MATH 1228 FINAL EXAM BOOKLET SOLUTIONS (WINTER 2024)

Test One Material

A. Basic Set Theory (1.1)

Example 1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$160 = 90 + 100 - n(A \cap B)$

$160 = 190 - n(A \cap B)$

$\therefore n(A \cap B) = 30$

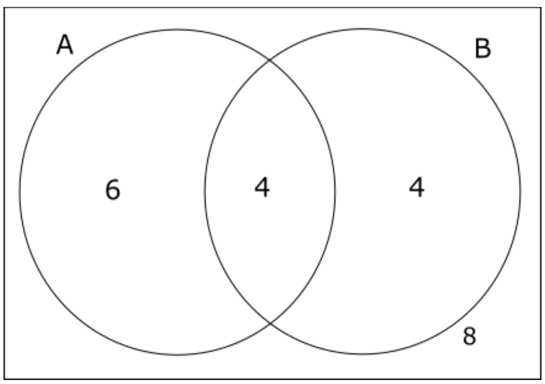
Example 2. $n(A^c) = 140 - 50 = 90$

Example 3. a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

= 10+8-4=14 OR $n(A \cup B) = 6 + 4 + 4 = 14$

b) $n(B^c \cup A)$ = not in B or in A (or both)

=8+6+4=18



Example 4. a) $A^c = \{7,8\}$ $B^c = \{1,3,5,7\}$

$n(A^c \cup B) = not\ in\ A\ or\ in\ B\ (or\ both)$

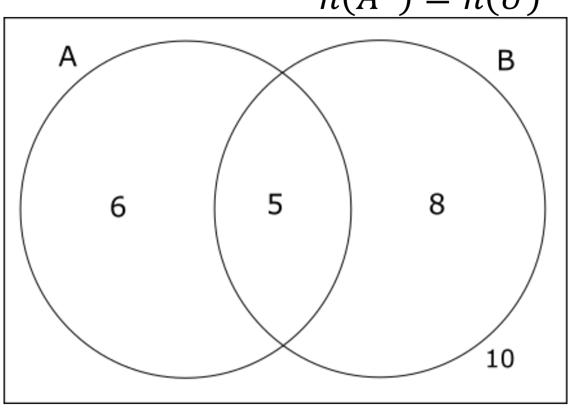
$= n\{2,4,6,7,8\} = 5$

b) $n(A \cap B^c) = n\{1,3,5\} = 3$

c) $n(B \cap A^c) = n\{8\} = 1$

Example 5. $n(A \cap B) = 29 - 6 - 8 - 10 = 5$

$n(A^C) = n(U) - n(A) = 29 - 11 = 18$



Example 6. $n(A \cup B) = 140 - 50 = 90$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

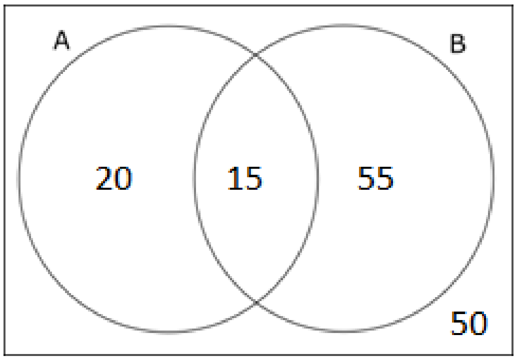
$90 = 35 + 70 - n(A \cap B)$

$90 = 105 - n(A \cap B)$

$n(A \cap B) = 15$

$n(A \cup B^C) = in\ A\ or\ not\ in\ B\ or\ both = n(A) + n(B^C) - n(A \cap B^C)$

$= 35 + 70 - 20 = 85$



Or use shading=20+15+50=85

Example 7.

A. $F \cup M$	B. M∩ F^C	C. $F^C \cup (P \cup M)$	D. $F^C \cup M \cup P$	E.$F \cap P \cap M$

$F \cup M = Female\ or\ has\ a\ meal - plan$
He is neither of these so he does not belong to this group

M∩ F^C= has meal-plan and is male. He is not in this set as he doesn't have a meal plan

$F^C \cup (P \cup M)$ is an "or" so he is in this set since his is not female.

$F^C \cup M \cup T$ says male or has a meal-plan or he is in psych and since he is in at least one of these, he is in this set

$F \cap P \cap M$ says "and" so he must be female and in psych and have a meal-plan to be in this set, so he is not in this set.

He is not in A, B or E.

Example 8.

I. $(S \cup L)^C$	II. $(M \cap L)^C$	III. $M \cap (L^C \cap S^C)$	IV. $M^C \cup S \cup L$

I. $(S \cup L)^C = S^C \cap L^C$ means not in second year and not from London. She is in this set as she is in first year and from PEI

II. $(M \cap L)^C = M^C \cup L^C$ means she is not taking finite on main or she is not from London. She is not taking it on Main, so she is in this set.

III. $M \cap (L^C \cap S^C)$ means she has to be taking it on main and she isn't so she isn't in this set

IV. $M^C \cup S \cup L$ means she is taking it at an affiliate or she is in second year, or she is from London, so since she is taking finite at an affiliate she is in this set.

So, she is in I, II and IV.

Example 9.

A. $F^C \cup M$
B. $B^C \cap M \cap F^C$
C. $F \cup B$
D. $(F \cap B^C)^C$
E. $(F^C \cap B)^C$
F. $B \cup M^C \cup F^C$

A. says either male or has a meal plan...True since he has a meal plan and he is male. (he is in both, but only needs to be in one)
B. says not taking business and has a meal plan and is male...True he is in all of these groups
C. says he is female or taking business which, he is not in as he isn't female and he isn't taking business.
D. Use the formula $(F \cap B^C)^C = F^C \cup B$ says he is in the set of "not female=male" or in the set of taking business...true, since he is male
E. $(F^C \cap B)^C = (F^C)^C \cup B^C$=$F \cup B^C$ $= female\ or\ not\ in\ business = true$ since he is in the set "not in business"
F is true since it says in business or no meal-plan or male and he is male.

The answer is C).

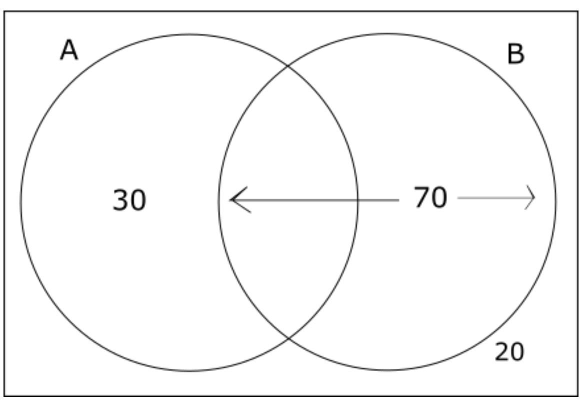
Practice Exam Questions on Set Theory and Venn Diagrams

A1.
a) n(A)= 5 since there are 5 numbers or elements in set A.
b) n(B)= 8 since there are 8 numbers or elements in set B.
c) AUB={1,2,3,4,5,6,7,8,9.10}
d) n(AUB)= 10 since there are 10 elements in the union between A and B
e) $n(A \cap B) = 3 \; since \; A \cap B = \{5,7,9\}$

A2.
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
75 = 30 + 50 - $n(A \cap B)$
$n(A \cap B) = 80 - 75 = 5$

Therefore, n(A∩ B)=5.

Use the following information for questions A3, A4 and A5.



A3. How many elements are in B?

There are 30 elements in A, but not in B and there are 20 elements outside the circles of both A and B, which are not contained in either of them.

The universe contains 120 elements, so the number in B is 120-30-20=70
The answer is B).

A4. Find n(AUB).
The number in B is 70 and there are 30 elements in A, that are not in B, so the total number of elements in the union of A and B is 30+70=100
The answer is B).

A5. Find $n(A^C \cup B)$.

To find the number not in A, or in B, or in both of these sets, we take all of the elements in the universe, except those in A alone.

Therefore, 120-30=90
The answer is B).

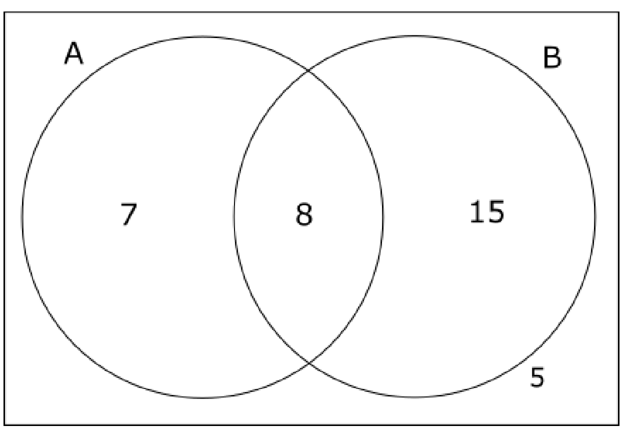
A6.

Draw a Venn diagram and label the information.

First, label the centre where 8 elements are in A and B
Second, label 7 in A, but not in B
Next, label 5 outside the circles4
The number in B, but not in A will be 35-7-8-5 = 15

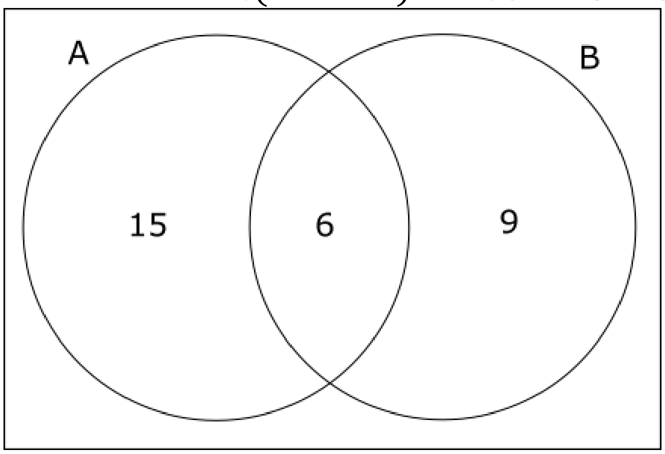
The number in B will be 8+15=23

The answer is D).



A7.

$$n(A^C \cap B^C) = 100 - 15 - 6 - 9 = 100 - 30 = 70$$



A8.

A. $M \cap (W \cap D)$	B. $(M \cup D)^C$	C. $M^C \cap (W \cup D)$	D.$M \cap (W \cup D)$	E. none of the above

The answer is D.

A9.

A. $M^C \cap B^C$	B. $M \cap B$	C. $(M \cap B)^C$	D. $M^C \cap B^C$	E. $none\ of\ the\ above$

A. is the set not in mos who don't take the bus.

B. is the set who are in mos and take the bus

C. is $(M \cap B)^C = M^C \cup B^C$ is the set who are not in mos or do not take the bus

D. is the set not in mos and do not take the bus

The answer is E. It should be $M^C \cap B$

A10. $M \cup \ (W \cap D)^C$=$M \cup (W^C \cup D^C)$

The answer is E. This is the set of students either in mos or the set of students who don't walk or don't drive themselves.

A11. $A \cap B = \{2,4\}$ $\quad A \cup B = \{1,2,3,4,5,6\}$

a) $(A \cup B) \cap C$ =$\{1,2,3,4,5,6\} \cap \{1,2,3,4,5\} = \{1,2,3,4,5\}$

b) $(A \cup B)^C = A^C \cap B^C = \{6,7,8,9,10,11,12\} \cap \{1,3,5,7,8,9,10,11,12\} =$
$\{7,8,9,10,11,12\}$

c) $(A \cap B)^C = \{1,3,5,6,7,8,9,10,11,12\}$

A12. n(U)=100
$n(A \cup B) = 100 - n(A^C \cap B^C) = 100 - 30 = 70$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
70=25+50- $n(A \cap B)$
$n(A \cap B) = 5$

A13. When shading the regions, remember that Union= or, so it can be in either place, but Intersection =AND which means it has to be in both places at the same time
The answer is B). It represents the students who knit and crochet and do NOT cross-stitch.

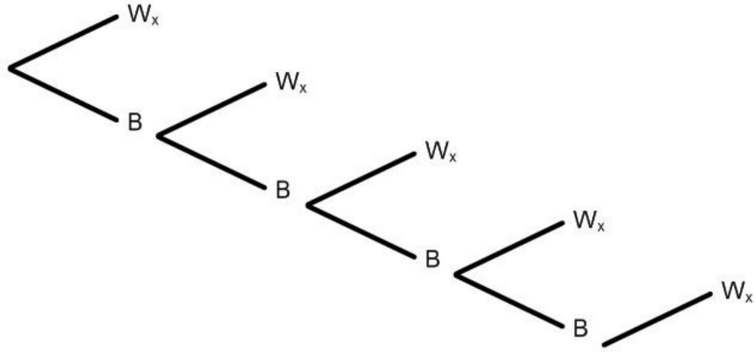
A14.

A. P$\cap$ $(M \cup H^C)$	B. $H \cap M$	C. $(H \cap M) \cap P$	D. $H \cup M \cup P$	E. none of the above

A. $P \cap (M \cup H^C)$
Means he is in psych and…and since he is not in psych, he can't be in this set
B. $\boldsymbol{H \cap M}$ means he is in philosophy and mathematics and he is in both of these, so he does belong to this set.
C. $(H \cap M) \cap P$ says he is in philosophy and math and in psych, but he is not taking psychology, so he is not in this set
D. $\boldsymbol{H \cup M \cup P}$ says he is in philosophy or math or psych and since he is in at least one of these, he is in this set.
The answer is both B and D.

B. Tree Diagrams (1.2)

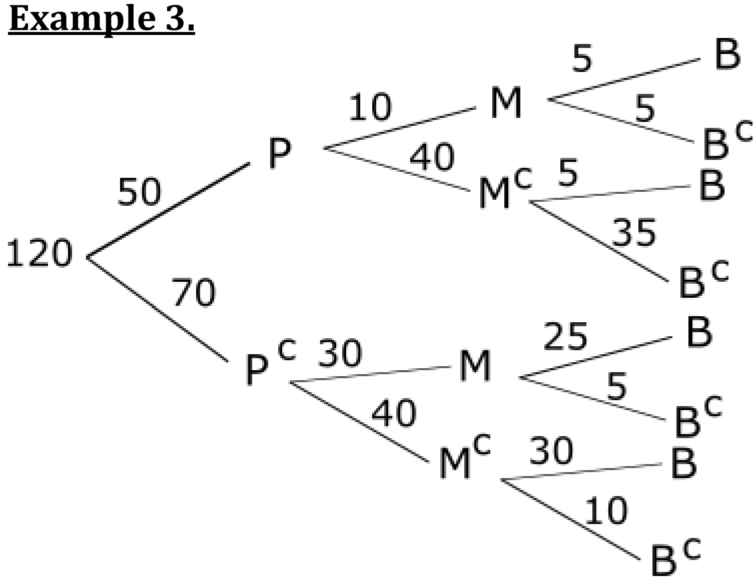
Example 1. 4B, 5W



So, there are 5 possible sequences.
They are W,BW, BBW, BBBW, BBBBW

Example 2.
There are 6 sequences. WW, WLW, WLL, LWW, LWL and LL that result in a best two out of three.

Example 3.

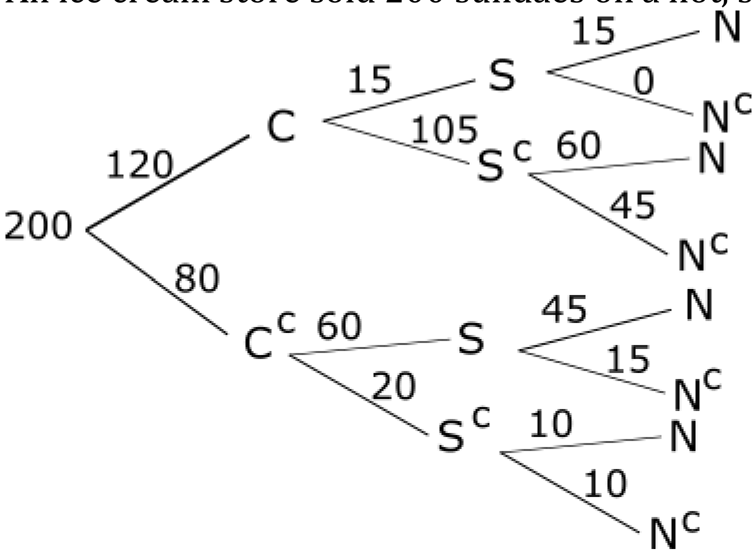


65 -5 -25 – 30 = 5 Bacon
A pizza place sells 120 pizzas on a particular evening... , find:

a) 5+5=10
b) Total – no toppings= 120 -10 = 110

Example 4.

An ice cream store sold 200 sundaes on a hot, summer afternoon.



So, 15 people have chocolate with sprinkles and nuts…along the top of the tree, the last entry

Example 5.

a) $n(C^C) = 10 + 16 + 12 + 12 = 50$

b) $n(B \cap C) = 5 + 18 = 23$

c) $n(A \cup C) = \textit{in } A \textit{ or in } C \textit{ or both}$

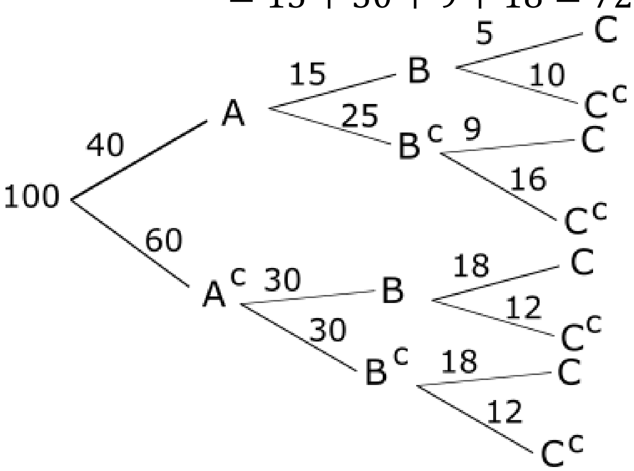
$= 40 + 18 + 18 = 76$

Add any C's not yet counted in A

d) $n(B) = 15 + 30 = 45$

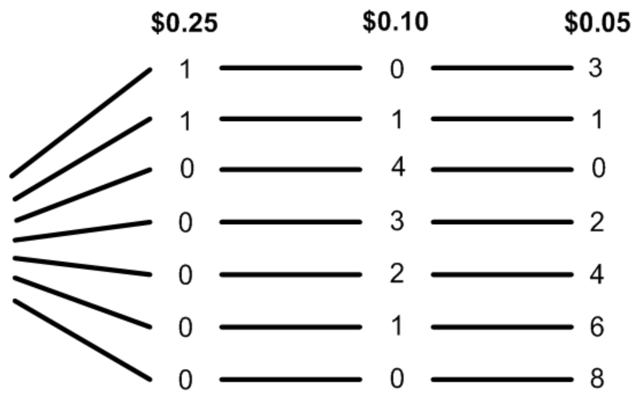
e) $n(B \cup C) = \textit{in } B \textit{ or } C \textit{ or both}$

$= 15 + 30 + 9 + 18 = 72$



Example 6.

Draw a tree as shown below. There are 7 different ways to make 40 cents using quarters, dimes and nickels only.

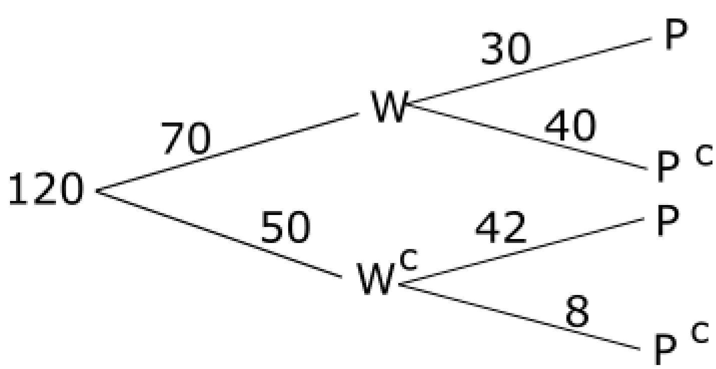


Practice Exam Questions on Tree Diagrams

B1.

Draw a tree diagram, to help you see what is going on.
Let W be the women and W^C is the women
Let P be the students passing the course and P^C is the students not passing the course

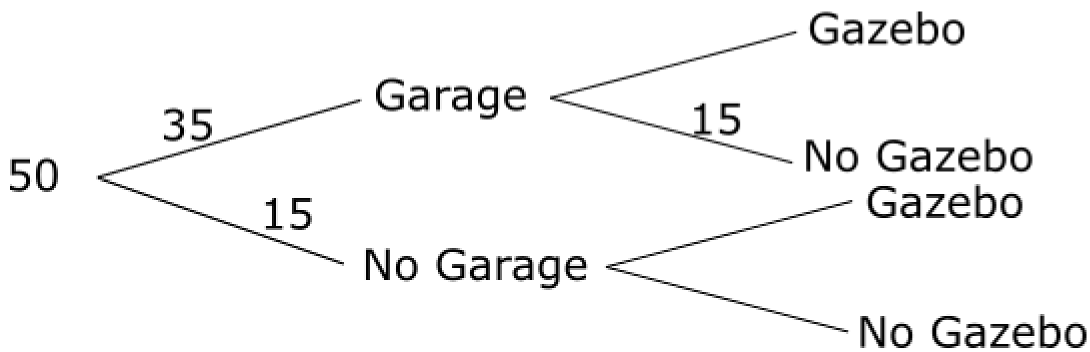
0.60(120)=72



From the tree diagram, the number of men failing the course is 8.

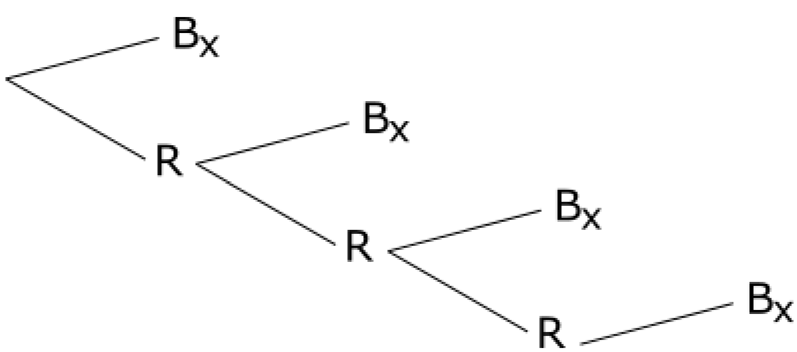
The answer is A)

B2. There are 50 houses on Creston Street...



30-15=15 is the number in G^C
Since there are 50 houses total, 50-15 = 35
So, the number of houses that have a garage is 35.
The answer is A).

B3. Draw a tree diagram where B represents drawing a black marble and R represents drawing a red marble



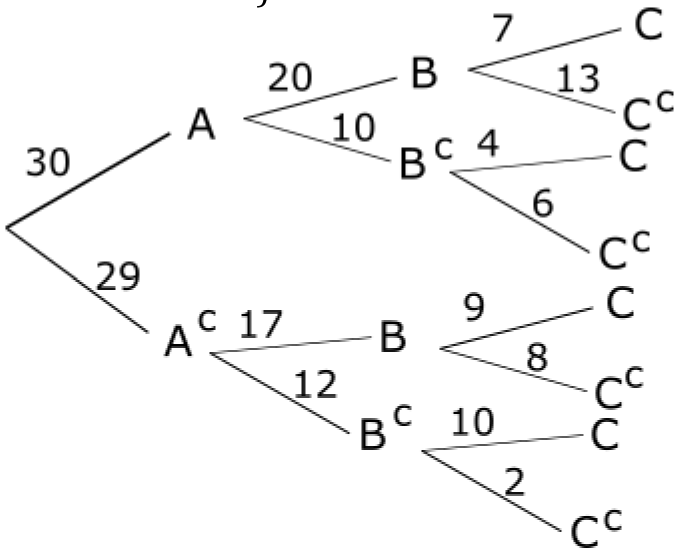
From the tree, there are 4 branches, so there are 4 possibilities.

B, RB, RRB, RRRB, so the answer is C).

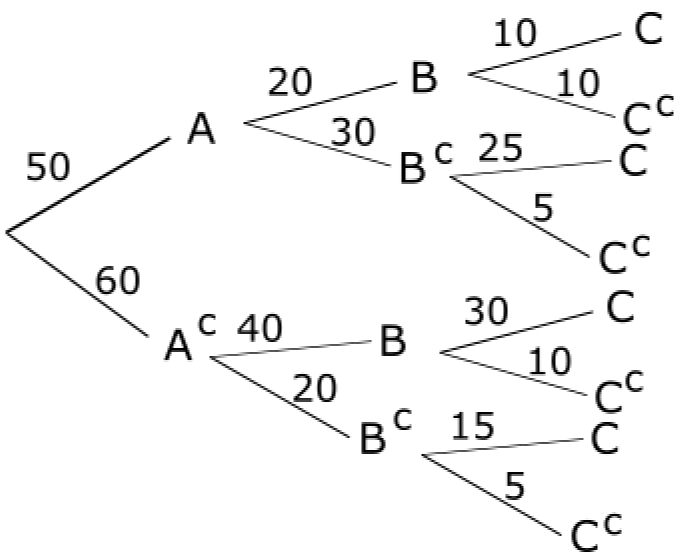
B4. Fill in all of the unknowns directly on the tree.

n(B)=20+17=37

The answer is C).

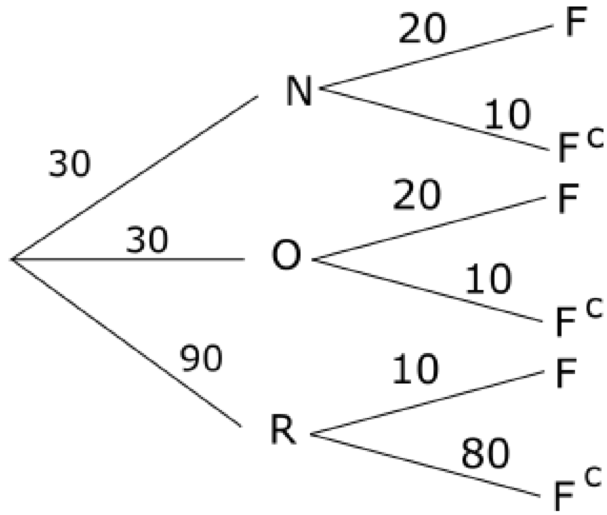


B5. Given the tree below, find n(C^C).



Fill in the unknowns using the information given in the tree diagram above
find n(C^C)=10+5+10+5=30
The answer is C).

B6.
Draw a tree diagram

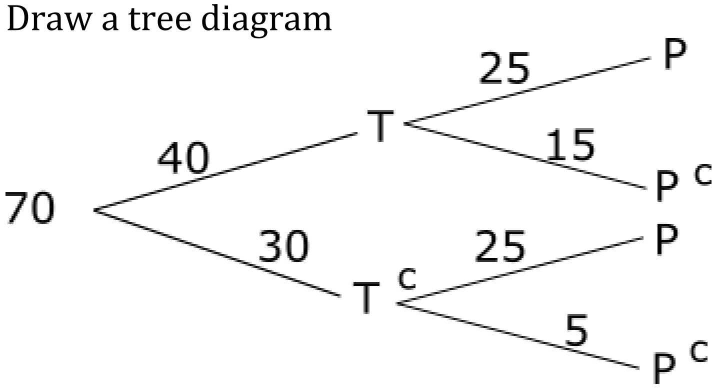


Let, N=number of students who never attended class
O=number of students who occasionally attended class
R= number of students who regularly attended class
F^C=number of students who passed
F=number of students who failed

Occasionally and failed=20. The answer is B.

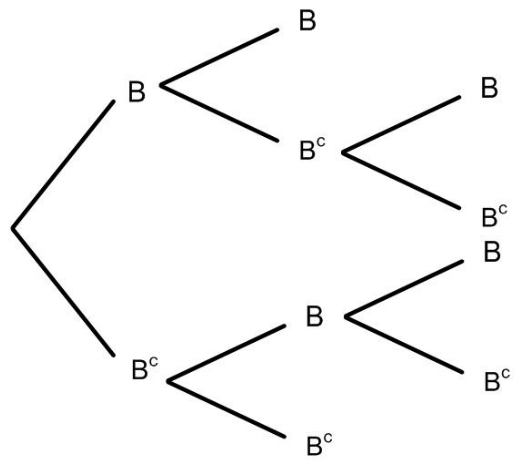
B7.
Let T be the number of students who have taken math before
Let T^C be the number of students who have taken math before
Let P be the number of students who passed the exam
Let P^C be the number of students who failed the exam

Draw a tree diagram



25 have taken it before and passed
The answer is B).

B8.



Draw a tree diagram
Let B be the games that Dane wins
Let B^C be the games that Dom wins or Dane loses

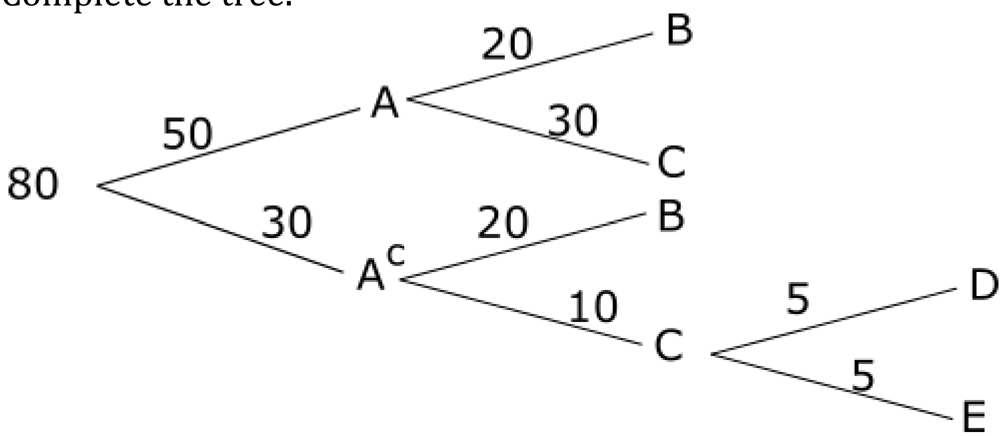
There are 6 branches on the far right of the tree, therefore there are 6 possible sequences. They are BB, B B^CB, BB^C B^C , B^C BB, B^C B B^C **,** B^C B^C

The answer is d).

<u>Consider the counting tree shown below to answer B9-12.</u>

B9. Find $n(E \cap C)$= in E and in C= go to the far right of both of those letters and we get 5.

Complete the tree.



The answer is B).

B10. Find

$n(A^C \cap C \cap E)$=5 not in A and in C and in E
The answer is D) because we take the number to the far right in the tree to find the number in A^C and C and E.

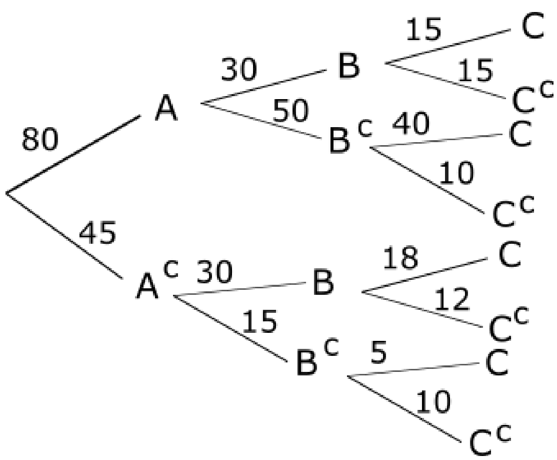
B11. $n(A \cup B) = 50 + 20 = 70$
** don't count the B at the top because you already added it when you added n(A).

B12. $n(B \cup C) = 20 + 30 + 20 + 10 = 80$

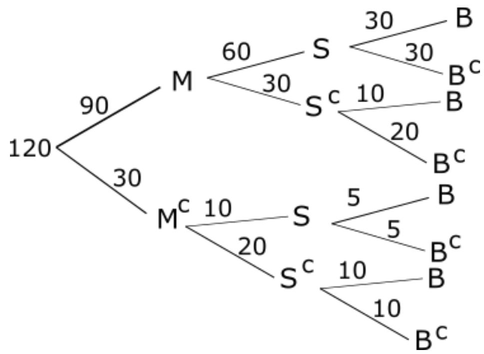
B13. Consider the tree given below:
Draw a tree diagram and fill in all of the unknowns.

a) $n(C^C)$=15+10+12+10=47

b) $n(B \cap C^C)$ =15+12=27



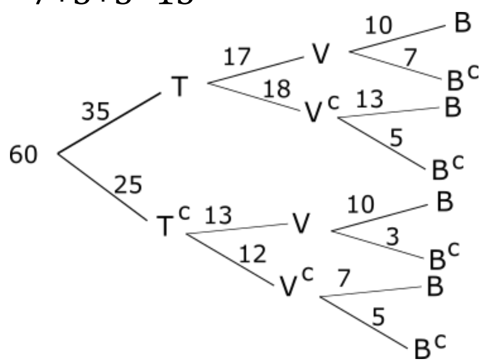
B14.



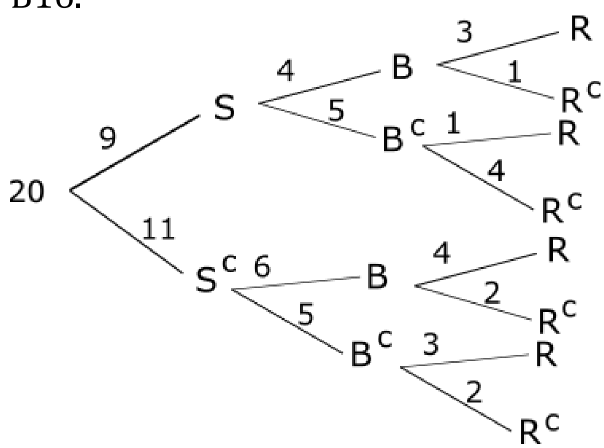
From the tree:
Number taking math and bus, but not science=10

B15.

Exactly one=$n(TV^CB^C) + n(T^CVB^C) + n(T^CV^CB)$
=7+5+3=15



B16.



∴they made 9+11=20 smoothies!

C. The Fundamental Counting Principle (1.3)

Review of Subsets

a) 2^5 b) 2^3 c)$2^5 - 1$

c) $2^5 - 2$ e) $\binom{5}{4}$ f) $\binom{3}{3} = 1$

Example 1.

a) $2^9 - \ yes\ or\ no\ to\ each\ letter$

b) $2^9 - 1$ (subtract the case of the null set or empty set that contains no letters)

$c)2^9 - 2$ $(at\ least\ 1, not\ all,$
$means\ take\ out\ the\ null\ set\ and\ the\ set\ of\ all\ of\ the\ letters)$

d) $\{i\ love\ math\}$ $h\ is\ in$
$i, e, a, o\ are\ all\ out$
$\therefore l, v, m, t\ \ are\ the\ 4\ letters\ left$ $\therefore 2^4 = 16$

e) $\binom{9}{4} = \frac{9!}{4!5!}$

f) $\{i\ love\ math\}\ a, t\ in$ $\therefore only\ need\ 2\ more\ letters$
$m\ out$ $\therefore letters\ left\ are\ \{iloveh\}$
$\binom{6}{2}\ to\ choose\ the\ other\ 2\ letters$
$= \frac{6!}{2!4!} = \frac{6\times5\times4!}{2!4!} = \frac{30}{2} = 15$

Example 2.

There are 5 possible numbers for the first digit to be even (0,2,4,6, 8) and 5 possible numbers for the last digit to be odd (1,3,5,7,9) and the middle digit can't repeat either of the other digits, so once you pick a number for the first and last digits, there will be 8 numbers left for the middle
ie. $5 \times 8 \times 5 = 200$

Example 3.

Odd numbers less than 300 can be a 1, 2 or 3 digit number

=_+_ _+_ _ _

3 choices + 5× 3 + 2×5× 3

The one digit number can be a "1" or a "3" or a "5"

The two digit number can end in a "1" or a "3" or a "5"to be odd and since repetition is allowed, there are 5 choices for the first digit.

The three digit number must end in a "1" or "3" or a "5" to be odd. Then, it must be less than 300

∴ can only start with a "1" or "2"

$\therefore\ \ 3 + 15 + 30 = 48\ choices$

Example 4.

$6 \times 5 \times 10 = 300\ choices\ he\ is\ having\ one\ of\ each\ type\ of\ dessert$

So, this is the fundamental counting principle

Example 5.

$6 + 5 + 10 = 21\ desserts$

$\therefore 2^{21}\ choices$

This is like a subset question as he can say "yes" or "no" to each dessert and there are 21 of them!

Example 6.

$2^{21} - 1\ \ take\ away\ case\ of\ choosing\ no\ desserts$

$ie)\ saying\ no\ to\ all\ 19\ desserts$

Example 7.

$2^{21} - 2$

Example 8.

a) 4^{10} choices for each of 10 different candies to go to

b) $4^{10} - 4$

$since\ we\ take\ away\ each\ case\ where\ all$

$candies\ go\ to\ each\ of\ 4\ children$

Example 9. $\binom{10}{4} = \frac{10!}{4!6!}$

Example 10.
There are 5 choices for each candy – 4 kids or Courtney

$\therefore 5^{10}\ ways$

Example 11.
$4 \times 2 \times 4 = 32\, ways$
Here we have 4 choices of destination, 2 choices of mode of travel and 4 choices of what to do on vacation

Example 12.
$4 \times 2 \times 3 \times 3 \times 2 = 144\, ways$

Here there are 4 choices of daily special, two choices of soup or salad and 3 choices of potatoe and 3 choices of dessert and two choices of yes or no to ice cream

Example 13.

She can say yes or no to each of 8 toppings for her pizza
no salad=2^8 $choices\ of\ topping\ for\ pizza$
Caesar salad = $2^8 \times 1 \times 1 = 2^8$
House salad = $2^8 \times 1 \times 4$
Total = $1\times 2^8 + 1\times 2^8 + 4 \times 2^8 = 6 \times 2^8$

Example 14. {6,7,8,9}

At most twice= used 2,1 or 0 times but not all 3 times
-either do a bunch of cases and add them up, or to
Total ways - # ways that all numbers are the same

=$4^3 - \binom{4}{1} \times 1 \times 1 \times 1 = 64 - 4 = 60$

where the 4 choose 1 is to choose the one number and then all of the numbers have 1 choice

Example 15. 5^{100} is the number of ways, since there are 5 choices of mark for each of 100 students

Example 16. 1 2 3 4 5 6 7

Start with a 3 or start with a 4,5, 6, 7

$\underline{1} \times \underline{3} \times \underline{7} \times \underline{7} \quad + \quad \underline{4} \times \underline{7} \times \underline{7} \times \underline{7}$

↑ ↖ ↑ ↖↑↗

Start with a 3 | can be a 5,6,or 7 | start with 4, 5, 6, 7 | repetition is allowed so it can be any of the 7 digits

$= 3 \times 7^2 + 4 \times 7^3$

Example 17. $\{S\ U\ C\ E\ F\ L\}$ *are the only letters*

$\therefore 2^6 = 64$

Example 18. *There are* 6 *letters that are not vowels*.

$\therefore 2^6 - 1 = 63$ *since we take away the empty set*.

Practice Exam Questions on The Fundamental Counting Principle

C1. The dessert table at a buffet has...
There are 19 desserts and a customer can either say yes or no to each one.
The answer is A).

C2. Refer to Question C1...In how many different ways can Matt select his dessert?

He has 5 choices for cheesecake, 3 choices for pie, 4 choices for cake, 3 choices of scones and 4 choices of yogurt.

Therefore, he can select his dessert in 5x3x4x3x4 ways.

The answer is D).

C3... is going to the movies tonight... how many different ways could ____________ dress for the dance?

She has 5 choices of outfit, 4 choices of shoes and 2 choices of whether or not to wear a scarf

Therefore, she has $5x4x2=40$ ways to dress for the dance.

The answer is B).

C4. The number of subsets is 2^n and here n=0,1,2,3,4,5,6,7,8,9,10,11,12 =13 elements

The answer is D).

C5.
There are $\binom{10}{4}$ subsets....
The answer is D).

C6. ...is also going out to the movies tonight...

Purple…black…shoes= 1x1x4

Other sweater…pants…shoes= 4x3x4=48

The total ways is 4+48=52. The answer is C).

C7. How many (positive) even integers less than 5000 can be formed using only digits which are in the set {2, 3, 4, 5}, if repetition is allowed?
For 4-digit numbers, we have 3x4x4x2=96 ***choices*** since the first digit has to be less than 5 for 5000 and the last must be even, so 2 or 4
Now, three-digit numbers can be...
___ ____ ____ the last number has to be even, so it can be a "2" or a "4" so 2 choices. Then, we can repeat, so there are 4 choices for the first number and then 4 for the middle number ie. 4x4x2=**32 *choices***

Now, for a two-digit number, we have 2 choices for the last digit and then we can use the number we chose again for the first digit, so it will be 4x2=***8 choices***

For a 1-digit number, it can be only the "2" or "4" to be even, so 2 ***choices***.

The final answer is 2+8+32+96=138 ways

C8. There are 5x4x2x2=80 ways to order his ice cream.

The answer is B).

C9. You can have 5x4x3 different passwords since repetition is not allowed.
The answer is D).

C10. There are 5x5x5 passwords, if repetition is allowed because there are five letters to choose from for each digit of the 3-letter password.

The answer is A).

C11. Grampa has 16 different hockey cards...
Grampa has two choices for each hockey card of who to give it to, that is 2^{16} possibilities.

However, we must subtract the case where all of the hockey cards go to one grandson as well as the case where all of the hockey cards could go to the other.

The final answer is $2^{16} - 2$ ways.

C12. Since there are 16 hockey cards, Grampa just has to choose 5 to give to Joshua.

The answer is $\binom{16}{12}$ways.

C13. There are 20 students in a theatre group...
Each student has 9 choices and there are 20 students, so there are 9^{20} ways.

C14. Kate is a first-year student is choosing her courses for next year...

They can select their courses in $\binom{5}{3} \times \binom{10}{1} \times \binom{2}{2}$ ways.

C15. 2^6 is the number of subsets of these 6 letters.

C16. Put "a" in 1 way...put "f" out of the set in 1 way....so 2^4 ways=16 ways to say "in" or "out" to each of the other 4 letters.

C17. $2^6 - 1$

C18. {a,b,c,d,e,f}
b-in
a,e - out

So, we need 2 more letters to have a subset of size 3 and there are 3 letters left ie. {b,c,d}
So, the answer is $\binom{3}{2} = 3\ ways$

D. Permutations (1.4)

Example 1. 6!

Example 2. (6-1)!=5!

Example 3. SOPHIE has 6 different letters $\therefore$ 6!

Example 4. COMPUTER has 8 different letters

a) Begin with C $\therefore$ *put c in front*

C _ _ _ _ _ _ _

$\therefore$ 7 *other letters to arrange* $\therefore$ 7!

b) COMPUTER -8 letters

Vowels are o,u and e $\therefore$ *count them as one spot* $\rightarrow$ *all together*

OUE _ _ _ _ _ $\therefore$ 5 *other letters to be arranged*

Total of 6 "units" to arrange = 6!

Then, we can arrange the 3 vowels in their group of 3

$\therefore$ 6! 3! *is the answer*.

c) E, R apart

ways E,R apart = Total ways - # ways E, R together

Total ways = 8!

E , R together ER _ _ _ _ _ _

Put E,R together, then there are 6 letters left to arrange

$\therefore$ there are 7 "units" to arrange

$\therefore$ E, R together =7! $\times$2! Since ER can be RE

$\therefore$ # ways E,R apart = 8! -7!2!

Example 5. 8 – circle

G, T and D together

Place Grant, Jason and Daniel together to start the circle

∴ 5 people left to actually be arranged in the circle

∴ 5! 3! since the 3 boys can be arranged in their group of 3 in 3! ways

b) place the three boys together with Jason in the centre...so, 2! ways for Grant and Daniel to switch places ∴ 5! 2! ways

Example 6. $k = 6 \quad n = 10$ (only arranging some of objects in a line)

Formula p. 39

$$\frac{n!}{(n-k)!} = \frac{10!}{(10-6)!} = \frac{10!}{4!}$$

OR choose 6 of 10 books first, and then arrange them.

ie. $\binom{10}{6} \times 6!$

$$= \frac{10!}{4!6!} \times 6! \quad = \frac{10!}{4!}$$

Example 7. 8 girls – circle

R, E across from each other

- Place Rosemary to start the circle and then there is 1 way to place Elizabeth across from her NOTE: it doesn't matter if you placed Elizabeth first as it would still be the same arrangement

  Now, there are only 6 girls left to arrange = 6!

  NOTE: do not multiply by 2! Since switching R to E creates an identical arrangement

Therefore, the total ways R, E are not across from one another=
Total ways to arrange the girls in a circle - # ways R, E are across
=7! - 6!

Example 8. 7 girls, 6 boys

a) To alternate, you must start with girls since it you start with boys, you would have 2 girls left at the end and it wouldn't be alternating.
GBGBGBGBGBGBG = 7!6!

b) 6 girls, 6 boys
GBGBGBGBGBGB OR BGBGBGBGBGBG
= 6!6!2! since starting with a boy and starting with a girl is a different arrangement in a line

c) If you did have 6 girls and 6 boys for example, place a girl to start the circle and then arrange the rest of the kids so that they alternate
∴ 5!6! since there are 5 girls left to arrange and also there are 6 boys

Example 9. $k = 5 \quad n = 20$

Formula p. 39 – in a circle

$$\frac{1}{k}\frac{n!}{(n-k)!} = \frac{1}{5}\left(\frac{20!}{(20-5)!}\right) = \frac{1}{5}\left(\frac{20!}{15!}\right) \quad \text{OR}$$

Choose which 5 trophies of the 20 to arrange
Then, to arrange 5 trophies in a circle is 4!

$$\therefore \binom{20}{5} \times 4! = \frac{20!}{5!15!} \times 4! = \frac{20!}{15!} \times \frac{4!}{5!} = \frac{20!}{15!} \times \frac{4!}{5\times 4!}$$
$$= \frac{20!}{15!} \times \frac{1}{5} = \frac{1}{5}\left(\frac{20!}{15!}\right)$$

Example 10.

First split the group of 10 into groups of 4 and 6 people for each table. This can be done in $\binom{10}{4\quad 6}=\frac{10!}{4!6!}$ ways
Then, the 4 people can be arranged in a circle in 3! ways and the 6 people can be arranged in another circle in 5! ways, so the final answer is:

$$\binom{10}{4\quad 6} 3!\,5!$$

Practice Exam Questions on Permutations

D1.
There are 7 different letters
So, there are 7! ways to arrange the letters.

The answer is B).

D2.
There are 6 different letters in the name “SAMUEL”.
So, there are 6! ways to arrange the letters.
The answer is B).

D3. There are 6 different books, so the answer is C).

D4. See Question D3.

First, put the 4 Calculus books together
$\underline{C_1\ C_2\ C_3 C_4}$ _____ _____

There are 3 “groups” to arrange and then we have to arrange the calc books in their own group in 4! Ways, so the answer is 3!4!
Therefore, the answer is D).

D5. There are 8 people, around a table, so there are 7! ways to arrange them.
The answer is C).

D6. **** draw a circle with M in one spot and 7 other spots

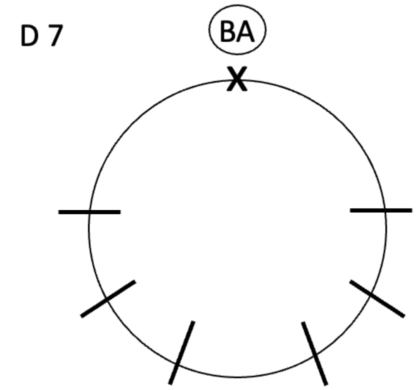
Place a man M_1 at the round table to occupy the first spot.
Then, beside the man, can be any one of 4 females. There are 4! ways to arrange the females in every second spot

There are 3! ways to arrange the 3 men left in every second spot
Therefore, there are 4!3! ways to arrange them if men and women must alternate.
The answer is B).

Note: It would be the same answer if we had placed a female first.

D7. Whenever you want people to be apart, always find the total ways without any restrictions and then subtract the number of ways to put them together

Total ways= 7! (8 people in a circle)



Ben and Alyssa together...place Ben and Alyssa together to mark the start of the circle

There are 6! ways to arrange the remaining people and 2! ways for Ben and Alyssa to switch spots

Total ways together= 6!2!
The number of ways with Ben and Alyssa apart= 7! - 6!2!
The answer is C).

D8. There are 13 people total to arrange. The answer is B).

D9.

Put all of the girls together and count them as one spot

$\underline{G_1\ G_2\ G_3\ G_4\ G_5\ G_6}$ ___ ___ ___ ___ ___ ___ ____

Since the girls don't have to be first, there are 8 "things" to arrange.

Then, the girls can stay together, but can be arranged amongst themselves in 6! ways.

Therefore, there are 8!6! ways to arrange them so the girls are all together.

The answer is E).

D10.
Since there are 7 girls and only 6 boys, we must start with a girl. If we started with a boy, there wouldn't be enough of them to alternate all the way down the line

G B G B G B G B G B G
Therefore, 7!6! ways to arrange them so they alternate.
The answer is A).

D11. A pharmacy has eight new employees to assign...

Each employee can either be assigned to front cash or stocking shelves, so there are two choices for each employee. This would mean 2^8 ways.

The trick is that each area must have at least one employee, so we must subtract the case where all employees are assigned to "front cash" as well as the case where all were assigned to "stock shelves".
The answer is then D).

D12. Put the Calc book at the end in 1 way
__ __ __ C
There are three other books to arrange...3! ways or 6 ways

D13. Place the three different math books together.

$\underline{M_1 M_2 M_3}$ ____ ____ ____ ____ ____

There are 6 "things" or "units" to arrange, because the math can move along the line as they only need to be together, but not necessarily first.

These 6 units can be arranged in 6! ways.

Then, the 3 math can be arranged amongst themselves in their group of 3.
This can be done in 3! ways.
The total number of arrangements is 6!3! ways.

D14.The number of arrangements of "k" of a total of "n" distinct items in a circle is $\frac{1}{k}\frac{n!}{(n-k)!}$

In this question, n=5 and k=3, so the number of ways is
$\frac{1}{3}\frac{5!}{(5-3)!} = \frac{1}{3}\frac{5!}{2!} = \frac{5(4)(3!)}{3!}=20$

D15.

4 dogs ___ ___ ___

So, the 4 dog pictures count as one "unit", so there are 4 things to arrange, in 4! Ways and then the four dog pictures can also be arranged in their group of 4, amongst themselves.

Therefore, there are 4!4! ways

D16.

4 dog and 3 cat pictures

4 dog 3 cat

There are two "units" to arrange because the dog pictures are all stuck together as one unit and so are all of the cat pictures.

So, there are 4!3!2! ways because after the 2! For the two units to be arranged, we can arrange the 4 dog pictures amongst themselves and we can also arrange the 3cat pictures amongst themselves

(2! is because the dog ones are first or cat ones first)

D17. Place one of them and then there is only 1 way to place the other one, right across. So, there are 6-2=4 people to arrange in 4! ways. You do NOT multiply by 2, because it doesn't matter who you place first, if you shift the circle around, it is the exact same arrangement of Helen and Tina.

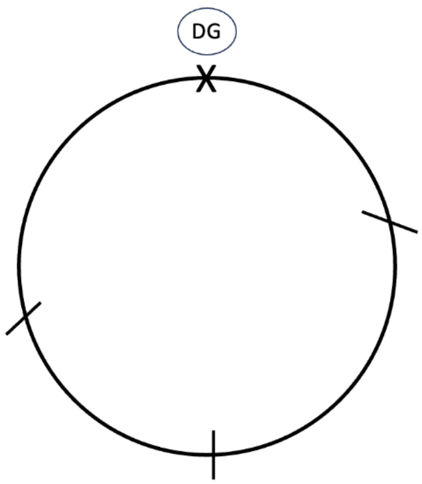
D18. In how many ways can they sit in a circle so that Helen and Tina are NOT side-by-side?

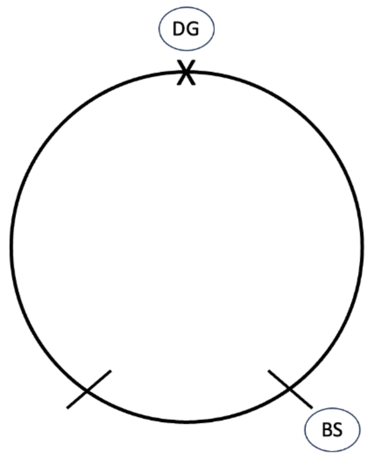
Total ways - # ways together= 5! – 4!2! (same idea as example 8)

*D19. To find this we do:
ways David and Gabriel are together - # ways both David and Gabriel are together AND Benjamin and Samuel are together

=3!2! – 2!2!2!= 6(2) – 8 = 4 ways

NOTE: We get a 2! For the two spots left after David and Gabriel are placed to start the circle together and we can arrange Benjamin and Samuel in one spot and the 5th person in the other, so 2! ways for who goes in which spot
Then, we get a 2! For DG to switch to GD and another 2! For BS or SB for Benjamin and Samuel to switch places





E. Combinations (1.5)

Example 1. 5 women, 8 men

a) $\binom{13}{3} \times 3!$ since you have to put the 3 people into the 3 positions in 3! ways

$= \frac{13!}{3!10!} \times 3!$

$= \frac{13!}{10!}$ *or* $\frac{13\times12\times11\times10!}{10!} = 13 \times 12 \times 11$

b)subcommittee of 4 people
- at least one man = 1,2,3 or 4 men
= too many cases

$\therefore$ we do:

\# ways to get at least 1 man= total ways - # ways to have no men

$= \binom{13}{4} - \binom{8}{0}\binom{5}{4}$

remember $\binom{8}{0} = 1$

c) Austin - on Kristen - off

$\therefore 8 - 1 = 7$ *men left*

$5 - 1 = 4$ *women left* $\therefore 7 + 4 = 11$ *people left*

BUT, Austin is <u>on</u> the committee therefore we only need to choose 3 more people

$\therefore \binom{11}{3}$

Example 2. 15 questions, 2 answers

At most 13 correct = 0, 1, 2, 3, 4, 5, 6, 7 or 8, 9,10, 11,12 or 13 correct
= too many cases!

$\therefore$ At most 13 correct = total ways – 14 correct – 15 correct

$= 2^{15} - \binom{15}{14} - \binom{15}{15} = 2^{15} - 15 - 1 = 2^{15} - 16$

Example 3. 10 – DVD 12 – CD

Buy 6 DVD, 4 CD $= \binom{10}{6}$ *and* $\binom{12}{4}$

$= \binom{10}{6} \times \binom{12}{4}$

Example 4. 10 close friends – invite 4
3 cases: Sarah comes, Kerstyn doesn't OR Kerstyn comes, Sarah doesn't OR neither are invited

If Sarah comes, you need to choose 3 more friends but there are 10-2=8 left since Kerstyn can't come

$$\binom{8}{3}$$

Same $\binom{8}{3}$ $for\ case$ 2

Case 3 = Kerstyn and Sarah both are not coming

$\therefore$ 8 left and choose 4 to invite $=\binom{8}{4}$

$$2\binom{8}{3}+\binom{8}{4}$$

Example 5. 5DS, 3SK, 2JG
Choose 4 books with at least one of each
= 2DS, 1SK, 1JG OR 1DS, 2SK, 1JG OR 1DS, 1SK, 2JG

$$=\binom{5}{2}\binom{3}{1}\binom{2}{1}+\binom{5}{1}\binom{3}{2}\binom{2}{1}+\binom{5}{1}\binom{3}{1}\binom{2}{2}$$
$$=\frac{5!}{2!3!}\times 3\times 2+5\times\frac{3!}{2!1!}\times 2+5\times 3\times 1$$
$$=\frac{5\times4\times3!}{2!3!}\times 6+10\times\frac{3!}{2!}+15$$
$$=\frac{20}{2}\times 6+\frac{10\times3\times2!}{2!}+15$$
$$=60+30+15$$
$$=105$$

Example 6.

a) *true*

b) *true*

c) $\frac{7!}{3!} = \frac{7!}{4!}$ $\quad false$

$\frac{7!}{3!} = \frac{7\times6\times5\times4\times3\times2\times1}{3!} \qquad \frac{7!}{2!} = \frac{7\times6\times5\times4\times3\times2\times1}{2!}$

d) $\binom{7}{3} = \binom{7}{4}$ $\quad true$

$\binom{7}{3} = \frac{7!}{3!4!} \quad and \quad \binom{7}{4} = \frac{7!}{4!3!}$

e) $\binom{10}{4} = \binom{5}{2} \times \binom{2}{2}$ $\quad false$

$\binom{10}{4} = \frac{10!}{4!6!} \qquad \binom{5}{2} \times \binom{2}{2} = \frac{5!}{2!3!} \times 1$

Example 7. 5 of 12 pens

Pink and purple are 2 of the colours

$\therefore$ 10 colours left and need 3 more $\therefore \binom{10}{3}$

Example 8. a) $\binom{4}{2}\binom{48}{1}$

b) # ways to get at least 1 ace=total ways− no aces=$\binom{52}{3} - \binom{4}{0}\binom{48}{3}$

Example 9.

a)$\binom{52}{5}$

b) 13 hearts in the deck $\binom{13}{5}$

c) All hearts or all diamonds or all spades or all clubs

$4 \times \binom{13}{5}$

d) $52 - 8 = 44\ left$

$\binom{4}{2}\binom{4}{1}\binom{44}{2}$

e) $\binom{13}{3}\binom{13}{1}\binom{26}{1}$

f) 3 hearts, 1 diamond, 1 other or 3 hearts, 2 diamonds

$\binom{13}{3}\binom{13}{1}\binom{26}{1} + \binom{13}{3}\binom{13}{2}$

Example 10. $\binom{12}{4\ 4\ 4} \div 3! = \frac{1}{3!}\binom{12}{4\ 4\ 4} = \frac{12!}{4!4!4!} \div 3!$

We divide by 3! since the three groups are indistinguishable ie doing the same thing

Example 11. a) $\binom{12}{4\ 4\ 4}$ Here the groups are working on different papers, so they are distinguishable, so we don't divide

b) $\frac{\binom{12}{4\ 4\ 4}\times 3}{3!}$ since if groups are doing the same thing, we divide by 3! and if they have to all do the same topic then there are 3 topics and we pick one for everyone to do

Example 12. 12 different coloured balloons (just like 12 different people!)
4 equal groups

$$\binom{12}{3\quad 3\quad 3\quad 3} \div 4! = \frac{12!}{3!3!3!3!4!}$$

Since all groups of balloons are the same size and the groups are not distinguishable, we divide by 4!

Example 13. 9 flute players – 2 First, 3 second, 4 third

$$\binom{9}{2\ 3\ 4} = \frac{9!}{2!3!4!}$$

These groups are all different since they play different parts of songs, so we do not divide

Example 14. Alina and Halee play First flute
∴ 7 left – 3 second, 4 third

$$\binom{7}{3\ 4} = \frac{7!}{3!4!}$$

Example 15. We divide by 5! Since the five groups are not distinguishable from each other

$$\binom{20}{4\quad 4\quad 4\quad 4\quad 4} \div 5!$$

Example 16. First, we divide the books into 5 piles of 5 books. We don't need to divide by 5! Since the people are distinguishable, so the groups of books are distinguishable. ie. We need a group of books for student A, a group for student B, etc.

Then, to pick a report, each student has 5 choices of which book to pick and there are 5 students, so we have 5x5x5x5x5 total choices, or 5^5 choices.

$$\binom{25}{5\quad 5\quad 5\quad 5\quad 5} \times 5^5$$

Example 17. $\binom{15}{4\ \ 5\ \ 6} \times 3!$

$We\ have\ to\ multiply\ by\ 3!\ to\ assign\ 3\ different\ duties$
$to\ 3\ different\ groups, since\ the\ groups\ are\ all\ different\ sizes.$

Example 18. $\binom{3}{1}\ \frac{\binom{15}{5\ 5\ 5}}{3!}$

$$= \frac{3}{6}\ \binom{15}{5\ \ 5\ \ 5} = \frac{1}{2}\binom{15}{5\ \ 5\ \ 5}$$

They are all doing the same duty. ∴ We must divide by 3! for 3 indistinguishable groups. Then, multiply by $\binom{3}{1}$ to choose which duty all 3 groups will do.

Example 19. $\binom{15}{5\ \ 5\ \ 5}$

Here, the groups are all distinguishable so this divides the 15 officers into 3 distinguishable groups and there is no need to divide or multiply.

Example 20. $\binom{5}{3} \times 2^4$ since he can choose yes or no for each science book on whether or not to donate each one.

Example 21. 7 people
First person shakes with 6 people
6+5+4+3+2+1=21 handshakes

Example 22. 5 marks, 50 students
At least 1 A, but not more than 3 A's = 1A, 2A's or 3A's

$$= \binom{50}{1} \times 4^{49} + \binom{50}{2} \times 4^{48} + \binom{50}{3} \times 4^{47}$$

*After you choose the 1 person to get an A, there are only 4 grades left (B, C, D, F) for the other 49 people, etc.

Practice Exam Questions on Combinations

E1. **Any time you need to find the ways containing at least one of anything, take the total ways and subtract the ways containing none (in this case no women)
The answer is C) since the total ways to choose a committee of 5 is 13 choose 5 and we take away the case of getting 0 women, which means to choose 5 of the 6 men.

E2. This means there are 3 men and 2 women on the committee
The answer is B).

E3. At a dance there are 12 women and 15 men without dance partners...

Since there are too many men, we must choose 12 of them from the 15 men

We do this in $\binom{15}{12}$ ways. Then, we can arrange the 12 women with 12 of the men in 12! ways.
The answer is B).

E4.... has 4 mystery novels and 5 romance books he wants to read...

There are 9 books and he wants to choose 3 of them to take with him.
This can be done in $\binom{9}{3} = \frac{9!}{6!3!}$ ways

E5. See Question E4. In how many distinct ways can he choose 3 of these books if...

Now, he has to choose 1 of 4 mystery novels AND 2 of 5 romance books OR he can choose 2 of 4 mystery and 1 of 5 romance books

Whenever we want one thing AND another, we multiply the two answers.
Therefore, there are $\binom{4}{1} \times \binom{5}{2} + \binom{4}{2} \times \binom{5}{1}$ ways

E6.... has to choose 4 courses for next term...

There are $\binom{5}{2}$ways to choose 2 of 5 math courses

There are $\binom{2}{1}$ways to choose 1 of 2 applied math courses

There are $\binom{6}{1}$ways to choose an 1 of 6 science courses

Therefore, there are$\binom{5}{2}\binom{2}{1}\binom{6}{1}$ =$\frac{5!}{3!2!}(2)(6) = \frac{5\times4\times3!(12)}{3!2!} = 10(12) = 120$ to choose all of the courses.
The answer is D).

E7. It doesn't say the two groups of equal size are doing different tasks, so we assume these two groups of six are indistinguishable.

So, there are$\binom{12}{6\ \ 6} \div 2!$ ways to divide the 12 students into two groups of 6 students. We divide by 2! since there are two groups of 6
The answer is A).

E8. The two groups of 4 students are not distinguishable from each other and neither are the two groups of 2 students.

The answer is D).

E9. In how many ways can 12 students be divided into two groups of 4 students and two groups of 2 students, if each group is doing a different duty?

Since the groups are now distinguishable because they have different duties, we don't divide by 2! 2! for the two groups of 4 and two groups of 2 students.

The answer is A).

E10. In how many ways can a president, secretary and treasurer be chosen...

First, we have to choose 3 of the 15 students to take on a position.

Then, we have to arrange the 3 students into the 3 positions of president, secretary and treasurer.

The total ways is
$\binom{15}{3} \times 3! = \frac{15!}{3!12!} 3! = \frac{15!}{12!} = \frac{15 \times 14 \times 13 \times 12!}{12!} = 15 \times 14 \times 13$

E11. In how many ways can _______ invite 6 of his 9 friends...

Let the two who are not on speaking terms be A and B

He can either invite neither A or B in $\binom{7}{6}$ ways because if he doesn't include A or B, there are 7 people left to choose from.

OR

He can invite A and NOT B in $\binom{7}{5}$ ways since A is already going, so only 5 need to be chosen from the 7 remaining people, not including A and B...

OR

He can invite B and NOT A in $\binom{7}{5}$ ways since B is already going, so only 5 need to be chosen from the 7 remaining people, not including A and B

Therefore, the total ways is $2\binom{7}{5} + \binom{7}{6}$.
The answer is D).

E12. In how many distinct ways can 20 soccer players...
The answer is D) since the 4 groups are indistinguishable from each other.

E13. In how many ways can the 21 employees be divided into three groups of 7 and all assigned to “front cash”.

They all have the same duty, so the 3 groups are indistinguishable.

So, the number of ways is $\binom{21}{7\ 7\ 7} \div 3!$

The answer is C).

E14. In how many ways can the 21 employees be divided into three groups of 7 and be assigned to different duties?

Since the groups are assigned different duties, they are distinguishable from one another, so we don’t divide by 3!. We don’t multiply be 3! either as they are all distinguishable groups, so dividing them up is done in the grouping into 3 groups of 5. NOTE: if the groups were of 3,4,5 people in each and they were assigned different duties, you would multiply be 3! Since the groups are now distinguishable. $\binom{21}{7\ 7\ 7}$
The answer is A).

E15. In how many ways can the 21 employees be divided into three groups of 7 and each be assigned the same duty in a given week?

They are assigned the same duty, but we don’t know which duty.

First, because the 21 employees are divided into three groups of 7 that are indistinguishable in $\binom{21}{7\ 7\ 7} \div 3!$ ways.
Now, there are 3 duties to choose from or $\binom{3}{1} = 3$ ways to choose the duty

they will all do.

Therefore, there are $\binom{21}{7\ 7\ 7} \times \frac{3}{3!} = \binom{21}{7\ 7\ 7} \times \frac{3}{6} = \frac{\binom{21}{7\ 7\ 7}}{2}$ ways.
The answer is E).

E16. Since each group is assigned a different duty, the groups are distinguishable from one another.

Therefore, there are $\binom{10}{5\quad 5}$ ways.

E17. In how many ways can 12 soccer players be divided into 6 groups of 2 and all groups run the same practice drill?

Since the five groups are the same size and they are doing the same thing, they are NOT distinguishable from one another. Since there are "6" groups of the same size, we divide by 6!

Therefore, there are $\frac{\binom{12}{2\quad 2\quad 2\quad 2\quad 2\quad 2}}{6!}$ ways to arrange the players into 6 indistinguishable groups of 2 players.

E18. Since it doesn't tell us the two groups of 4 are given different jobs, these two groups are indistinguishable. However, we can tell the groups of 6 from the group of 2, because they are different sizes. Since there are two indistinguishable groups, we divide by 2!

So, there are $\frac{\binom{14}{2\quad 6\quad 6}}{2!}$ ways.

E19. This time, the groups are all different sizes, so they are all distinguishable from one another. After we divide them into groups, there are 3! ways to arrange the three coaches.

The total number of ways is $\binom{12}{4\quad 6\quad 2} \times 3!$

E20. B) is the answer since v) is the only one that is false.

I. True

II. True $\binom{8}{3} = \frac{8!}{3!5!}$ and $\binom{8}{5} = \frac{8!}{5!3!}$

III. True $\binom{5}{2} = \frac{5!}{2!3!}$ and $\binom{5}{2\quad 3\quad 1} = \frac{5!}{2!3!1!}$

IV. $\binom{5}{3} = \frac{5!}{3!2!} = \frac{5(4)3!}{3!2!} = \frac{20}{2} = 10$

E21. There are 7 men and 8 women in a particular class. We need to choose 6 people to form a committee. In how many ways can this be done if...

In order to find the number of ways to have at least one man, you find the total ways subtract the ways to have NO men. This is much faster than adding the cases of 1, 2, 3, 4, 5, or 6 men.

Total ways to select 6 people from 15 people is $\binom{15}{6}$

If there are NO men, we choose all 6 from the 8 women in $\binom{8}{6}$ ways

The answer is $\binom{15}{6} - \binom{7}{0}\binom{8}{6}$

**Note: $\binom{7}{0} = 1$ so you don't have to include it!!!

E22. See E21. In how many ways can a committee of 6 people be chosen if there is to be at least three men, but...

The choices here are 3 or 4 men.

We have to do both cases:

The answer is $\binom{7}{3} \times \binom{8}{3} + \binom{7}{4} \times \binom{8}{2}$
**Note: We multiply these answers because we need 3 men AND 3 women to make a total of 6 people. The word "AND" always means MULTIPLY!!!

F. Labeling Problems (1.6)

Example 1.

a) $\frac{7!}{2!}$

b) $\frac{11!}{2!2!2!}$

Example 2.

T _ _ _ _ _ _ _ _ _ _

$\frac{10!}{2!2!}$ put the T first – 2 M's

- 2 A's

Example 3. Put the IC together first

IC _ _ _ _ _ _ _ _ _ _ multiply by 2! for IC or CI

$\frac{10!}{2!2!2!}2! = \frac{10!}{2!2!}$

Example 4. $\frac{10!}{2!3!4!}$

Example 5. PIN / EA /PP /LE = 9 letters

a) Put 2 E's together therefore 6 other letters left

EE _ _ _ _ _ _ _ _

$\therefore$ 8 "$units$" to arrange, 3 repeating PPP's free to move throughout the line

Total ways with 2 E's together = $\frac{8!}{3!}$

NOTE: we do not multiply by 2! for EE to switch as it would always stay the same

b) # ways N,L apart = total ways - # ways N, L together

ways N, L together = NL _ _ _ _ _ _ _

$$\frac{8!}{3!2!} \times 2! = \frac{8!}{3!}$$

*multiply by 2! since NL can be LN

ways N,L apart = total ways - # ways N, L together $= \frac{9!}{3!2!} - \frac{8!}{3!}$

Example 6. POSSUM= 6 letters

4 letter words= no S, 1-S or 2-S, so three cases
No "S", choose all 4 of POUM …1 way…arrange the 4 different letters in 4! ways
1 letter "S"= S plus choose 3 of POUM…so $\binom{4}{3} \times 4!$ since you can arrange all the 4 different letters in 4! ways
2 letter "S" plus choose 2 from POUM…so $\binom{4}{2} \times \frac{4!}{2!}$ since there are 4 letters and 2 of them are the same, ie. SS PM for example

$4! + \binom{4}{3} \times 4! + \binom{4}{2} \times \frac{4!}{2!}$

Example 7. MATHEMATICS
M's together = $\frac{10!}{2!2!}$ <u>MM</u> _ _ _ _ _ _ _ _ _ _
M's together and T's together = $\frac{9!}{2!}$ <u>MM</u> <u>TT</u> _ _ _ _ _ _ _

# ways M's together And T's apart	=	total ways M's together	-	# ways M's and T's together
	=	$\frac{10!}{2!2!}$	$-$	$\frac{9!}{2!}$

Example 8. MATHEMATICS

Vowels before consonants – put vowels first
AAEI _ _ _ _ _ _ _
$\frac{7!}{2!2!} \times \frac{4!}{2!}$
↑ ↘
2 T's, 2 M's 4 vowels, 2 A's repeat

Example 9. $11\ letters$

$\underline{TT}\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$

$\frac{10!}{2!2!}$

M A

Practice Exam Questions on Labeling Problems

F1. In how many ways can the letters of the word "BABBOON" be arranged so that...
Put the "A" in the first spot in 1 way.

A ___ ___ ___ ___ ___ ___

There are 6 letters left two arrange, and "3" are B's and "2" are O's.
Therefore, there are $\frac{6!}{2!3!}$ ways.

F2. How many different 11 digit codes can be made by permuting the digits …
There are $\frac{11!}{2!3!3!2!}$ ways to permute these digits.
The answer is B).

F3. There are 7 letters in BABBOON with 3B's and 2 O's
Put the B's together and there are 4 other letter to arrange, so 5 "units" total to arrange

BBB _ _ _ _

$\frac{5!}{2!}$ we divide by 2! For the two O's that are free to switch about with each other. We don't multiply by 3! as BBB looks the same as BBB
The answer is D).

F4. There are 11 letters in "MATHEMATICS" with 2 M's, 2 T's, and 2 A's

The number of ways is $\frac{11!}{2!2!2!}$
The answer is C).

F5.
There are 9 letters with 3P's and 2 E's
The number of ways is $\frac{9!}{3!2!}$
The answer is C).

F6. There are 9 letters with 3 P's, 2 E's

First , we put the 3P's together

PPP __ __ __ __ __ ___
No, since the 3P's don't have to be first, they can move too!

So we have 7! for 7 "units" to arrange...then we divide by 2! for the 2E's. We don't multiply by 3! because no matter how you switch around the 3P's they still give the exact same arrangement.
The answer is 7!/2!

F7. Find the number of permutations of the word "MATHEMATICS" that begin...

First, put an "T" at the beginning and end spots (these T's look alike, so there is only one way to do this)

T __ __ __ __ __ __ __ __ __ T
Now, there are 9 letters to arrange, with 2M's and 2 A's that look alike

The number of ways is $\frac{9!}{2!2!}$

F8. Find the number of permutations of the word "CANADA" that have....

First, find the total number of permutations without any restrictions=6!/3!

Second, find the number of permutations with the "C" and "D" together
CD __ __ __ __

$= \frac{5!2!}{3!}$

Multiply by 2! for CD or DC and divide by 3! for 3 A's that are identical
Then, the number of permutations with the "C" and "D" apart is $\frac{6!}{3!} - \frac{5!2!}{3!}$

F9. Find the number of permutations of the word "OTTAWA" which begin...

First, put the "T" in the first and last spots in one way

T ___ ___ ___ ___ T

Now, there are 4 letters left to arrange, with 2 A's.

The total number of permutations is $\frac{4!}{2!}$

F10. Find the number of permutations of the word "STATISTICS" that have...

There are 10 letters in the word STATISTICS, with 3 T's, 3 S's and 2 I's

When letter have to be apart, we find the total ways - # ways together.

The <u>total number of permutations</u> with no restrictions is $\frac{10!}{3!3!2!}$

<u>Number of ways with A, C together</u>

First, put the AC together.

AC ___ ___ ___ ___ ___ ___ ___ ___

There are 9 units to arrange, because Ac are stuck together as one "unit"

There are 3T's, 2S's and 2 I's AND we can arrange AC in 2! ways

The answer is $\frac{9!}{2!3!3!}2! = \frac{9!}{3!3!}$ ways

The total # ways for A, C apart= $\frac{10!}{3!3!2!} - \frac{9!}{3!3!}$

F11. There are 8 paintings to be hung on a very long wall. If there are 5 different landscape paintings and 3 different portraits...

Put all of the landscape pictures together

$\underline{L_1L_2L_3L_4L_5}$ ___ ___ ___

The landscape paintings don't have to be first, they just have to be together. So, they can move down the line, as long as they are all beside each other.

Therefore, there are 4 "things" or "units" to arrange.

Now, the 5 landscape can be rearranged amongst themselves in 5! ways.

The total number of ways is 5! 4!

F12. A dorm has four positions to fill: President, Vice-President, Secretary and Treasurer. At this point, they are only looking to fill two of the four positions...

First, we choose 2 students of 100 to fill positions in $\binom{100}{2}$ ways

Since they have 4 positions and only want to fill two, we must choose which two positions to fill in $\binom{4}{2}$ ways

Now, we arrange these 2 people into the 2 positions, for example P, VP in 2! ways

The total # of ways is $\binom{100}{2} \times \binom{4}{2} \times 2!$

F13. A class at Huron contains only 15 students. The prof can assign grades ...

At most two A's means he can assign no A's, one A, or two A's.

We do each of these cases and then add them up.
If he assigns no A's, there are 4 choices of grades for each of 15 students, so 4^{15} ways.
If he assigns 1 A, we have to choose which of the 15 students gets the A, and then there are still 4 choices of grades for the other 14 students. Therefore, $\binom{15}{1} \times 4^{14}$

If he assigns 2 A's, we have to choose which of the 15 students get the A's, and then there are 4 choices for the remaining 18 students. Therefore, $\binom{15}{2} \times 4^{13}$
The final answer is $4^{15} + \binom{15}{1} \times 4^{14} + \binom{15}{2} \times 4^{13}$

F14. A class of 30 students is running a race for a fundraiser...
First, we have to choose which of the 30 students will win the race in $\binom{30}{3}$ ways

Now, we have the three students chosen to win first, second and third, but we don't know which student wins first, which wins second, etc.

We can order these three students into these three places in 3! ways.
The total number of ways is $\binom{30}{3} \times 3!$

F15...has six lunches to pack and he has 3 chocolate chip cookies, 2 oatmeal cookies...

Since all cookies of the same type are indistinguishable, it is like 3 C's, 2 P's and 2 O's
The number of ways to do this is $\frac{7!}{3!2!2!}$
The answer is E).

F16. Consider the letters of the word "GLASSES".

a) How many 7- letter words can be made from these letters? (Each letter is used as many times as it appears in the word)
7 letters, with 3 S's identical
So, there are $\frac{7!}{3!}$ possible words

b) How many 5- letter words can be made?
** this is tricky, because a 5-letter word can have one "S", two "S"'s, or three "S"'s
Case 1 one S…then we must have all of the other letters GLAE, so 5 different letters can be arranged in 5! Ways

Case 2 SS plus 3 of the letters left of GLAE
So, choose which 3 of the other 4 we have and then arrange the five letters, 2 are alike
$\binom{4}{3} \times \frac{5!}{2!}$

Case 3
SSS plus 2 of the letters left of GLAE
So, choose which 2 of the other 4 letters we have and then arrange them, 3 are alike
$\binom{4}{2} \times \frac{5!}{3!}$
Add them all up and you get 120 + 240+120=480 words

c) Put the 3 S's together and count them as one "spot"
<u>SSS</u> ___ ___ ___ ____ So, we have 5 "things" to arrange
5! and we DO NOT multiply by 3! to mix up the 3S's because they all look identical, so there is only one arrangement of them

d) To find the total ways the 3S's are NOT all together, we take:
Total ways - # ways (3S together)
Therefore, $\frac{7!}{3!} - 5!$

F17. DESSERTS

a) There are 2 E's and 3S's...So, $\frac{8!}{2!3!}$ ways to arrange all of the letters

b) Put the 3S's together

SSS __ ___ ___ ___ ___
Now, there are 6 "things" to arrange in 6! ways, and we still divide by the repeating E's.
We don't multiply by 3! for arranging the 3S's because they all look the same, so there is only one arrangement
So, we get: $\frac{6!}{2!}$

c) The total ways for the 2E's apart= total ways from a) - # ways the 2E's are together
2E's together....

EE __ ___ ___ ___ ___ ___ and then we divide by 3! for the 3S's

So, we get: Total ways - $\frac{7!}{3!}$ =

$$\frac{8!}{2!\,3!} - \frac{7!}{3!}$$

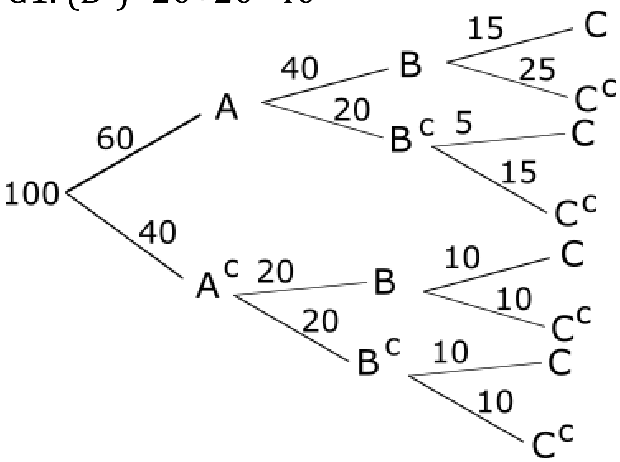
d) To find the 3S's together and the 2E's NOT together, use the total ways as your answer to part b) and then subtract the ways the 3S's and the 2E's are together.

The 3S's and 2E's together would be... SSS EE ___ ___ ____ So, 5! and we don't divide by anything, because the 2E's are not spread out through the line. We also don't multiply by anything because the 3S and the 2E look the same.

So, we get: $\frac{6!}{2!} - 5!$

G. Practice Test (Ch. 1)

G1. (B^C)=20+20=40



G2. $n(A \cap B \cap C) = 15$
far right branch at the top

G3. Find $n(A \cup C^C)$.
$n(A \cup C) = \{in\ A\ or\ not\ in\ C\ or\ in\ both\} = 60 + 10 + 10 = 80$
** don't add any C^c that have already been counted in A at the top of the tree.... otherwise you will be double counting

G4. Find $n(B \cap C^C)$=25+10=35

G5. $\{choose\ Alia\ not\ Erica\ \ OR\ \ choose\ Erica\ not\ Alia\ \ OR\ \ choose\ neither\}$
$= \binom{1}{1}\binom{4}{2} + \binom{1}{1}\binom{4}{2} + \binom{4}{3}$

or $2\binom{4}{2} + \binom{4}{3}$

G6. The candies are different...you have 3 children for each of the 15 candies to go to so 3x3x3x...x3 fifteen times.
Therefore, 3^{15}

G7. In how many ways can 15 different candies be distributed among three children if any number of candies can be given to each child and all of the candies are given away?
So, 3^{15}-3 since there are 3 kids and none of them can get all of the candies

G8. In how many ways can 18 students be divided into 2 groups of 6 students and 2 groups of 3 students?

$\frac{\binom{18}{3\ \ 3\ \ 6\ \ 6}}{2!2!}$ (divide by 2! For 2 groups of 6 that are indistinguishable and 2! For two groups of 3 that are indistinguishable)

G9. In how many ways can 14 students be divided into 2 groups of 6 students and 2 groups of 3 students if each group is assigned to a different task?

$\binom{18}{3\ \ 3\ \ 6\ \ 6}$ They are assigned different tasks, so they are distinguishable and we don't divide.

G10.

Find the number of subsets of the letters {a,b,c,d,e,f}.

We can either put each letter in or out of the set, so 2 choices for each letter…

\# subsets = 2^6

G11. "a"- in (1 way)

"f"- out (1 way)

Therefore, 2^4 ways to say "in" or "out" to each of the other 4 letters

G12. Find the number of permutations of the letters of the word "STATISTICS" that have the 3T's together and the 3S's not all together.

The number of ways to put the 3T's together is...

TTT __ __ __ __ __ __ __

$\frac{8!}{3!2!}$ where we divide by 2! for the 2 I's and 3! for the 3 S's.

To find the 3T's together and the 3S's not all together, we find the ways the 3T's are together and the 3S's are together as well and then subtract from our first answer

SSS TTT _ _ __ __

$\frac{6!}{2!}$ ways because we divide by 2! for the 2I's. We don't multiply by anything else, because the SSS will only be one arrangement and the same goes for the TTT.

So, we get: $\frac{8!}{3!2!} - \frac{6!}{2!}$

G13. Find the number of permutations of the letters of the word "MATHEMATICS" that begin and end with "M"

M __ __ __ __ __ ___ __ __ __ M

Place the M at the start and at the end.
There are now, 9 letters left to arrange and there are 2T's and 2A's.
We don't multiply by 2! for the two M's because they would still be the same arrangement.

We get: $\frac{9!}{2!2!}$

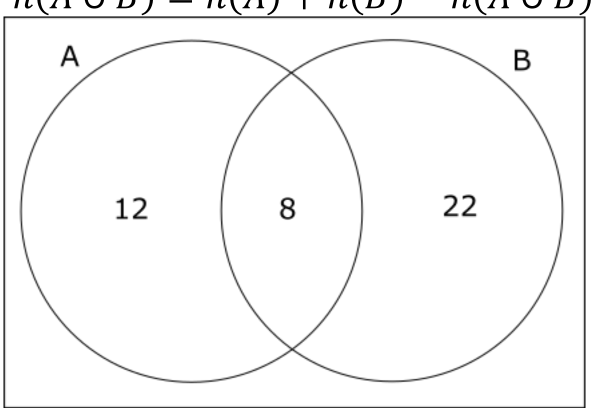
G14. How many permutations of the letters of the word "WESTERN" have the "R"and "N" apart?

RN _ _ _ _ _ 6! for 6 "units" to arrange and then divide by 2! for the 2E's...then multiply by 2! for RN and NR switching

Total ways - # ways together
$= \frac{7!}{2!} - \frac{6!}{2!} \times 2!$
$= \frac{7!}{2!} - 6!$

G15. Given A and B are subsets of a universal set U with n($A \cap B$)=8, n(A)=20 and n(B)=30. Find n(AUB).

$n(A \cup B) = n(A) + n(B) - n(A \cup B) = 20 + 30 - 8 = 42$



G16.
Given A and B are subsets of a universal set U with $n(A^C \cap B^C) = 25$, n(U)=100, n(A)=40 and n(B)=50, find $n(A \cap B)$.

$n(A \cup B) = 100 - 25 = 75$ ...since the union is everything except the "outside" of the circles in the Venn diagram

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
75=40+50-$n(A \cap B)$
$n(A \cap B) = 15$

G17. How many even 3-digit numbers less than 400 can be formed using only the digits 2,5,6 and 8, if repetition is allowed?

It has to start with a "2" otherwise it can't be less than 400. Now, it also must end in a "2" or a "6" or an "8" to be an even number

So, there are 3 choices for the last digit and 1 choice for the first digit. The middle digit can be any of the 4 numbers since repetition is allowed

1x4x3=12 numbers

G18. How many numbers less than 5000 can be made using the digits 1,2,3,4,5,6 if repetition is NOT allowed?

One-digit numbers= 6 choices

2-digit numbers=6x5=30 since there are 6 choices for the first digit and 5 for the second since there is no repetition

3-digit numbers=6x5x4=120 since there is no repetition

4-digit numbers=must be less than 5000...start with a 1,2,3 or 4...second digit can't repeat, so there will be 5 numbers still left

4x5x4x3=240
The total numbers= 6+30+120+240=396

Test Two Material

A. Sample Spaces and Events (2.1)

Sample Space

1. complete with all possible outcomes
and
2. no overlapping outcomes, ie. mutually exclusive

Example 1. An experiment consists of tossing an ordinary coin 3 times. Which of the following sets would qualify as possible sample spaces for this experiment?

S1={zero heads, one head, two heads, three heads}

Yes, this is complete with all possible outcomes and mutually exclusive as there is no overlap

S2={no heads, no tails, at least one of each}

Yes, this is complete and mutually exclusive...remember at least one of each is NOT the same as "at least one head", "at least one tail" which would result in tons of overlap

S3={no heads, no tails, at least one head, at least one tail}

No, these overlap...at least one head contains the outcome 1H and 2T and at least one tail contains that same outcome...not mutually exclusive so not a sample space

S4={more heads than tails, more tails than heads}

Yes, since the coin was tossed an odd number of times, there is no "equal number of heads and tails needed".

S5={ at most one head, exactly one tail, three heads}
0H, 1H   1T (2H)   3H
It is complete and mutually exclusive.
$\therefore$ *It is a sample space*.

So, S1, S2, S4 and S5 are all sample spaces.

Example 2. A box contains 5 blue, 2 green and 4 purple marbles. Three marbles are selected without replacement. Which of the following are NOT sample spaces?

S1={all blue, all green, all purple}

No, it is missing lots of outcomes...ex 1 blue, 1 green, 1 purple...incomplete

S2={at least one blue, at least one green, at least one purple}

No, there is overlap because at least one blue could be the outcome 2 blue and 1 green and so could "at least one green"....not mutually exclusive

S3={no green, one green, two green, three green}

No, there is no replacement so it is not possible to get 3 green since there are only 2 green in the box...not possible

S4={no blue, one blue, two blue, three blue }

Yes, it is complete and mutually exclusive

S5={no blue, at least one blue}

Yes, it is complete and mutually exclusive

So, S1, S2 and S3 are NOT sample spaces

Example 3. An experiment consists of tossing an ordinary coin eight times. Which of the following are possible sample spaces for this experiment?

S1={more heads than tails, fewer heads than tails}

No, it is missing the case of getting an equal number of heads and tails, since the coin is tossed an even number of times....incomplete

S2={at least 2 heads, more than 2 tails}

No, the outcome 3T, 5H occurs in both of these...overlap...not mutually exclusive

S3={no heads, at least one head}

Yes, this is complete and mutually exclusive

S4={no heads, no tails, all heads, all tails}

No, this is missing several outcomes, ex. 2H,6T

Therefore, only S3 is a sample space.

Example 4. S1 not a sample space as it is missing outcomes and it overlaps
S2 there is overlap since 1 red and 4 black are in both
S3 yes, it is a sample space
S4 0 Queen, 1 Queen, 0 Jack, 1 Jack overlap, so not a sample space
S5 is not complete, so not a sample space
S6 yes, it is a sample space

Example 5.

S2 is an equiprobable sample space meaning all sets have equal probabilities. The answer is D).

Example 6.

1.Set of all marks out of 25 of all the students. It is not equiprobable since more than one student could get the same mark

Equiprobable
Set of all student numbers of students
Set of all individual students

Practice Exam Questions on Sample Spaces and Events

A1.

S1 contains all possible outcomes and there is no overlap, so it is correct.

S2 is missing lots of possible rolls, as it only contains the rolls of all odd or all even and you could roll 3 even, 2 odd number, etc.

S3 has overlap since at least one even number and at least one odd number are not mutually exclusive. For example, 2 even, 3 odd would belong to both of these sets.

The answer is a).

A2. S1 contains all possible outcomes and they are mutually exclusive.

S2 is a sample space

S3 contains all outcomes and there is no overlap

The answer is a). All of them are sample spaces.

A3. S1 = {zero heads, exactly one head, exactly two heads, exactly four heads, five heads} The outcome of getting exactly 3 H is missing

S2 = {no heads, no tails, at least one head} At least one head contains the outcome of all 5H and this outcome is the same as 0T, so they are NOT mutually exclusive

S3, S4, S5 and S6 are possible sample spaces.

A4. S1={ no 4, one 4, two 4's, three 4's} is a sample space

S2= { no 5's, one 5, two 3's, three 3's} is not because there is overlap since it talks about the number "5" and then switches to the number "3".

S3={ sum of three dice is an odd number, sum of the three dice is an even number} is a sample space
The answer is d).

A5. A coin is tossed four times. Which of the following are possible sample spaces?

S1={even number of heads, odd number of heads} is a sample space because even and odd are the only two possibilities.

S2={ more heads than tails, more tails than heads}is NOT a sample space because there are an even number of tosses, there could also be "equal number of heads and tails".

S3={ all heads, all tails}is NOT a sample space because it is missing lots of outcomes, for ex. 1H, 3T, etc.

S4= is 2,3,4H or 3T(1H) or 4T(0H) so it is complete and mutually exclusive
The answer is d).

A6. A die is rolled 3 times. Which of the following is NOT a possible sample space for this experiment?
S1={sum of the numbers on all rolls is odd, sum of numbers on all rolls is even}

S2={6 does not appear on any rolls, 6 appears on at least one roll}

S3={sum of all rolls is divisible by 5, sum of all rolls is not divisible by52}

S4={sum of all rolls is divisible by 2, sum of all rolls is not divisible by 5}

S5={first roll is less than the second roll, first roll is greater than or equal to the second roll}
The answer is a) S4 only

A7. A box contains 5 purple, 3 red and 6 green marbles. Three marbles are drawn without replacement. Which of the following is NOT a possible sample space for this experiment?

S1={all purple, all red, all green} is ***not*** a sample space because there are lots of combinations of three marbles missing, ex. 1P, 2R

S2={at least one red, no red} is a sample space because you either get no red, or at least one red and they are mutually exclusive.

S3={0 red, 1 red, 2 red, 3 red}is a sample space

S4={ more green than purple, more purple than green, same number of purple and green}is a sample space because it contains all possibilities and there is no overlap.

S5={at least one red, more than one purple}is ***not*** a sample space because these are not mutually exclusive. At least one red can give you the same outcome as more than one purple. For example, 1R, 2P belongs to both.

The answer is a).

A8. S3 is not a sample space because it is not mutually exclusive. At least 1 club contains the event (1 club, 1 spade). At least one black card contains the same event. Remember, if there is any overlap, it is NOT a sample space.

The answer is d).

A9. In S1, at least one head contains the event (1H, 3T). More than one tail contains this same event, so they are NOT mutually exclusive.

S2, the second option is correct as a sample space.

S3 is a sample space.

The answer is c).

A10. S1 is not because you could have 2 loonies and 2 toonies...not complete
S2 is not because there is overlap...2 loonies is the same case as 2 toonies...not mutually exclusive and you also can't get four loonies as there is no replacement.
S3 is not because an odd number of loonies contains the outcome 3 loonies and 1 toonie and it overlaps with another case...not mutually exclusive
S4 is a sample space since it is mutually exclusive and complete...odd number of loonies contains the outcomes [1 loonies, 3 tooonies] and [3 loonies, 1 toonie], while even number of toonies contains the rest of the possible outcomes.

Only S4 is a sample space.

A11. S1 doesn't represent this experiment of drawing two cards
S2 not all of these are equally likely
S3 not a sample space...missing outcomes
S4 is a sample space, but they are not equally likely
The answer is E).

A12. A box contains 3 loonies and 3 quarters. Four coins are selected without replacement from the box. Which of the following are sample spaces?

S1={even number of loonies, more loonies than quarters, odd number of quarters}
No, for example odd number of loonies could be 1 loonie and that is a case that also comes up in more loonies than quarters ie. 3 loonies, 1 quarter

S2={one loonie, two loonies, three loonies}
Yes, since there is no replacement and there are only 3 quarters, 0 loonies is impossible, so this contains all possibilities and there is no overlap

S3={four quarters, four loonies, no quarters, no loonies}
No, they are missing the outcome 2 quarters and 2 loonies...not complete

S4={more loonies than quarters, more quarters than loonies}
No, it is missing the outcome where you could get 2 of each coin

S5={even number of loonies, odd number of loonies}

Yes, it is a sample space because you can only have even or odd and they are not the same

So, only S2 and S5 is a sample space

B. Basic Properties of Probability (2.1, 2.2)

Review of Subsets
Example 1.

a) 2^8
b) a- in and e-out
2^6
c) 2^8-1 with at least one letter
d) 2^8-2
e) $\binom{8}{4}$
f) a- in f-out
So, we need three more letters and there are {b,c,d,e,g,h} left since the a is already in the set and f is out
$\binom{6}{3}$

Example 2.
a)The total number of subsets for five numbers, with no restrictions is 2^6

The number of subsets that contain the number 3 and 4 and don't contain the 6, is 2^3 (since the 3 is in, the 4 is in and the 6 is out)

Probability=$\frac{2^3}{2^6} = \frac{1}{8}$

b) here we can have all cases except the empty set and the set that contains all 7 numbers
So, the probability is $\frac{2^6-2}{2^6} = \frac{62}{64}$

c) here the number "3" is in the set and "6" is not, so we need to pick 2 more numbers from {1,2,4,5 }

So, the probability is $\frac{\binom{1}{1}\binom{4}{2}}{\binom{6}{3}}$

Example 3.

(a) Since $\Pr(A^C \cap B^C) = 0.20$, we know that Pr(AUB)= 1 – 0.20=0.80
From the union formula,

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

0.80=0.7+0.4-$\Pr(A \cap B)$

$\Pr(A \cap B)$=0.30

b) $Pr(B \cup A^C)^C = \Pr[B^C \cap (A^C)^C] = \Pr(B^C \cap A) = 0.7 - 0.3 = 0.4$

Example 4.

Let x be the probability that "b" occurs
Note: the total of c and d is 0.4

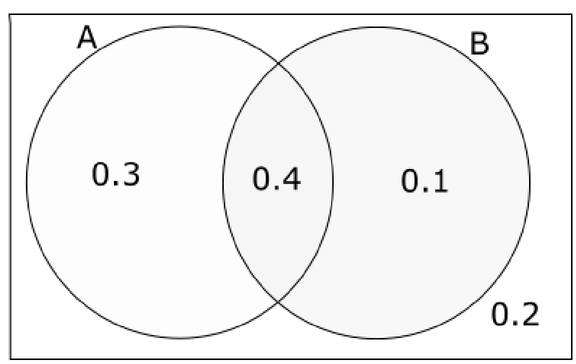
Event	Probability
A	3x
B	x
C	0.4
D	
E	6x

Since Pr(b)=1/3 Pr(a), then Pr(a)=3Pr(b)

Total probability =1

3x + x + 0.4+6x=1
10x+0.4=1
10x=0.6
x=0.06
Pr(a)= 3x = 3(0.06)=0.18

Example 5.



Draw a Venn diagram

$Pr(A^C) = 1 - \Pr(A) = 1 - (0.4 + 0.3)$
$= 1 - 0.7 = 0.3$

$\Pr(A^C \cup B) = not\ in\ A\ or\ in\ B\ or\ in\ both\ of\ these$

$=0.2 + 0.3 = 0.5$ *or use formula for union*

$= \Pr(A^C) + \Pr(B) - \Pr(A^C \cap B) = 0.3 + 0.5 - 0.1 = 0.7$

Example 6.

a) Pr(ace)=4/52=1/13

b) Pr(red face card)= 6/52=3/26 since there are only 6 red face cards= queen of diamonds, queen of hearts, king of diamonds, king of hearts, jack of diamonds, jack of hearts

Example 7. a) Find the probability of getting two diamonds and one club.

$$\frac{\binom{13}{2}\binom{13}{1}}{\binom{52}{3}}$$

b) Find the probability of getting at least one club.

1- Pr(no clubs)= 1 - $\frac{\binom{13}{0}\binom{39}{3}}{\binom{52}{3}}$

c) Find the probability of getting exactly one ace.

$$\frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}}$$

d) one diamond, 1 heart, 1 other or 1 diamond and 2 hearts

$$= \frac{\binom{13}{1}\binom{13}{1}\binom{26}{1} + \binom{13}{1}\binom{13}{2}}{\binom{52}{3}}$$

Example 8.

Given $\Pr(A \cup B) = 0.9, \Pr(A) = 0.6$ *and* $\Pr(B) = 0.4$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
0.9 = 0.6 + 0.4 - $\Pr(A \cap B)$
$\Pr(A \cap B) = 0.1$

a) No, because if they were mutually exclusive, Pr(A and B)=0.

b) for independence, we need that Pr(A and B)=Pr(A)xPr(B)

Pr(A and B)=0.1

Pr(A)xPr(B)=0.6x0.4=0.24

Therefore, they are NOT independent.

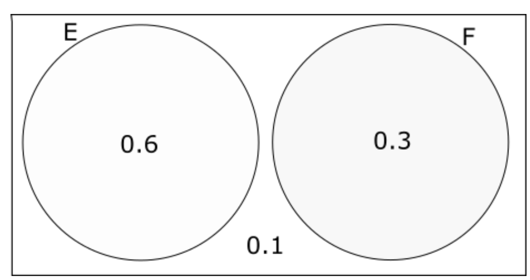
Example 9.

a) Pr(E and F)=Pr(E)xPr(F) since they are independent

Pr(E and F)=0.3(0.5)=0.15

b) $\Pr(E \cup F^C) = \Pr(E) + \Pr(F^C) - \Pr(E \cap F^C)$
=Pr(E) +$\Pr(F^C) - \Pr(E)\Pr(F^C)$
=0.3+0.5 - 0.3(0.5) = 0.65

c) $\Pr(E^C \cap F^C) = (0.7)(0.5) = 0.35$



Example 10.

Since they are mutually exclusive, Pr(E and F)=0.

$$\Pr(E^C \cap F^C) = 1 - 0.6 - 0.3 = 0.1$$

$\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$
=0.6+0.3 – 0 =0.9
$\Pr(E \cap F^C) = \Pr(E) = 0.6$

Example 11.

a)Pr(exactly 2 hearts)=$\frac{\binom{13}{2}\times\binom{39}{3}}{\binom{52}{5}}$

b) Pr(at least one club)= 1 – Pr(no clubs) = 1 - $\frac{\binom{13}{0}\times\binom{39}{5}}{\binom{52}{5}}$

c) Pr(at least two hearts)= 1- Pr(no hearts) - Pr(1 heart)

=1 - $\frac{\binom{13}{0}\times\binom{39}{5}}{\binom{52}{5}}$ - $\frac{\binom{13}{1}\times\binom{39}{4}}{\binom{52}{5}}$

Example 12.

Pr(2 clubs) = 13/52(13/52) or (1/4) (1/4) = 1/16 since there is replacement you CANNOT use choose

Example 13. Draw a circle and place Laura and Vesna must go across from her. They don't count in the arrangement, because someone has to "start" the circle. So, there are 12 people to arrange around the circle and remember, if you had placed Vesna first, instead of Laura, they would still be in the same arrangement with one another.

The denominator is the total ways to arrange the 12 people in a circle, 11!

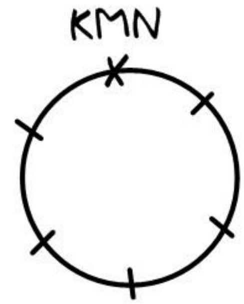
So, we get $\frac{10!}{11!}=\frac{1}{11}$

Example 14.

Pr(Iris, Frances and Hailey not all together)= 1 – Pr(they are all together)

There are 3! ways to arrange the three girls together in their own group of 3.

$=1 - \frac{5!3!}{7!} = 1 - \frac{6}{42} = 1 - \frac{1}{7} = \frac{6}{7}$



Example 15.

$\underline{G1G2G3G4G5G6}$ _ _ _ _ _

$\frac{6!6!}{11!}$

Practice Exam Questions on Probability

B1.There are 13 clubs and we want to choose 3 of them. There are 13 hearts and we want to choose 2 of them. The total number in the denominator is 52 cards and we want to choose 5 of them.

Therefore, the answer is $\frac{\binom{13}{3}\binom{13}{2}}{\binom{52}{5}}$

B2. Five of the same suit… Choose one of four suits and then choose 5 of those 13 cards.

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}$$

B3. If Jason and Daniel are both on the committee, there are 6 people left and we need to choose one more person to have a committee of 3 people. The denominator is the total ways with no restrictions to choose a group of 3 people from a total of 8 people.

The answer is $\frac{\binom{2}{2}\binom{6}{1}}{\binom{8}{3}}$

B4. Place ______ and ________ together to start the circle.

Then, there are 8 people to arrange. Also, _______ and ________ can switch places, but still be together.

The total ways to arrange 10 people in a circle with no restrictions is 9!

The answer is $\frac{8!2!}{9!} = \frac{8!2!}{9\times 8!} = \frac{2}{9}$

B5 .

	Probability
A	x
B	3x
C	4x
Total	1

Since the total probability must always add to 1, we get
8x=1
So, x=1/8

The Pr(b)=3/8

The answer is c).

B6. Whenever we need people apart, we do 1- Pr(together).

Pr(M and J)=1 - Pr(M and J together)

$=1-\frac{18!2!}{19!}$

$=1-\frac{18!2!}{19(18!)}$

$= 1-\frac{2}{19}=\frac{17}{19}$

B7. Pr(at least one diamond)
$1-\Pr\,(no\ diamonds)$

$=1-\frac{\binom{39}{3}}{\binom{52}{3}}$

B8.
Pr(at least 2H)= Pr(2H) + Pr(3H)

$=\frac{\binom{13}{2}\binom{39}{1}+\binom{13}{3}}{\binom{52}{3}}$

B9.
Put Rachel in a spot to start the circle. Then, there is only one spot to put Bianca right across from her. The total ways with no restrictions to arrange 10 people in a circle is 9!.

Pr(Rachel across and Bianca)=$\frac{8!}{9!} = \frac{8!}{9\times 8!} = \frac{1}{9}$

The answer is b).

B10.

Event	Probability
A	0.6
D	
B	3x
C	x
Total	1

Since the total of A and D is 0.6, it doesn't matter what each of them are equal to, just the total.

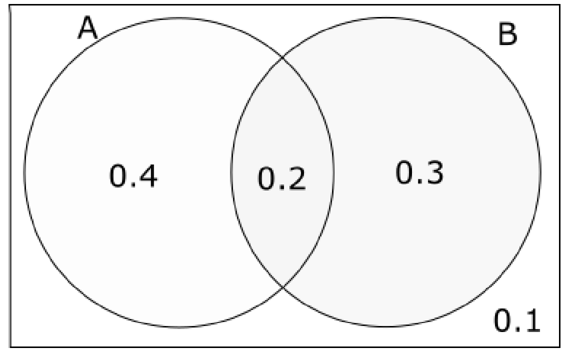
0.6 + 3x + x = 1
4x = 0.4
x=0.1

Pr(b)=3x= 3(0.1)=0.3

The answer is b).

B11.
Since A and B are mutually exclusive, $\Pr(A \cap B) = 0$

Therefore, $\Pr(A \cup B) = \Pr(A) + \Pr(B) = 0.3 + 0.5 = 0.8$
The answer is d).



B12.

$\Pr(A \cap B^C) = 1 - 0.2 - 0.3 - 0.1$
$= 0.4$

Pr(A)=0.4+0.2=0.6

The answer is d).

B13.

There are 2 red aces and 52 cards total.
The probability is 2/52=1/26

B14.

<u>GGGGG</u> ___ ___ ___ ___

There are 5 units to arrange, because the girls are stuck together as one unit.
Also, the girls can be arranged amongst themselves
The answer is $\frac{5!5!}{9!}$

B15.

To find the probability of Erin and Kaitlyn apart, we do 1 - Pr(together)

There are 9 people, so the total ways to arrange them in a line is 9!

<u>EK</u> ___ ___ ___ ___ ___ ___ ___

There are eight units to arrange since Erin and Kaitlyn are together as one unit.
Also, Erin and Kaitlyn can switch places in 2! ways.

$$1 - \Pr(together) = 1 - \frac{8!\,2!}{9!} = 1 - \frac{2}{9} = \frac{7}{9}$$

B16. The total ways to arrange six people in a circle is 5!.

Place Logan to start the circle. Then, Ryan has only one spot to go in, across from Logan.

There are only 4 people to arrange in 4! ways.

The answer is $\frac{4!}{5!} = \frac{4!}{5\times 4!} = \frac{1}{5}$

B17.
a) There are 7 women and we want 4 of them.
The denominator is 11 people and we want to choose 4 of them.

$$\frac{\binom{7}{4}}{\binom{12}{4}}$$

b) An equal number means 2 men and 2 women

$$\frac{\binom{5}{2}\binom{7}{2}}{\binom{12}{4}}$$

c) There are only 1 way to put Jeff on the committee. If Jeff is on and Kate is not on, there are 10 people left and we need to choose 3 more people, in addition to Jeff.
The answer is $\frac{\binom{10}{3}}{\binom{12}{4}}$

B18. Pr(at least 2 heads)=Pr(2heads)+Pr(3 heads)

= draw a tree or write out cases!!
HHT, THH, HTH, HHH
Pr(at least 2 heads)=4/8=1/2 or 0.50

B19. Pr(ABC together)= # ways together/# ways total to arrange the letters $= \frac{6!3!}{8!}$ the 3! is to arrange the ABC, while still keeping the 3 letters together

<u>ABC</u> __ __ __ __ __ 6!3!

B20. Pr(2 customers choose Tiena)=$\frac{\binom{5}{2}3^3}{4^5}$

The denominator is 4x4x4x4x4 since each of the five customers can choose any of the four salespeople to deal with

On the numerator, we have 5 customers and we want two of them to choose Tiena and then the other 3 can choose from the other 3 salespeople

B21. Pr(E and F)=0

$\Pr(E^C \cap F^C) = 1 - 0.5 - 0.1 = 0.4$

B22.
a) No, if they're independent, then Pr(E and F)=Pr(E)xPr(F)≠0 since neither Pr(E) or Pr(F) are equal to 0.

b)
$\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$
= 0.4 + 0.5 - (0.4)(0.5) since they are independent
=0.90 - 0.20
= 0.70
$\Pr(E^C \cap F^C) = 1 - \Pr(E \cup F) = 1 - 0.70 = 0.30$

$\Pr(E^C \cap F^C) = \Pr(E^C)\Pr(F^C)$=0.6(0.5)=0.30 Multiply since they are independent

B23.
Since E and F are mutually exclusive, Pr(E and F)=0.
$\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$

=Pr(E) + Pr(F) - 0
=0.5 + 0.3
=0.8

$\Pr(E^C \cap F^C) = 1 - 0.8 = 0.2$

B24. Since A and B are independent, $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

0.2= 0.4 Pr(B)
Pr(B)=0.5

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
$\Pr(A \cup B) = \Pr(A) + \Pr(B)$ - $\Pr(A) \times \Pr(B)$
=0.4 + 0.5 - 0.2
=0.9 - 0.2
=0.70

B25. The letters of "SOPHIE" can be arranged in alphabetical order in only 1 way.

Pr(alphabetical order)=$\frac{1}{6!} = \frac{1}{720}$

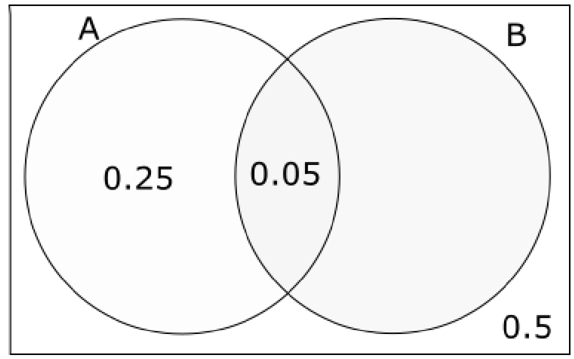
B26. Pr(3 different numbers)=$\frac{6\times5\times4}{6^3} = \frac{20}{36} = \frac{5}{9}$

B27. Pr(Liam and Cayden are on the same team)=$\frac{\binom{10}{6\ \ 4}}{\frac{\binom{12}{6\ \ 6}}{2!}}$

* The denominator is divided by 2! since the 2 groups are indistinguishable from one another
* The numerator involves putting ______ and________ on one team...so that team only needs 4 more people and the other team needs 6 people...these are distinguishable because one team has Liam and Cayden and the other team doesn't.

B28.

Let A=event that interviewer A will pass a candidate
Let B= event that interviewer B will pass a candidate



Pr(A)=0.30 and Pr(B)=0.25 and
$\Pr(A^C \cap B^C) = 0.50$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
$1 - 0.50 = 0.30 + 0.25 - \Pr(A \cap B)$
$\Pr(A \cap B) = 0.05$

$\Pr(A \cap B^C) = \Pr(A) - \Pr(A \cap B) = 0.30 - 0.05 = 0.25$
Therefore, the probability a candidate will be passed by A but not by B is 0.25.

B29. Put them together and count them as one spot

CTC ___ ___ ___ ___ ___

So, there are 6 "things" to arrange and then Tanya must be in the middle, but Celine and Chad can switch....so 2!

The total for the denominator is the total ways to arrange the 8 people in a line with no restrictions

So, we get $\frac{6!2!}{8!} = \frac{6!2!}{8(7)6!} = \frac{2}{56} = \frac{1}{28}$

B30.
$\Pr(A \cup B)$= 1 - 0.3 = 0.7

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
0.7 = 0.3 + 0.5 - $\Pr(A \cap B)$
$\Pr(A \cap B) = 0.1$

B31.

a) $\frac{\binom{4}{1}\binom{4}{2}\binom{12}{2}}{\binom{52}{5}}$

b) Pr(four of a kind)=$\frac{\binom{13}{1}\times\binom{4}{4}\times\binom{48}{1}}{\binom{52}{5}}$

First, choose one of 13 types of cards to have three of…then of those cards, there are 4 and we want all four of them.

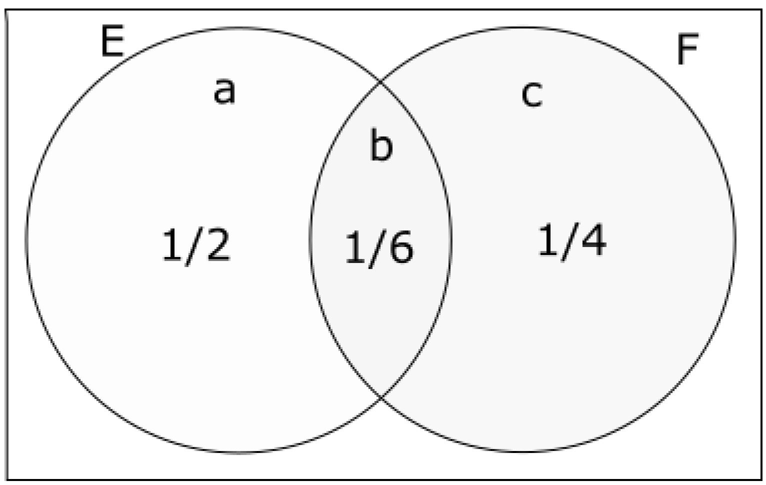
Now, there are 48 cards left and we need one more card

c) Full house is like KKKAA

Pr(full house)=$\frac{\binom{13}{1}\times\binom{4}{2}\times\binom{12}{1}\binom{4}{3}}{\binom{52}{5}}$

B32.

Event	Prob
A	1/2
B	1/6=2/12
C	¼=3/12



E=a, b and F=b, c...draw a Venn diagram

$$\Pr(E^C \cup F^C) = \Pr\,(E \cap F)^C = 1 - \Pr(E \cap F) = 1 - \frac{2}{12} = \frac{12}{12} - \frac{2}{12} = \frac{10}{12} = \frac{5}{6}$$

C. Conditional Probability (2.3)

Example 1.

Pr $(E/F)=\frac{\Pr(E\cap F)}{\Pr(F)}$

$$\frac{1}{2}=\frac{0.1}{\Pr(F)}$$

0.5Pr(F)=0.1

Pr(F)=1/5

Example 2.

$$\Pr(A/B)=\frac{\Pr(A\cap B)}{\Pr(B)}$$

$$\frac{1}{3}=\frac{\Pr(A\cap B)}{\Pr(B)}$$

$$\frac{1}{3}=\frac{\Pr(A\cap B)}{1/2}$$

$$\Pr(A\cap B)=\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$$

$$\Pr(B/A)=\frac{\Pr(A\cap B)}{\Pr(A)}=\frac{\frac{1}{6}}{\frac{1}{4}}=\frac{1}{6}\times\frac{4}{1}=\frac{2}{3}$$

Example 3.

Since A and B are independent, we know Pr(A and B)=Pr(A)xPr(B)...substitute into the top of the equation

Pr(B)=0.6

Pr(B^C)= 1 – 0.6 = 0.40

$$\Pr(A/B)=\frac{\Pr(A\cap B)}{\Pr(B)}=\frac{\Pr(A)\,xPr(B)}{\Pr(B)}=\Pr(A)=0.35$$

$Pr(A\cup B^C)=Pr(A)+Pr(B^C)-Pr(A\cap B^C)$

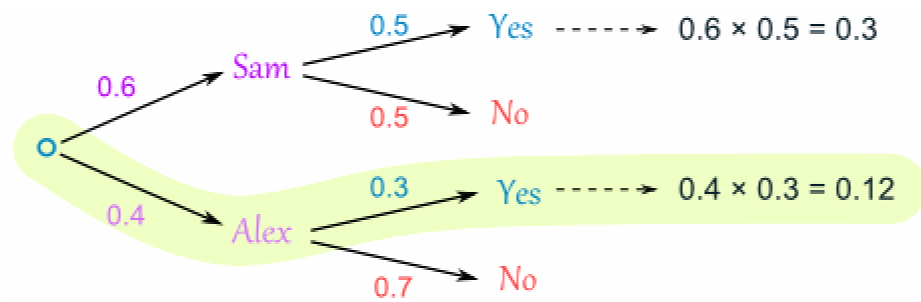
=Pr(A) + $Pr(B^C)-\Pr(A)\,Pr(B^C)$ since they are independent

=0.35+0.40- 0.35(0.4)

=0.75 - 0.14=0.61

Example 4. Remember that:

P(A and B) = P(A) x P(B|A)



Pr(play goal)=Pr(Sam and goal) + Pr(Alex coaches and play goal)

=0.3 + 0.12 = **0.42 probability** of being a Goalkeeper today

Example 5.

$\Pr(G\ or\ H) = \Pr(G) + \Pr(H) - \Pr(G \cap H)$

$0.7 = 0.40 + 0.45 - \Pr(G \cap H)$

$\Pr(G \cap H)$=0.15

$\Pr\ (G/H) = \frac{\Pr(G \cap H)}{\Pr(H)} = \frac{0.15}{0.45} = \frac{15}{45} = \frac{1}{3}$

Example 6.

You can draw a tree and/or use the conditional formula. But, you can just reason your way through it by reducing your sample space. If the first card is a heart, then you then have only 51 cards left and 12 hearts (one has been removed).

So, the prob. the second card is a heart is 12/51

Example 7.
Here we will reduce our sample space…since it is known or given the first roll is a 3, we only look at the sample space of the first roll is a 3 ie. S={ (3,1) (3,2) (3,3,) (3,4) **(3,5)** **(3,6)**}

Now, we circle which ones have a sum greater than 7, which would be the last three, so we get 2/6 or 1/3.

Example 8.
We know that they have a least one girl...so the only possible outcomes for the 3 kids are the ones that have at least one girl.

Reduced sample space={*BGG*, ***GGG***, *GBG*, GBB, GGB, BGB, BBG}}...exclude BBB

So, look at these 7 outcomes and our answer is the ones that have all 3 girls...So, the probability is 1/7.

Example 9. Let E=sum is odd = 1/2

F=5 on first toss = 6/36 = 1/6

G=sum greater than 8 = sum9, 10, 11 or 12=
P(3,6)(6,3)(4,5)(5,4)(5,5)(4,6)(6,4)(5,6)(6,5)(6,6)}=10/36 = 5/18

E and F= 5 first and odd...Pr(E and F)=3/36=1/12
{(5,2)(5,4)(5,6)}

$\Pr(E\ and\ F) = \frac{1}{12}$

$\Pr(E) = \frac{1}{2} and \Pr(F) = \frac{1}{6}$
So, $\Pr(E) \times \Pr(F) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

So, $\Pr(E\ and\ F) = \Pr(E) \times \Pr(F)$ and E and F are independent

Practice Exam Questions on Conditional Probability

C1.

Pr(2 tens)=$\frac{\binom{4}{2}}{\binom{52}{2}}$

C2.

Pr(fail stop/not signal)= $\frac{\Pr(fail\ stop\ and\ not\ signal)}{\Pr(not\ signal)} = \frac{0.15}{0.20} = \frac{15}{20} = \frac{3}{4}$

C3.

$\mathrm{n}(E \cup B) = 200 - 50 = 150$

Pr(E)=100/200=1/2
Pr(B)=90/200=9/20

$\mathrm{n}(E \cup B) = \mathrm{n}(E) + \mathrm{n}(B) - \mathrm{n}(E \cap B)$
150=100 +90 – $n(E \cap B)$
$n(E \cap B) = 40$

$\Pr(E^C \cap B^C) = \frac{50}{200} = 1/4$

a) Pr(E/B)=$\frac{\Pr(E\ and\ B)}{\Pr(B)} = \frac{40/200}{90/200} = \frac{4}{9}$

b) Pr(E or B but not both)= 60+50/200=110/200=11/20

C4.

$\Pr(A/B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{0.1}{0.6} = 1/6$

C5.

Pr (E/F)=$\frac{\text{Pr }(E \cap F)}{\text{Pr }(F)}$

2/3=$\frac{Pr\ (E \cap F)}{1/3}$

$\Pr(E \cap F) = 2/9$
The answer is a).

C6.

Pr (E/F)=$\frac{\text{Pr }(E \cap F)}{\text{Pr }(F)}$

$0.40 = \frac{0.2}{\text{Pr }(F)}$

Pr(F)=1/2

C7.

Pr(sum 12 given same #)=Pr{(6,6,6)}/Pr$\{(1,1,1)(2,2,2)\big)(3,3,3)(4,4,4)(5,5,5,)(6,6,6)\}$

=$\frac{\frac{1}{216}}{\frac{6}{216}} = \frac{1}{6}$

or reduce sample space S={(1,1,1)(2,2,2)(3,3,3**,)(4,4,4**)(5,5,5)(6,6,6)} since they have to have the same number on all three dice...circle ones with a sum of 12=1/6

C8.

$$\Pr(2nd\ girl\ given\ at\ least\ 1\ boy) = \frac{\Pr(BG)}{1 - \text{Pr (no boys)}} = \frac{1/4}{1 - 1/4} = \frac{1}{3}$$

or reduce the sample space S={at least 1 B}={ GB, ***BG,*** BB***}...***three outcomes and you want the prob. the 2nd is a girl, so, the prob. is 1/3

C9.

Pr(2 boys given one son)=$\frac{(\Pr 2\ boys\ and\ \ has\ one\ son)}{\Pr\ (has\ a\ son)} = \frac{1/4}{3/4} = \frac{1}{3}$

or reduce the sample space S={GB, BG, ***BB***} since you know they have a son, so it can't be GG...then circle the outcomes with two boys...ie 1/3

C10.

Pr(2nd ace/ 1st ace)= $\frac{\Pr(2nd\ ace\ and\ 1st\ ace)}{\Pr(1st\ ace)} = \left(\frac{\left(\frac{4}{52}\right)\frac{3}{51}}{\frac{4}{52}}\right) = \frac{3}{51} = \frac{1}{17}$

C11.

Pr ($sum\ greater\ than$ 9/ same # on dice)=$\frac{\Pr\ (sum\ greater\ than\ 9\ and\ same\ \#)}{\Pr\ (same\ \#\ on\ all\ dice)} =$

$\frac{\frac{2}{36}}{\frac{6}{36}} = \frac{1}{3}$

or reduce sample space S={(1,1)(2,2)(3,3)(4,4)(***5,5)(6,6***)} and circle the ones that have a sum greater than 9 = 2/6=1/3

C12.

Pr(1 spade/2hearts)=$\frac{\Pr\ (1\ spade\ and\ 2\ hearts)}{\Pr\ (2hearts)} = \frac{\binom{13}{1}\binom{13}{2}/\binom{52}{3}}{\binom{13}{2}\binom{39}{1}/\binom{52}{3}}$=$\frac{13}{39} = \frac{1}{3}$

C13.

Pr(F/E)=$\frac{\Pr\ (E \cap F)}{\Pr\ (E)}$

0.30=$\frac{\Pr\ (E \cap F)}{0.20}$

Therefore, the probability both E and F occur is 0.2x0.3=0.06
The answer is a).

C14.
not in F and also in E is just E, since they are mutually exclusive
Pr(E^C /F)=$\frac{\Pr(E^C \cap F)}{\Pr(F)}$=$\frac{\Pr(F)}{\Pr(F)} = 1$ since E, F are mutually exclusive

The answer is c).

C15.

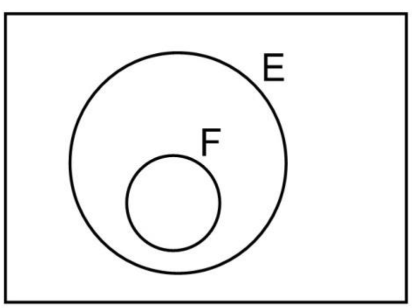
Pr(F/E)= $\frac{\Pr(E \cap F)}{\Pr(E)} = \frac{0.15}{0.60} = \frac{15}{60} = \frac{1}{4}$

C16.
$\Pr(E \cup F) = \Pr(E) + \Pr(F) = 0.1 + 0.9 = 1$ since E, F are mutually exclusive.

The answer is d).

C17.
Since F occurs whenever E fails to occur, Pr(F)=1-Pr(E)=1 - 0.60 = 0.40
The Pr(EUF)=0.4 + 0.6 = 1
The answer is b).

C18.



Since F cannot occur unless E also occurs, F is inside of E.

$\Pr(E \cup F) = 0.65$

C19.
If A and B are independent, $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$=0.30x0.20=0.06
Pr(AUB)=0.3 + 0.2 - 0.06 = 0.44

C20. They are mutually exclusive, so there is no overlap of circles
Pr(B) = 1 - 0.3 - 0.25 = 0.45

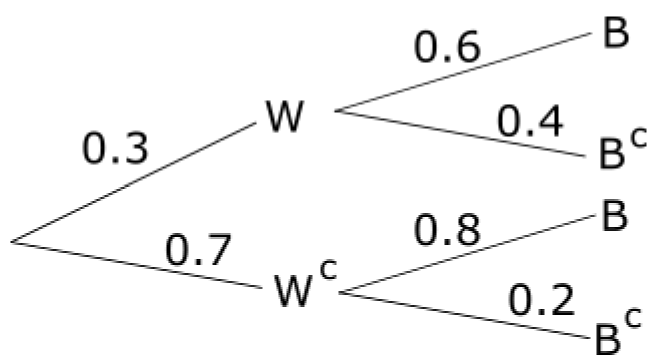
The answer is b).

C21.

Pr(spade drawn/not a diamond)=$\frac{\Pr\,(spade\ and\ not\ a\ diamond}{\Pr\,(not\ a\ diamond)} = \frac{\Pr\,(spade)}{3/4} = \frac{1/4}{3/4} =$ 1/3

Or reduce your sample space and once you take a card that isn't a diamond, you have 39 cards left and 13 of them are spades, so we have 13/29 = 1/3

C22. Pr(B)= 0.3(0.6) + (0.70)(0.80)= 0.18 + 0.56 = 0.74



C23.
Pr(W^C/B)=$\frac{\Pr(W^c \cap B)}{\Pr(B)} = \frac{0.7(0.8)}{0.74} = \frac{56}{74} = \frac{28}{37}$

C24.
Pr(E/F)=$\frac{\Pr\,(E \cap F)}{\Pr\,(F)}$

Pr$(E \cap F)$ =Pr(E/F) Pr(F)
$= \frac{1}{5} \times \frac{1}{4} = \frac{1}{20}$

The answer is b).

C25. If $\Pr(A \cup B) = 0.85\ and\ \Pr(A) = 0.5\ and\ \Pr(B^C) = 0.6,\ find\ \Pr(A \cap B)$.
Pr(B)=1 - 0.60= 0.40

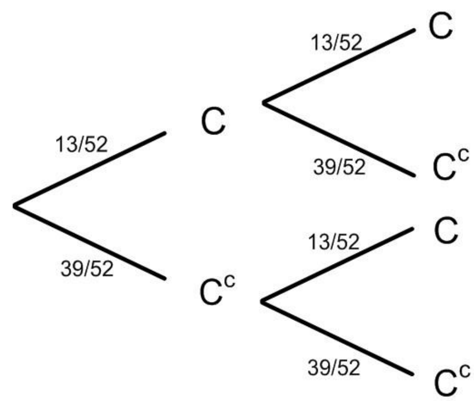
$PrA \cup B) = \Pr(A) + \Pr(B) - \Pr\,(A \cap B)$

0.85 = 0.50 + 0.40 - Pr$(A \cap B)$

Pr(A$\cap B) = 0.05$

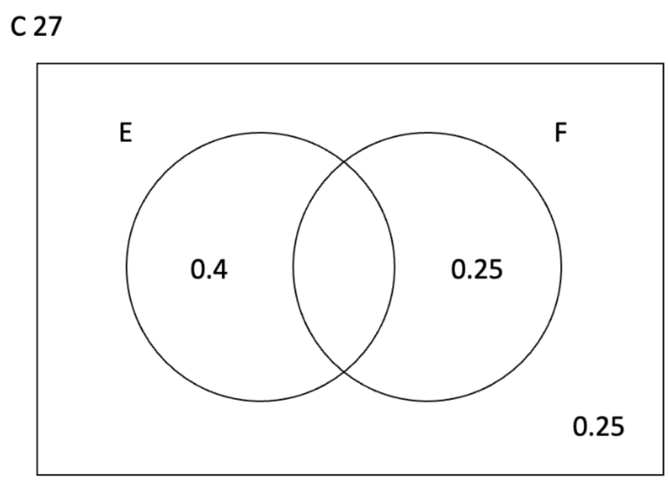
C26. Pr(2nd diamond/1st diamond)=$\frac{\Pr(both\ diamonds)}{\Pr(1st\ card\ diamond)} = \frac{\frac{13}{52}\left(\frac{13}{52}\right)}{\frac{13}{52}\left(\frac{13}{52}\right)+\frac{13}{52}\left(\frac{39}{52}\right)}=\frac{\frac{1}{16}}{\frac{4}{16}} = \frac{1}{4}$

or after you take the first card out, you replace it...so there are still 52 cards and still 13 diamonds...prob. of getting another diamond is just 13/52 = 1/4



C27. a) $\Pr(E \cap F)$

Draw a Venn diagram.



$\Pr(E \cap F) = 1 - 0.4 - 0.25 - 0.25$
$= 0.10$

b) Pr(E/F^C)=$\frac{\Pr(E \cap F^C)}{\Pr(F^C)} = \frac{0.40}{0.65} = \frac{40}{65} = \frac{8}{13}$

c) Pr(F/E)=$\frac{\Pr(F \cap E)}{\Pr(E)} = \frac{0.1}{0.5} = \frac{1}{5}$

C28. A and B are independent, so Pr(A and B)=Pr(A)xPr(B)

Pr(B/A)=$\frac{\Pr(A \cap B)}{\Pr(A)} = \frac{\Pr(A) \times \Pr(B)}{\Pr(A)} = \Pr(B) = 1/6$

So, Pr(B)=1/6

C29. Pr(A U B)=Pr(A) + Pr(B) - Pr(A and B)
0.50=0.2 + 0.4 - Pr(A and B)
Pr(A and B)=0.1
Pr(B/A)=$\frac{\Pr(A \cap B)}{\Pr(A)} = \frac{0.1}{0.2} = \frac{1}{2}$

C30. Brutal Question...find the probability that Leah and Jena are across from one another in a circle of 6 people, if you know they are not side-by-side

This is a conditional probability...
Pr(across/not side by side)= Pr(Across and not side by side)/Pr(not side by side)
=Pr(across)/1-Pr(side-by-side)
=$\frac{\frac{4!}{5!}}{1-\frac{4!2!}{5!}}$=$\frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{5} \times \frac{5}{3} = \frac{1}{3}$

To find the probability they are across, place Lindsay and then there is only one space across to place Jessica, so there are 4 people left to arrange. The denominator is the total ways to arrange 6 people in a circle, which is 5!
The Pr(side by side)= put them together to start your circle, so there are 4 people left to arrange and then you multiply by 2! for LJ and JL to switch.

$$\frac{4!\,2!}{5!}$$

C31. Let E=sum is odd = 1/2

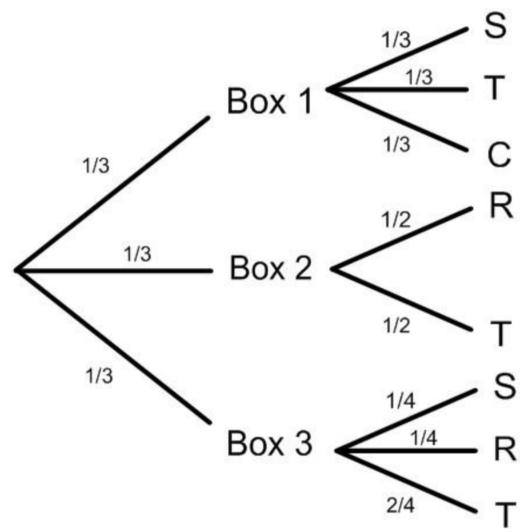
F=5 on first toss = 6/36 = 1/6
G=sum greater than 8 = sum 9,10, 11 or 12=
P(3,6)(6,3)(4,5)(5,4)(5,5)(4,6)(6,4)(5,6)(6,5)(6,6)}=10/36 = 5/18

$\Pr(E \text{ and } G) = \frac{6}{36} = \frac{1}{6}$ ({(3,6)(6,3)(4,5)(5,4)(5,6)(6,5)}

$\Pr(E) = \frac{1}{2}$ *and* $\Pr(G) = \frac{5}{18}$
So, $\Pr(E) \times \Pr(G) = \frac{1}{2} \times \frac{5}{18} = \frac{5}{36}$
So, $\Pr(E \text{ and } G) \neq \Pr(E) \times \Pr(G)$ and E and G are DEPENDENT

D. Probability Trees and Conditional Probability (2.4)

Example 1.



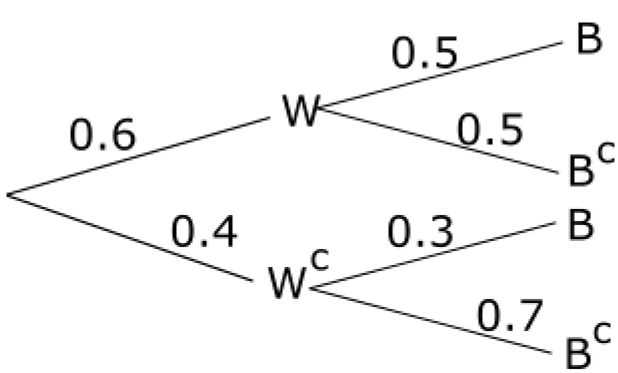
a) $\Pr(T) = Tr(Box\ 1\ and\ T) +$
$\Pr(Box\ 2\ and\ T) + \Pr(Box\ 3\ and\ T)$
$= \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{2}{4}$
$= \frac{1}{9} + \frac{1}{6} + \frac{1}{6} = \frac{4}{36} + \frac{6}{36} + \frac{6}{36} = \frac{16}{36} = \frac{8}{18} = \frac{4}{9}$

b) $\Pr(2nd|T) = \frac{\Pr(2nd \cap T)}{\Pr(T)} = \frac{1/3 \times 1/2}{4/9} = \frac{1/6}{4/9} =$
$\frac{1}{6}\left(\frac{9}{4}\right) = \frac{9}{24} = \frac{3}{8}$

Example 2.

a) $\Pr(B) = \Pr(W \cap B) + \Pr(W^C \cap B)$
$= 0.60 \times 0.50 + 0.40 \times 0.30$
$= 0.30 + 0.12$
$= 0.42$

$b)\ \Pr(M|B) = \frac{\Pr\ (M \cap B)}{\Pr\ (B)} = \frac{0.40 \times 0.30}{0.42} = \frac{0.12}{0.42} = \frac{12}{42} = \frac{2}{7}$



Example 3.

$\Pr(k) + \Pr(k^c k) + \Pr(k^c k^c k) + \Pr(k^c k^c k^c k)\ OR$

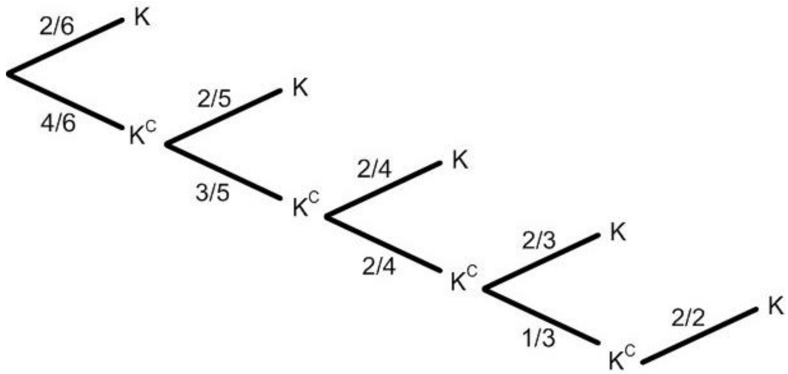
$1 - \Pr(k^c k^c k^c k^c k)$

$= 1 - \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} \times 1$

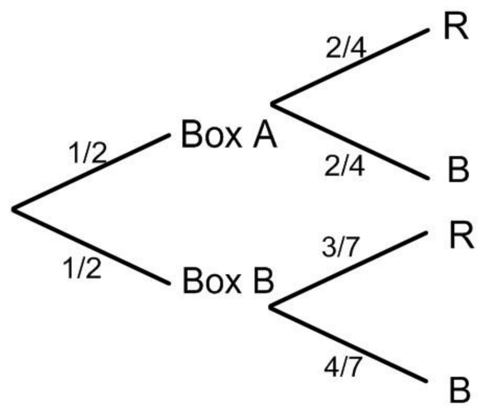
$= 1 - \frac{4}{5 \times 4 \times 3}$

$= 1 - \frac{1}{15}$

$= \frac{14}{15}$



Example 4.



a)

$\Pr(R) = \Pr(Box\ A\ and\ R)$
$+ \Pr(Box\ B\ and\ R)$

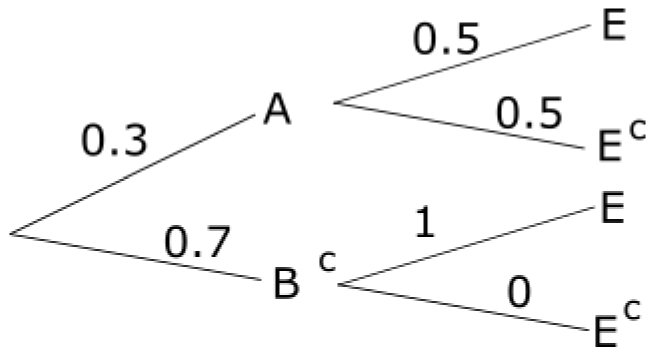
$= \frac{1}{2} \times \frac{2}{4} + \frac{1}{2} \times \frac{3}{7}$

$= \frac{1}{4} + \frac{3}{14}$

b)

$$\Pr(BoxA|B) = \frac{\Pr\ (BoxAand\ B)}{\Pr\ (B)} = \frac{1/2\left(\frac{2}{4}\right)}{1/2\,(2/4) + 1/2\,(4/7)} = \frac{1/4}{1/4 + 4/14}$$

E. Bayes' Theorem (2.5)



<u>Example 1.</u>

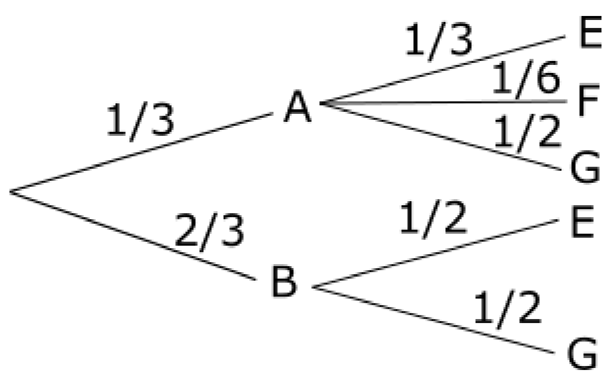
a)
$\Pr(E^c) = \Pr(A \cap E^c) + \Pr(B \cap E^c)$
$= 0.3 \times 0.5 + 0.7 \times 0$
$= 0.15$

b)
$\Pr(E|A) = 0.5$

c) $\Pr(E \cup A^c) = \Pr(A^c \cup E) = 0.7 + 0.3(0.5) = 0.85$

d)$\Pr(A|E) = \frac{\Pr(E|A)\Pr(A)}{\Pr(E)}$

$$= \frac{0.5(0.3)}{0.3(0.5) + 0.7(1)}$$
$$= \frac{0.15}{0.15 + 0.70} = \frac{0.15}{0.85} = \frac{15}{85} = \frac{3}{17}$$



<u>Example 2</u>.

a) $Pr(A \cap G) = \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6}$

b) $\Pr(A \cup G) = \frac{1}{3} + \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

c) $\Pr(E|A) = \frac{1}{3}$

d) $\Pr(A|E) = \frac{\Pr(E|A)\Pr(A)}{\Pr(E)} = \frac{1/3\left(\frac{1}{3}\right)}{\frac{1}{3}(1/3)+\frac{2}{3}(1/2)} = \frac{\frac{1}{9}}{\frac{1}{9}+\frac{1}{3}} = \frac{\frac{1}{9}}{\frac{4}{9}} = \frac{1}{9} \times \frac{9}{4} = \frac{1}{4}$

e) $\Pr(A \cap G^C) = \Pr(A \cap E) + \Pr(A \cap F)$

$$= \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{1}{6}\right) = \frac{1}{9} + \frac{1}{18} = \frac{2}{18} + \frac{1}{18} = \frac{3}{18} = \frac{1}{6}$$

Example 3.

a) $\Pr(D/G) = \frac{\Pr(G/D)\Pr(D)}{\Pr(G)} = \frac{\frac{3}{5}\left(\frac{1}{3}\right)}{\frac{1}{3}\left(\frac{3}{5}\right)+\frac{2}{3}\left(\frac{1}{4}\right)} = \frac{\frac{1}{5}}{\frac{1}{5}+\frac{1}{6}}$

b) $\Pr(D^C/G^C)=3/4$

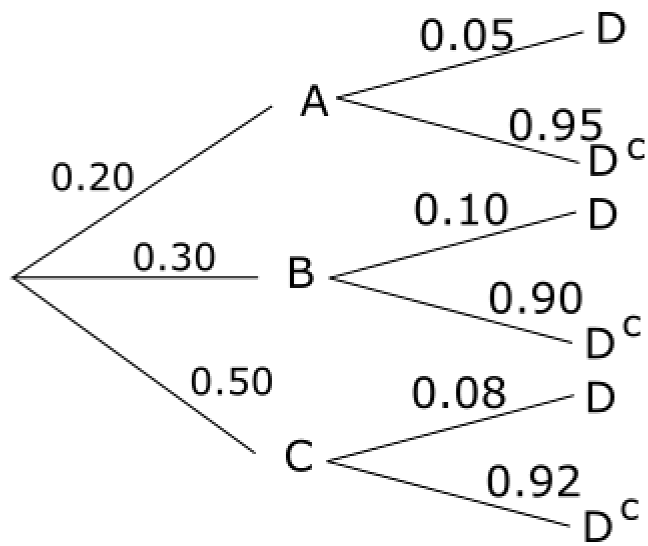
c) $\Pr(D \cup G^C) = \frac{1}{3} + \frac{3}{4}\left(\frac{2}{3}\right) = \frac{1}{3} + \frac{6}{12} = \frac{4}{12} + \frac{6}{12} = \frac{10}{12} = \frac{5}{6}$

d) $\Pr(F \cup D^C) = \frac{1}{3}\left(\frac{3}{5}\right)(1) + \frac{2}{3} = \frac{1}{5} + \frac{2}{3} = \frac{3}{15} + \frac{10}{15} = \frac{13}{15}$

Example 4:

$$\Pr(A|D) = \frac{\Pr(D|A)\Pr(A)}{\Pr(D)}$$

$$= \frac{(0.05)(0.20)}{0.2(0.05) + 0.3(0.10) + 0.5(0.08)}$$



Example 5:

a)

$\Pr(G) = \Pr(B \cap G) + \Pr(P \cap G)$

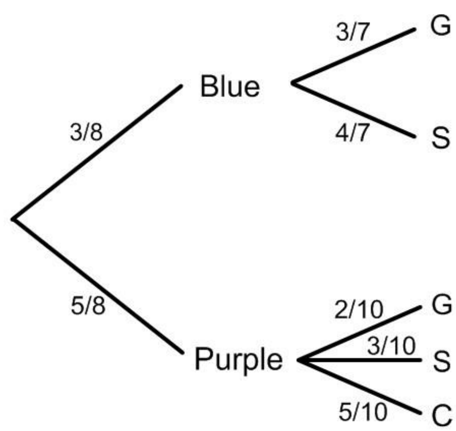
$= \frac{3}{8} \times \frac{3}{7} + \frac{5}{8} \times \frac{2}{10}$

$= \frac{9}{56} + \frac{10}{80}$

b)

$\Pr(P|S) = \frac{\Pr(S|P)\Pr(P)}{\Pr(S)}$

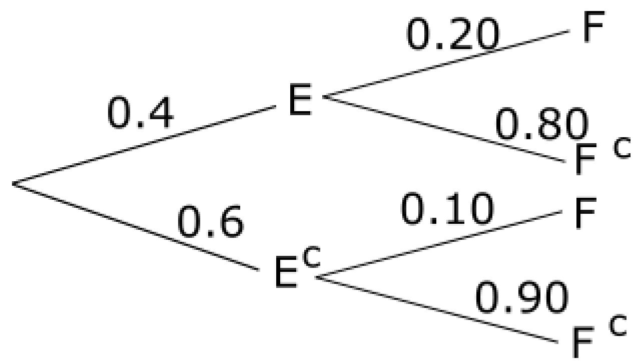
$= \frac{{}^{3}/_{10}\left({}^{5}/_{8}\right)}{{}^{3}/_{8}\left({}^{4}/_{7}\right) + {}^{5}/_{8}\left({}^{3}/_{10}\right)} = \frac{{}^{15}/_{80}}{{}^{12}/_{56} + {}^{15}/_{80}}$



Practice Exam Questions on Conditional Probability/Bayes Theorem

E1.

a) Pr(F/E)=0.20



Pr(F/E^C)=0.10
Pr(E)=0.40

Find Pr($E \cap F$)

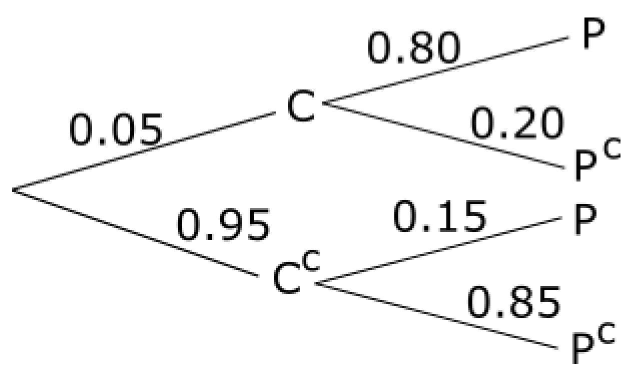
Pr ($E \cap F$)=(0.40)(0.20)=0.08

The answer is C.

b) Pr(E/F)=$\frac{\Pr(E\cap F)}{\Pr(F)}$

$=\frac{0.08}{0.4(0.2)+0.6(0.10)} = \frac{0.08}{0.08+0.06} = \frac{0.08}{0.14} = \frac{8}{14} = \frac{4}{7}$

E2.



Draw a Tree diagram

Pr(C/P)=$\frac{\Pr(C\cap P)}{\Pr(P)} = \frac{0.05(0.80)}{0.05(0.80)+0.95(0.15)}$

E3.

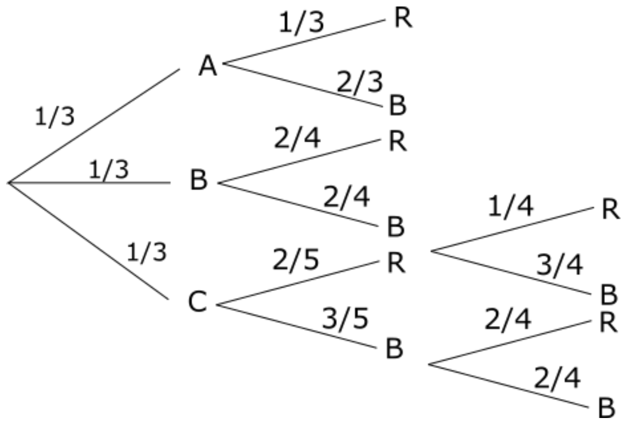
Pr(red) = Pr(red from Box A) + Pr(red from Box B) + Pr(red from Box C)

$=\frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{2}{4}\right) + \frac{1}{3}\left(\frac{2}{5}\right) = \frac{1}{9} + \frac{1}{6} + \frac{2}{15}$

E4.

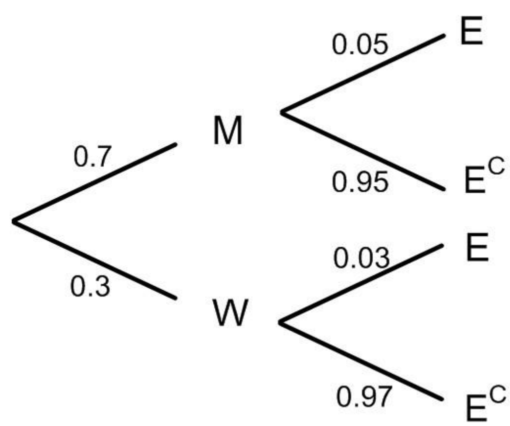
Pr(C and 2nd red) = Pr(C and BR) + Pr(C and RR)

$=\frac{1}{3}\times\frac{3}{5}\times\frac{2}{4}+\frac{1}{3}\times\frac{2}{5}\times\frac{1}{4}=\frac{1}{10}+\frac{1}{30}=\frac{3}{30}+\frac{1}{30}=\frac{4}{30}=\frac{2}{15}$



E5.

Let E=earn more than $30000

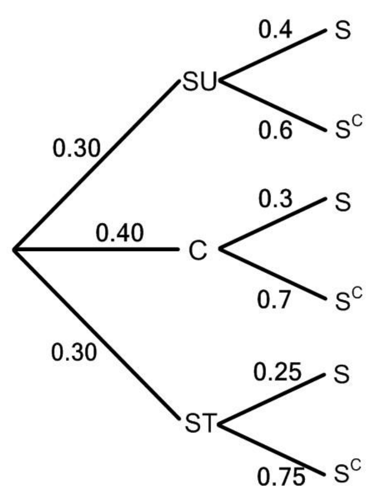


$\Pr(M/E)=\frac{\Pr(M\cap E)}{\Pr(E)}=\frac{0.70(0.05)}{0.3(0.03)+0.70(0.05)}$

E6.

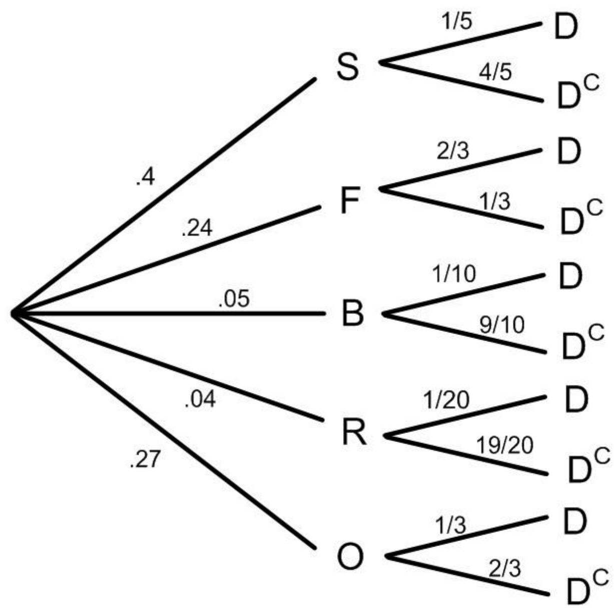
$\Pr(SU/S)=\frac{\Pr(SU\cap S)}{\Pr(S)}=\frac{0.30(0.40)}{0.3(0.25)+0.4(0.3)+0.3(0.4)}$

or use Bayes $\Pr(ST/S)=\frac{\Pr(S/SU)Pr(SU)}{\Pr(S)}$=same answer



E7.

a) Had a fall. (Do NOT simplify)



$$\Pr(F/D)=\frac{\Pr(D/F)Pr(F)}{\Pr(D)}=\frac{0.24(\frac{2}{3})}{0.4\left(\frac{1}{5}\right)+0.24\left(\frac{2}{3}\right)+0.05\left(\frac{1}{10}\right)+0.04\left(\frac{1}{20}\right)+0.27(\frac{1}{3})}$$

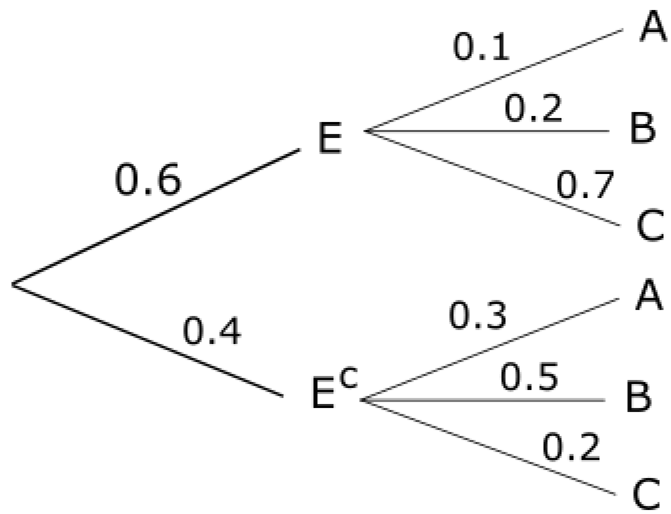
b) Stung by a bee. (Do NOT simplify)

$$\Pr(B/D)=\frac{\Pr(D/B)\Pr(B)}{\Pr(D)}=\frac{0.05(\frac{1}{10})}{same\ as\ a)}$$

E8.

$$\Pr(\text{all 4 cheese/at least 1 did})=\frac{\Pr(all\ 4\ and\ at\ least\ 1)}{\Pr(at\ least\ 1)}=\frac{\Pr(all\ 4)}{1-\Pr(none\ did)}=\frac{0.6(0.6)(0.6)(0.60)}{1-(0.4)^4}$$

E9. Find Pr(A).



Pr(E and A) + Pr(not E and A)
=0.6(0.1) + (0.40)(0.30)
=0.06 + 0.12=0.18

E10. Find Pr(A/E).

Pr(A/E) = 0.1 directly from the tree diagram

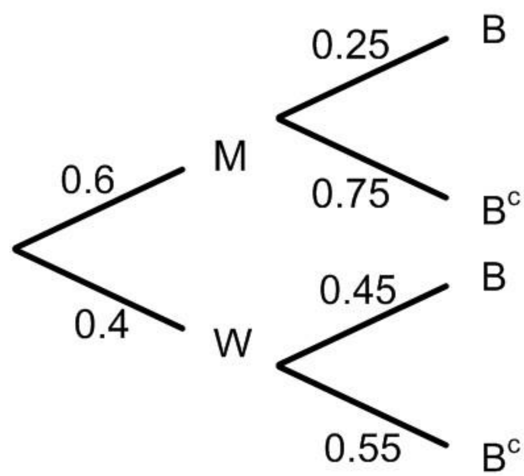
E11. Find Pr(E/A)
Use Pr(A) from E9.

Use Baye's Theorem

Pr(E/A)=$\frac{\Pr(A/E)Pr(E)}{\Pr(A)} = \frac{0.10(0.6)}{0.26} = \frac{0.06}{0.26} = \frac{6}{26} = \frac{3}{13}$

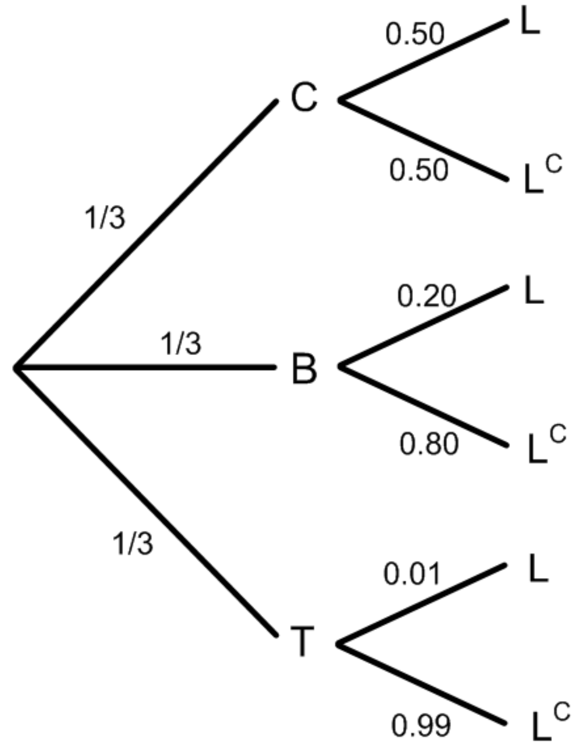
E12. Pr(EUB)= Pr(E) + Pr($E^C \cap B) = 0.6 + (0.40)(0.5) = 0.6 + 0.2 = 0.8$

E13.



$$\Pr(W|B) = \frac{\Pr(B|W)\Pr(W)}{\Pr(B)}$$
$$= \frac{0.45(0.4)}{0.6(0.25) + 0.4(0.45)}$$

E14.



$$\Pr(B|L) = \frac{\Pr(L|B)\Pr(B)}{\Pr(L)}$$

$$= \frac{0.2(1/3)}{1/3\,(0.5) + 1/3\,(0.2) + 1/3\,(0.01)}$$

E15.

a) $\frac{\binom{4}{2}}{\binom{13}{2}}$

b) 0 (no diamonds)

c) $\Pr(both\ clubs \mid 1st\ card\ club)$

$$= \frac{\Pr(both\ clubs + 1st\ card\ club)}{\Pr(1st\ card\ club)}$$

$$= \frac{\Pr(both\ clubs)}{\Pr(1st\ card\ club)} = \frac{\binom{6}{1}\binom{5}{1}/\binom{13}{2}}{\binom{6}{1}\binom{12}{1}/\binom{13}{2}} = \frac{6 \times 5}{6 \times 12} = \frac{30}{72} = \frac{5}{12}$$

or reduce sample space...1st card club and take it out...leaves 5 clubs, 4 hearts and 3 spades

Now, the prob. of getting a club = 5/12

E16.

a) 0.2

b) Pr(B)=Pr(R)Pr(B/R) + Pr(R^C)Pr(B/R^C)

=0.4(0.2)+0.6(0.3)

=0.08 + 0.18 = 0.26

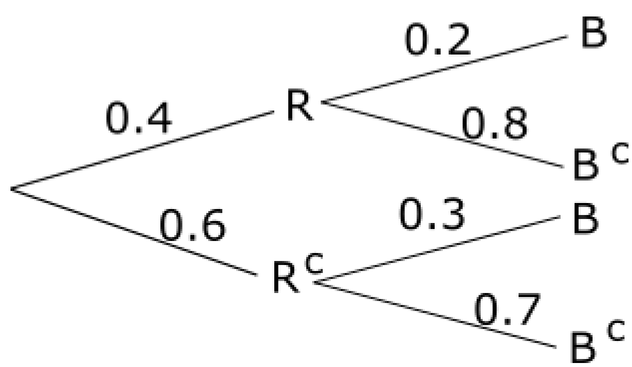
c) Pr(R and B) = 0.4(0.2) = 0.08

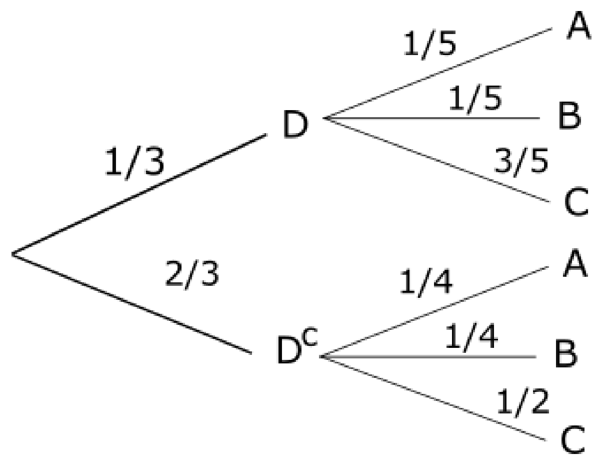
d)Pr(R/B) $= \frac{\Pr(B|R)\Pr(R)}{\Pr(B)}$

$$= \frac{0.2(0.4)}{0.26}$$

$$= \frac{0.08}{0.26}$$

$$= \frac{8}{26} = \frac{4}{13}$$





E17. a) $\Pr(D/B) = \frac{\Pr(B|D)\Pr(D)}{\Pr(B)}$

$$= \frac{\frac{1}{5}\left(\frac{1}{3}\right)}{\frac{1}{3}\left(\frac{1}{5}\right) + \frac{2}{3}\left(\frac{1}{4}\right)}$$

$$= \frac{\frac{1}{15}}{\frac{1}{15} + \frac{1}{6}} = \frac{\frac{2}{30}}{\frac{2}{30} + \frac{5}{30}} = \frac{2}{30}\left(\frac{30}{7}\right) = \frac{2}{7}$$

b) Pr(B/D)= $^1/_5$

c) Pr(A)= $\frac{1}{3}\left(\frac{1}{5}\right) + \frac{2}{3} \times \frac{1}{4} = \frac{1}{15} + \frac{1}{6} = \frac{2}{30} + \frac{5}{30} = \frac{7}{30}$

d) $\Pr(D \ or \ B) = \frac{1}{3} + \frac{2}{3} \times \frac{1}{4} = \frac{1}{3} + \frac{2}{12} = \frac{6}{12} = \frac{1}{2}$

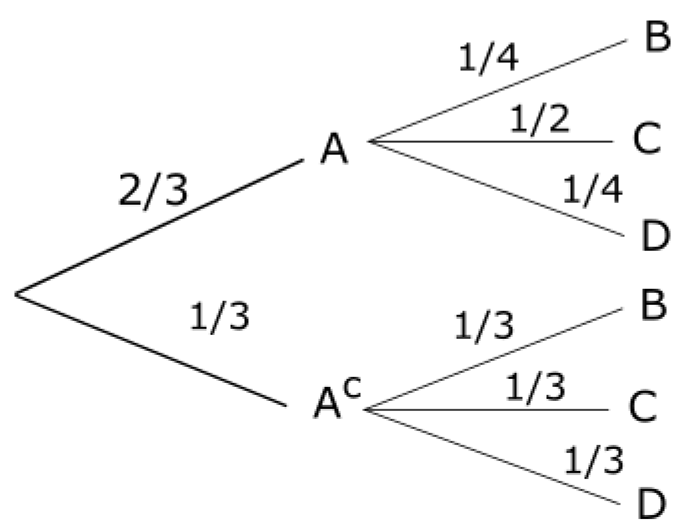
e) $\Pr(A \cup B) = in \ A \ or \ in \ B \ or \ in \ both = \frac{1}{3}\left(\frac{1}{5}\right) + \frac{1}{3}\left(\frac{1}{5}\right) + \frac{2}{3}\left(\frac{1}{4}\right) + \frac{2}{3}\left(\frac{1}{4}\right) = \frac{2}{15} + \frac{4}{12} = \frac{2}{15} + \frac{1}{3} = \frac{2}{15} + \frac{5}{15} = \frac{7}{15}$
(this is in D and A, in D and B, in D^C and in A, in D^C and in B)

E18.
a) Pr(D/A)=$^1/_4$
b) $\Pr(A/C) = \frac{\Pr(C|A)\Pr(A)}{\Pr(C)}$

$$= \frac{\frac{1}{2}\left(\frac{2}{3}\right)}{\frac{2}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{9}} = \frac{\frac{3}{9}}{\frac{4}{9}} = \frac{3}{9}\left(\frac{9}{4}\right) = \frac{3}{4}$$

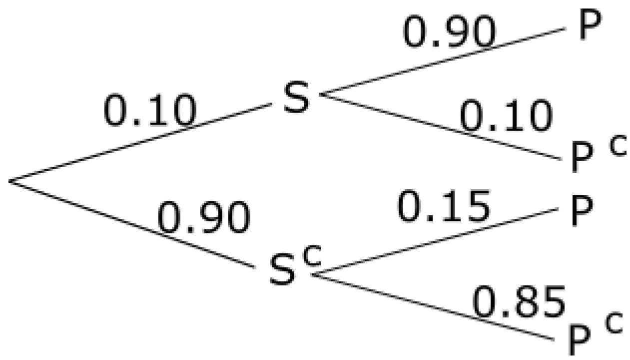
c) $\Pr(D \cap A^C) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

d) $\Pr(A \cap B^C) = \Pr(A \ and \ C) + \Pr(A \ and \ D)$
=2/3 (1/2) + 2/3 (1/4)
=1/3 +1/6 = 2/6 + 1/6 = 3/6 = ½



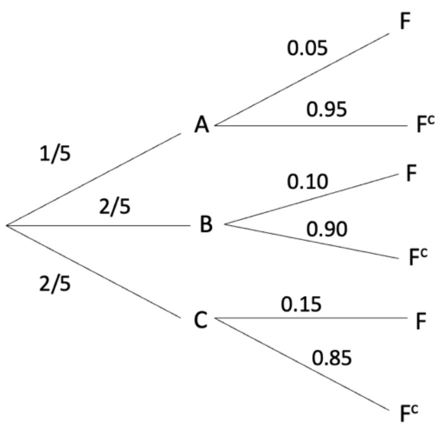
E19. P=tests positive

S= use steroids



$\Pr(S/P) = \frac{\Pr(P/S)Pr(S)}{\Pr(P)} = \frac{(0.90)(0.10)}{0.10(0.90)+0.90(0.15)}$

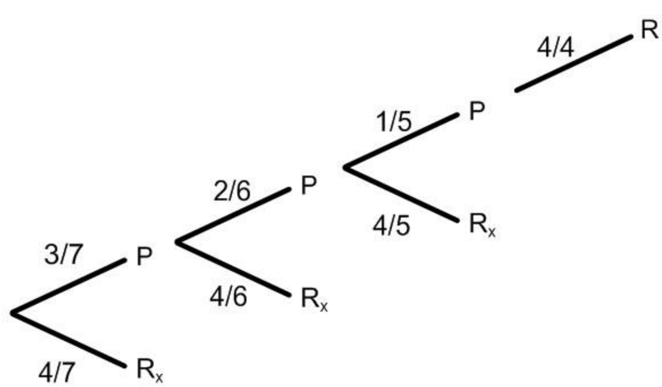
E20.



$\Pr(A/F^C) = \frac{\Pr(F^C/A)Pr(A)}{\Pr(F^C)} = \frac{(0.95)(\frac{1}{5})}{\frac{1}{5}(0.95)+\frac{2}{5}(0.9)+\left(\frac{2}{5}\right)(0.85)}$

E21.

$$\Pr(PPR) + \Pr(PR)$$
$$= \frac{3}{7} \times \frac{2}{6} \times \frac{4}{5} + \frac{3}{7} \times \frac{4}{6}$$
$$= \frac{4}{35} + \frac{12}{42}$$
$$= \frac{4}{35} + \frac{2}{7} = \frac{4}{35} + \frac{10}{35} = \frac{14}{35}$$



F. Bernoulli Trials...Independent Trials (2.6)

Example 1.

$\binom{50}{20}(0.5)^{20}(0.5)^{30}$=$\binom{50}{20}(0.5)^{50}$...multiplying with same base=add exponents

Example 2.

Pr(at least 1 head)=1-Pr(no heads)= $1-\binom{25}{0}(0.5)^{0}(0.5)^{25}=1-(0.5)^{25}$

Example 3

n=10

p=1/4 correct

q=3/4 incorrect

Pr(X=9)=$\binom{10}{9}\left(\frac{1}{4}\right)^{9}\left(\frac{3}{4}\right)^{1}$

Pr(at least 9 correct) = Pr(X=9) + Pr(X=10)

=$\binom{10}{9}\left(\frac{1}{4}\right)^{9}\left(\frac{3}{4}\right)^{1}+\binom{10}{10}\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^{0}$

=$\binom{10}{9}\left(\frac{1}{4}\right)^{9}\left(\frac{3}{4}\right)^{1}+\left(\frac{1}{4}\right)^{10}$

Example 4.

p=0.40 heads

q=0.6 tails

n=10

x= 4 tails...x and p must be the same thing...both heads! or you can do both tails!

Pr(4 tails)=Pr(6 heads)=Pr(X=4)=$\binom{10}{4}(0.6)^{4}(0.4)^{6}$ NOTE: $\binom{10}{4}=\binom{10}{6}=\frac{10!}{4!6!}$

Example 5.

n = 30

$$\binom{30}{5\quad 3\quad 2\quad 20}\left(1/6\right)^5\left(1/6\right)^3\left(1/6\right)^2\left(3/6\right)^{20}$$

Example 6.

p=0.02 defective

q=0.98 not defective

n=10

$\Pr(X \leq 8) = 0{,}1{,}2{,}3{,}4{,}5{,}6{,}7 \ or \ 8$

$= too\ difficult, so\ use\ 1 - \Pr(X = 9) - \Pr\,(X = 10)$

$$= 1 - \binom{10}{9}(0.02)^9(0.98)^1 - \binom{10}{10}(0.02)^{10}(0.98)^0$$

$$= 1 - \binom{10}{9}(0.02)^9(0.98)^1 - (0.02)^{10}$$

Example 7.

3 red, 3 purple and 4 blue...Pr(red)=3/10, Pr(purple)=3/10 and Pr(blue)=4/10 or 2/5

n=20

Pr(5 red and 5 purple...means of the 20 draws, 10 are red)

$$\binom{20}{5\quad 5\quad 10}\left(\frac{3}{10}\right)^5\left(\frac{3}{10}\right)^5\left(\frac{2}{5}\right)^{10}$$

Example 8.

n=10

$$\binom{10}{4\quad 4\quad 1\quad 1}\left(\frac{1}{4}\right)^4\left(\frac{1}{4}\right)^4\left(\frac{1}{4}\right)^1\left(\frac{1}{4}\right)^1 \ or \ \binom{10}{4\quad 4\quad 1\quad 1}\left(\frac{1}{4}\right)^{10}$$

Here, we want 4 hearts, 4 diamonds, 1 club and 1 spade and the probability of each is 1/4.

Example 9.
p=1/5=0.2 correct
q=0.8 incorrect
n=?

1-0.4= 0.6
at least 40% chance of getting at least one question correct=
less than 60% (or equal) chance of getting no questions correct

n	0.8^n
1	0.8
2	0.64
3	0.512

Therefore, to be at least 40% sure, the student would need to answer at least 3 questions.

Practice Exam Questions on Bernoulli Trials

F1.

a) p=0.8
q=0.2
one graduate = 4 that do graduate
Pr(1 graduate)= $\binom{5}{1}(0.80)^1(0.20)^4$ or $\binom{5}{4}(0.80)^1(0.20)$ (same answer)

b)Pr(3 not graduate) = Pr(2 do graduate)=
$\binom{5}{2}(0.8)^2(0.2)^3$ *or* $\binom{5}{3}(0.8)^2(0.2)^3$

F2.

n=10 p=4/5 q=1/5

$\Pr(X \geq 8) = \Pr(X = 8) + \Pr(X = 9) + \Pr\,(X = 10)$
$=\binom{10}{8}\left(\frac{4}{5}\right)^8\left(\frac{1}{5}\right)^2 + \binom{10}{9}\left(\frac{4}{5}\right)^9\left(\frac{1}{5}\right)^1 + \binom{10}{10}\left(\frac{4}{5}\right)^{10}\left(\frac{1}{5}\right)^0$

F3. a)

p=1/3
q=2/3
n=6
Pr(no hits)=$\binom{6}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^6=\left(\frac{2}{3}\right)^6$
b)

Pr(at least 1 hit)= 1 - Pr(no hits)
$=1-\left(\frac{2}{3}\right)^6$
c) Pr(X=2)= $\binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4$

F4.
n=10 Bernoulli
p=ace=4/52 = 1/13
q=12/13
Pr(exactly 1 ace) =$\binom{10}{1}\left(\frac{1}{13}\right)^1\left(\frac{12}{13}\right)^9$

The answer is b).

F5. Pr(at least 1 heart = 1 - Pr(no hearts)

= $1-\binom{10}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10}$
The answer is d).

F6. Bernoulli with n=10 p=4/52=1/13 four and q=48/52=12/13 (not a four)
Pr(at least one four)= 1 - Pr (no fours)
$1-\binom{10}{0}\left(\frac{1}{13}\right)^0\left(\frac{12}{13}\right)^{10}$=1 - $\left(\frac{12}{13}\right)^{10}$

F7. n=?

p=0.10 prize
q=0.90 lose

It is too difficult to figure out the probability directly to be at least 70% sure to win at least once because we could win once, twice, three times, etc...
1- 0.30= 0.70

We need less than or equal to 70% chance of not winning any prize at all.

Complete a chart with n and q^n and keep increasing n until the answer is less than 0.70

n	q^n=(0.90)^n
1	0.90
2	0.81
3	0.729
4	0.6561 $\leq$0.70

Therefore, since it took until n=4 to get equal or below 70%, she must play for 4 weeks in order to be at least 30% certain of winning a prize at least once.

F8.

a) n=10 questions, p=1/5 correct answer and q=4/5 incorrect answer
Pr(x=5 correct)=$\binom{10}{5}\left(\frac{1}{5}\right)^5\left(\frac{4}{5}\right)^5$

b) Pr(at least one of five)= 1 - Pr(none of five correct)
$=1-\binom{5}{0}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^5 = 1 - \left(\frac{4}{5}\right)^5$

F9.
n=?
p=0.20 prize
q=0.80 lose

It is too difficult to figure out the probability directly to be at least 50% sure to win at least once because we could win once, twice, three times, etc...

1- 0.55= 0.45

We need less than or equal to 45% chance of not winning any prize at all.

Complete a chart with n and q^n and keep increasing n until the answer is less than 0.45

n	q^n=(0.80)^n
1	0.80>0.45
2	0.64
3	0.512
4	0.4096≤ 0.45

Therefore, since it took until n=4 to get below 45%, she must buy 4 tickets in order to be at least 55% certain of winning a prize at least once.

F10.
a) she will get exactly 4 shots in

n = 12
p = 0.4
q = 0.6
x = 4

$$\Pr(x = 4) = \binom{12}{4}(0.4)^4(0.6)^8$$

b) she will get at least 2 shots in (means 2, 3, …., 10,11 or 12)

$$\Pr(x \geq 2) = 1 - \Pr(x = 0) - \Pr(x = 1)$$
$$= 1 - \binom{10}{0}(0.4)^0(0.6)^{12} - \binom{10}{1}(0.4)^1(0.6)^{11}$$

c) How many shots should he take on the net to be at least 80% certain of making at least one shot? from a) q=0.6 prob. of failure on any one shot

≤0.20 certain to make no shots

n	q^n **=0.6^n**
1	0.6
2	0.36
3	0.216
4	0.1296≤0.2

Therefore 4 shots on net

F11.
At least 80% sure means ≤ 0.20 certain of no bullseye

p = 0.30
q = 0.70

n	**0.7^n**
1	0.7
2	0.49
3	0.343
4	0.2401
5	0.16807 ≤ 0.20

$\therefore 5\ shots$

F12.
n = 30
p = 0.5
q = 0.5
x = 10

$$\Pr(x = 10) = \binom{30}{10}(0.5)^{10}(0.5)^{20}$$

F13.

n = 20

p = ½

q = ½

x = 12

$$\Pr(x = 12) = \binom{20}{12}\left(1/2\right)^{12}\left(1/2\right)^{8}$$

F14.

n = 80

$$\binom{80}{50 \quad 10 \quad 20}\left(1/2\right)^{50}\left(1/6\right)^{10}\left(1/3\right)^{20}$$

$$P_1 = odd = \frac{3}{6} = \frac{1}{2}$$

$$P_2 = get\ a\ 2 = \frac{1}{6}$$

$$P_3 = get\ a\ 4\ or\ a\ 6 = \frac{2}{6} = \frac{1}{3}$$

F15.

n=10

$$\binom{10}{4 \quad 3 \quad 3}\left(\frac{5}{15}\right)^{4}\left(\frac{4}{15}\right)^{3}\left(\frac{6}{15}\right)^{3}$$

F16. n=60

$$\binom{70}{20 \quad 10 \quad 25 \quad 15}\left(\frac{1}{2}\right)^{20}\left(\frac{1}{6}\right)^{10}\left(\frac{1}{6}\right)^{25}\left(\frac{1}{6}\right)^{15}$$

F17.

We have 6 red, 4 green, 3 orange and 8 purple...so 21 total

(a) no replacement= CHOOSE

$$\frac{\binom{6}{2}\binom{4}{2}\binom{3}{2}\binom{8}{2}}{\binom{21}{8}}$$

b) with replacement= BERNOULLI

$$\binom{8}{2\quad 2\quad 2\quad 2}\left(\frac{6}{21}\right)^2\left(\frac{4}{21}\right)^2\left(\frac{3}{21}\right)^2\left(\frac{8}{21}\right)^2$$

F18. n=10

a) p= heart = 1/4

q= not heart = 3/4

$$\Pr(X=6)=\binom{10}{6}\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^4$$

b) Pr(at least one ace)= 1 - Pr(no aces)

$$=1-\binom{10}{0}\left(\frac{4}{52}\right)^0\left(\frac{48}{52}\right)^{10}=1-\left(\frac{12}{13}\right)^{10}$$

c) $\binom{10}{1\quad 3\quad 2\quad 4}\left(\frac{4}{52}\right)^1\left(\frac{12}{52}\right)^3\left(\frac{4}{52}\right)^2\left(\frac{32}{52}\right)^4$

Here, we want one 10, 3 face cards, and two sixes.

G. Extra Practice Exam Questions (Ch. 2)

G1. Pr(2B's together)=# ways 2 B's together/total ways to arrange the word

Total ways to arrange the word "probability" is $\frac{11!}{2!2!}$

Total ways to have the 2B's together= BB _ __ __ __ __ __ __ __ __

=$\frac{10!}{2!}$

Therefore, the probability the 2B's are together is: $\frac{\frac{10!}{2!}}{\frac{11!}{2!2!}}$

G2. The answer is A). The rest either have overlap or they are incomplete.

G3. C) since there are only 5 green, you can't get all 6 balls green, so you must have at least 1 ball that is red or purple.

G4. If Pr(E)=0.30, Pr(F)=0.70 and $\Pr(E^C \cap F^C) = 0.2, find \Pr(E \cap F)$.
$\Pr(E \cup F) = 1 - 0.2 = 0.8$
$\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$
$0.8 = 0.3 + 0.7 - \Pr(E \cap F)$
$\Pr(E \cap F) = 0.2$

G5. If E and F are mutually exclusive, with Pr(E)=0.4 and Pr(F)=0.2, find $\Pr(E \cap F^C)$.
$\Pr(E \cap F^C) = \Pr(E)$ *since they are mutually exclusive*
$= 0.4$

G6. There are 5 red and 4 green balls in a box. Two balls are drawn out of the box at random, without replacement. What is the probability they are different colours?
$\Pr(RG) + \Pr(GR)$

$$= \frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{5}{8} = \frac{20 + 20}{72} = \frac{40}{72} = \frac{20}{36} = \frac{5}{9}$$

G7. The letters of the word “dentist” are arranged in a line at random. If the first 4 letters placed in the line spell out, in order, the word “dent”, what is the probability that when the line of letters is completed the letters spell out the word “dentist” in that order?

dent _ _ _ $\quad\quad$ $\Pr('\text{ist'}) = \frac{1}{3!} = \frac{1}{6}$

already spelled
out (1 way)

G8. Gail and Tina are sitting in a circle with 6 other people. What is the probability that if everyone chooses their seats randomly, that Tina and Gail do NOT sit beside one another?

$\Pr(not\ together) = 1 - \Pr\ (together)$ $\quad\quad$ Put Gail and Tina to start circle, 6 to arrange

$$= 1 - \frac{6!\,2!}{7!} = 1 - \frac{2}{7} = \frac{5}{7}$$

G9. To check independence...Pr(E and F)=Pr(E)xPr(F)
Or in this case look for the one where E involves a suit and F involves the denomination of a card or vice versa. If both involve a suit, like diamonds and hearts, they won’t be independent and if both involve the number on a card, like queens and “not aces” they won’t be independent. The answer is D).

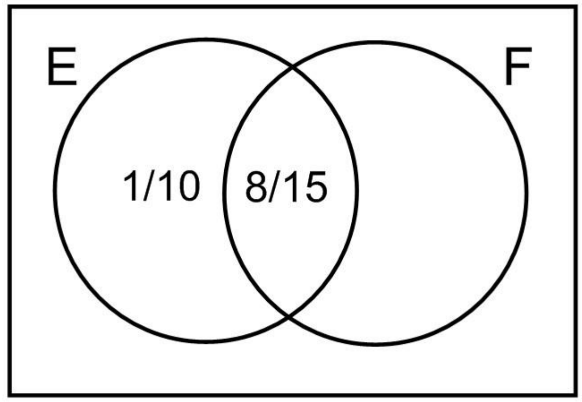
G10. If $\Pr(E^c) = \frac{1}{5}\, and\ \Pr(F) = \frac{1}{2}, and\ E\ and\ F\ are\ independent\ events, find\ \Pr(E \cup F)$.

$\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr\ (E \cap F)$
$\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E) \times \Pr(F)$ *since independent*

$$= \frac{4}{5} + \frac{1}{2} - \frac{4}{5} \times \frac{1}{2}$$

$$= \frac{8}{10} + \frac{5}{10} - \frac{4}{10} = \frac{9}{10}$$

G11. If Pr(E/F)=2/3 and Pr(E/F^C)=1/2 and Pr(F)=3/5, find Pr(E).



$\Pr(E/F) = \frac{\Pr(E \cap F)}{\Pr(F)}$

$2/3 = \frac{\Pr\,(E \cap F)}{3/5}$

$\Pr(E \cap F) = \frac{6}{15} = \frac{2}{5}$

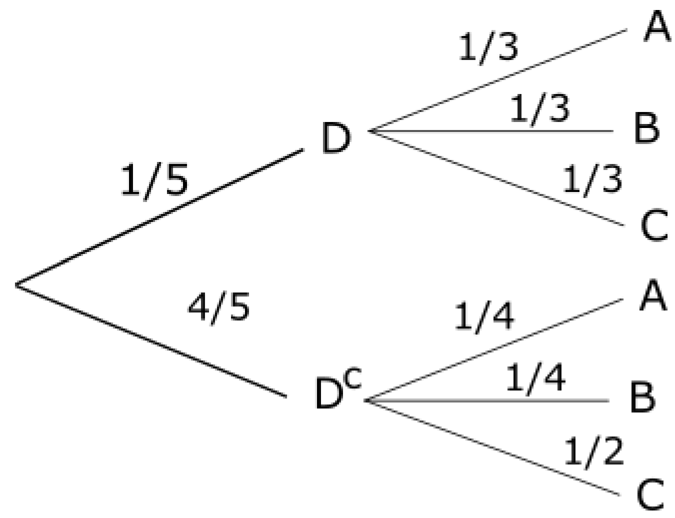
$\Pr(E/F^C) = \frac{\Pr(E \cap F^C)}{\Pr(F^C)}$

$1/2 = \frac{\Pr\,(E \cap F^C)}{2/5}$

$\Pr(E \cap F^C) = \frac{1}{5}$

Pr(E)= $\Pr(E \cap F) + \Pr(E \cap F^C) = \frac{2}{5} + \frac{1}{5} = \frac{3}{5}$

G12. Use the following diagram to answer the questions below:



a) Find Pr(A) = $\frac{1}{5} \times \frac{1}{3} + \frac{4}{5} \times \frac{1}{4} = \frac{1}{15} + \frac{1}{5} = \frac{1}{15} + \frac{3}{15} = \frac{4}{15}$

b) Find the probability that C occurs, given that event D occurs

$\Pr(C|D) = {}^{1}/_{3}$

c) Find $\Pr(C \cap D)$

$= \frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$

d) Find Pr (D^C/A)

$$= \frac{\Pr(A|D^C)\Pr(D^C)}{\Pr(A)} = \frac{{}^{1}/_{4}\left(\frac{4}{5}\right)}{\frac{4}{15}} = \frac{\frac{1}{5}}{{}^{4}/_{15}} = \frac{1}{5}\left(\frac{15}{4}\right) = \frac{3}{4}$$

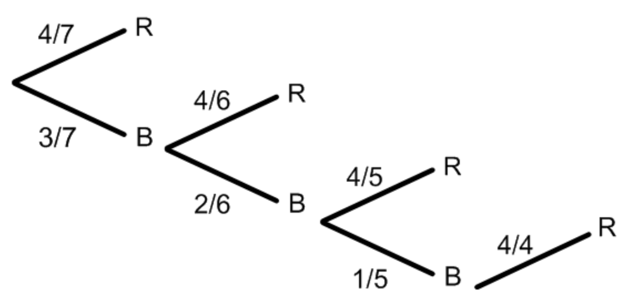
G13.

n = 10
p = 0.30 tail
q = 0.70 heads
x = 7

$\Pr(7 \text{ heads}) = \Pr(3 \text{ tails}) = \Pr(x = 3) = \binom{10}{3}(0.30)^3(0.70)^7$

or $\binom{10}{7}(0.30)^3(0.70)^7$ (same answer)

G14.



Pr $(three\ draws\ to\ get\ red)$
$= \Pr(BBR)$

$= \frac{3}{7}\left(\frac{2}{6}\right)\left(\frac{4}{5}\right) = \frac{4}{35}$

G15.

n = 10
p=sum 6 = 5/36 =
q = 31/36

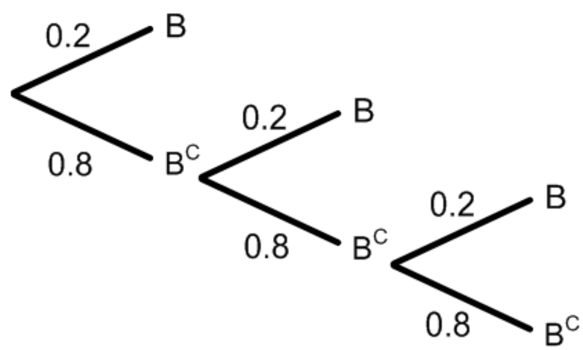
$$Pr(sum\ 6\ at\ least\ once) = 1 - Pr\ (no\ sum\ 6)$$
$$= 1 - \binom{10}{0}\left(\frac{5}{36}\right)^0\left(\frac{31}{36}\right)^{10}$$
$$= 1 - \left(\frac{31}{36}\right)^{10}$$

G16. $Pr(F, U\ Team\ A) + Pr(F, U\ Team\ B) + \ Pr\ (F, U\ Team\ C)$

$$= \frac{\begin{pmatrix} \square & 11 & \square \\ 1 & 5 & 5 \end{pmatrix}}{\begin{pmatrix} \square & 13 & \square \\ 3 & 5 & 5 \end{pmatrix}} + \frac{\begin{pmatrix} \square & 11 & \square \\ 3 & 3 & 5 \end{pmatrix}}{\begin{pmatrix} \square & 13 & \square \\ 3 & 5 & 5 \end{pmatrix}} + \frac{\begin{pmatrix} \square & 11 & \square \\ 3 & 5 & 3 \end{pmatrix}}{\begin{pmatrix} \square & 13 & \square \\ 3 & 5 & 5 \end{pmatrix}}$$

Distinguishable teams, so we don't divide by 2!

G17. The probability that _________ will see a bear....



$Pr\ (no\ more\ than\ 3\ visits)$

$= Pr(1\ visit) + Pr(2\ visits)$
$\quad + \ Pr\ (3\ visits)$

$= 0.2 + \ 0.8(0.2) + \ (0.8)(0.8)(0.2)$

$= 0.2 + \ 0.16 + 0.128 = 0.488$

G18.

n = 8
P_1 = 0.01 (win $15000)
P_2 = 0.05 (win $500)
P_3 = 0.10 (win ticket)
P_4 = 1 – 0.16 = 0.84 (not win)

$$\binom{8}{1\quad 1\quad 1\quad 5}(0.01)^1(0.05)^1(0.10)^1(0.84)^5$$

G19. p=0.20 to win a prize and q=0.80

a) Pr(win at least one prize)= 1 - Pr(win no prize)=1 $- \binom{7}{0}(0.20)^0(0.80)^7 =$ $1-(0.80)^7$

b) at least 60% sure of winning at least one prize= 1-0.60 or less than 0.40 sure to win NO prize

n	0.80^n
1	0.80
2	0.64
3	0.512
4	0.4096
5	0.32768 <0.40

So, you have to be 5 coffees to be at least 60% sure to win at least one prize

G20.

Pr(Jess and Sam together) =$\frac{4!2!}{5!} = \frac{2}{5}$

5! is on the denominator because in a circle there are (n-1)! ways to arrange the people

After we place Jess and Sam together on the circle to start the circle, there are 4 people left to arrange in 4! ways. We multiply by 2! because JS and SJ can switch places.

G21.

Event	Prob
A	8x
B	4x
C	X

Since probabilities add to 1, 8x+4x+x=1

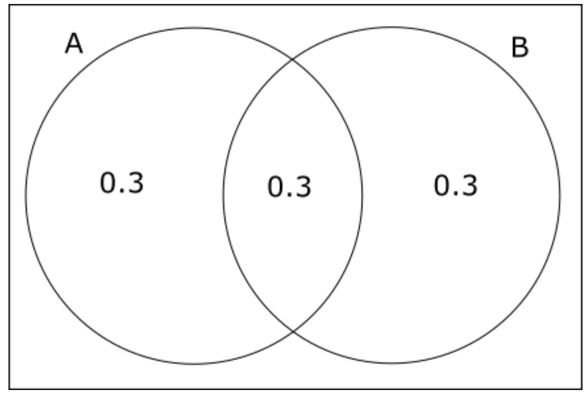
X=1/13

Pr(A)=8/13

G22.

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

0.45=0.5+0.2-Pr(A and B)
$\Pr(A \cap B) = 0.25$



G23. $\Pr(A \cup B) = 1 - 0.15 = 0.85$
$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
$0.9 = 0.60 + \Pr(B) - 0.30$
$\Pr(\mathrm{B}) = 0.60$

$\Pr(B \cap A^C) = 0.6 - 0.3 = 0.3$

G24.
Pr(sum 7) ={ Pr((1,6(6,1)(2,5)(5,2)(3,4)(4,3)}= 6/36=1/6

G25.

Pr(sum greater than 9/first roll was a "5")=

$$=\frac{\Pr(\mathit{sum\ greater\ than}\ 9\ \mathit{and\ first\ roll}\ 5)}{\Pr(\mathit{first\ roll}\ 5)} = \frac{\Pr\{(5,5)(5,6)\}}{\Pr\{(5,1)(5,2)(5,3)(5,4)(5,5,)(5,6)\}}$$

$$= \frac{2/36}{6/36} = \frac{1}{3}$$

OR Reduce your sample space and say that since the first die was a 5, we only look at the cases that start with a 5.ie...{ (5,1)(5,2)(5,3)(5,4)(5,5,)(5,6)Now, we see how many of these have a sum greater than 9...2 of these 6...so, 2/6=1/3

G26. $\Pr(M \cap C) = \Pr(M) \times \Pr(C) = 0.7 \times 0.8 = 0.56$ pass both

Pr(fails both) = 0.3 x 0.2 = 0.06

G27. A box contains 4 red balls, 2 green balls and 4 yellow balls. Three balls are drawn without replacement. Find:

a) The probability of getting all three yellow

$$\frac{3}{10}\times\frac{2}{9}\times\frac{1}{8}=\frac{6}{720}=\frac{1}{120}$$

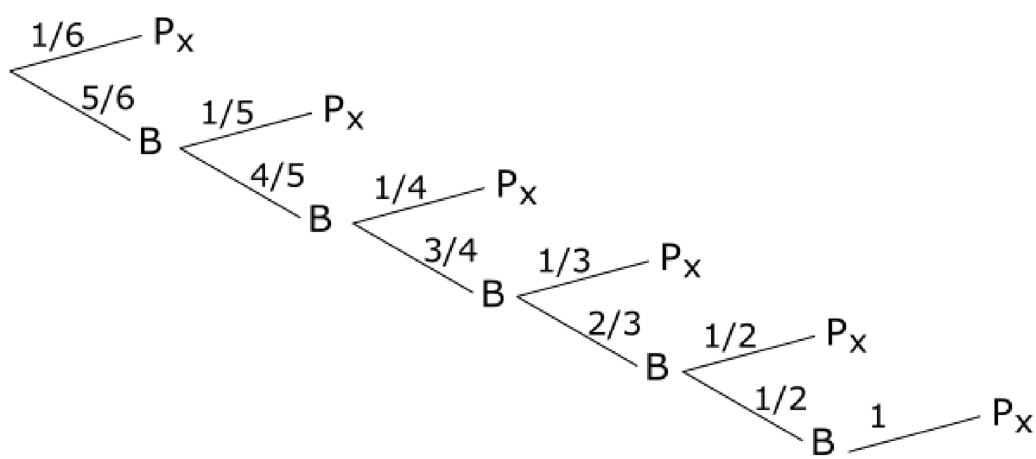
b) Pr(2 red and 1 yellow)=Pr(RRY) + Pr(RYR) + Pr(YRR)

=(4/10)(3/9)(4/8)+(4/10)(4/9)(3/8)+(4/10)(4/9)(3/8)=48/720 + 48/720 +48/720 = 144/720=1/5

c) Pr(RRR) + Pr(YYY) + Pr(GGG)

= (4/10)(3/9)(2/8) + (4/10)(3/9)(2/8) + 0 (can't get all G)
=24/720 + 24/720
=48/720=1/15

*G28.
Pr(P) + Pr(BP) + Pr(BBP)
=1/6 +(5/6)(1/5) + 5/6(4/5)(1/4) = 1/6 + 1/6 + 1/6 = 1/2
No more than 3 pens means she either got purple on the first try, or second try or third try. So, she got purple right away, or one blue and then a purple, or blue, blue and then purple.



G29. n=10

Pr(5 comes up at least once) = 1- Pr(no 5 comes up)
$=1-\binom{10}{0}\left(\frac{1}{6}\right)^{0}\left(\frac{5}{6}\right)^{10}=1-\left(\frac{5}{6}\right)^{10}$

G30. $\binom{10}{2\quad 3\quad 5}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{6}\right)^{3}\left(\frac{2}{6}\right)^{5}$

This is the more complicated form of the Bernoulli equation...Here, we have more than one "success" because we want an even number twice and a "5" three times.

n1=2 even=2,4,6
n2=3 get a 5
n3=10-2-3=5 rolls get a 1 or a 3
The numbers in the brackets are the probabilities of getting an even number, a "5" and a "1 or 3".

G31. $\Pr(A/B)=\frac{\Pr(A\cap B)}{\Pr(B)}=\frac{0.2}{0.3}=\frac{2}{3}$

G32. If Pr(E/F)=2/5 and Pr(F)=1/4, where E and F are independent events, what is Pr(E)?
Recall, for independence, Pr(E and F)=Pr(E)xPr(F)
So, the conditional formula becomes...

$\Pr(E/F)=\frac{\Pr(E)\times\Pr(F)}{\Pr(F)}$ **Here the Pr(F) cancels
$\Pr(E)=\frac{2}{5}$

G33. If E and F are mutually exclusive events with Pr(E)=0.35 and Pr(F)=0.2, find Pr(E/F).

They are mutually exclusive, so Pr(E/F)= $\frac{\Pr(E\cap F)}{\Pr(F)}=\frac{0}{0.2}=0$
So, Pr(E/F)=0

Post-Test Two Material

A. Discrete Random Variables (3.1)

Example 2

X	Pr(X)
2	$^1/_{36}$
3	$^2/_{36}$
4	$^3/_{36}$
5	$^4/_{36}$
6	$^5/_{36}$
7	$^6/_{36}$
8	$^5/_{36}$
9	$^4/_{36}$
10	$^3/_{36}$
11	$^2/_{36}$
12	$^1/_{36}$

a) Pr(X=4)=3/36=1/12 from table

b) Pr(X<4)=Pr(X=2) + Pr(X=3)=3/36=1/12

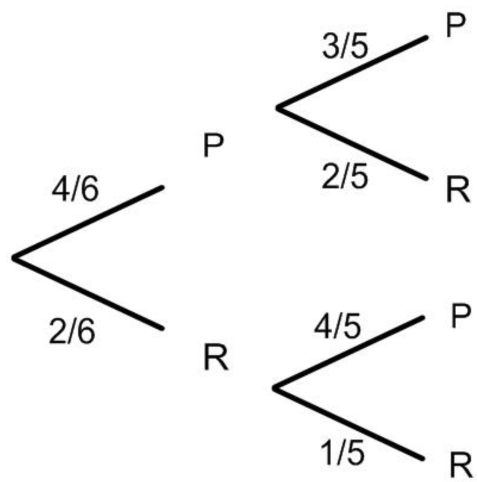
Example 3.

X	Pr[X=x]	Cumulative Distribution Function (c.d.f)
0	1/16	1/16
1	1/4=4/16	116+4/16=5/16
2	¼=4/16	5/16+4/16=9/16
3	6/16	9/16+6/16=15/16
4	1/16	1

Example 4.

X= number of red=0,1,2

X	Pr[X=x]
0	PP=4/6(3/5)=12/30=2/5
1	16/30=8/15
2	RR= 2/6(1/5)=2/30=1/15



Example 5.

a) Pr(Y=5)=F(5)-F(4)=0.9-0.55=0.35

b)Pr(Y<4)=Pr(Y≤ 3) =F(3)=0.45

c)Pr(Y=4)+Pr(Y=5)=(F(4) - F(3) +0.35=0.55-0.45+0.35=0.45

Example 6.

$\Pr(x \geq 7) = 1 - \Pr\,(x \leq 6)$
$= 1 - F(6)$
$= 1 - 0.80$
$= 0.20$

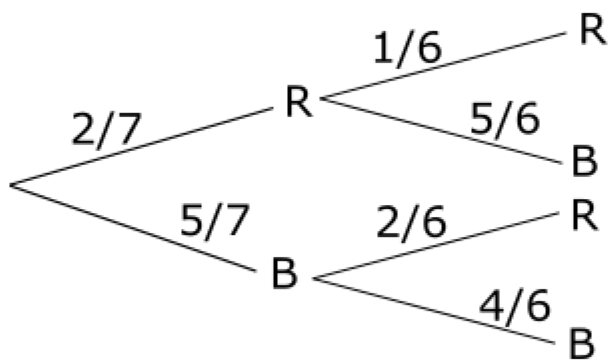
$\therefore$ The answer is $\boxed{C}$.

Pr(X>7)=$1 - \Pr\,(x \leq 7)$
$= 1 - F(6)$
$= 1 - 0.85$
$= 0.15$

Practice Exam Questions on Discrete Random Variables

A1. X=Number of Red Disks

$$\Pr(X = 1) = \Pr(1R) = \Pr(RB) + \Pr(BR)$$
$$= \frac{2}{7} \times \frac{5}{6} + \frac{5}{7} \times \frac{2}{6} = \frac{20}{42} = \frac{10}{21}$$



A2. Given the following probability distribution function, find Pr(X=2).

X	Pr(X)
1	0.1
2	0.2
3	0.4
4	0.2
5	0.1

$\Pr(X = 2) = 1 - 0.1 - 0.4 - 0.2 - 0.1 = 1 - 0.8 = 0.20$

Prob(X at most 3) =F(3)=0.1+0.2+0.4=0.7

A3.See A2. Find F(2).

F(2) = 0.1+0.2=0.3

*A4. If X is a discrete random variable with Pr(X=0)=0.2, Pr(X=1)=0.1, Pr(X=2)=0.4 and Pr(X=3)=0.3 and E is the event that X has the value of 0 or 1, find Pr(E).

X	Pr(X)	F(X)
0	0.2	0.2
1	0.1	0.3
2	0.5	0.8
3	0.2	1

Pr(E)= Pr(X=0) + Pr(X=1)= 0.20 + 0.1 = 0.3

A5. See A4. Find F(2).
F(2)=0.80

*A6. A 6-card hand is dealt from a standard deck of cards. The discrete random variable X counts the number of diamonds in the hand.
a) Find n(S).

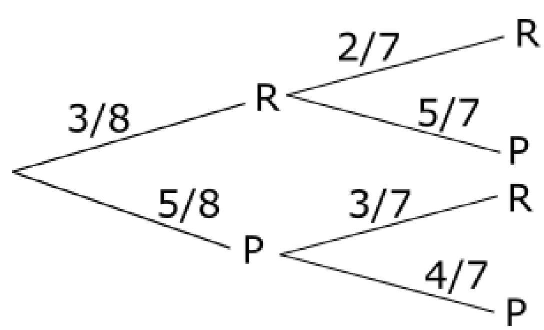
$n(S) = \binom{52}{6}$

b) Find the number of sample points in the event (X=2).
x=2 means we get 2 diamonds and four other cards that are NOT diamonds

n(E)=$\binom{13}{2}\binom{39}{4}$

A7. The discrete random variable X has the probability distribution function shown below:

X	Pr(X=x)
1	1/9
2	2/9
3	6/9



Find Pr(X<3).

=Pr(X=1) + Pr(X=2)=1/9 + 2/9= 3/9=1/3

A8. 3 red, 5 purple
X= number of purple=0,1,2...can get RR, PR, RP or PP

Pr(X=1)= Pr(get one purple)
=Pr(PR) + Pr(RP)
$=\frac{5}{8}\times\frac{3}{7}+\frac{3}{8}\times\frac{5}{7}$

$=\frac{30}{56}$

$=\frac{15}{28}$

A9.

Y	Pr(Y)	F(Y)
2		
3		0.40
4	0.15	0.55
5	0.30	0.85
6	0.15	1

* Remember the last entry in the cumulative column is 1, but in prob. the entries all add up to 1

a) Pr(Y=4)=F(4)-F(3)=0.55-0.40=0.15
b) Pr(Y=5)=F(5)-F(4)=0.85-0.55=0.30

Pr(Y<5)=Pr(Y$\leq$ 4) =F(4)=0.55

c) Pr(Y=4)+Pr(Y=5)+Pr(Y=6)=0.15+0.3+0.15=0.60

A10.

W	Pr(W)
1	$1/4$
3	$1/2$
-1	$1/4$

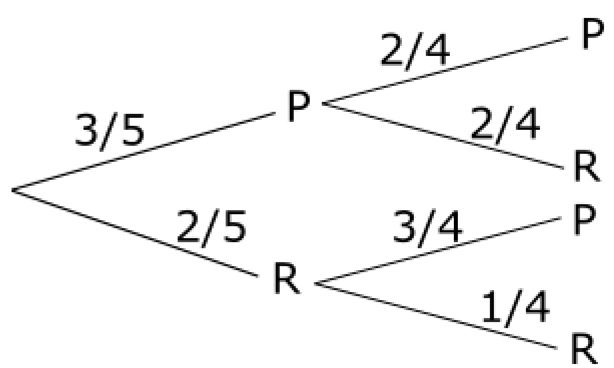
A11.

X	Pr[X=x]	Cumulative Distribution Function (c.d.f)
0	0.15	0.15
1	0.25	0.15+0.25=0.4
2	0.20	0.4+0.2=0.6
3	0.35	0.6+0.35=0.95
4	0.05	1

Pr(X<3)=Pr(X$\leq$ $2) = F(2) = 0.6$

A12.
X= number of purple=0,1,2

X	Pr[X=x]
0	RR= 2/5 x 1/4= 2/20=1/10
1	6/10
2	PP= 3/5x 2/4 = 6/20=3/10



A13. Find the cumulative distribution for the probability distribution function given below:

X	Pr(X)	F(X)
1	0.1	0.10
2	0.25	0.10+0.25=0.35
3	0.45	0.35+0.45=0.80
4	0.20	1

b) $\Pr(X < 4) = 0.1 + 0.25 + 0.45 = 0.8$

c) F(3)= 0.1+0.25+0.45= 0.8

A14. Pr(X=6)= F(6)-F(5)=0.60 - 0.40 = 0.20

A15.

X	Pr[X=x]	Cumulative Distribution Function (c.d.f)
0	1/16	1/16
1	3/8=6/16	1/16+6/16=7/16
2	4/16	7/16+4/16=11/16
3	4/16	11/16+4/16=15/16
4	1/16	1

A16.
Pr(X=6)= F(6)-F(5)=0.67 - 0.5 = 0.17

A17.
a) Pr(X=3)=F(3)-F(2)=0.75 - 0.35=0.40

b) Pr(2<X<5)=Pr(X=3)+Pr(X=4)
=F(3)-F(2) + F(4)-F(3)
=0.40 + 0.85-0.75 from a)
=0.50

B. The Mean and Standard Deviation (3.2)

Example 1. net winnings, so you take away the $1 you pay from your winnings

X	Pr[X=x]
2-1=1	3/6 (even #)
-2-1 = -3	2/6 (roll 1 or 3)
4-1 = 3	1/6 (roll a 5)

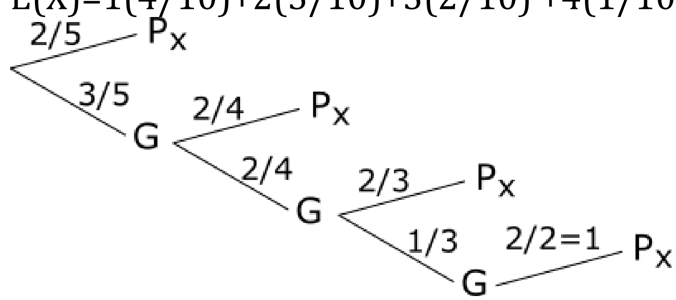
E(X)=1(3/6)+(-3)(2/6)+3(1/6)=$ 0 So, it is a fair game!

Example 2.

X= number of draws = 1, 2, 3 or 4 draws to get a purple

X	Pr[X=x]
1	P=2/5=4/10
2	GP= 3/5(2/4)=6/20=3/10
3	GGP= (3/5)(2/4)(2/3)=1/5= 2/10
4	GGGP= (3/5)(2/4)(1/3)(1)=2/ 20=1/10

E(X)=1(4/10)+2(3/10)+3(2/10) +4(1/10)=(4+6+6+4)/10=2



Example 3.

X	Pr(X)	X^2
0	1/3	0
1	1/3	1
2	1/3	4

Total = 1

$$E(X) = 0 \times \frac{1}{3} + 1 \times \frac{1}{3} + 2 \times \frac{1}{3} = \frac{3}{3} = 1$$

$$E(X^2) = 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) + 4\left(\frac{1}{3}\right) = \frac{5}{3}$$

$$V(X) = E(X^2) - \left(E(X)\right)^2 = \frac{5}{3} - (1)^2 = \frac{5}{3} - \frac{3}{3} = \frac{2}{3}$$

$$\sigma = \sqrt{\frac{2}{3}}$$

Example 4.

X	Pr(X)	X^2
-1	½=3/6	1
0	1/3=2/6	0
2	1/6	4

Total=1

$$E(X) = -1\left(\frac{3}{6}\right) + 0\left(\frac{2}{6}\right) + 2\left(\frac{1}{6}\right) = -\frac{1}{6}$$

$$E(X^2) = 1\left(\frac{3}{6}\right) + 0\left(\frac{2}{6}\right) + 4(\frac{1}{6}) = \frac{7}{6}$$

$$V(X) = \frac{7}{6} - \left(-\frac{1}{6}\right)^2 = \frac{7}{6} - \frac{1}{36} = \frac{42}{6} - \frac{1}{36} = \frac{41}{36}$$

$$\sigma = \sqrt{\frac{41}{36}} = \sqrt{41}/6$$

Example 5. Given E(X)=5, E(Y)=2 and V(X)=5, find each of the following:

a) E(2X-3Y)=2E(X) - 3 E(Y)= 2(5) -3(2) = 4
b) E(4X-1)=4E(X) - 1= 4(5) - 1 = 19
c) V(2X-3)=2^2V(X)= 4(5)= 20
d) V(-X+2)= $(-1)^2$V(X)=1(5)=5
$\sigma(-X + 2)$=$\sqrt{5}$

Example 6.

$Y=-2X+4$ $E(Y)=E(-2X+4)$ $3=-2E(X)+4$ $3-4=-2E(X)$ $-1=-2E(X)$ $E(X)=1/2$	$Y=-2X+4$ $V(Y)=V(-2X+4)$ $V(Y)=(-2)^2V(X)$ $V(Y)=4(5)=20$ $\sigma(Y)=\sqrt{20}$

Example 7.

$Y=3X-5$ $E(Y)=E(3X-5)$ $E(Y)=3E(X)-5$ $2=3E(X)-5$ $7=3E(X)$ $E(X)=\frac{7}{3}$	$Y=3X-5$ $V(Y)=V(3X-5)$ $V(Y)=3^2V(X)$ $10=9V(X)$ $V(X)=\frac{10}{9}$

Example 8.

$V(x)=E(x^2)-\left(E(x)\right)^2=10-2^2=10-4=6$
$\sigma(x)=\sqrt{6}$

Example 9.

X	Pr(X=x)
0	0
6-a	0.4
6+a	0.6

If E(X)=6.4, find the value of "a".
6.4=0(0)+(6-a)(0.4)+(6+a)(0.6)
6.4=2.4 – 0.4 a + 3.6 + 0.6a
6.4 – 2.4 – 3.6 = 0.2 a
0.4 =0.2a
a=2

Example 10. $\sigma(X)=9,\ so\ V(X)=81$
$V(X)=E(X^2)-\left(E(X)\right)^2$
81= $E(X^2)-(5)^2$
$E(X^2)=81+25=106$

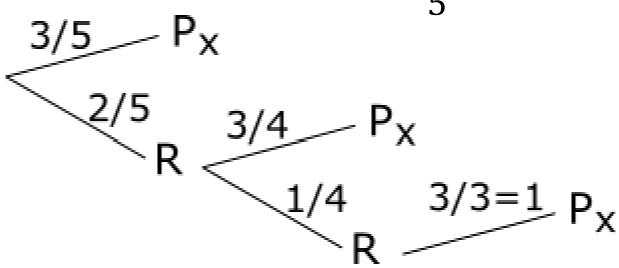
B. Practice Exam Questions on Mean and Standard Deviation

B1.

X	Pr(X)
1 {P}	$\frac{3}{5} = \frac{6}{10}$
2 {RP}	$\frac{2}{5} \times \frac{3}{4} = \frac{6}{20} = \frac{3}{10}$
3 {RRP}	$\frac{2}{5}\left(\frac{1}{4}\right)\left(\frac{3}{3}\right) = \frac{2}{20} = \frac{1}{10}$

$E(X) = 1\left(\frac{6}{10}\right) + 2\left(\frac{3}{10}\right) + 3\left(\frac{1}{10}\right) = \frac{6+6+3}{10} = \frac{15}{10} = 1.5$

$\Pr(X < 2) = \Pr(X = 1) = \frac{3}{5}$



B2.

$Y = -2X + 5$ $E(Y) = -2E(X) + 5$ $5 = -2E(X) + 5$ $0 = -2E(X)$ $E(X) = 0$	$Y = -2X + 5$ $V(Y) = (-2)^2 V(X)$ $V(Y) = 4(6)$ $V(Y) = 24$

B3.

$Y = 3X - 4$ $E(Y) = 3E(X) - 4$ $3 = 3E(X) - 4$ $7 = 3E(X)$ $E(X) = {}^{7}/_{3}$	$Y = 3X - 4$ $V(Y) = 3^2 V(X)$ $7 = 9V(X)$ $V(X) = \frac{7}{9}$ $\sigma(X) = \sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{3}$

B4.

$V(X) = E(X^2) - \left(E(X)\right)^2 = 15 - 2^2 = 15 - 4 = 11$

$\sigma(X) = \sqrt{11}$

B5.

X	Pr(X)
2	$^1/_{36}$
3	$^2/_{36}$
4	$^3/_{36}$
5	$^4/_{36}$
6	$^5/_{36}$
7	$^6/_{36}$
8	$^5/_{36}$
9	$^4/_{36}$
10	$^3/_{36}$
11	$^2/_{36}$
12	$^1/_{36}$

$E(X) = 2(^1/_{36}) + 3(^2/_{36}) + 4(^3/_{36}) + \cdots + 12(^1/_{36})$

$E(X) = \frac{252}{36} = 7$

Or $E(X+Y) = E(X) + E(Y)$ {X = # on 1st die and Y = # on 2nd die}

$= \frac{7}{2} + \frac{7}{2} = \frac{14}{2} = 7$ $(using\ chart\ from\ ex. 1\ p. 130\ \ booklet)$

B6.

X	Pr(X)	X^2
1	0.3	1
2	0.2	4
0	0.5	0

$\mu = 1(0.3) + 2(0.2) + 0(0.5) = 0.3 + 0.4 = 0.7$

B7.

X	Pr(X)	X^2
1	0.3	1
2	0.2	4
0	0.5	0

E(X)= $\mu = 1(0.3) + 2(0.2) + 0(0.5) = 0.3 + 0.4 = 0.7$
$E(X^2) = 0.3(1) + 4(0.2) + 0(0.5) = 0.3 + 0.8 = 1.1$

$V(X)=E(X^2) - (E(X))^2$
$=1.1 – 0.7^2$
=1.1-0.49
=0.61

B8.

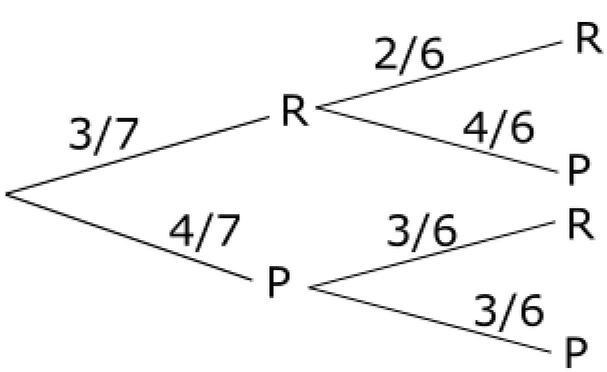
X	Pr(X)
0	3/7x2/6=6/42
1	3/7x4/6+4/7x3/6=24/42
2	4/7x3/6=12/42

X=0 means we got no purple, which means RR

X=1 means we got one purple, which could be PR or RP

X=2 means we got two purple, or PP

E(X)=0+1(24/42)+2(12/42)=48/42=8/7



*B9. $V(X)=7^2=49$

$V(X)=E(X^2)-(E(X))^2$
$49=E(X^2) - 3^2$
$E(X^2)=9+49=58$

B10. $V(-2X+1)=(-2)^2V(X)=4(10)=40$
$\sigma(-2X+1) = \sqrt{40}$

*B11. $V(-2X+5)=(-2)^2V(X)=4(5)=20$

B12. Given E(X)=2, E(Y)=4 and V(X)=5, find each of the following:

a) $E(2X-5Y)=2(2)-5(4)= 4-20=-16$
b) $E(-2X-1)= -2(2) - 1 = -5$
c) $V(-2X-5)=(-2)^2V(X)= 4(5)=20$
d) $V(-2X+1)\ )=(-2)^2V(X)= 4(5)=20$
$\sigma(-2X+1)=\sqrt{20}$

*B13. Given E(X)=5, E(Y)=2 and V(X)=6, find each of the following:

a) $E(2X-4Y)=2E(X) - 4\,E(Y)= 2(5) -4(2) = 10-8=2$
b) $E(-X+3)= -1E(X) + 3= -1(5) + 3 = -2$
c) $V(4X-3)=4^2V(X)=16(6)= 96$
d) $V(-X+5)= (-1)^2V(X)=1(6)=6$
$\sigma(-X+2)=\sqrt{6}$

*B14. Given $E(X^2)=10$, E(X)=3, E(Y)=5, find each of the following:

$V(X)=E(X^2) - (E(X))^2$
$V(X)= 10 - 3^2$
$V(X)=1$

a) $E(-2X-3Y)=-2E(X) -3E(Y)= -2(3) -3\,(5)= -6 -15 = -21$
b) $V(-2X+3)=(-2)^2V(X)= 4(1)=4$

B15.

X	Pr(X)	X^2
-1	$^1/_2$	1
0	$^1/_4$	0
1	$^1/_4$	1

$$E(X) = -1 \times \frac{1}{2} + 0 \times \frac{1}{4} + 1 \times \frac{1}{4} = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$$

$$E(X^2) = 1\left(\frac{1}{2}\right) + 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{4}\right) = \frac{3}{4}$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{3}{4} - \left(\frac{-1}{4}\right)^2 = \frac{12}{16} - \frac{1}{16} = \frac{11}{16}$$

$$\sigma(X) = \sqrt{\frac{11}{16}}$$

B16.

X	Pr[X=x]
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

E(X)=1(1/6)+2(1/6)+…6(1/6)=21/6 = 7/2 = $3.50

*B17.

X	Pr(X)
0 RR	2/5 (1/4)=2/20
1 RP, PR	12/20
2 PP	3/5 (2/4)=6/20

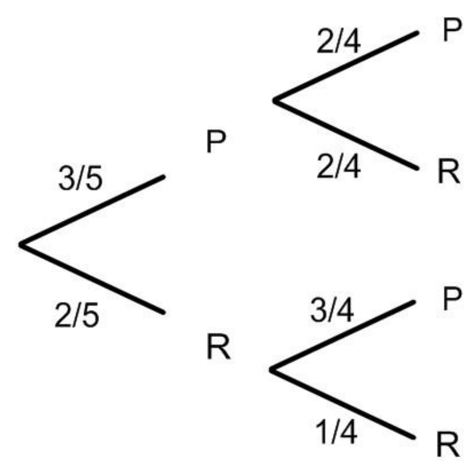
X=0 means we got no purple, which means RR

X=1 means we got one purple, which could be PR or RP

X=2 means we got two purple, or PP

Pr(1 Purple) = 1 - 2/20 - 6/20 = 12/20

E(X)=0+ 1 (12/20) + 2(6/20) = 24/20 = 12/10 = 6/5



B18.

$$\mu = E(X) = 1\left(\frac{3}{9}\right) + \ 2\left(\frac{5}{9}\right) + 3\left(\frac{1}{9}\right) = \frac{3+10+3}{9} = \frac{16}{9}$$

$$V(X) = E(X^2) - \mu^2 = 1\left(\frac{3}{9}\right) + 4\left(\frac{5}{9}\right) + 9\left(\frac{1}{9}\right) - \left(\frac{16}{9}\right)^2 = \frac{32}{9} - \left(\frac{16}{9}\right)^2$$

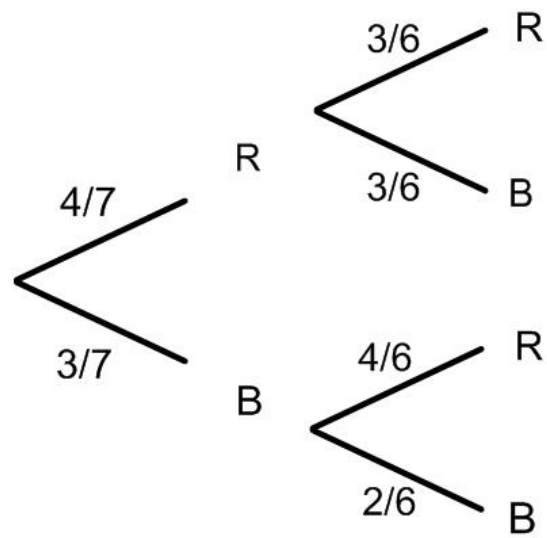
$$\sigma(X) = \sqrt{\frac{32}{9} - \left(\frac{16}{9}\right)^2}$$

B19.

X	Pr(x)	X^2
0 BB	6/42=1/7	0
1 BR, RB	4/7	1
2 RR	12/42=2/7	4

E(X)= mean= 0(1/7)+1(4/7)+2(2/7)=8/7

V(X)= 0(1/7)+1(4/7)+4(2/7) - $(8/7)^2$=
12/7 - 64/49 = 84/49 – 64/49 = 20/49



B20.

X	Pr(X=x)	X^2
-1	3a=3/9	1
0	2a=2/9	0
2	2a=2/9	4
4	2a=2/9	16
TOTAL	1	

3a + 2a + 2a + 2a = 1
a=1/9

a) Pr(X=2)=2/9

b) E(X)=$-1\left(\frac{3}{9}\right)+0\left(\frac{2}{9}\right)+2\left(\frac{2}{9}\right)+4\left(\frac{2}{9}\right)=\frac{9}{9}=1$

c) E(X^2)=$1\left(\frac{3}{9}\right)+0\left(\frac{2}{9}\right)+4\left(\frac{2}{9}\right)+16\left(\frac{2}{9}\right)=\frac{43}{9}$

$$V(X) = E(X^2) - \left(E(X)\right)^2 = \frac{43}{9} - (1)^2 = \frac{43}{9} - \frac{9}{9} = \frac{34}{9}$$

B21. Given E(X)=4, E(Y)=3 and V(X)=7, find each of the following:

a) E(3X-4Y)=3E(X) - 4 E(Y)= 3(4) - 4(3) = 0

b) E(-X+6)= -1E(X) + 6= -1(4) + 6 = 2

c) V(- 4 + X)=V(X -4)=1^2V(X)= 1(7)=7

d) V(-X+8)= $(-1)^2$V(X)=1(7)=7
$\sigma(-X+8)=\sqrt{7}$

B22.

a) Y= -2X+7
E(Y)= E(-2X+7)
5= -2E(X)+7
-2 = -2E(X)
E(X)=1

b) Y= -2X+7
V(Y)=V(-2X+7)
V(Y)=4V(X)
V(Y)=4(6)=24
$\sigma(Y)=\sqrt{24}$

B23.

a) Y= 5X -3
E(Y)= E(5X- 3)
5= 5E(X) - 3
8 =5E(X)
E(X)=8/5

b) Y= 5X -3
V(Y)=V(5X -3)
V(Y)=25V(X)
16=25V(X)
V(X)=16/25

B24.

$V(X) = E(X^2) - \left(E(X)\right)^2 = 20 - 4^2 = 20 - 16 = 4$

$\sigma(X) = \sqrt{4} = 2$

B25.

X	Pr(X)	X²
-1	0	1
0	$1/2$	0
1	1/2	1

$E(X) = -1 \times 0 + 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$

$E(X^2) = 1(0) + 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = \frac{1}{2}$

$V(X) = E(X^2) - (E(X))^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\sigma = \sqrt{\frac{1}{4}} = \frac{1}{2}$

C. Binomial Random Variables (3.1)

Example 1.

$a)\ n = 20, p = \frac{1}{5}, q = \frac{4}{5}$

$E(X) = np = 20 \times \frac{1}{5} = 4$

$b)\ V(X) = npq = 4 \times \frac{4}{5} = \frac{16}{5}$

$c)\ \sigma(X) = \sqrt{V(X)} = \sqrt{\frac{16}{5}}$

Example 2.

$n = 48, p = \frac{1}{4}, q = \frac{3}{4}$

$\mu = np = 48(\frac{1}{4}) = 12$

$V(X) = npq = 48 \times \frac{1}{4} \times \frac{3}{4} = 9$

$V(X)=E(X^2)-(E(X))^2$

$9= E(X^2)-12^2$

$E(X^2)=9+144=153$

Practice Exam Questions on Binomial Random Variables

C1. p=1/3
q=2/3
n=18
X= # baskets she gets in

a)E(X)=np=18(1/3)=6

b) V(X)=npq=18(1/3)(2/3)=4
$\sigma(X) = \sqrt{npq} = \sqrt{4} = 2$

C2. n=60
p=get a "4"=1/6
q=5/6
a) Pr(X=3)=$\binom{60}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{57}$ this is Bernoulli
b) Pr(4 at least once)= 1 - Pr(no 4)=$1-\binom{60}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{60}$
c) E(x)=np=60(1/6)=10 times
d) V(X)=npq=60(1/6)(5/6)=50/6=25/3

$$\sigma(X) = \sqrt{npq} = \sqrt{\frac{25}{3}} = \frac{5}{\sqrt{3}}$$

C3. n=100
X= odd #
p=1/2 (odd is a 1,3,5)
q=1/2 (even is 2,4,6)

a) E(X)=np=100(1/2)=50

b) V(X)=npq=50(1/2)=25

c) $\sigma(X) = \sqrt{npq} = \sqrt{25} = 5$

C4. X=# times a 3 comes up
n=60
p=1/6 (get a "3")
E(X)=np=60(1/6)=10

$C5. p = \frac{1}{5} = 0.2, q = 0.8, n = 75$

$E(X) = np = 75\left(\frac{1}{5}\right) = 15$

C6. $n = 72,\ p = \frac{1}{2}, q = \frac{1}{2}, x = \#\ of\ times\ odd\ number\ comes\ up$

a) $E(X) = np = 72 \times \frac{1}{2} = 36$

b) $V(X) = npq = 72 \times \frac{1}{2} \times \frac{1}{2} = 18$

$c)$ $\sigma(X) = \sqrt{V(X)} = \sqrt{18}$

D. Independent Random Variables (3.3)

Example 1.

$E(XY) = E(X)E(Y)$

X	Pr(X)
1	$1/6$
2	$1/6$
3	$1/6$
4	$1/6$
5	$1/6$
6	$1/6$

$$E(X) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = \frac{21}{6} = \frac{7}{2}$$

$$E(XY) = \frac{7}{2} \times \frac{7}{2} = \frac{49}{4}$$

Example 2.

Let X and Y be two independent random variables where $E(X^2)=10$, $E(X)=2$, $E(Y)=5$ and $V(Y)=4$.

a) Find $\sigma(X)$

$V(X)=E(X^2) - (E(X))^2= 10 - 2^2=6$

$\sigma(X) = \sqrt{6}$

b) Find E(2X-4)

E(2X-4)=2E(X) - 4 = 2(2) - 4 =0

c) Find V(9 - X)

$V(-X+9)=(-1)^2V(X)=(1)(6)=6$

d) Find V(-X+4Y)

$V(-X+4Y)=(-1)^2V(X)+(4)^2V(Y)= (1)(6) + 16(4)=70$

*e) Find $E(X^2Y)=E(X^2)(E(Y)=(10)(5)=50$

Example 3.

X is binomial with n=4, p=4/10=0.4
a) E(x)=np=4(0.4)=1.6

Y is binomial with n=12, p=0.5 and q=0.5
b) E(Y)=np=12(0.5)=6

c) E(XY)=E(X)E(Y) since they are independent
=1.6(6)= 9.6

Example 4.

V(X)=15, E(Y)=3, $E(Y^2)=18$

Find $V(Y)=E(Y^2)-[E(Y)]^2= 18 - 3^2=18- 9=9$

X, Y independent, so $V(X-Y)=1^2V(X) + (-1)^2V(Y)=1(15)+1(9)=24$

Example 5.

Since they are independent, E(XY)=E(X)E(Y)

a) 12=4E(Y)
E(Y)=3

b) $V(Y) = E(Y^2) - \left(E(Y)\right)^2 = 4 - (3)^2 = 4 - 9 = -5$

c) $V(-1X + 3Y) = (-1)^2V(X) + 3^2V(Y) = 1(3) + 9(-5)$
$= 3 - 45 = -42$

$\sigma(-1X + 3Y) = \sqrt{-42}$
$= undefined\ since\ we\ can't take\ the\ square\ root\ of\ a\ negative\ number$

Example 6.

X	Pr(X)	Y	XY	X+Y	$(X+Y)^2$
1	0.4	-1	-1	0	0
2	0	0	0	2	4
3	0.6	1	3	4	16

$a)\ E(X) = 1(0.4) + 2(0) + 3(0.6) = 0.4 + 1.8 = 2.2$
$b)\ E(Y) = -1(0.4) + 0(0) + 1(0.6) = -0.4 + 0.6 = 0.2$
$c)\ E(XY) = (XY)\,\Pr(XY) = -1(0.4) + 0(0) + 3(0.6) = -0.4 + 1.8 = 1.4$
No, they are not independent.

d) $E(X+Y)=E(X)+E(Y)=2.2+0.2 = 2.4$
$E(X+Y)^2=0(0.4)+4(0)+16(0.6)= 9.6$

$V(X+Y)=E(X+Y)^2- [E(X+Y)]^2$
$=9.6 - 2.4^2=9.6 - 5.76=3.84$ NOT independent

Example 7.

$E(aW + b) = aE(W) + b$
$\therefore E(XY + 12) = 1\big(E(XY)\big) + 12$
$= E(X)\cdot E(Y) + 12$ since X,Y are independent
$=(-4)(6) + 12= -12$ The answer is D.

Practice Exam Questions on Independent Random Variables

$D1. V(X) = E(X^2) - \left(E(X)\right)^2 = 12 - 2^2 = 8$

a)$V(2X + 4Y) = 2^2V(X) + 4^2V(Y)$

$= 4V(X) + 16V(Y)$

$= 4(8) + 16(3) = 32 + 48 = 80$

b) $V(X + 2Y) = 1^2V(X) + 2^2V(Y) = 1(8) + 4(3) = 20$

$\sigma(X + 2Y) = \sqrt{20}$

c) $E(XY) = E(X)E(Y) = 2 \times 3 = 6$

*D2.Find each of the following:

a) $E(X^2)$

$V(X)=E(X^2) - (E(X))^2$

$5=E(X^2) - 2^2$

$E(X^2)=9$

b) V(4X - Y)

$V(4X-Y)=(4)^2V(X) + (-1)^2V(Y)=16(5) + (1)(6)= 80+6=86$

c) $\sigma(-2X + 5Y)$

$V(-2X+5Y)=(-2)^2V(X) + (5)^2V(Y)=4(5)+25(6)=20+150=170$

$\sigma(-2X + 5Y) = \sqrt{170}$

D3. a) E(XY)=E(X)E(Y)=(6)(2)=12

b) E(2X+3Y)=2E(X) + 3E(Y)= (2)(6) + (3)(2)=18

c) $V(X- Y)= 1^2V(X) + (-1)^2V(Y)= (1)(1/2) + (1)(1/3) =3/6 + 2/6 = 5/6$

d) $\sigma(X + 2Y)$

$V(X + 2Y) = (1)^2V(X) + (2)^2V(Y)$

=(1)(1/2) + (4)(1/3)=3/6 + 4/3 = 3/6 +8/6 = 11/6

$\sigma(X + 2Y) = \sqrt{\frac{11}{6}}$

D4.

a) $E(-3X+4) = -3E(X) + 4 = -3(2) + 2 = -6 + 4 = -2$

$b)$ $E(Y^2) = ?$

$V(Y)=E(Y^2) - (E(Y))^2$

$8=E(Y^2) - 4^2$

$E(Y^2)=8+16=24$

$c) E(XY) = E(X)E(Y) = 2(4) = 8$

d) $V(3X-1) = 3^2V(X) = 9(6) = 54$

$e)$ $V(-2X - Y) = (-2)^2V(X) + (-1)^2V(Y)$

$= 4(6) + 1(8) = 24 + 8 = 32$

D5.

$V(3X - 4Y) = 3^2V(X) + (-4)^2V(Y) = 9(6) + 16(5) = 54 + 80 = 134$

$\sigma(3X - 4Y) = \sqrt{134}$

D6. ...For example, when event E occurs, X has value -1 and Y has value 2.

Outcome	Probability	X	Y	X-Y	$(X-Y)^2$
E	0	-1	2	-3	9
F	½	0	1	-1	1
G	½	2	0	2	4

If E(X)= 1, E(Y)= 1/2, V(X)=1 and V(Y)=1/4, find V(X-Y).

****They are NOT independent so you can't use**
$V(X-Y)=(1)^2V(X) + (-1)^2V(Y)$
So, we use: $V(X-Y)= E(X-Y)^2 - (E(X-Y))^2$

a) E(X-Y)=(-3)(0) + (-1)(1/2) + 2(1/2)= -1/2 + 2/2= 1/2

b) $E(X-Y)^2$

=9(0) + 1(1/2) + 4(1/2)

=5/2

$V(X-Y)= E(X-Y)^2 - (E(X-Y))^2$

$=5/2 - (1/2)^2$

=10/4 – 1/4

=9/4

D7. X is binomial with X=# white with n=4 and p=4/9

$E(X) = 4\left(\frac{4}{9}\right)$=16/9

$E(Y) = np = 10 \times \frac{1}{2} = 5$

$E(XY) = E(X)E(Y) = \frac{16}{9}(5) = \frac{80}{9}$

D8.

X is binomial with n=60, p=2/6=1/3 and q=2/3
E(x)=np=60(1/3)=20

Y is binomial with n=20, p=0.4 tails and q=0.6 heads
E(Y)=np=30(0.4)=12

E(XY)=E(X)E(Y) since they are independent
=20(12)=240

D9.

X	Prob	Y	X+Y	$(X+Y)^2$
1	0.3	0	1	1
2	0.5	1	3	9
3	0.2	2	5	25

a)E(X+Y)=0.3(1)+0.5(3)+0.2(5)= 0.3+1.5+1=2.8

b)$E(X+Y)^2$=1(0.3)+9(0.5)+25(0.2)=0.3+4.5+5=9.8

V(X+Y)=9.8 - $(2.8)^2$

*D10. E(XY – 4X).

$E(aX + bY) = a\,E(X) + b\,E(Y)$

$\therefore E(1XY - 4X) = 1E(XY) - 4E(X)$

$= E(X)\cdot E(Y) - 4E(X)$ since X & Y are independent

$= 6\cdot 5 - 4(6)$

$= 30 - 24$

$= 6$

*D11. $find\ \sigma(X-Y)$.

$V(X)=5^2=25 \qquad V(Y)=3^2=9$

$\therefore V(aX+bY)=a^2V(X)+b^2V(Y)$

$V(X-Y)=1^2V(X)+(-1)^2V(Y)$ since X, Y are independent

$V(X-Y)=1(25)+1(9)=34$

$\sigma(X-Y)=\sqrt{V(X-Y)}=\sqrt{34}$

D12. If X and Y are independent random variables with E(X) = -7 and E(Y)=5, find E(2XY+3).

$E(aW+b)=aE(W)+b$

$\therefore E(2XY+2)=2\big(E(XY)\big)+3$

$=2E(X)\cdot E(Y)+3$ since X,Y are independent

=2(-7)(5) +3= -70+3= -67

E. Final Exam Questions on Sections A to D (3.1 to 3.3)

E1.

$\mu = 4, V(X) = 12$

$\sigma = \sqrt{V(X)} = \sqrt{12}$

E2.

$V(X) = E(X^2) - \mu^2$

$25 = E(X^2) - 5^2$

$E(X^2) = 50$

E3.

$Y = 10 - 2X$

$Y = -2X + 10$

$E(Y) = -2E(X) + 10$

$E(Y) = -2(6) + 10 = -12 + 10 = -2$

E4.

$V(Y) = V(-2X + 10)$

$V(Y) = (-2)^2 V(X) = 4(2) = 8$

$\sigma(Y) = \sqrt{V(Y)} = \sqrt{8}$

E5.

$\sigma = 3, V(X) = 3^2 = 9$

$V(X) = E(X^2) - \mu^2$

$E(X^2) = V(X) + \mu^2$

$E(X^2) = 9 + 64 = 73$

E6. $V(5X + 1) = 5^2 V(X) = 25(3) = 75$

E7.

i) $E(XY) = E(X)E(Y) = 5(6) = 30\ true$

ii) $E(X + Y) = E(1X + 1Y) = 1E(X) + 1E(Y) = 5 + 6 = 11$ true

iii) $E(X - Y) = 1E(X) - 1E(Y) = 5 - 6 = -1$ true

So, all of them are always true and the answer is A).

E8.

X	Pr(X)	X^2
-1	0	1
0	1/2	0
1	1/2	1

$\mu = -1(0) + 0\left(\frac{1}{2}\right) + 1(1/2) = \frac{1}{2}$

$E(X^2) = 1(0) + 0(\frac{1}{2}) + 1\left(\frac{1}{2}\right) = \frac{1}{2}$

$V(X) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

E9.

$E(X) = 1.2$

$0(0.7) + 0.1(5 + a) + 0.2(5 - a) = 1.2$

$0 + 0.5 + 0.1a + 1 - 0.2a = 1.2$

$-0.1a = 1.2 - 1.5$

-0.1a= -0.3

$a = 3$

E10.

Roll	W	Pr(W)
1	-2	$^1/_6$
2	3	$^1/_6$
3	-2	$^1/_6$
4	3	$^1/_6$
5	6	$^1/_6$
6	3	$^1/_6$

$E(W) = -2(^1/_6) + 3(^1/_6) + (-2)(^1/_6) + 3(^1/_6) + 6(^1/_6) + 3(^1/_6)$

$E(W) = \$\frac{11}{6}$

E11. If Y=7X-3 and V(X)=1, find $\sigma(Y)$.
V(Y)=V(7X-3)
V(Y)=7^2V(X)
V(Y)=(49)(1)
V(Y)=49
$\sigma(Y) = \sqrt{49} = 7$

*E12. Use the following table below to answer the questions below:

Outcome	Probability	Value of X	Value of Y	XY
E	0	-1	2	-2
F	0.3	2	1	2
G	0.7	5	0	0

a) Find the value of Pr[(X=-1)∩(Pr(Y=2)]=0
b) Find the value of Pr(XY=0)
Pr(XY=0) = 0.7
c) Find E(XY) ** not independent, so you can't use E(XY)=E(X)E(Y)
E(XY)= (-2)(0) + 2(0.3) + 0(0.7)=0.6

E13.
a) $E(X) = -1(1/3) + 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) \quad = 0$
b) $E(Y) = 0(1/3) + 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) \quad = 1$
$E(Y^2) = 0^2(1/3) + 1^2\left(\frac{1}{3}\right) + 2^2\left(\frac{1}{3}\right) \quad = \frac{5}{3}$
$\therefore V(Y) = E(Y^2) - [E(Y)]^2$
$V(Y) = \frac{5}{3} - (1)^2 = \frac{5}{3} - \frac{3}{3} = \frac{2}{3}$

Outcome	Probability	Value of X	Value of Y	X-Y
E	1/3	-1	0	-1
F	1/3	0	1	-1
G	1/3	1	2	-1

c) $E(X-Y) = E(X) - E(Y) = 0 - 1 = -1$
$E(X-Y)^2 = (-1)^2\left(\frac{1}{3}\right) + (-1)^2\left(\frac{1}{3}\right) + (-1)^2\left(\frac{1}{3}\right)$
$= 1$
$\therefore V(X-Y) = 1 - (-1)^2 = 1 - 1 = 0$
$\therefore \ \sigma(X-Y) = \sqrt{0} = 0$

F. Probability Density Functions (4.1)

Example 1.
Find the probability the salad weight is between 12 and 15 ounces.

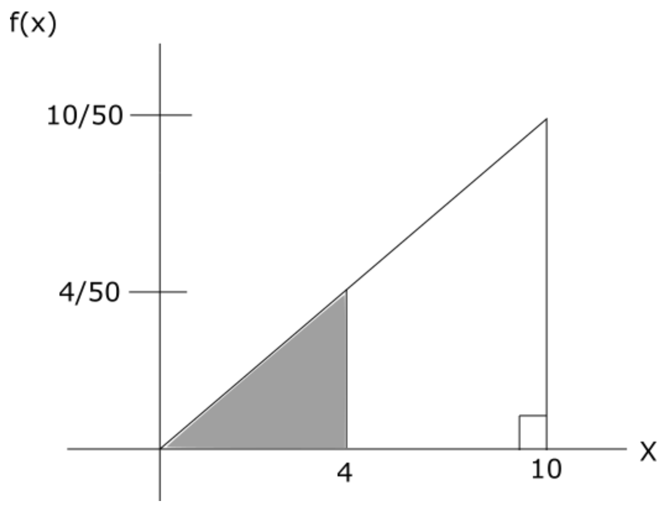
Pr(12<X<15)= area under the rectangle= Length x Width= (3)(1/10)=0.30

Example 2. Given the graph, find the Pr(X<8).
Pr(X<8)= 1 - Pr(X>8)= 1 - Area of the triangle= $1-\frac{bh}{2}=1-\frac{(2)(0.08)}{2}=$
$1-0.08=0.92$

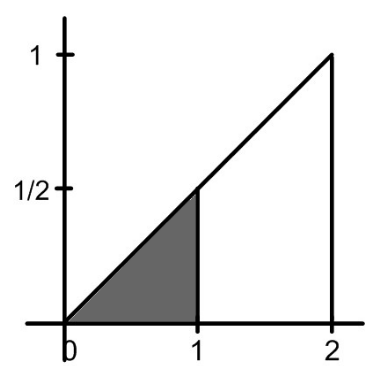
Example 3. Graph f(x)=x/50 for $0 \leq x \leq 10$, where f(x)=0 otherwise

Use the graph to find Pr(X<4).

PrX<4)= area of triangle=$\frac{bh}{2}=\frac{(4)(\frac{4}{50})}{2}=0.16$



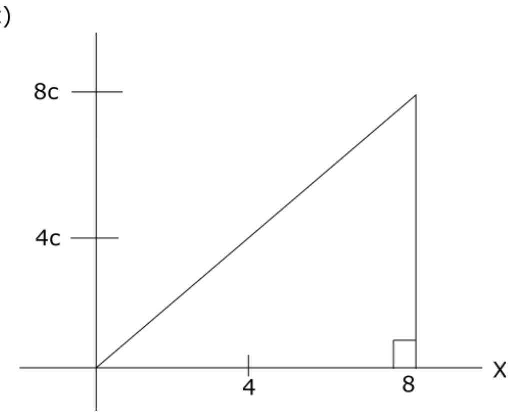
Example 4. Continuous random variable X has probability density function f(x)=x/2
if $0 \leq x \leq 2$ and f(x)=0 otherwise. Find Pr(1<X<2).



Pr(X<1)=$\frac{bh}{2}=\frac{(1)(\frac{1}{2})}{2}=0.25$

Pr(1<X<2)=Pr(X>1)=1-0.25=0.75

***Example 5.** Continuous random variable X has probability density function f(x)=cx for $0 \leq x \leq 8$. Find the value of c.



Total Area=1

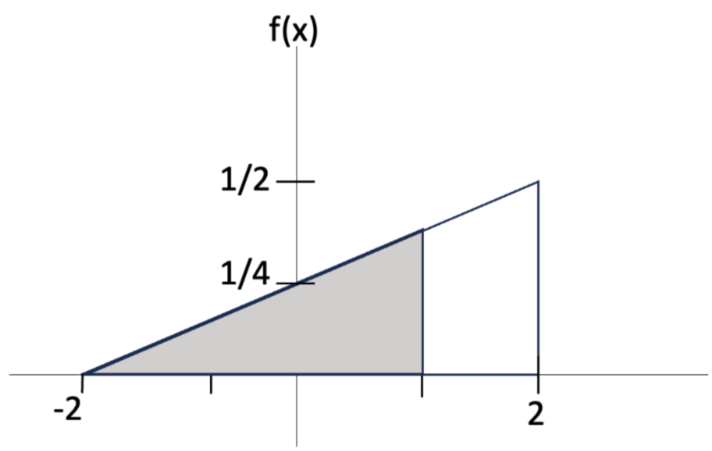
$$\frac{bh}{2} = 1$$

$$\frac{(8)(8c)}{2} = 1$$

64c=2

c=2/64=1/32

***Example 6.**



x	f(x)=$\frac{x+2}{8}$
-2	0
0	¼
2	1/2
1	3/8

Pr(X<1)=$\frac{1}{2}bh = \frac{1}{2}(3)\left(\frac{3}{8}\right) = \frac{9}{16}$

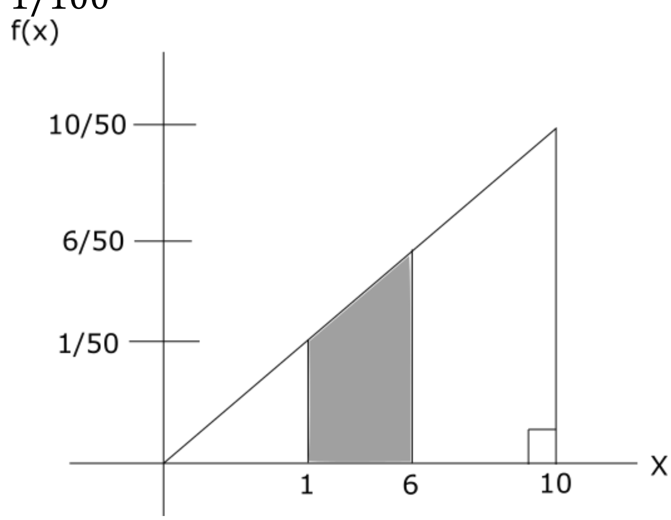
<u>Practice Exam Questions on Probability Density Functions</u>

F1. Using the graph below, find the Pr(X<0.75).
Pr(X<0.75)=0.5

F2. Using the graph below, find Pr(X=1) and Pr(X<1).
Pr(X=1)=0 (can't equal one number, there is no area to find!)
Pr(X<1)= 1 - Pr(X>1)= 1 - bxh/2
=1 - (1)(0.5)/2
=1-0.25
=0.75

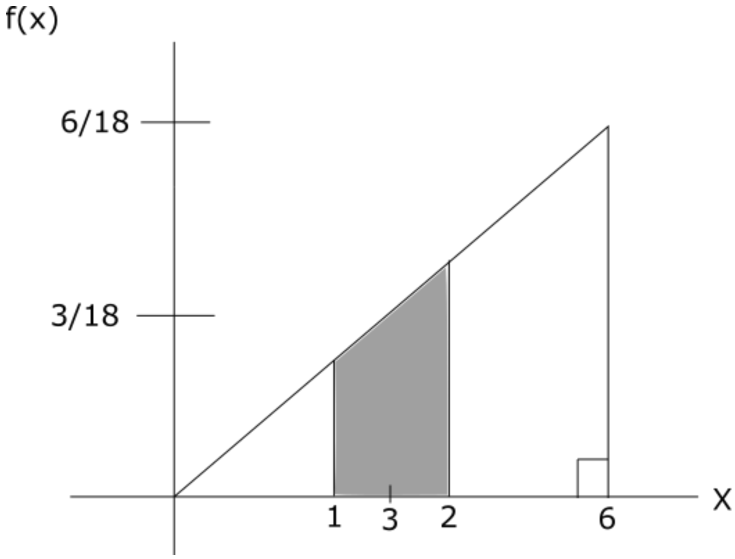
F3. Using the graph below and the probability density function f(x) = 1/8, find Pr(3<X<6).
Pr(3<X<6)=LxW=(3)(1/8)= 3/8

F4. Continuous random variable X has probability density function $f(x) = \frac{x}{50}$ if 0<x<10 and f(x)=0 otherwise. Find Pr(1<X<6).
Pr(1<X<6)= Area of Triangle + Area of rectangle
Or Area of triangle between 0 and 6 - Area of triangle from 0 to 1
=(6)(6/50)/2 - (1)(1/50)/2
=36/100 - 1/100
=0.35



F5. Continuous random variable X has probability density function $f(x) = \frac{x}{18}$, if 0<x<6 and f(x)=0 otherwise. Find Pr(1<X<2).

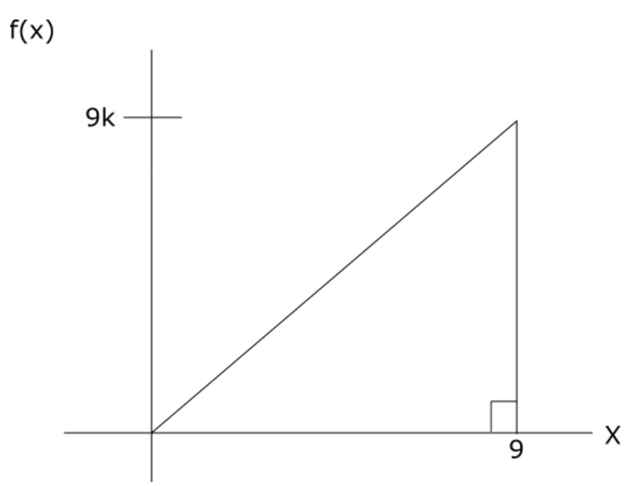
Find Pr(1<X<2)= Area Triangle 0 to 2- Area Triangle 0 to 1
=(2)(2/18)/2 - (1)(1/18)/2
=4/36 - 1/36
=3/36 = 1/12



*F6. The probability density function f(x) of a continuous random variable X is defined by f(x)=kx if 0<x<9 and f(x)=0 otherwise. Find the value of k.

The total area under the graph must be equal to 1.

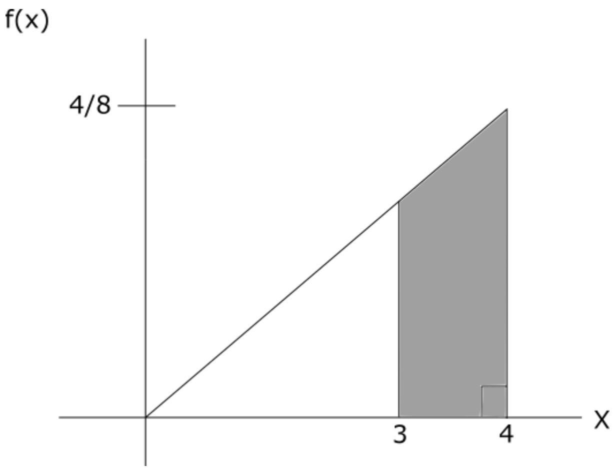
bh/2 = 1
(9)(9k)/2 = 1
81k/2=1
k=2/81



F7. Continuous random variable X has probability density function $f(x) = \frac{x}{8}$, if 0<x<4 and f(x)=0 otherwise. Find Pr(X>3).
Pr(X>3)= 1 - Pr(X<3)
= 1 - bh/2

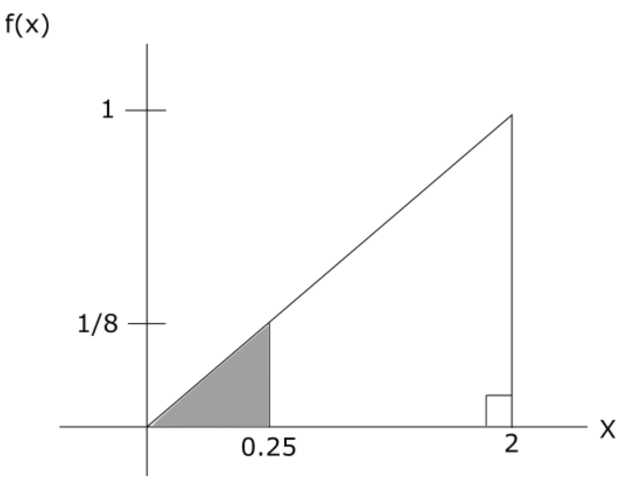
= 1- (3)(3/8)/2

= 1 – 9/16
=7/16



F8. Continuous random variable X has probability density function $f(x) = \frac{x}{2}$, if 0<x<2 and f(x)=0 otherwise. Find Pr(X<0.25).

Pr(X<0.25)= area of a triangle= bh/2= (1/4)(1/8)/2= 1/32(1/2) = 1/64



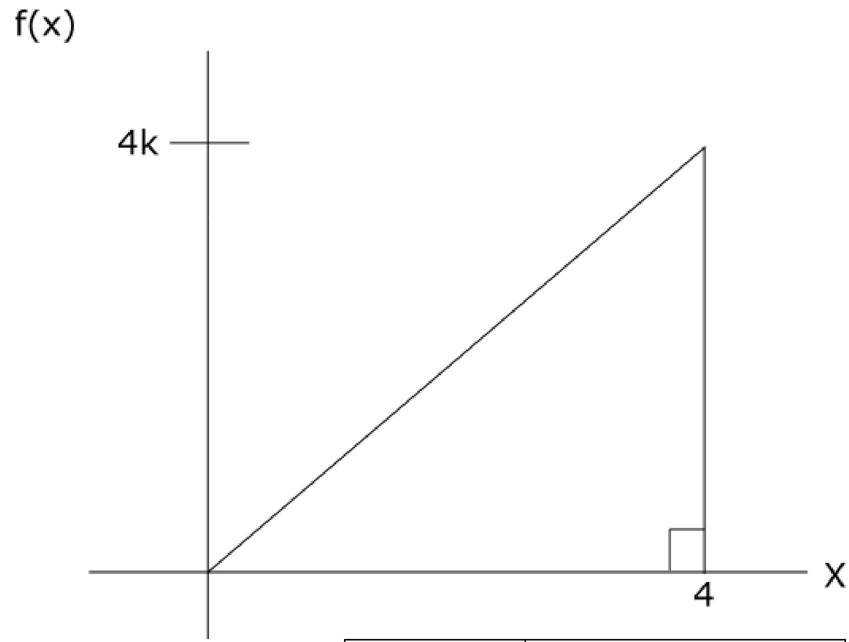
*F9. Find $\Pr(X \leq 4k)$.

$Total\ Area = 1$

$\frac{bh}{2} = 1$

$\frac{4(4k)}{2} = 1$

$k = \frac{1}{8}$ $f(x) = \frac{1}{8}\ x$



x	$f(x) = kx$
0	0
2	$2k$
4	4k

x	$f(x) = kx$=1/8 (x)
0	0
1	1/8
2	2/8
3	3/8
4	4/8
½	1/8(1/2)=1/16

$f(x) = \frac{1}{8}x$

$\Pr(x \leq 4k) = \Pr(x \leq 4/8) = \Pr\ (x \leq 1/2)$ =(1/2)(1/16)/2=1/64

G. Normal Distributions (4.2)

Empirical Rule

Example 1.

$right = 50\% - 34\% = 16\%$

$left = 50\% - \frac{95}{2} = 2.5\%$

$total = 18.5\%$

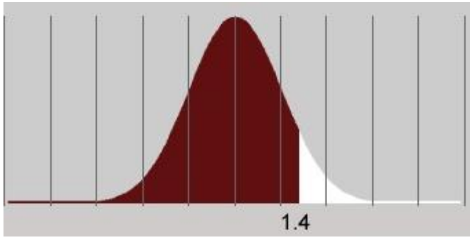
Example 2.

$left = \frac{99.7}{2} = 49.85$ $\qquad$ $right = \frac{95}{2} = 47.5$

$total = 97.35\%$

Normal Random Variables

Example 1.

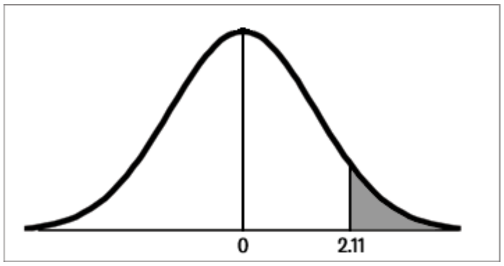


b) Pr[Z>1.4]= 1-Pr(Z<1.4)=1-0.9192=0.0808

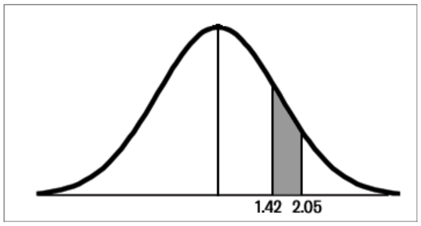
c) Pr[Z>-2]

Pr[Z>-2] = same area as Pr[Z<2] = 0.9772

Example 2.



a) Pr(1.42<Z<2.05)= Pr(Z<2.05) - Pr(Z<1.42)= 0.9798-0.9222=0.0576

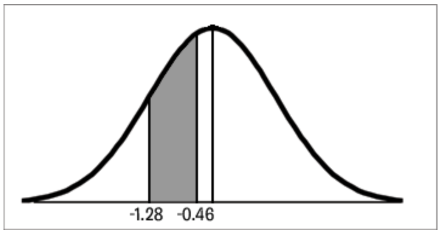


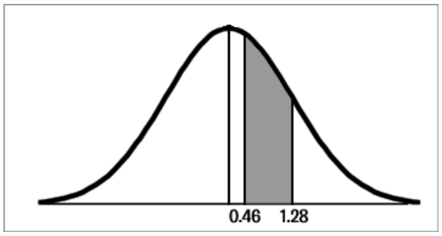
b) Pr(Z>2.11)= 1- Pr(Z<2.11)= 1 - 0.9826=0.0174

c) Pr(-1.28<Z<-0.46) re-draw on the positive side

Pr(0.46<Z<1.28)

=Pr(Z<1.28) - Pr(Z<0.46)
=0.8997 - 0.6772=0.2225

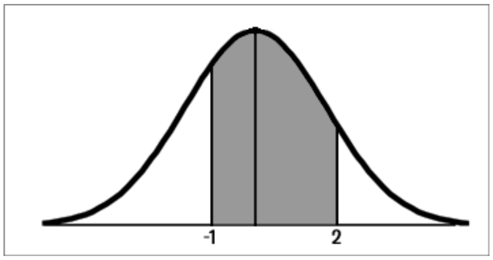




d) Pr(Z<-0.48)= re-draw as the same shaded area on the positive side
=Pr(Z>0.48)
=1 - Pr(Z<0.48)
= 1 -0.6844
=0.3156

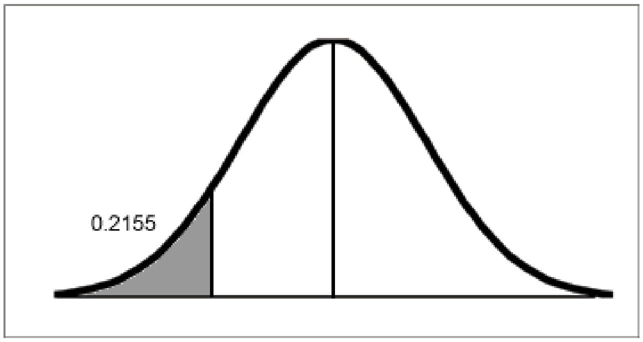
e) Pr(-2.11<Z<0.86)
=Pr(Z<0.86) - Pr(Z<-2.11)
=0.8051 - (1-Pr(Z<2.11)
=0.8051 - (1-0.9826)
=0.8051 -0.0174
=0.7877

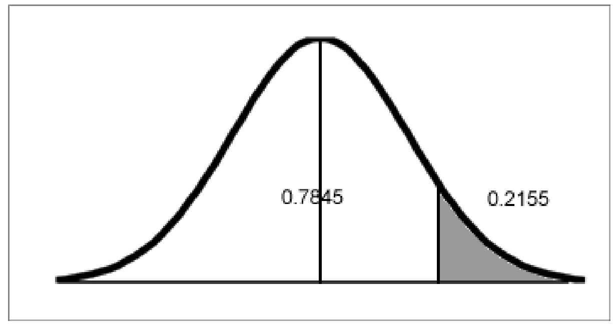
f) Pr(-1<Z<2)= Pr(Z<2) - Pr(Z<-1)= 0.9772 - 0.1587=0.8185



Example 3.

a) Pr(Z<k)=0.2155...look up this number in the body of the area chart and find the corresponding "k" or "Z" value along the left...we know k is a negative number since the area less than it is below 0.50.





Since we aren't given the chart with negative numbers, we can re-draw it in the positive and then just remember in your final answer that "k" was originally negative.

k= -0.79 is the closest value we can get (0.7852)

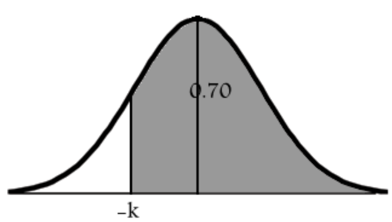
b) Pr(Z>k)=0.2119

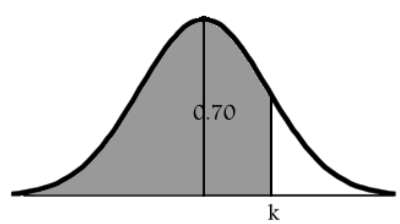
Look up 1 - 0.2119=0.7881
k=Z=0.8

Example 4. Find k such that Pr(Z>k)=0.7

Look up the area 0.7 and find "k" along the left side of the table.
Remember, k is a negative value since the original z-score diagram had k on the left of zero

Therefore, k= -0.525





Example 5.

$\Pr(X<14)=\Pr\left(Z<\frac{14-8}{3}\right) = \Pr(Z < 2)$

$=0.9772$

Example 6.

mean= μ=5

$\sigma = 3$

$\Pr(X > 10) = \Pr\left(Z > \frac{10-5}{3}\right) = \Pr\left(Z > \frac{5}{3}\right) = \Pr(Z > 1.67)$

$= 1 - 0.9525 = 0.0475$

Example 7.

mean= μ=11

$\sigma = 2$

$= \Pr\left(\frac{14-11}{2} < Z < \frac{15-11}{2}\right)$

$= \Pr(1.5 < Z < 2)$

$=\Pr(Z<2) - \Pr(Z<1.5)$

$=0.9772 - 0.9332$

$=0.044$

Example 8.

$Z=-2$

$Z = \frac{X-\mu}{\sigma}$

$-2 = \frac{X-4}{6}$

$X- 4= -12$

$X=-8$

$Z=3$

$Z = \frac{X-\mu}{\sigma}$

$3 = \frac{X-4}{6}$

$X- 4= 18$

$X=22$

So, $-8<X<22$

Example 9. X is a normal random variable with unknown mean μ and standard
deviation σ=3. If Pr[X<25]=0.9772, what is the value of μ?

Look up the area 0.9772 in the body and you get Z=2.

$$Z = \frac{X - \mu}{\sigma}$$

$$2 = \frac{25 - \mu}{3}$$

μ =19

Example 10. ...in the top 4% of all students taking this exam?

$\mu = 60$
$\sigma = 10$

We have 0.04 area above the line, so the area below our line is 1-0.04 = 0.96

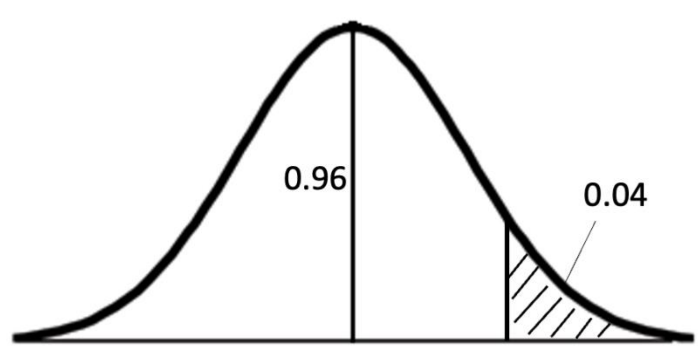
Pr(Z<k)=0.96
Look up the area 0.96 in the body of the table and k=1.75(the closest is 0.9599)

$$Z = \frac{X - \mu}{\sigma}$$

$$1.75 = \frac{X - 60}{10}$$

X=17.5+60=77.5

Therefore, a student must score 77.5



Practice Exam Questions on Normal Distributions

G1. For the standard normal random variable Z, find the value of Pr[Z<1.6].

A. 0.0548	B. 0.9452	C. 0.8554	D. 0.1446	E. None of the above

Pr (Z<1.6)=0.9452
The answer is b).

G2. For the standard normal random variable Z, find the value of Pr[Z>-0.80].

A.0.7881	B. 0.80	C. 0.2119	D. 0.5319	E. None of the above

Pr(Z> - 0.80) = Pr(Z<0.80) = 0.7881
The answer is a).

G3. For the standard normal random variable Z, find the value of Pr[-1.20<Z<1.20].

A. 2.4	B. 0.9918	C. 0.1151	D. 0.7698	E. None of the above

Pr (-1.20<Z<1.20)= Pr(Z<1.2) - Pr(Z<-1.2)
=0.8849 – (1-0.8849)
=0.7698
The answer is d).

G4. For the standard normal random variable Z, what is the value of Pr[Z>1.76]?

A.0.0392	B. 0.9608	C. 0.9554	D. 0.0446	E. None of the above

Pr (Z> 1.76)= 1 - Pr(Z< 1.76)
= 1 - 0.9608
=0.0392
The answer is a).

G5. If Z is the standard normal random variable, find Pr[-1 < Z < 1].

A. 0.0228	B. 0.9772	C. 0.1587	D. 0.8413	E. None of the above

Pr (-1<Z<1)= Pr(Z<1) - Pr(Z<-1)
=0.8413 – (1-0.8413)
=0.6826
The answer is e).

G6. Find the value of k if it is known that Pr[k<Z<1.5]= 0.0483 , where Z is the standard normal random variable.

A. 1.2	B. -1.2	C. 0.8849	D. 1.66	E. none of the above

Pr(Z<1.5)= 0.9332
0.9332 – 0.0483=0.8849
Look up area 0.8849 and you get k=1.2

The answer is a).

G7. Use the table for the standard normal random variable Z to find Pr[-0.65<Z<1.92].

A. 0.6226	B. 0.2284	C. 0.7148	D. 0.2852	E. None of the above

Pr (-0.65<Z<1.92)=Pr(Z<1.92) - Pr(Z<-0.65)
Pr(Z<1.92) - (1-Pr(Z<0.65)
=0.9726 - (1-0.7422)
=0.7148

The answer is c).

G8. Use the table for the standard normal random variable Z to find a value of k for which Pr[Z<k]= 0.9495

A. 0.9495	B. 0.8264	C. -1.64	D. 1.64	E. None of the above

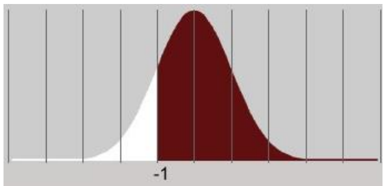
Pr(Z<k)=0.9495 same area as Pr(Z>k)

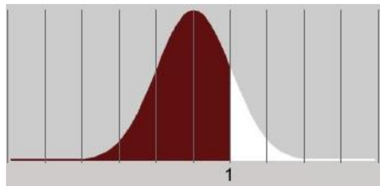
Find the area=0.9495 by looking in the body of the chart...we get k=1.64
The answer is d).

G9. X is a normal random variable with mean μ=90 and standard deviation

σ=10. Find Pr[X>80].

A. 0.8413	B. 0.1587	C. 0.9222	D. 0.0778	E. None of the above





$$Z = \frac{80-90}{10} = -1$$

Pr(X>80)=Pr(Z>-1)= Pr(Z<1)=0.8413...the answer is a).

G10. X is a normal random variable with mean μ=8 and standard deviation σ =2. Find Pr[12<X<14].

A. 1	B. 0.9987	C. 0.9772	D. 0.8413	E. None of the above

Pr(12<X<14)=Pr $(\frac{12-8}{2} < Z < \frac{14-8}{2})$
=Pr(2<Z<3)
=Pr(Z<3) - Pr(Z<2)
=0.9987 - 0.9772
=0.0215
The answer is e).

G11. Pr(X<520)=Pr(Z<$\frac{520-500}{20}$)=Pr(Z<1)= 0.8413

G12. X is a normal random variable with mean 35 and standard deviation 5.

Pr(30<X<40)=Pr$(\frac{30-35}{5} < Z < \frac{40-35}{5}) = \Pr(-1 < Z < 1)$
=Pr(Z<1) - Pr(Z<-1)
=0.8413 – (1-0.8413)
=0.8413 - 0.1587
=0.6826

G13. ...greater than 630?

Let X be the test score. Then $X \sim N(\mu,\sigma)$

with $\mu = 500$, $\sigma = 100$. So, $\Pr(X > 630) = \Pr\left(Z = \frac{X-\mu}{\sigma} > \frac{630-500}{100}\right)$

$$= \Pr(Z > 1.3) = 1 - \Pr(Z < 1.3) = 1 - 0.9032 = 0.0968.$$

G14. ... and standard deviation 10.

(a) What percentage of the children have IQ's greater than 125?

Let X be the child's IQ. Then $X \sim N(\mu,\sigma)$

with $\mu = 100, \sigma = 10$. So

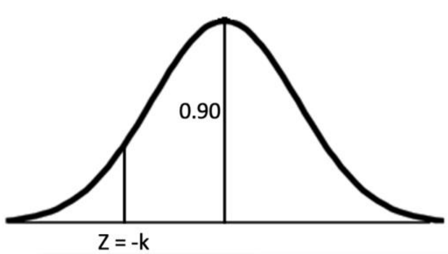
$$\Pr(X > 125) = \Pr\left(Z = \frac{X-\mu}{\sigma} > \frac{125-100}{10}\right)$$

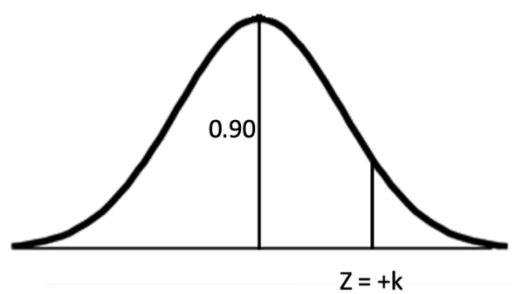
$$= \Pr(Z > 2.5) = 1 - \Pr(Z < 2.5) = 1 - 0.9938 = 0.0062$$

(b) About 90% of the children have IQ's greater than what value?

Draw the area 0.90 in the body and we need Z=-k, but we don't have the negative Z table. So, we have to re-draw the area on the positive side and find Z=+k and then just remember our original picture was a negative Z value.

Look up the area 0.90 in the body and we get Z=1.28, but we know that Z is really -1.28





$$Z = \frac{X-\mu}{\sigma}$$

$$-1.28 = \frac{X-100}{10}$$

X=-12.8+100=87.2

Thus, 90% of the children have IQ's greater than 87.2.

G15. (a) What is the probability of getting a 81 or less on the exam?

Let X be the final grade. Then $X \sim N(\mu, \sigma)$ with $\mu = 73$, $\sigma = 8$. Then

$$\Pr(X \le 81) = \Pr\left(Z = \frac{X-\mu}{\sigma} \le \frac{81-73}{8}\right) = \Pr(Z < 1) = 0.8413.$$

(b) What percentage of students scored between 65 and 89?

$$\Pr(65 < X < 89) = \Pr\left(\frac{65-73}{8} < Z < \frac{89-73}{8}\right) = \Pr(-1 < Z < 2)$$

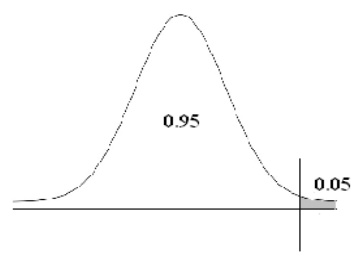
$$= \Pr(Z < 2) - \Pr(Z < -1) = 0.9772 - 0.1587 = 0.8185.$$

c) Only 5% of the students taking the test scored higher than what grade?

Look up area=0.95 in the body of the chart and get Z=1.645

$$\Pr(Z > z) = 0.05 \Rightarrow z = 1.645.$$

$$z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + z\sigma = \mu + 1.645\sigma$$



So $x = 73 + 1.645 \cdot (8) = 86.16.$

Therefore, 5% of the students scored higher than 86%.

G16. Pr(X<41)=Pr $\left(Z < \frac{41-45}{5}\right) = \Pr(Z < -0.8) = \Pr(Z > 0.8) =$
$= 1 - \Pr(Z < 0.8)$
=1-0.7881
=0.2119

G17. Pr(9<Z<10)=Pr $\left(\frac{9-8}{2} < Z < \frac{10-8}{2}\right) = (0.5 < Z < 1)$
=Pr(Z<1) - Pr(Z<0.5)
=0.8413 – 0.6915 =0.1498

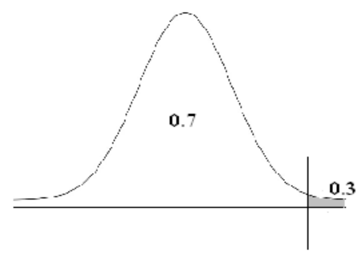
G18. ... earn less than $3400 per month?
$\mu = 3200$
$\sigma = 200$

Pr(X<3200)=Pr(Z<$\frac{3200-3600}{400}$) $= \Pr(Z < -1) = \Pr(Z > 1) = 1 - 0.8413 =$ 0.1587

G19. Let X be the waiting time (in days). Then

$X \sim N(\mu, \sigma)$ with $\mu = 120$, $\sigma = 20$.

Look up the area 0.70 in the body of the table (area below your line) and get Z=0.52



So $\Pr(Z > z) = 0.30 \Rightarrow z = 0.52$.

$$z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + z\sigma = \mu + 0.52\sigma$$

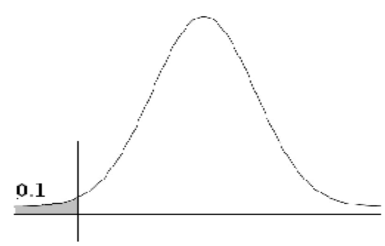
$$= 120 + 0.52 \cdot (20) = 130.4.$$

Therefore, 30% of the patients must wait for more than 130 days for a heart transplant.

(b) What is the longest time spent waiting for a heart transplant that would still place a patient in the bottom 10% of waiting times?

Look up area 0.10 in the body! We can't as it isn't on our chart. But, we know from our picture that Z is negative.

So, look up the area 0.90 and get Z=1.28 and by symmetry this is the same area, but on the opposite side. So, our Z is -1.28.



$\Pr(Z < z) = 0.10 \Rightarrow z = -1.28$.

$$z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + z\sigma = \mu - 1.28\sigma$$

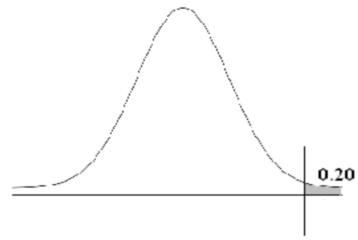
$$= 120 - 1.28 \cdot (20) = 94.4.$$

Therefore, 10% of the patients have to wait for less than 94 days for a heart transplant.

G20.

How long do the longest 20% of pregnancies last?

Look up the area below your line, which is 0.80 in the body of the table and we get Z=0.84



$$\Pr(Z > z) = 0.20 \Rightarrow z = 0.84.$$

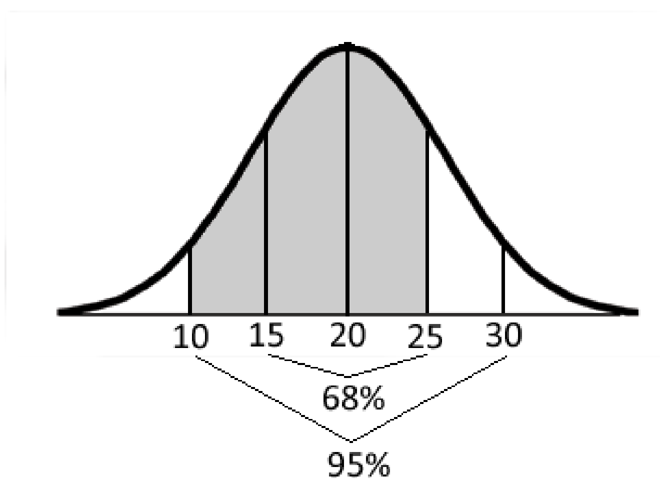
$$z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + z\sigma = \mu + 0.84\sigma$$
$$= 266 + 0.84 \cdot (16) = 279.44.$$

Therefore, the longest 20% of pregnancies last more than 279 days.

G21.Mean =20 and standard dev=5

Find % between 10 and 25

See diagram to the right= 95/2 + 68/2 = 81.5%



H. Approximations to the Discrete Random Variable (4.3)

Example 1. If X=10,20,30,40,50, then k=10
To find Pr[X=20] we use the approximation Pr(15<Y<25)
To find Pr[X=50] we use the approximation Pr(45<Y<55)
To find Pr[$20 \leq X \leq 40$] we use the approximation Pr(15<Y<45)

Example 2. Let the discrete random variable X have possible values 0,2,4,6,8,10,12 and 14.

Let Y be a continuous random variable which approximates X. Find an expression in terms of Y which approximates each of the following:

a) Pr $(4 \leq X < 10)$
=Pr(3<Y<9)

b) Pr $(4 < X < 12)$
=Pr(5<Y<11)

c) Pr $(2 \leq X \leq 8)$
=Pr(1<Y<9)

Example 3. The discrete random variable X has been approximated by a continuous random variable Y. The possible values of X are 9, 12, 15, 18, 21 and 24. Find an approximation expressed in terms of Y to the following probabilities:

a) Pr(X<15)

=Pr(Y<13.5)

b) Pr $(X \geq 15)$
=Pr(Y>13.5)

c) Pr(X≠18)
=1 - Pr(X=18)

=1 - Pr(16.5<Y<19.5)

***Example 4**. The answer is C) since the points coloured in are 5, 7 and 9.

I. Normal Approximations to the Binomial Distribution (4.3)

Example 1.
$\mu = 100$
$\sigma = 10$

Convert X to Y to Z for a normal approximation
$\Pr(X \geq 120) = \Pr(Y > 119.5) \ldots convert\ to\ Z\ using\ Z = \frac{Y-\mu}{\sigma}$

We want $\Pr(Y > 119.5) = \Pr\left(Z > \frac{119.5-100}{10}\right) = \Pr\left(Z > \frac{19.5}{10}\right) = \Pr(Z > 1.95)$

use the Z-table to find the probability Z is less than 1.95

$\Pr(Z > 1.95) = 1 - 0.9744 = 0.0256$

Example 2. Find a normal approximation to Pr[(B(144,0.5)>76]
a) check np and nq and make sure both are greater than or equal to 5

np=144(0.5)=72
nq=144(0.5)=72
so, since both are >/5 we can use the normal approx. to the binomial

b) Here, this notation means than n=144 and p=0.5

$\mu = np$ =144(0.5)=72
$\sigma = \sqrt{npq} = \sqrt{72(0.5)} = 6$

We want Pr(X>76)...from our number line, the approximation is...=Pr(Y>76.5)
=Pr $(Z > \frac{76.5-72}{6})$=Pr(Z>4.5/6)=Pr(Z>45/60)=Pr(Z>3/4)=Pr(Z>0.75)=
1-Pr(Z<0.75)=1- 0.7734=0.2266

Example 3.

$\mu = 20$
$\sigma = 5$

Pr(X<24)...draw a number line and shade 28, 26, 24, ...all numbers less than 30, even numbers as it states in the question

Find the approximation by extending the amount of area by 1/2 way between the two numbers, ie. by 1

Pr(X<25)=Pr(Y<23)=Pr $\left(Z < \frac{23-20}{5}\right) = \Pr\left(Z < \frac{3}{5}\right) = \Pr(Z < 0.6) = 0.7257$

Example 4. Darren plays soccer and has a probability of 2/3 of missing each shot he takes on net. He takes 18 shots on net. Let X be the random variable that counts the number of goals he gets.

np=6≥ 5 and nq=18(2/3) ≥ 5*, so we can approximate*
a) Find E(X).

q=2/3 miss
p=1/3 get shot in
n=18

E(X)=np=18(1/3)=6

b) Find $\sigma(X)$.

$\sigma = \sqrt{npq} = \sqrt{6(\frac{2}{3})} = 2$

c) Find the probability of getting between 5 and 10 shots (inclusive) in the net.

$\Pr(5 \leq X \leq 10) = \Pr(4.5 < Y < 10.5) =$
Pr $(\frac{4.5-6}{2} < Z < \frac{10.5-6}{2}$)=Pr(-0.75 <Z<2.25)
= Pr(Z<2.25) - Pr(Z<-0.75)=Pr(Z<2.25)–(1-Pr(Z<0.75))
= 0.9878 –(1-0.7734)=0.7612

Practice Exam Questions on the Normal Approximation to the Binomial

*I1. Assuming an equal chance of a new baby being a boy or a girl, what is the likelihood that more than 65 out of the next 100 births at a local hospital will be boys?

n=100 check np=50≥ 5 $and\ nq = 50 \geq 5$, so we can approximate

p=0.5

q=0.5

$\mu = np = 100(0.5) = 50$
$\sigma = \sqrt{npq} = \sqrt{50(0.5)} = 5$
Pr(X>65)=Pr(Y>65.5)=Pr $\left(Z > \frac{65.5-50}{5}\right) = \Pr(Z > 3.1) = 1 - \Pr(Z < 3.1)$
$= 1 - 0.999 = 0.001$

*I2. Let X be a discrete random variable whose possible values are consecutive integers. If X is approximately normal with mean 100 and standard deviation 10, find Pr(X$\leq$ 115).

$\mu = 100$
$\sigma = 10$
Pr(X$\leq$ 115) $= \Pr(Y < 115.5) = \Pr\left(Z < \frac{115.5-100}{10}\right) = \Pr(Z < 1.55) = 0.9394$

*I3. Alyssa rolls a single die 180 times.

a) Find the expected number of times she will roll a "6".
check np=30≥ 5 $and\ nq = 180(\frac{5}{6}) \geq 5$, so we can approximate

$\mu = np = 180(1/6) = 30$

b) Find the standard deviation of the number of times she will roll a "6".

$\sigma = \sqrt{npq} = \sqrt{30(5/6)} = 5$

c) Find the probability Alyssa will roll a "6" at most 15 times.

$$\Pr(X \leq 15) = \Pr(Y < 15.5) = \Pr\left(Z < \frac{15.5 - 30}{5}\right) = \Pr(Z < -14.5/5)$$
$$= \Pr(Z < -2.9)$$
=Pr(Z>2.9)
=1-Pr(Z<2.9)
=1-0.9981=0.0019

*I4.A single die is tossed 100 times. Let X be the number of times an even number comes up.
a) Find E(X)
check np=50≥ 5 $and\ nq = 50 \geq 5$, so we can approximate

$$\mu = np = 100(0.5) = 50$$

b) Find the standard deviation

$$\sigma = \sqrt{npq} = \sqrt{50(0.5)} = 5$$

c) Find the probability that an even number comes up at least 45 times.

$$\Pr(\mathrm{X} \geq 40) = \Pr(\mathrm{Y} > 44.5) = \Pr\left(\mathrm{Z} > \frac{44.5 - 50}{5}\right) = \Pr(\mathrm{Z} > -1.1)$$
$$= \Pr(\mathrm{Z} < 1.1) = 0.8643$$

I5. $x = \#\ tails$
$n = 18$
$q = heads = \dfrac{2}{3}$
$p = tails = \frac{1}{3}$ (p must be tails as X=# of tails)

check np=6≥ 5 $and\ nq = 18\left(\frac{2}{3}\right) = 12 \geq 5$, so we can approximate

a) $= np$
$= 18\left(\dfrac{1}{3}\right) = 6$

b)

$= \sqrt{npq}$

$= \sqrt{6\left(\frac{2}{3}\right)} = \sqrt{4} = 2$

c)

$Z = \frac{Y-\mu}{\sigma} = \frac{12.5-6}{2} = 3.25$

$Pr(X \le 12)$
$= Pr(Y < 12.5)$
$= Pr(Z < 3.25)$
$= 0.9994$

I6.

$p = get\ red = 0.5$
$q = 0.5$

check np=50≥ 5 $and\ nq = 50 \ge 5$, so we can approximate

a) $= np$
$= 100(0.5) = 50$

b) $= \sqrt{npq}$
$= \sqrt{50(0.5)} = 5$

c) $Pr(40 \le X \le 55)$
$= Pr(39.5 < Y < 55.5)$
$= Pr(-2.1 < Z < 1.1)$
$= 0.8643 - (1 - 0.9821)$
$= 0.8643 - 0.0179 = 0.8464$

$Z_1 = \frac{39.5-50}{5} = -2.1$
$Z_2 = \frac{55.5-50}{5} = 1.1$

I7.

$n = 100$
$p = q = 0.5$

check np=50≥ 5 $and\ nq = 50 \geq 5$, so we can approximate

$\mu = np = 50$
$\sigma = \sqrt{npq} = \sqrt{25} = 5$

$Pr(X \leq 55)$
$= Pr(Y < 55.5)$

$$Z = \frac{Y - \mu}{\sigma} = \frac{55.5 - 50}{5} = \frac{5.5}{5} = 1.1$$

$\therefore Pr(Z < 1.1) = 0.8643$

J. Final Exam Questions on Sections F to I (4.1 to 4.3)

J1. .... Continuous random variable Y is a good approximation for X. Which one of the following gives the value of Pr(X=9)?

The answer is B).

*J2. n=100 It is a Binomial question, so we use the normal approximation

p=0.2

q=0.8

$\mu = np = 100(0.2) = 20$

$\sigma = \sqrt{npq} = \sqrt{20(0.8)} = 4$

$\Pr(25 \leq X \leq 28) = \Pr(24.5 < Y < 28.5) = \Pr\left(\frac{24.5-20}{4} < Z < \frac{28.5-20}{4}\right) =$

$\Pr\ (1.125 < Z < 2.125)$

=Pr(1.13<Z<2.13)

=Pr(Z<2.13) - Pr(Z<1.13)=0.9834 - 0.8708=0.1126

J3. If Z is the standard normal random variable, find Pr(Z<1.3).
Pr(Z<1.3)=0.9032 from the table

J4. If Z is the standard normal random variable, find Pr(Z<-1.3).
Pr(Z<-1.3)= Pr(Z>1.3)= 1 - Pr(Z<1.3)= 1 - 0.9032=0.0968

J5. If Z is the standard normal random variable, find Pr(Z>-2.25).
Pr(Z>-2.25)= Pr(Z<2.25)=0.9878

J6. If X is a normal random variable with mean 10 and standard deviation 5, find Pr(X<12).

Pr(X<12)=Pr($Z < \frac{12-10}{5}$) $= \Pr(Z < 0.4) = 0.6554$

J7. If X is a normal random variable with mean 15 and standard deviation 2.5, find Pr(X>20).

Pr(X>20)=Pr$\left(Z > \frac{20-15}{2.5}\right) = \Pr(Z > 2) = 1 - \Pr(Z < 2)$
$= 1 - 0.9772 = 0.0228$

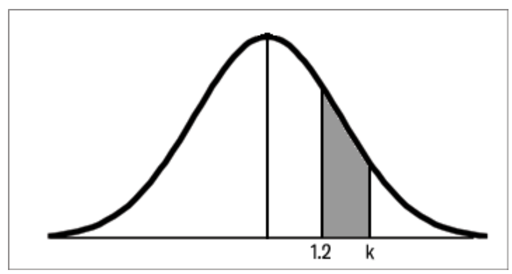
J8. Consider the standard normal random variable. Find Pr(0.6<Z<2.2).

Pr(0.6<Z<2.2)=Pr(Z<2.2) - Pr(Z<0.6)=0.9861 – 0.7257 = 0.2604

*J9. Consider the standard normal random variable Z. If it is known that Pr(1.2<Z<k)=0.0522, what is the value of k?

Pr(Z<k) - Pr(Z<1.3)=0.0522
Pr(Z<k) - 0.9032= 0.0.0522
Pr(Z<k)=0.9554 ...look up this area in the body of the table and find the corresponding value of k along the left

k=1.7



J10. Let X be a normal random variable with mean 8 and standard deviation 2. Find Pr($X \geq 4$).

Pr($X \geq 4$)=Pr$(Z > \frac{4-8}{2}) = \Pr(Z > -2) = \Pr(Z < 2) = 0.9772$

J11. Find Pr(12<X<21).

Pr(12<X<21)=Pr$\left(\frac{12-15}{3} < Z < \frac{21-15}{3}\right) = \Pr(-1 < Z < 2)$
$= \Pr(Z < 2) - \Pr(Z < -1)$
=0.9772 -(1-Pr(Z<1)
=0.9772 - (1-0.8413)
=0.9772- 0.1587
=0.8185

*J12.
p=0.50
q=0.50
n=100
check np=50≥ 5 $and\ nq = 100(0.5) = 50 \geq 5$, so we can approximate

$\mu = np = 100(0.5) = 50$

$\sigma = \sqrt{npq} = \sqrt{50(0.5)} = 5$

$\Pr(X \geq 60) = \Pr(Y > 59.5) = \Pr\left(Z > \frac{59.5 - 50}{5}\right) = \Pr(Z > 1.9)$
$= 1 - Pr(Z < 1.9) = 1 - \ 0.9713 = 0.0287$

*J13. n=30
p=0.3

E(X)=np=30(0.3)=9

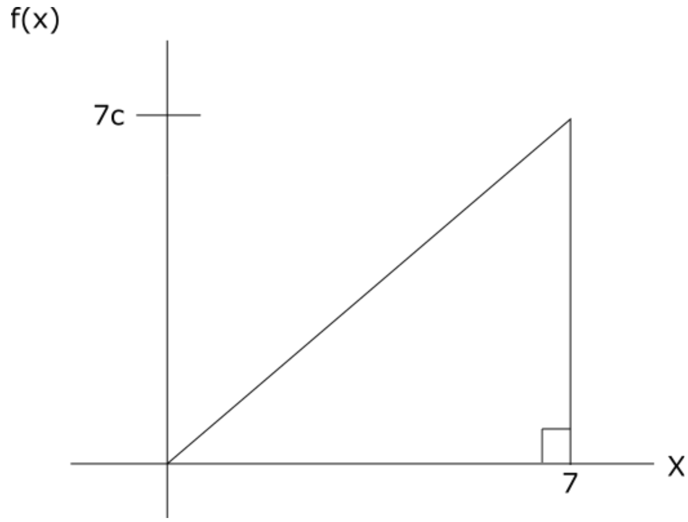
*J14. If X is a normal random variable with mean 14 and standard deviation 2, find k such that Pr(X<k)=0.8413.
Look up the area 0.8413 and you get Z=1
Sub into the formula for Z: $Z = \frac{X-\mu}{\sigma}$

$1 = \frac{X - 14}{2}$
X-14 = 2
X=16

*J15. Z is the standard normal random variable. Find k such that Pr(k<Z<0)=0.3849
Pr(Z>0)=0.50
Pr(Z<k) = 0.3849+0.5 = 0.8849 We know k is negative since Z=k is to the left of 0, so find Pr(Z<k) = 0.8849 and look up that area in the body and we get k=1.2. But, remember k is negative so we have that k= -1.2

*J16. If "c" is some possible value of X, which of the following gives a good approximation of the value of Pr(X>c)? You want to colour in c+1, c+2, etc. so your approximation starts halfway between c and c+1. So, we do Pr(Y>c+1/2). The answer is e).

*J17. Continuous random variable. X has probability density function f(x)=cx if 0≤x≤ 7 and f(x)=0 otherwise. What is the value of c?



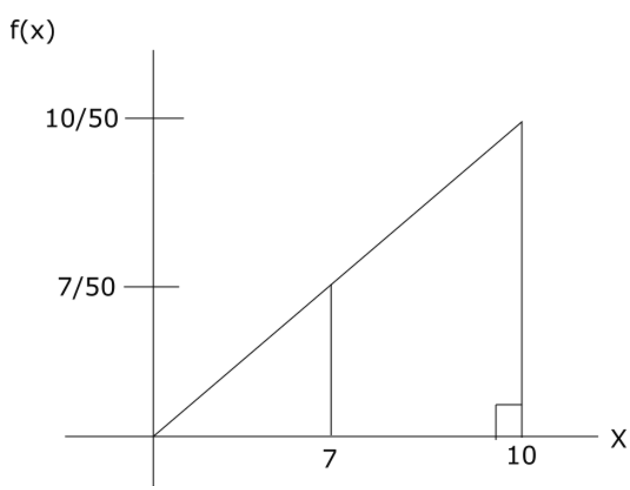
Draw a graph.

Total area under the triangle = 1
bh/2=1

(7)(7c)/2 = 1
49c=2
c=2/49

J18. Continuous random variable X has probability density function f(x)=x/50 if 0<x<10 and f(x)=0 otherwise. Find the value of Pr[X<7].

Draw a graph.
Pr(X<7)=Area of a triangle=bh/2= (7)(7/50)/2=49/100 = 0.49



*J19. If X and Y are independent random variables such that E(X)=6, E(Y)=4. Which of the following must always be true?

i) E(X-Y)=2
ii) E(XY)=24
iii) E(3X+1)=19

E(X - Y)=E(X) - 1 E(Y)=6 - 4 = 2 true

E(XY)=E(X)E(Y) since they are independent...true

E(3X+1)=3E(X) + 1=3(6)+1=19 true

The answer is d).

*J20. If X and Y are independent random variables such that V(X)=25, V(Y)=10. Which of the following must always be true?

i) V(X+Y)=35
ii) V(X-Y)=15
iii) σ(X-Y)=$\sqrt{15}$

V(X+Y)=1^2V(X) + $(1)^2$V(Y)=25+10=35 true

V(X-Y)= $(1)^2$V(X) + $(-1)^2$V(Y)=35 false

iii) false, σ(X-Y)=$\sqrt{35}$

The answer is e).

*J21. The value of a discrete random variable Y is given by the equation Y=6X-5, where X has probability distribution function given below. Find Pr(Y=7).

X	Pr(X)
0	0
1	0.2
2	0.8

If we know Y=7, sub it into the equation to find X.

Y=6X-5
7=6X-5
6X=12
X=2

Pr(X=2)=0.8 from the table

*J22. See J21. If E is the event that X has value 0 or 1, find Pr(E).
Pr(E)=0 +0.2 = 0.2

J23. The answer is c).

J24. The answer is D).

J25. The discrete random variable X has values 0,2,4,6,8,10 and 12. If Y is a continuous random variable that approximates X, which of the following is approximated by (5<Y<9)?

The answer is e).

J26. $\Pr(x \geq 7) = 1 - \Pr\,(x \leq 6)$

$$= 1 - F(6)$$
$$= 1 - 0.85$$
$$= 0.15$$

$\therefore$ The answer is $\boxed{A}$.

J27. $E(aW+b)=aE(W)+b$

$\therefore E(XY+12)=1\big(E(XY)\big)+12$

$= E(X)\cdot E(Y)+12$ since X,Y are independent

$= -7(6)+12$

$= -42+12$

$= -30$

∴The answer is $\boxed{B}$.

J28. $E(aX+bY)=a\,E(X)+b\,E(Y)$

$\therefore E(1XY-3X)=1E(XY)-4E(X)$

$= E(X)\cdot E(Y)-4E(X)$ since X & Y are independent

$= 6\cdot 5-4(6)$

$= 30-24$

$= 6$

J29. $V(X)=5^2=25$ $V(Y)=8^2=64$

$\therefore V(aX+bY)=a^2V(X)+b^2V(Y)$

$V(X-Y)=1^2V(X)+(-1)^2V(Y)$ since X, Y are independent

$V(X-Y)=1(25)+1(64)=89$

$\sigma(X-Y)=\sqrt{V(X-Y)}=\sqrt{89}$

J30. Find the area below the line= 1- 0.0102=0.9898
Look up the area 0.9898 in the body of the table and get Z=2.32
The answer is $\boxed{B}$.

J31.

$Total\ Area=1$

$\frac{bh}{2}=1$

$\frac{3(3k)}{2}=1$

$k=\frac{2}{9}$ $f(x)=\frac{2}{9}x$

x	$F(x)=kx$
0	0
1	k
2	$2k$
3	3k

$\Pr(x\le k)=\Pr(x\le 2/9)=\frac{\frac{2}{9}(\frac{4}{81})}{2}$

$=\frac{4}{729}$

J32. $\mu = 4a$ $\sigma = 4a$ $z = \frac{x-\mu}{\sigma}$

$$= \frac{8a-4a}{4a}$$

$$= 1$$

$$\Pr(x < 8a) = \Pr(z < 1)$$

$$= 0.8413$$

Best of luck on the exam!!!!