution. If AX=0, and there is no inverse, then there are infinitely many solutions because there can't be a unique (trivial) solutions.

So, ii) and iv) are false
The answer is a).

B10. # parameters= n-r = 15 - 9 = 6. The answer is b).

B11.
Row-reduce and count the number of non-zero rows
-2R1+R2→ $R2$
-3R1+R3→ $R3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R1+R2→ $R1$
R2 x (-1) → $R2$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ rank=2. The answer is c).

B12. d) is false because if it is invertible, it must be a unique (trivial) solution.
e) is also false as if it is invertible, it cannot be "no solution".

B13. Given the matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & a^2-36 & 6+a \end{array}\right]$

What value of "a" results in no solution? Infinitely many solutions? Unique solution?

a=6 gives 0 0 0 /12 no solution

a= -6 gives 0 0 0/ 0 infinitely many solutions

a≠6,-6 gives a unique solution

B14.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-16+12}\begin{bmatrix} -4 & 2 \\ -6 & 4 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -4 & 2 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ 3/2 & -1 \end{bmatrix}$$

B15.

Row-reduce $[A|I]$ ... $[I|A^{-1}]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$....R3 – R2→ $R3$ RREF...$\left[\begin{array}{ccc|ccc} \square & \square & \square & \square & \square & \square \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & -1 & 1 \end{array}\right]$ R3 $\div 4 \rightarrow R3$

$\left[\begin{array}{ccc|ccc} \square & \square & \square & \square & \square & \square \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & 1/4 \end{array}\right]$ don't need to finish since you only need the third row

b32= -1/4

B16. A=$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$.

Consider the system: ax+by+cz=3
dx+ey+fz=1
gx+hy+iz=4

Find the value of z.

X=$A^{-1}b$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

z=3 – 1 + 0 = 2

The answer is c).

B17.

AB=$\begin{bmatrix} 2 & 3 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

c31=2+3+1=6

C. Minor and Co-Factors (4.1)

Example 1. Delete second row and third column

$$A_{23} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Example 2. Delete the first row and second column

$$B_{12} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

Example 3.

Delete the first row and second column

1,2 minor = det $\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} = 0 - 2 = -2$

1,2 cofactor= $(-1)^{1+2}(-2) = 2$

D. Determinants (4.1)

Example 2. Find the determinant of each of the following:

a) det A=7

b) detB=ad - bc = (2)(-7) - (-1)(3) = -14 + 3 = -11

c) C=$\begin{bmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

Expanding along the last row...
detA= $(-1)^{3+1}(4)det\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} + (-1)^{3+2}(2)det\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$
=4(3+2) + (-2)(2+4)
=20-12
=8

d) D=$\begin{bmatrix} 3 & -1 & 0 \\ 1 & -2 & 2 \\ -1 & 0 & 3 \end{bmatrix}$

expand along the third column
det B=$(-1)^{2+3}(2)det\begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix} +(-1)^{3+3}(3)det\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$

= -2(-1) + 3(-5)=2 - 15
= -13

e) E=$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & -1 \\ 3 & 4 & 0 \end{bmatrix}$

Expand along the second column because it has the most zeros

det C=$(-1)^{3+2}(4)det\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$
= -4(-1-2)
=12

f) F= $\begin{bmatrix} 3 & -1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 4 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

expand along the first column

det F= $(-1)^{1+1}(3)det\begin{bmatrix} 0 & 2 & 2 \\ 4 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

expand along the first column

det F=3 $\left[(-1)^{2+1}(4)det\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} + (-1)^{3+1}(1)det\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}\right]$

=3[(-4)(4)+1(-4)]

= 3(-16-4)

= - 60

g) G= $\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

Expand along the first column

det G= $(-1)^{1+1}(2)det\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

expand along third column

det G= 2[$(-1)^{2+3}(2)det\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$]

=2[(-2)(1)]

= -4

h)

$let\ A =$ (1,1) (1,2) (1,3)

$$\begin{bmatrix} 1 & 2 & -3 \\ a & b & c \\ d & e & f \end{bmatrix} \text{and given}$$

$$\det \begin{bmatrix} a & b \\ d & e \end{bmatrix} = 4$$

$$\det \begin{bmatrix} a & c \\ d & f \end{bmatrix} = 2$$

$$\det \begin{bmatrix} b & c \\ e & f \end{bmatrix} = 3$$

Find $\det A$

$$detA = (-1)^{1+1}(1)\det \begin{bmatrix} b & c \\ e & f \end{bmatrix} + (-1)^{1+2}(2)\det \begin{bmatrix} a & c \\ d & f \end{bmatrix} + (-1)^{1+3}(-3)\det \begin{bmatrix} a & b \\ d & e \end{bmatrix}$$

$$= (1)(3) + (-2)(2) + (-3)(4)$$

$$= 3 - 4 - 12$$

$$= -13$$

Example 3. Find the 3,2 minor and 3,2 cofactor of C

Delete the third row and second column

3,2 minor = det $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ expanding along top row

$=(-1)^{1+1}(2)\det\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} + (-1)^{1+2}(1)det\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

.=(2)(-3)+(-1)(0)= -6

Det C= -6

3,2 cofactor= $(-1)^{3+2}(-6) = 6$

Example 4. Find the 2,2 minor and the 2,2 cofactor of D= $\begin{bmatrix} 3 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & 4 & 0 & 1 \\ 0 & -1 & 1 & 3 \end{bmatrix}$

Delete the second row and second column

2,2 minor= det $\begin{bmatrix} 3 & 2 & 3 \\ -1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

expand along last row...

= 0 + $(-1)^{3+2}$(1) det $\begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix} + (-1)^{3+3}(3)det\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

= -1(3+3) +3(0+2)
=0

2,2 cofactor= $(-1)^{2+2}(0) = 0$

Example 5.

1,2 minor=detA_{12}= det $\begin{bmatrix} 0 & 2 & -1 \\ -1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

=$(-1)^{2+1}(-1)det\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + (-1)^{3+1}(2)det\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

=(-1)(-1)(7)+(1)(2)(2)=7+4=11

1,2 cofactor = $(-1)^{1+2}[1{,}2\ minor] = (-1)(11) = -11$

E. Triangular Form Method (4.1)

Example 3.

a) A= $\begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 3 \\ -3 & -1 & 4 \end{bmatrix}$

R3 + R1→ $R3$

$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

detA=(3)(1)(3)=9 multiply along main diagonal

b) B= $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 4 \\ -1 & -2 & 1 \end{bmatrix}$ R3+R1 → $R3$ $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix}$ R3-R2 → $R3$ $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix}$ sign of the determinant)

detB=1(2)(-1)= -2 (multiply along the main diagonal)

c) C= $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 1 & 4 & 6 \end{bmatrix}$ R3-R1 → $R3$

$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix}$ R3-2R2 → $R3$

$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & -6 \end{bmatrix}$

detC= -6

d) switch row 1 and 2 and that switches the sign of the determinant:

det D=$-det\begin{bmatrix} 2 & -2 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 4 & 5 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ R3-2R2 $\rightarrow R3$

$$= -det\begin{bmatrix} 2 & -2 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

det D= (-1)2(2)(1)(1) = -4

e) switch the top two rows: E=$\begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ R3-2R2 $\rightarrow R3$, R4-R2 $\rightarrow R4$

$$\begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

det E= (2)(1)(-3)(-1) = 6

<u>Example 4.</u> Find the determinant of each of the following matrices:

a) det A=0 since there are two equal columns

b) det B= (1)(6)(3)= 18 since it is in lower triangular form

c) detC=0 since there is a row of 0's

d) $\begin{bmatrix} 1 & 2 & 4 \\ 5 & -1 & 2 \\ -2 & -4 & -8 \end{bmatrix}$

det D=0 since column row 1 times (-2) is equal to row 3

e) E= $\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 3 & 0 & 5 \end{bmatrix}$

R3 - 2R2→R3 and R4 - 3R2→ $R4$

$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix}$ R4 - R3 → $R4$

$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

det E= (1)(-3)= -3

f) F= $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 0 & 0 & 3 \\ 3 & 1 & 1 & 2 \\ -1 & -1 & 0 & -1 \end{bmatrix}$ R2-2R1 → $R2$ and R3-3R1 → $R3\ and\ R4 + R1 \to R4$

$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 1 & 4 & -4 \\ 0 & -1 & -1 & 1 \end{bmatrix}$ R4 + R3→ $R4$ $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 1 & 4 & -4 \\ 0 & 0 & 3 & -3 \end{bmatrix}$

you can switch to expanding along the first column, to avoid the fractions

det F= $(-1)^{1+1}(1)det \begin{bmatrix} 0 & 2 & -1 \\ 1 & 4 & -4 \\ 0 & 3 & -3 \end{bmatrix}$

expand along first column

det F= $(-1)^{2+1}(1)det \begin{bmatrix} 2 & -1 \\ 3 & -3 \end{bmatrix}$

= (-1) (-6+3)

=3

F. Basket-Weave Method (4.1)

Example 2.

R= 0 + 24 + 12 = 36
L= 0 + 16 - 20= -4

R - L = 36 - (-4) = 40

Example 3.

R=-2+12+12=22

L= -9 + 8 + 4 = 3

R - L = 22 – 3 = 19

det A = 19

Example 4.

First, expand along second row...

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 1 & 3 \\ 1 & -1 & 0 & 4 \end{bmatrix}$$

det = $(-1)^{2+4}$ (3) det$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

det= (1)(3) $[basket\ weave\ the\ 3x3]$
=(1)(3) $[right\ products - left\ products]$
=(1)(3) $[(0 + 1 + 0) - (-2 - 1 + 0)]$
=(3)(1+3)
=12

G. Properties of Determinants (4.2)

Example 1.
a) det(AB)= detA det B=(3)(4)=12
b) det (A^TB)= detA^T det B = (detA)(detB) =12
c) det(A^{-1})= 1/detA = 1/3
d) det(3A)= 3^3detA= 27(detA)=27(3) =81
e) det(2B)= 2^3 det B = 8(4) = 32

Example 2.
detA=3, detB=2 and both are 3x3 matrices

a) det (AB)= detAdetB= (3)(2)= 6
b) det (B^TA)= det B det A= (2)(3) =6
c) det (A^{-1} B)=$\frac{1}{detA}\, detB$=$\frac{1}{3}(2) = \frac{2}{3}$
d) det(2A)= 2^3 (3)= 24
e) det (-3B)= $(-3)^3(2) = -54$

Example 3. multiplied row 1 by 2 and column 2 by 4...multiplies determinant by 2 and by 4..switch column 1 and 3 which multiplies the determinant by -1.
new det = old det (2)(4)(-1) = 2(2)(4)(-1) = -16

Example 4. detA = - 5

A. det A^T = det A = - 5 true

B. det A^{-1} = 1/-5 = 1/5 false it should be -1/5

C. det(A^2) = det(AA)=det A (detA) = (-5)(-5)=25 true

D. true, since det A is NOT equal to 0

E. det (2A)= 2^3 det A = 8 (-5) = -40 false

The answer is b and e) are false.

Example 5. C...det C is not equal to 12, while A, B and C are. (it is not the main diagonal, down and to the right)

Example 6. A + 2B = $\begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

det (A + 2B)= 4(6)(-2)= - 48

Example 7. det B = det A (-2)
detB = 36 (-2) = -72

Remember, g-2d back into g doesn't affect the determinant.
If it were g-2d back into d, it would multiply the determinant by -2.

Example 8. expand along bottom row (or you can basket weave)
det A = 0

$0 = (-1)^{3+2}(3)\det\begin{bmatrix} 4 & 0 \\ k & k \end{bmatrix} + (-1)^{3+3}(2)det\begin{bmatrix} 4 & 3 \\ k & 1 \end{bmatrix}$

0 = -3 (4k - 0) + 2 (4- 3k)
0 = -12k+ 8 – 6k
18k = 8
k = 8/18 = 4/9

Example 9.. 3x3 det A = 2

$\det(3A)\det(A^T A^{-1} A)$
$=\det(3A)\det(IA)$
$=\det(3A)\det A$
$=3^3 \det A \det A$
=27(2)(2)
=108

Example 10. det A = (-1)(8)(3) = -24

det A^{-1} = -1/24

Example 11. 4k - 2 = -36k+2
4k= -36k+2 +2
4k + 36k= 4
40k=4
k=1/10

Example 12. switch row 1 and row 2 will multiply the det by -1...3 times row 2 will multiply the det by 3 and -2 times row 3 will multiply the original det by -2

So, new det= old det (-1)(2)(3)= 4(-1)(3)(-2) since the original det was 4...final answer = 24

Example 13. it is in triangular form, so multiply along the main diagonal to find the determinant

2(k) (-3) = 72
-6k = 72
k = - 12

Example 14. When you do R2 - R1→ $R2$
and R3-R1→ $R3$ you get:

$\begin{bmatrix} a & b & c \\ -1 & -1 & -1 \\ 5 & 5 & 5 \end{bmatrix}$ if you then do R3+5R2 → $R3$ you get a row of 0's, so detA=0.

Example 15. $\det(2A^{-1}) = 5$
$2^4 detA^{-1} = 5$ since A is a 4x4 matrix
$16\left(\frac{1}{detA}\right) = 5$
$detA = \frac{16}{5}$
The answer is B).

Example 16.
$\det A + (\det B)\,(\det D) + \det C$
$= 0 + (14)(1) + 0 = 14$
NOTE: det A=0 since A has a column of 0’s and det C=0 since matrix C has two equal rows. Matrix B is in upper triangular, so the det B= (2)(-1)(-7)= 14. D is an identity matrix, so det D = 1 (1)(1)(1)(1)=1.

H. Practice Exam Questions on Determinants (4.1, 4.2)

H1.
multiply along the main diagonal since it is already in upper triangular
detA= (2)(2)(4)= 16

H2. Find the 1,2 cofactor of the matrix A= $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

delete the first row and second column
1,2 cofactor= $(-1)^{1+2} det \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = (-1)(2+3) = -5$

H3.
The matrix is in lower triangular, so just multiply along the main diagonal to find the determinant

detA= (2)(3)(2)(6)= 72

H4. det $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & -1 & -1 \end{bmatrix}$ R2↔R3

$= -det \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix} = -det \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -5 \\ 0 & -1 & 0 & -3 \end{bmatrix} = -det \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
R4+R3→R4

R4 -R1→R4 R4+R2→R4

$-det \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & -5 \end{bmatrix}$ = (-1)(1)(1)(-1)(-5)= -5

H5. Find the 3,2 minor of B= $\begin{bmatrix} 5 & 1 & 2 \\ 4 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ delete the third row and second column

$= detA_{32} = det\begin{bmatrix} 5 & 2 \\ 4 & 1 \end{bmatrix} = (5-8) = -3$

H6. Find det (A+B) where A= $\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and B= $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 7 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

****Find (A+B) first!!!**
note: you **cannot say det(A+B)=det A + det B...it is not true for any matrices A and B

A+B= $\begin{bmatrix} 2 & -1 & 6 \\ 0 & 8 & 3 \\ 1 & 1 & 5 \end{bmatrix}$ expand along the first column

det(A + B)= $(-1)^{1+1}(2)det\begin{bmatrix} 8 & 3 \\ 1 & 5 \end{bmatrix} + (-1)^{3+1}(1)det\begin{bmatrix} -1 & 6 \\ 8 & 3 \end{bmatrix}$
=(2)(40-3) + (1)(-3 -48)
=74 - 51
=23

H7.

det(A^TB^{-1})= detA (1/detB)= 2 (1/5) = 2/5

H8. If det $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$=4

find the determinant of $\begin{bmatrix} 2a & 2b & 2c \\ 3g & 3h & 3i \\ d & e & f \end{bmatrix}$.

switched rows x (-1) and then multiplied one row by 2, one row by 3 so det gets multiplied by 2 and 3

det=4(-1)(2)(3)= -24
The answer is d).

H9. Delete the third row and first column

3,1 cofactor=$(-1)^{3+1}detB_{31} = (1)det\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = 1 + 2 = 3$

H10. Delete the second row and first column

2,1 minor =$detA_{21} = det\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$= 9 + 4 = 13

H11.

matrix is in upper triangular form, so multiply along main diagonal to find the determinant

det= $2(\sqrt{2}(4\sqrt{2}) = 2(4)(2) = 16$ since a square root times itself, is just the number under the root

H12.

matrix is in lower triangular form, so multiply along main diagonal to find determinant

det=(1)(6)(1)(2)= 12

H13.

switch row 1 and row 4, and multiply det by (-1)

$= -det\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ row reduce to get zero in row 4

R4-R3 $\rightarrow R4$

$=-det\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

= -(1)(2)(1)(4)= -8

H14.

a)det=ad-bc= 4 + 3 = 7

b) det=$-\sqrt{5}$

c) det=(2)(-1)(-15)(3)= 90

H15. $\begin{bmatrix} 2x & x & -2x \\ x & x+1 & -2x \\ 0 & 0 & 3x+1 \end{bmatrix}$

expand along third row

det=$(-1)^{3+3}(3x+1)det\begin{bmatrix} 2x & x \\ x & x+1 \end{bmatrix} = (3x+1)(2x^2+2x-x^2) = (3x+1)(x^2+2x)$

H16.
det= ad-bc= -4 – 5 = -9
The answer is b).

H17.
det= ad - bc = 3 - 10 = -7

The answer is c).

H18.
$det\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 4 \\ 2 & 2 & 4 \end{bmatrix}$ using R3-2R1→R3 and R2-2R1→R2, then expand along the first column

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & -5 & 4 \\ 0 & -4 & 4 \end{bmatrix}$$

$= (-1)^{1+1}(1)det\begin{bmatrix} -5 & 4 \\ -4 & 4 \end{bmatrix} = -20 + 16 = -4$

H19. The matrix A=$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 4 & 2 & 2 \end{bmatrix}$
has detA= -4. Which of the following is ***false***?

A. is false since the transpose has the same det as the original matrix
B. is true since det (inverse)=1/detA
C. is false since det(2A)=2^3(detA)=8(-4)= -32
D. is true since det(A^2)=det(AA)=detA(detA)= 16

The answer is e).

H20.
$\det(A^T B^{-1}) = \det A\ (1/\det B) = 2/5$
The answer is e).

H21.
$\det(C^{-1} D^T) = (1/\det C)(\det D) = 1/4\ (3) = 3/4$

The answer is b).

H22.
$\det(2A)^T = 2^3(\det A) = 8(4) = 32$

The answer is c).

H23. If det $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$=4, find the determinant of $\begin{bmatrix} 2a & 2d & 2g \\ b & e & h \\ c & f & i \end{bmatrix}$.

The matrix is transposed, but this doesn't change the determinant
The first row is multiplied by 2, so the determinant gets multiplied by 2

det= 2(4)= 8

The answer is c).

H24.

The first elementary row-operation doesn't change the det, so just multiply it by 2 because the third row is multiplied by 2, so the new determinant is 5(2) =10.

The answer is c).

H25. If A is a 4x4 matrix with det A= 2, which of the following statements are true?
I) $\det A^T=2$ true

II) det (-A)= -2
$\det(-A)= (-1)^4(\det A)=2$ so II is false

III) $\det(AA^{-1})=1$
$\det(AA^{-1})=\det(I)=1$ is true

IV) $\det(2A)=2^4(\det A)=16(2)=32$ so IV) is false

The answer is d).

H26.
det(CD)=detCdetD= (3-10)(9-0)= (-7)(9)= -63. The answer is b).

H27.
The answer is c). since if $\det B\neq 0$, there must be a unique solution

H28.
for A, if you multiply one column by 3, you multiply det by 3, but then when you multiply another column by 1/3, you multiply det by 1/3 and get back where you started!

B, has same det, transpose doesn't change determinant

C, elementary row operations don't change det

The answer is d).

H29.
It won't have an inverse if detA=0

(3)(6) - 3k = 0
18 - 3k=0
k=6
The answer is a).

H30. Determine the value of x so that matrix A=$\begin{bmatrix} 2 & 0 & 4 \\ x & 1 & 3 \\ 2 & 0 & x \end{bmatrix}$ has an inverse.

it has an inverse if detA≠0

If det A=0, 2x-8 = 0 so x=4 and this would be the value for which there is NO inverse
The answer is c). Basket weave and get left = 8+0+0=8 and right=2x+0+0=2x and right – left = 2x – 8 $\neq 0$,so we get x$\neq$ 4,since you know the determinant can't be 0 since it has an inverse. The answer is c).

H31.The answer is e). because if detA=0, there can't be a unique solution or an inverse, or the identity when you row-reduce. Also, if det A =0, there is at least one row of 0's so the rank is less than n. The homogeneous system can't be a unique solution, so it must be infinitely many solutions.

H32. Suppose A is an nxn matrix with det A≠0. Which of the following statements is true?

A homogeneous system can't have no solution anyway, but since det A is not zero, it must have a unique solution, which for a homogeneous system is the trivial solution..x=y=z=0. A is invertible since the determinant is not 0, and Ax=b must have a unique solution so a) is false. (There are no rows of 0's as if there were det A =0). d) is also false as the homogeneous system cannot be "no solution". The rank A=n since there are no rows of 0's, so e) is true.

The answer is e).

H33. Find $det\begin{bmatrix}1 & 0 & 5 & 3\\0 & -1 & 0 & 4\\0 & 0 & -3 & -2\\2 & 2 & 0 & 3\end{bmatrix}$ $R4 - 2R1 \rightarrow R4$

start row-reducing...

$$\begin{bmatrix}1 & 0 & 5 & 3\\0 & -1 & 0 & 4\\0 & 0 & -3 & -2\\0 & 2 & -10 & -3\end{bmatrix}$$ R4+2R2→ R4

$$\begin{bmatrix}1 & 0 & 5 & 3\\0 & -1 & 0 & 4\\0 & 0 & -3 & -2\\0 & 0 & -10 & 5\end{bmatrix}$$

expand along the first column...detA $(-1)^{1+1}(1)\det\begin{bmatrix}-1 & 0 & 4\\0 & -3 & -2\\0 & -10 & 5\end{bmatrix}$

expand along the first column of the 3x3...note: the numbers left in front of the 3x3 are just equal to 1

det A = $(-1)^{1+1}(-1)\det\begin{bmatrix}-3 & -2\\-10 & 5\end{bmatrix} = -1(-15 - 20) = 35$

H34. Delete the first row and second column

1,2 cofactor= $(-1)^{1+2}det\begin{bmatrix}3 & 3\\3 & 8\end{bmatrix} = (-1)(24 - 9) = -15$

H35. Delete the 2nd row and 1st column

2,1 minor= $\det A_{12}=\det\begin{bmatrix}1 & 1\\6 & 8\end{bmatrix} = 8 - 6 = 2$

H36. If A= $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and det A = 4,

a) Find det $\begin{bmatrix} a & d & g \\ 4b & 4e & 4h \\ c & f & i \end{bmatrix}$

NEW det=4(4)=16

b) Find det $\begin{bmatrix} 3a & 3b & 3c \\ 4d & 4e & 4f \\ 2g-6a & 2h-6b & 2i-6c \end{bmatrix}$

NEW det=(3)(4)(2)(4)=96

c) Find det $\begin{bmatrix} a-3d & b-3e & c-3f \\ g & h & i \\ 3d & 3e & 3f \end{bmatrix}$

NEW det=(-1)(3)(4)= -12

d) Find det $\begin{bmatrix} 2a & 2d & 2g \\ b & e & h \\ c & f & i \end{bmatrix}$

NEW det=2(4)=8

H37. Let A= $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. If det $\begin{bmatrix} 2a & 2d & 2g \\ b & e & h \\ c & f & i \end{bmatrix}$ = -40, find det A.

This one is backwards! You are finding det A, the original determinant and you know the final answer is -40.

2 det A = -40

detA=1/2(-40)= -20

H38.
$16=2^3$ detB

det B=16/8 = 2

The answer is a).

H39.
det(-3A)=$(-3)^2$detA= 9(-2)= -18

The answer is e).

H40.

(i) det (AB)= detA detB= 5(3)=15
(ii) det(A^TB^TA)=(det A)(detB) (detA)=5(3)(5)=75
(iii) $\det\left(A^{-1}B\right)$
=det=(1/5)(3)=3/5

H41. Find the det (A^{-1}B) if $A = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 3 & \sqrt[3]{5} \\ 0 & 0 & -2 \end{bmatrix}$ and B=$\begin{bmatrix} 8 & 0 & 0 \\ b & 3 & 0 \\ c & a & 1 \end{bmatrix}$

det(A^{-1}B)= (1/detA)(detB)=(1/-24)(24)=- 1 since they are both in triangular form so we can just multiply long main diagonal

H42. Let A = $\begin{bmatrix} 2 & 3 & 2 \\ 0 & c & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Find c if det $A^{-1} = \frac{1}{8}$.

if det A^{-1}=1/8, then det A=8

detA=8= (2)(c)(2)...multiply along main diagonal since A is in triangular form

4c=8
c=2

H43. Find all values of k for which A has an inverse if $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & k & 4 \\ 0 & 2k & k \end{bmatrix}$.

If A has an inverse, detA≠0...expand along first column

detA=$(-1)^{1+1}(2)det\begin{bmatrix} k & 4 \\ 2k & k \end{bmatrix} = (2)(k^2-8k) = 2k^2-16k = 2k(k-8)$=0

No inverse if k=0, 8

So, it will have an inverse as long as k≠0,8

H44. For what value of k is the rank of A=$\begin{bmatrix} k & -10 \\ 1 & k+7 \end{bmatrix}$ equal to 1? Hint: Use determinants.

Given rank =1, (there is a row of 0's), so det A=0 since that means it is not invertible and not unique

k(k+7) +10 = 0

k^2 +7k +10 = 0
(k+2)(k+5)=0

k=-2, -5

H45. det A=3, detB=2 and both are 4x4 matrices

a)det(AB)= detA detB= (3)(2)= 6

b) det (B^TA)= detB detA= (2)(3) = 6

c) det (A^{-1} B)= $\frac{1}{detA}det = \frac{1}{3}(2) = \frac{2}{3}$

d) det (3A)=3^4 (3)= 243

e) det (-2B)= $(-2)^4$ (2)= 32

H46. det(2A)= 2^3(det A) = 8(5) = 40
det(3A)=3^3(-2)= -54

*H47. det (- 6I) = $(-6)^{70}$ det I = $(-6)^{70}$ (1) = $(-6)^{70}$
since the determinant of any identity matrix is 1

H48.
det(2C)=2^4(2)= 32
det(3D)=3^4(3)=243

H49. First, row-reduce to make the matrix simpler...

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 2 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \text{R3+R1} \rightarrow R3 \; and \; R4 + R1 \rightarrow R4$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 3 & 3 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

then, expand along the first column

det A = $(-1)^{1+1}$(1) det A_{11}

$$=(1)(1) \det \begin{bmatrix} 0 & 1 & 1 \\ 3 & 3 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

Now, you can use the basket-weave method! (or another method of choice)

R= 0 -1 + 0 = -1

L= 3+0 -3 =0

R - L = -1 – 0 = -1

H50.
$detA = (k-3)(k+2)(k-4) = 0$ since it is in upper triangular form
So, k=3, -2 and 4 so that A is not invertible.

H51. If I is the 3x3 identity matrix, find $det(3I^{-1} - 7I^{T})$.
NOTE: the inverse and the transpose of I are the same matrix I, the identity matrix

$$det(3I^{-1} - 7I^{T}) = det(3I - 7I) = det(-4I) = (-4)^3 detI = -64(1) = -64$$

H52.
Given: $\det\begin{bmatrix} a & b \\ d & e \end{bmatrix} = 3$, $\det\begin{bmatrix} a & c \\ d & f \end{bmatrix} = 4$, $\det\begin{bmatrix} b & c \\ e & f \end{bmatrix} = -5$, Find $\det A$.
Expand along the top row of the original matrix:

$$detA = (-1)^{1+1}(1)\det\begin{bmatrix} b & c \\ e & f \end{bmatrix} + (-1)^{1+2}(2)\det\begin{bmatrix} a & c \\ d & f \end{bmatrix} + (-1)^{1+3}(-3)\det\begin{bmatrix} a & b \\ d & e \end{bmatrix}$$
$$= (1)(-5) + (-2)(4) + (-3)(3)$$
$$= -5 - 8 - 9$$
$$= -22$$

*H53. Find the (4,2) Adjoint of: $A = \begin{bmatrix} 2 & 1 & -1 & \mathbf{3} \\ \mathbf{-2} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ 0 & 1 & 3 & \mathbf{4} \\ 3 & -1 & 4 & \mathbf{5} \end{bmatrix}$.

(4,2) Adjoint = (2,4) cofactor, so we delete the 2nd row and 4th column of A first:

(4,2) Adjoint = (2,4) cofactor= $(-1)^{4+2} det\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 3 & -1 & 5 \end{bmatrix} = det\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 3 & -1 & 5 \end{bmatrix}$

Use any method to find the determinant.

Basket weaving gives us: down and left= -3 + (-6)+0= -9

Down and right = 10 + 9 + 0 = 19

Det = R – L = 19 + 9 = 28

The (4,2) Adjoint is 28.

H54. Use Basket Weaving:

R= 0 +20+12 = 32
L= -15 + 8 + 0= -7

R-L=32 – (-7) = 39
det B=39

H55.

a) A= $\begin{bmatrix} 2 & -1 & 2 \\ 0 & 2 & 3 \\ 0 & 3 & 4 \end{bmatrix}$

expand along the first column

det A= $(-1)^{1+1}(2)det\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$= 2(8 – 9) = -2

b) B=$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

switch row 2 and row 3 and multiply det by (-1)

$-det\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

= - (2)(2)(3)(4)

= - 48

H56. A= $\begin{bmatrix} 0 & 2 & 6 \\ 2 & 4 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

expand along the first column

$$detA = (-1)^{2+1}(2)det\begin{bmatrix} 2 & 6 \\ 2 & 2 \end{bmatrix} + (-1)^{3+1}(2)det\begin{bmatrix} 2 & 6 \\ 4 & 1 \end{bmatrix}$$

= -2 (4 - 12) + (2) (2 - 24)

=16 + 2(-22)
= -28

H57. $A = \begin{bmatrix} 2 & 6 & 4 & 8 \\ 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 \\ 2 & 6 & 4 & 8 \end{bmatrix}$

det A=0 since column 2 is three times column 1

H58. $\begin{bmatrix} 1 & 1 & -1 & 2 \\ -2 & 1 & 1 & 3 \\ 3 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 \end{bmatrix}$

R2 + 2R1 $\rightarrow R2$
R3-3R1$\rightarrow R3$
R4 - 2R1$\rightarrow R4$

$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & -1 & 7 \\ 0 & -2 & 4 & -6 \\ 0 & -3 & 2 & -4 \end{bmatrix}$ R4 + R2$\rightarrow R4$

$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & -1 & 7 \\ 0 & -2 & 4 & -6 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ ...now expand along first column

det=$(-1)^{1+1}$ (1) det$\begin{bmatrix} 3 & -1 & 7 \\ -2 & 4 & -6 \\ 0 & 1 & 3 \end{bmatrix}$ = det$\begin{bmatrix} 3 & -1 & 7 \\ -2 & 4 & -6 \\ 0 & 1 & 3 \end{bmatrix}$...use any method to finish!

Basket weaving...right products = 36+0 -14 = 22

Left products = 0 - 18+6 = - 12

Determinant=right products - left products = 22 +12 = 34

H59. If det $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = 4, find$ det $\begin{bmatrix} c & b & a \\ f & e & d \\ k-3c & h-3b & g-3a \end{bmatrix}$

New determinant = 4(-1) = -4 since they switched two columns
NOTE: k-4c is replacing k, so nothing is being done to the k, the spot being replaced, so this has no effect on the determinant

H60. New det=old det(3)(b)=3(3)(b)=9b

I. Cramer's Rule (4.3)

Example 1. Solve using Cramer's Rule: 2x + y = 3
x + 3y = 15

$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ detA=6-1=5

A(1)=$\begin{bmatrix} 3 & 1 \\ 15 & 3 \end{bmatrix}$ detA(1)=9-15= -6

A(2)=$\begin{bmatrix} 2 & 3 \\ 1 & 15 \end{bmatrix}$ detA(2)=30-3= 27

$x = \frac{detA(1)}{detA} = \frac{-6}{5}$

$y = \frac{detA(2)}{detA} = \frac{27}{5}$

The solution is (-6/5, 27/5).

Example 2.

$A = \begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix}$…detA=-2 using basket weaving..R= -3+9+4 = 10 and left= 12+3-3= 12 and so
det A = right - left = 10 - 12 = - 2

A(1)=$\begin{bmatrix} 10 & 3 & 4 \\ 2 & 1 & 1 \\ 4 & 1 & -1 \end{bmatrix}$ detA(1)=- 8 using basket weaving with R= -10 + 12 + 8 = 10 and left= 16+ 10 -6 = 20 and det A = right - left = 10 – 20 = -10

x=detA(1)/detA= -10/-2= 5

Example 3.

a) $z = \frac{detA(3)}{detA} = \frac{9}{3} = 3$ The answer is C).

b) $x = \frac{detA(1)}{detA} = -\frac{24}{3} = -8$ The answer is B).

c) $y = \frac{detA(2)}{detA} = -\frac{27}{3} = -9$ The answer is A).

Example 4.

det A= 6

A(1)=$\begin{bmatrix} 8 & 4 \\ 2 & -3 \end{bmatrix}$

det A(1)= - 24 - 8 = - 32

$$x = \frac{detA(1)}{detA} = -\frac{32}{6} = -\frac{16}{3}$$

Example 5.

det A = - 19

$$x2 = \frac{detA(2)}{detA}$$

$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ 4 & -3 & 2 \end{bmatrix}$ and b= $\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$

$A(2) = \begin{bmatrix} 1 & 4 & -1 \\ 3 & 6 & -2 \\ 4 & 2 & 2 \end{bmatrix}$ det A(2) = 22 by basket weaving...right products 12 -32 – 6 = -26 and left products = -24 -4 +24 = - 4

and det A(2) = right - left = –4 + 26 =22

So,

$$x2 = \frac{detA(2)}{detA} = \frac{22}{-19} = -\frac{22}{19}$$

Example 6. A= $\begin{bmatrix} 2 & 2 & a & -2 \\ 0 & 3 & b & -4 \\ 0 & 0 & c & 2 \\ 0 & 0 & d & 4 \end{bmatrix}$ det A=3

This involves determinants and finding a solution for a variable, so it is Cramer's Rule

To find x_3 we replace the third column of A with the b matrix (right hand side of the system of equations)

$2x_1+2x_2 + ax_3-2x_4=$ **-3**

$3x_2 + bx_3- 4x_4=$ **2**

$cx_3 + 2x_4=$ **-1**

$dx_3 + 4x_4=$ **0**

A(3)= $\begin{bmatrix} 2 & 2 & \mathbf{-3} & -2 \\ 0 & 3 & \mathbf{2} & -4 \\ 0 & 0 & \mathbf{-1} & 2 \\ 0 & 0 & \mathbf{0} & 4 \end{bmatrix}$

det A(3)= (2)(3)(-1)(4)= -24

$x_3= \frac{detA(3)}{\boldsymbol{det\ A}} = -\frac{24}{3} = -8$

J. Adjoint of a Matrix (4.3)

Example 1.
a) 2,1 minor = det[-1] = -1 (delete the second row and first column)

b) 2,1 cofactor= $(-1)^{2+1}$ [-1] = 1 for the 2,1 cofactor we delete the 2nd ROW and 1st column

c) 1,2 adj =2,1 cofactor= $(-1)^{1+2}\det A_{21}$...delete the second ROW and 1st column= (-1)(-1) = 1

Example 2. Find the 1,2 adjoint (Adj) for each of the following.

a) 2,1 Adjoint=1,2 cofactor= $(-1)^{2+1}\det A_{21}$= (-1) det[-1]= 1

Delete the 1st row, 2nd column

b) 2,2 Adjoint= 2,2 cofactor = delete the second row and second column = $(-1)^{2+2}\det A_{22}$= (1)det[5] = 5

Example 3. Find the adjoint matrix for $C=\begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix}$

$C_{11}= (-1)^{1+1}\det A_{11}= -1$

$C_{12}= (-1)^{1+2}\det A_{21}= -3$

$C_{21}= (-1)^{2+1}\det A_{12}= -2$

$C_{22}= (-1)^{2+2}\det A_{22}= 2$

Adjoint matrix=$\begin{bmatrix} -1 & -3 \\ -2 & 2 \end{bmatrix}=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example 4. Find the 1,3 entry of the adjoint matrix:

a) delete the 3rd row and 1st column

1,3 entry of adjoint= 3,1 cofactor=$(-1)^{1+3}$detA_{31}= (1) det$\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$= 1

b) delete the 3rd row and 2nd column

2,3 entry of adjoint=3,2 cofactor= $(-1)^{2+3}$detA_{31}= (-1) det$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$= -2

Example 5. 2,3 Adj D = (3,2) Cofactor D= delete the 3rd row and 2nd column

(2,3) Adj D= (3,2) Cof D=$(-1)^{3+2}det\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & -1 & -2 \end{bmatrix}$ =(-1)(right products – left products)

= -1 (-6 – (-5))= -1(-1) = 1

NOTE: I used basket-weaving, but you can find the determinant using any method

Right products= -4 + 0 + (-2) = -6

Left products = 0 -1 – 4 = -5

K. Finding the Inverse of a Matrix using the Determinant and the Adjoint (4.3)

Example 4.
detA= -4

3,1 Adjoint=1,3 cofactor=delete 1st row and 3rd column= $(-1)^{3+1}\det A_{13}$= (1)
$\det\begin{bmatrix}-1 & 1\\ 2 & 2\end{bmatrix}$=(1) (-2-2)= -4

$A^{-1} = \frac{1}{detA}AdjA = \frac{1}{-4}(-4) = 1$

Example 5.. detA= 3 since $\det A^{-1} = 1/3$

det(AdjA)= $(\det A)^{n-1} = (3)^{4-1} = 3^3 = 27$

Example 6.
$A^{-1} = \frac{1}{detA}AdjA = \frac{1}{-2}\begin{bmatrix}2 & 0 & 1\\ 2 & -1 & 2\\ 1 & 3 & 1\end{bmatrix}$
1,3 entry is -1/2 (1) = -1/2

Example 7. det(AdjA)= $(\det A)^{n-1}$
25 =$(\det A)^{n-1}$
n=3 since it is a 3x3 matrix... 25=$(\det A)^{3-1}$
25=$(\det A)^2$
det A = ± 5

Example 8.
det(AdjA)= $(\det A)^{n-1}$
243 =$(3)^{n-1}$
$(3)^5 = (3)^{n-1}$
5=n-1
n=6

So, it is a 6x6 matrix.

Example 9. If A is a 3x3 invertible matrix, find det $\left[2A\frac{1}{detA}\ AdjA\right]$.

Recall, $A^{-1} = \frac{1}{detA}AdjA,$ so substitute this into the determinant above

det $\left[2A\frac{1}{detA}\ AdjA\right]$ move 1/detA to be next to the AdjA…it is a constant so this is allowed

$= \det\left[2A\frac{1}{detA}\ AdjA\right]$

$= \det(2\text{A})\det[A^{-1}]$

= 2^3 detA det $[A^{-1}]$

=8 detA (1/detA)

=8

L. An Application to Find the Determinant (4.3)

Example 1.

$det\ A$ = (*ith row of A*)$\cdot$ $(ith\ column\ of\ Adj\ A)$

$$= \begin{bmatrix} a & b & c \\ \mathbf{3} & \mathbf{-3} & \mathbf{4} \\ d & e & f \end{bmatrix} \begin{bmatrix} -22 & \mathbf{8} & 17 \\ 8 & \mathbf{-7} & 2 \\ 17 & \mathbf{2} & -7 \end{bmatrix}$$

=(*2nd row of A*)$\cdot$ $(2nd\ column\ of\ Adj\ A)$

$$=\begin{bmatrix} 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ -7 \\ 2 \end{bmatrix} = (32)(8) + \ (-3)(-7) + (4)(2) = 24 + 21 + 8 = 53$$

Example 2.

$det\ A$ = (*ith row of A*)$\cdot$ $(ith\ column\ of\ Adj\ A)$

$$= \begin{bmatrix} a & b & c \\ d & e & f \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} -11 & g & 8 \\ 15 & h & -3 \\ -5 & i & 1 \end{bmatrix}$$

=(*3rd row of A*)$\cdot$ $(3rd\ column\ of\ Adj\ A)$

$$=\begin{bmatrix} 3 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ -3 \\ 1 \end{bmatrix} = (3)(8) + (4)(-3) + (1)(1) = 24 - 12 + 1 = 13$$

M. Practice Exam Questions on Determinant, Adjoint and Cramer's Rule (Ch. 4)

M1. det(AdjA)= $(\det A)^{n-1}=(3)^{3-1}=3^2=9$

M2. Since $\det(A^{-1}) = \frac{1}{3}$, detA=3
det(AdjA)= $(\det A)^{n-1}=(3)^{4-1}=3^3=27$
The answer is d).

M3. Since $\det A^{-1} = \frac{1}{2}$, detA=2
det(AdjA)= $(\det A)^{n-1}=(2)^{5-1}=2^4=16$
The answer is e).

M4. If $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 2 & 3 \\ 3 & 2 & 4 \end{bmatrix}$, find the (2,3)-entry of Adj A

2,3 Adjoint=3,2 cofactor= $(-1)^{2+3}\det A_{32}$...delete 3rd row and 2nd column
= (-1) det$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$
=(-1)(3-8)
=5

M5. If A= $\begin{bmatrix} 2 & 3 & 2 \\ -2 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, find the (2,1)-entry of Adj A.

2,1 Adjoint= 1,2 cofactor=$(-1)^{2+1}\det A_{12}$...delete 1st row and 2nd column
= (-1) det$\begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}$
=(-1)(-8-3)
=11

M6. If A is a 3x3 matrix with det A = -4 and Adj A = $\begin{bmatrix} 4 & 3 & 1 \\ 2 & 5 & 3 \\ \mathbf{6} & 7 & 4 \end{bmatrix}$, find the (3,1)-entry of A^{-1}.

(3,1) entry of $A^{-1} = \frac{1}{detA} AdjA = \frac{1}{-4}(6) = -3/2$

M7.If A = $\begin{bmatrix} 3 & 6 & 2 \\ d & e & f \\ 1 & 2 & 2 \end{bmatrix}$ and Adj A = $\begin{bmatrix} 0 & -8 & 4 \\ -1 & 4 & -1 \\ 1 & 0 & -3 \end{bmatrix}$, find det A.

or use the formula

det A = (ith row of A)· $(ith\ column\ of\ Adj\ A)$ *and use either the 1st or 3rd rows/columns*

det A = 1st row of A· 1$st\ column\ of\ AdjA$

=(3,6,2) $\cdot (0,-1,1) = 0 - 6 + 2 = -4$

M8.

This is Cramer's Rule

A(1)=$\begin{bmatrix} 4 & 4 \\ 3 & -3 \end{bmatrix}$ detA(1)= -12 – 12= -24

$$x = \frac{detA(1)}{detA} = -\frac{24}{48} = -\frac{1}{2}$$

M9. Let A=$\begin{bmatrix}1&0&1\\0&2&3\\3&0&5\end{bmatrix}$. If A^{-1} =M=$[m_{ij}]$, then determine the value of m_{21}.

Row-reduce to find the inverse
[A/I]….[I/A^{-1}]

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&2&3&0&1&0\\3&0&5&0&0&1\end{array}\right] \text{R3 – 3R1} \rightarrow R3 \left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&2&3&0&1&0\\0&0&2&-3&0&1\end{array}\right] \text{R2} \div 2 \rightarrow R2, R3 \div 2 \rightarrow R3$$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&3/2&0&1/2&0\\0&0&1&-\frac{3}{2}&0&\frac{1}{2}\end{array}\right] \text{R1 – R3} \rightarrow R1$$

$$\left[\begin{array}{ccc|ccc}1&0&0&\frac{5}{2}&0&\frac{-1}{2}\\0&1&3/2&0&1/2&0\\0&0&1&-\frac{3}{2}&0&\frac{1}{2}\end{array}\right] \text{R2 – 3/2 R3} \rightarrow R2$$

$$\text{...RREF} \left[\begin{array}{ccc|ccc}1&0&0&\frac{5}{2}&0&-\frac{1}{2}\\0&1&0&\frac{9}{4}&1/2&-3/4\\0&0&1&-\frac{3}{2}&0&\frac{1}{2}\end{array}\right]$$

2,1 entry is 9/4

M10. *det A = (ith row of A)·* $(ith\ column\ of\ Adj\ A)$
=(2nd row of A)· $(2nd\ column\ of\ Adj\ A)$

detA=$\begin{bmatrix}0&1&3\end{bmatrix}\begin{bmatrix}-9\\4\\7\end{bmatrix} = (0)(-9)+(1)(4)+(3)(7) = 0+4+21 = 25$

M11.
1,2 cofactor== delete 1st row and second column= $(-1)^{1+2}$detA_{12}= (-1)det$\begin{bmatrix}2&8\\1&2\end{bmatrix} =$
$-1(4-8) = 4$

2,1 adj= 1,2 cofactor=4

M12.
det(AdjA)= $(\det A)^{n-1} = (2)^{4-1} = 2^3 = 8$

The answer is a).

M13.
det(AdjA)= $(\det A)^{3-1} = (6)^2 = 36$

M14.
det(AdjB)= $(\det B)^{4-1}$
$125 = (\det B)^3$
detB=5

M15.
$$z = \frac{detA(3)}{detA} = \frac{50}{5} = 10$$
The answer is a).

M16.

$$x = \frac{detA(1)}{detA} = -\frac{125}{5} = -25$$
The answer is c).

M17.

$$y = \frac{detA(2)}{detA} = -\frac{25}{5} = -5$$
The answer is b).

M18. $x = \frac{detA(1)}{detA} = -\frac{6}{6} = -1$

$y = \frac{detA(2)}{detA} = \frac{36}{6} = 6$ The solution is (-1,6)

The answer is b). (Cramer’s Rule)

M19.
You can see that the matrix on top is A(1) since the second column in the original system has been replaced by the constant terms on the right of the system.

$$x = \frac{detA(1)}{detA}$$

Using Cramer's rule, the answer is a).

M20.

$\det(A^{-1}adjA) = \det(A^{-1}) \det(AdjA) = (1/detA)(detA)^{n-1} = (1/-3)(-3)^2 = -3$

The answer is e).

M21.
det A = 16 (multiply along main diagonal since it is in triangular form)

$$x = \frac{detA(1)}{detA}$$

$A(1) = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ and b= $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

det A(1)= (8+3+0) – (8+0+12)=11 – 20 = -9 by basket weaving

$$x = \frac{detA(1)}{detA} = \frac{-9}{16}$$

M22. a) is true
b) is false
c) is true, $\det(AB)^T = (\det(B^T A^T) = \det B^T \det A^T = \det B \det A = \det A \det B$
d) is false, many determinants are 2 but the matrices don't have to be equal for both matrices to have a determinant of 2
e) is false
f) is true $\det(AB)^{-1} = \det(B^{-1}A^{-1}) = \det B^{-1} \det A^{-1} = \frac{1}{detB}\left(\frac{1}{detA}\right) = \frac{1}{detAdetB}$

So, a, c and f) are all true.

*M23.For the system:
x+2y+3z=6
-x+2y+4z=8
2x+4y+5z=2

Find the value of y in the solution using Cramer's Rule

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

det A = -4 from basket weaving:
down and right = 10 + 16 – 12=14
down and left = 12 +16 -10= 18
det A = R – L = 14 – 18 = - 4

$$A(2) = \begin{bmatrix} 1 & 6 & 3 \\ -1 & 8 & 4 \\ 2 & 2 & 5 \end{bmatrix}$$

det A(2) = -40 from basket weaving:
down and right = 40+48 -6 = 82
down and left = 48 + 8 – 30 = 26
det A = R – L = 82 – 26 = 56

The solution for y is: $y = \frac{detA(2)}{detA} = \frac{56}{-4} = -14$

Find the value of z in the solution using Cramer's Rule.

$$A(3) = \begin{bmatrix} 1 & 2 & 6 \\ -1 & 2 & 8 \\ 2 & 4 & 2 \end{bmatrix}$$

det A(3) = -40 from basket weaving:
down and right = 4+32 – 24 = 12
down and left = 24+32 – 4 = 52
det A(3) = R – L = 12 – 52 = -40

$$z = \frac{detA(3)}{detA} = \frac{-40}{-4} = 10$$

*M24. For the system:
2x+5y+7z=2
-2x+y+3z=4
3y+9z=6

Find the value of x in the solution using Cramer's Rule

First, find det A:

$A = \begin{bmatrix} 2 & 5 & 7 \\ -2 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix}$ row reduce or expand or basket-weave and get det A= 48

If we row-reduce, we get:

$\begin{bmatrix} 2 & 5 & 7 \\ -2 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} R2 + R1 \to R2 \begin{bmatrix} 2 & 5 & 7 \\ 0 & 6 & 10 \\ 0 & 3 & 9 \end{bmatrix} R3 \leftrightarrow R2$(to get a smaller number on top, so we don't have to divide or use crazy fractions!)

= -det$\begin{bmatrix} 2 & 5 & 7 \\ 0 & 3 & 9 \\ 0 & 6 & 10 \end{bmatrix} R3 - 2R2 \to R3 = -det \begin{bmatrix} 2 & 5 & 7 \\ 0 & 3 & 9 \\ 0 & 0 & -8 \end{bmatrix}$ and we get det A=(-1)(2)(3)(-8)= 48

Remember, switching two rows switches the sign of the determinant

$A(1) = \begin{bmatrix} 2 & 5 & 7 \\ 4 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix}$

det A(1) = - 48 from basket weaving:
down and right = 18 + 90 + 84 = 192
down and left = 42 + 18 + 180= 240
det A(1) = R – L = 192 – 240 = - 48

$x = \frac{detA(1)}{detA} = -\frac{48}{48} = -1$

Find the value of y in the solution using Cramer's Rule

Now, $A(2) = \begin{bmatrix} 2 & 2 & 7 \\ -2 & 4 & 3 \\ 0 & 6 & 9 \end{bmatrix}$ Finding det A(2) we get: $\begin{bmatrix} 2 & 2 & 7 \\ -2 & 4 & 3 \\ 0 & 6 & 9 \end{bmatrix}$ $R2 + R1 \rightarrow R2 \begin{bmatrix} 2 & 2 & 7 \\ 0 & 6 & 10 \\ 0 & 6 & 9 \end{bmatrix}$

R3-R2→ $R3 \begin{bmatrix} 2 & 2 & 7 \\ 0 & 6 & 10 \\ 0 & 0 & -1 \end{bmatrix}$ and det A(2)= -12

The solution for y is: $y = \frac{detA(2)}{detA} = -\frac{12}{48} = -\frac{1}{4}$

*M25. If det A= -12, find the value of z in the solution of:

2x+2y+7z=5
-2x+4y + 3z = 1
6y + 9z = 3

$$A = \begin{bmatrix} 2 & 2 & 7 \\ -2 & 4 & 3 \\ 0 & 6 & 9 \end{bmatrix}$$

$A(3) = \begin{bmatrix} 2 & 2 & 5 \\ -2 & 4 & 1 \\ 0 & 6 & 3 \end{bmatrix}$…do the determinant using any method

Row reducing gives us:
$\begin{bmatrix} 2 & 2 & 5 \\ -2 & 4 & 1 \\ 0 & 6 & 3 \end{bmatrix}$ R2 + R1→ $R2 \begin{bmatrix} 2 & 2 & 5 \\ 0 & 6 & 6 \\ 0 & 6 & 3 \end{bmatrix}$ $R3 - R2 \rightarrow R3 \begin{bmatrix} 2 & 2 & 5 \\ 0 & 6 & 6 \\ 0 & 0 & -3 \end{bmatrix}$ Now, it is in upper triangular form, so multiply along the main diagonal to find det A = 2(6)(-3)= -36

det $A(3) = -36$

By Cramer's rule:
$z = detA(3)/\det A = -\frac{36}{-12} = 3$

*M26. If A is a 3x3 invertible matrix, find det $\left[A\frac{1}{detA}3AA^{-1}\,(AdjA)\right]$.

Recall, $A^{-1}=\frac{1}{detA}\,(AdjA)$ **we will use this near the last step**

det $\left[A\frac{1}{detA}3AA^{-1}\,(AdjA)\right]=$ det $\left[A\frac{1}{detA}3I\,(AdjA)\right]=det\left[3A\frac{1}{detA}\,I(AdjA)\right]$

$=det\left[3A\frac{1}{detA}\,(AdjA)\right]=\det(3AA^{-1})=3^3detA\,detA^{-1}=27\det A\left(\frac{1}{detA}\right)=27(1)=$
27

M27. det(AdjA)= $(\det A)^{n-1}$
$64=(2)^{n-1}$
$(2)^6(2)^{n-1}$
6=n-1
n=7

So, it is a 7x7 matrix.

Best of Luck on the exam!!