

CALC 1000 ACE Booklet Solutions (Fall 2024)

A. Solving Trig Equations	2
B. Graphing Exponential and Logarithmic Functions	8
C. Exponential Equations	12
D. Cancellation Laws	17
E. Inverse	25
F. Logarithms	33
G. Limits	42
H. Properties of Limits	43
I. Continuity	53
J. Limits at Infinity	57
K. Vertical and Horizontal Asymptotes	60
L. Squeeze Theorem	69
M. Intermediate Value Theorem	75
N. Definition of the Derivative	79
O. Evaluating Limits by recognizing them as derivatives	94
P. Average vs. Instantaneous Velocity and Acceleration	99
Q. Derivative Rules	100
R. Trig Derivatives	110
S. Derivatives of Inverse Trig	118
T. Equation of a Tangent Line	121
U. Second Derivatives	126
V. Tricky Domain Questions	129
W. Inverse Function Theorem	135
X. Implicit Differentiation	139
Y. Related Rates	155
NEW MATERIAL SINCE THE MIDTERM	168
A. L'Hospital's Rule	169
B. Critical Numbers	174
C. Maximum and Minimum	176
D. Concavity and Points of Inflection and The Second Derivative Test	181
E. Optimization	193
F. Sums and Sigma Notation	203
G. Riemann Sums	208
H. Indefinite Integrals or Antiderivatives	213
I. Definite Integrals and The Fundamental Theorem of Calculus	217
J. Substitution	234
K. Other Types of Integrals	249
L. Practice With All Types of Integration	251
M. Area Between Curves	258
N. Volume of Solids of Revolution	291

A. Solving Trig Equations

Example 1.

a) $\cos(x - y)$

b) $\frac{\sin 2x}{2}$

c) $\sin(x - y)$

Example 2. $\cos x + \sin 2x = 0$

$$\cos x + 2 \sin x \cos x = 0$$

$$\cos x(1 + 2 \sin x) = 0$$

$$\cos x = 0 \qquad 1 + 2 \sin x = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\sin x = \frac{-1}{2}$$

$$x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

$$x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

Example 3.

a) $\sin\theta \cot\theta = \cos\theta$

$$LS = \sin\theta \cot\theta$$

$$= \sin\theta \left(\frac{\cos\theta}{\sin\theta} \right)$$

$$= \cos\theta$$

$$RS = \cos\theta$$

$$LS=RS$$

$$b) (\sin x + \cos x)^2 = 1 + \sin 2x$$

$$LS = (\sin x + \cos x)(\sin x + \cos x)$$

$$= \sin^2 x + 2\sin x \cos x + \cos^2 x$$

$$= 1 + 2\sin x \cos x$$

$$= 1 + \sin 2x$$

$$LS=RS$$

$$A1. \cos(2x) = -\sin^2 x$$

$$\cos^2 x - \sin^2 x = -\sin^2 x$$

$$\cos^2 x - \sin^2 x + \sin^2 x = 0$$

$$\cos^2 x = 0$$

$$\cos x = 0 \quad x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$A2. 2\cos^2 x - \cos x - 1 = 0$$

$$(\cos x - 1)(2\cos x + 1) = 0$$

$$\cos x = 1 \quad \text{or} \quad 2\cos x + 1 = 0$$

$$x = 0, 2\pi \quad 2\cos x = -1$$

$$\cos x = \frac{-1}{2}$$

$$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \quad x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

$$A3. \quad \tan x = \sin 2x \quad [0, \pi]$$

$$\tan x = 2 \sin x \cos x$$

$$\frac{\sin x}{\cos x} = 2 \sin x \cos x$$

$$\sin x = 2 \sin x \cos^2 x$$

$$0 = 2 \sin x \cos^2 x - \sin x$$

$$0 = \sin x(2 \cos^2 x - 1)$$

$$\sin x = 0 \qquad 2 \cos^2 x - 1 = 0$$

$$x = 0, \pi \qquad \cos^2 x = \frac{1}{2}$$

$$\cos x = \frac{1}{\sqrt{2}} \quad \text{or} \quad \cos x = \frac{-1}{\sqrt{2}}$$

$$x = \frac{\pi}{4} \qquad x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

all other solutions from the CAST rule would be greater than π , so we don't include them

$$A4. \quad \tan x = \sin x \quad [0, \pi]$$

$$\frac{\sin x}{\cos x} = \sin x$$

$$\sin x = \sin x \cos x$$

$$\sin x - \sin x \cos x = 0 \quad [0, \pi]$$

$$\sin x(1 - \cos x) = 0$$

$$\sin x = 0 \qquad 1 = \cos x$$

$$x = 0, \pi, 2\pi \qquad x = 0, 2\pi \dots 2\pi \text{ is not in our interval}$$

\therefore solution is $x = 0, \pi$

A5.

$$a) (1 - \sin\theta)(1 + \sin\theta) = \cos^2\theta$$

$$LS = (1 - \sin\theta)(1 + \sin\theta)$$

$$= 1 + \sin\theta - \sin\theta - \sin^2\theta$$

$$= 1 - \sin^2\theta$$

$$= \cos^2\theta$$

$$LS=RS$$

$$b) \sin x \sin 2x + \cos x \cos 2x = \cos x$$

$$LS = \sin x \sin 2x + \cos x \cos 2x$$

$$LS = \sin x (2 \sin x \cos x) + \cos x (1 - 2 \sin^2 x)$$

$$= 2 \sin^2 x \cos x + \cos x (1 - 2 \sin^2 x)$$

$$= 2 \sin^2 x \cos x + \cos x - 2 \sin^2 x \cos x$$

$$= \cos x$$

$$LS=RS$$

$$c) \frac{1+\tan^2 x}{1+\cot^2 x} = \tan^2 x$$

$$\text{LS} = \frac{1+\tan^2 x}{1+\cot^2 x}$$

$$= \frac{1 + \frac{\sin^2 x}{\cos^2 x}}{1 + \frac{\cos^2 x}{\sin^2 x}}$$

$$= \frac{\frac{\cos^2 x + \sin^2 x}{\cos^2 x}}{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}$$

$$= \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}}$$

$$= \frac{1}{\cos^2 x} \left(\frac{\sin^2 x}{1} \right)$$

$$= \tan^2 x$$

$$\text{LS} = \text{RS}$$

$$d) \tan^2\vartheta - \sin^2\vartheta = \tan^2\vartheta \sin^2\vartheta$$

$$\begin{aligned} LS &= \frac{\sin^2\vartheta}{\cos^2\vartheta} - \frac{\sin^2\vartheta}{1} \\ &= \frac{\sin^2\vartheta}{\cos^2\vartheta} - \frac{\sin^2\vartheta \cos^2\vartheta}{\cos^2\vartheta} \\ &= \frac{\sin^2\vartheta(1-\cos^2\vartheta)}{\cos^2\vartheta} \\ &= \tan^2\vartheta \sin^2\vartheta \end{aligned}$$

$$LS=RS$$

$$e) 2\csc 2\vartheta = \sec\vartheta \csc\vartheta$$

$$LS=2\csc 2\vartheta$$

$$\begin{aligned} LS &= 2 \left(\frac{1}{\sin 2\vartheta} \right) \\ &= \frac{2}{2\sin\vartheta \cos\vartheta} \\ &= \frac{1}{\sin\vartheta} \left(\frac{1}{\cos\vartheta} \right) \\ &= \csc\vartheta \sec\vartheta \end{aligned}$$

$$LS=RS$$

B. Graphing Exponential and Logarithmic Functions

Example 1.

$$y = 2^{x+1}$$

$$D \ x \in R \text{ or } (-\infty, \infty)$$

$$R \ y \in R/y > 0 \text{ or } (0, \infty)$$

$$HA \ y = 0$$

$$x = int \ none$$

$$y = int \ (0,2)$$

Example 2.

$$y = 3^x$$

$$D \ x \in R \text{ or } (-\infty, \infty)$$

$$R \ y \in R/y > 0 \text{ or } (0, \infty)$$

$$HA \ y = 0$$

$$x = int \ none$$

$$y = int \ (0,1)$$

Example 3.

$$y = -2^x - 4$$

$$D \ x \in R \text{ or } (-\infty, \infty)$$

$$R \ y \in R/y < -4 \text{ or } (-\infty, -4)$$

$$HA \ y = -4$$

$$x = int = none \quad y = int = (0, -5)$$

Example 4.

$$y = 3(2^x) - 2$$

$$D \quad x \in R \quad (-\infty, \infty)$$

$$R \quad y \in R/y > -2 \quad (-2, \infty)$$

$$HA \quad y = -2$$

$$x - \text{int} = \frac{\ln \frac{2}{3}}{\ln 2}$$

$$y - \text{int} = (0, 1)$$

Work for how to get the intercepts...

$$x - \text{int} \quad \text{let } y = 0 \quad 0 = 3(2^x) - 2$$

$$3(2^x) = 2$$

$$2^x = \frac{2}{3}$$

$$\ln 2^x = \ln \frac{2}{3} \quad x = \frac{\ln \frac{2}{3}}{\ln 2}$$

$$y - \text{int} \quad \text{let } x = 0 \quad y = 3(2^0) - 2 = 3 - 2 = 1$$

Example 5.

$$y = (0.5)^x$$

$$D = x \in R \quad \text{or } (-\infty, \infty)$$

$$R = y \in R/y > 0 \quad \text{or } (0, \infty)$$

$$HA \quad y = 0$$

$$x - \text{int} = \text{none DNE} \quad y - \text{int} = (0, 1)$$

Example 6.

$$y = e^x$$

$$HA \quad y = 0$$

$$x - int = none \quad y - int = (0,1)$$

$$y = \ln x \quad HA \quad none$$

$$VA \quad x=0$$

$$x - int \quad (1,0) \quad y - int \quad (none)$$

Example 7.

$$y = 3 \log_2(x - 1) + 3$$

\uparrow \uparrow \uparrow

Vertical stretch right 1 up 3

(multiply y by 3)

Example 8. Sub. (1,7) into the equation $y=b^x+c$

$$7 = b^1 + c$$

$$c=7-b$$

Subst. (2,13) into the equation and get:

$$13 = b^2 + c \text{ subst. } c=b-7 \text{ into this equation}$$

$$13 = b^2 + (7 - b)$$

$$\text{Rearrange and we get: } b^2 - b - 6 = 0$$

$$\text{Factor and get } (b-3)(b+2)=0$$

$$\text{And } b=3 \text{ or } b= -2$$

If $b=3$, we get $c=7-b=7-3=4$ and the equation is $y=3^x+4$

If $b= -2$, we get $c=7-b=7-(-2)=9$ and the equation is $y=(-2)^x+9$

C. Exponential Equations

Example 1

a) $\frac{1}{4}b^5$

b) $9x^4y^2z^8$

c) $\frac{(4abc^2)(9a^2b^4)}{-6abc^2} = \frac{36a^3b^5c^2}{-6abc^2} = -6a^2b^4$

d) $(-2x^{-1}y^3)^2 = 4x^{-2}y^6 = \frac{4y^6}{x^2}$

Example 2. Solve each of the following:

a) $7^{5-2x} = 7^{3x+8} \therefore 5 - 2x = 3x + 8$

$$5 - 8 = 5x$$

$$-3 = 5x$$

$$x = -\frac{3}{5}$$

b) $6^{3x} = \left(\frac{1}{216}\right)^{3x+8}$

$$6^{3x} = (6^{-3})^{3x+8}$$

$$3x = -3(3x + 8)$$

$$3x = -9x - 24$$

$$12x = -24 \quad x = -2$$

$$c) 4^{3x-5} = \left(\frac{1}{64}\right)^3 = (4^{-3})^3$$

$$4^{3x-5} = 4^{-9}$$

$$3x - 5 = -9$$

$$3x = -9 + 5$$

$$3x = -4$$

$$x = \frac{-4}{3}$$

*d) If $3^{-2t} = e^{ct}$, find the value of c and simplify.

$$\text{Take the ln of both sides} \quad \ln 3^{-2t} = \ln e^{ct}$$

$$-2t \ln 3 = ct \ln e$$

$$-2 \ln 3 = c \quad \text{since } \ln e = 1$$

$$c = \ln 3^{-2} = \ln \left(\frac{1}{9}\right) = \ln 1 - \ln 9 = -\ln 9$$

$$e) (3x-1)(2x-1)=3(4x-1)-1$$

$$6x^2 - 3x - 2x + 1 = 12x - 4$$

$$6x^2 - 17x + 5 = 0$$

$$(2x-5)(3x-1)=0 \quad \text{so } x=5/2 \text{ or } x= 1/3$$

Example 3.

$$C_0 = 30$$

Tripling time = 3 hr

$$\rightarrow \therefore \begin{cases} 3 = 1e^{r(3)} \\ 3 = e^{3r} \\ \ln 3 = \ln e^{3r} \\ \ln 3 = 3r \overbrace{\ln e}^1 \\ r = \frac{\ln 3}{3} \end{cases}$$

Let $t=9\text{hr}$

$$\begin{aligned} \text{a) } C(t) &= C_0 e^{rt} \\ C(9) &= 20e^{\frac{\ln 3}{3}(9)} \\ &= 20e^{3 \ln 3} \\ &= 20e^{\ln 3^3} \\ &= 20e^{\ln 27} \\ &= 20(27) = 540 \end{aligned}$$

$$\begin{aligned} \text{b) } C(30) &= 20e^{\frac{\ln 3}{3}(30)} \\ &= 20e^{10 \ln 3} \\ &= 20e^{\ln 3^{10}} \\ &= 20(3^{10}) \end{aligned}$$

$$\text{C1. } 2^{4x-1} = 2^6$$

$$4x-1=6$$

$$4x=7$$

$$x=7/4$$

Therefore, answer is A) .

$$\text{C2. } 3^{3x-4} = 3^4$$

$$3x-4=4$$

$$3x=8$$

$x=8/3$. Therefore, answer is B) .

$$\text{C3. } 5^{6x} = 5^4$$

$$6x=4$$

$$x=2/3$$

Therefore, answer is C) .

$$\text{C4. } (2^{-6})^2 = 4^{3x}$$

$$2^{-12} = 2^{6x}$$

$$-12=6x$$

$$x=-2$$

$$C5. \quad e^{-x} = \ln 2$$

$$\ln e^{-x} = \ln(\ln 2)$$

$$-x \ln e = \ln(\ln 2)$$

$$x = -\ln(\ln 2)$$

$$C6. (3x-1)(2x-1) = 3(4x-1) - 1$$

$$6x^2 - 3x - 2x + 1 = 12x - 4$$

$$6x^2 - 17x + 5 = 0$$

$$(2x-5)(3x-1) = 0 \quad \text{so } x = 5/2 \text{ or } x = 1/3$$

D. Cancellation Laws

Example 1.

$$\cos(\cos^{-1}(\frac{-1}{2})) = \frac{-1}{2} \text{ since } x \in [-1,1]$$

Example 2.

$$\cos \left[2 \arccos \left(\frac{1}{4} \right) \right]$$

$$\text{Let } \arccos \left(\frac{1}{4} \right) =$$

$$\theta, \text{ then we want } \cos 2\theta = 2\cos^2\theta - 1$$

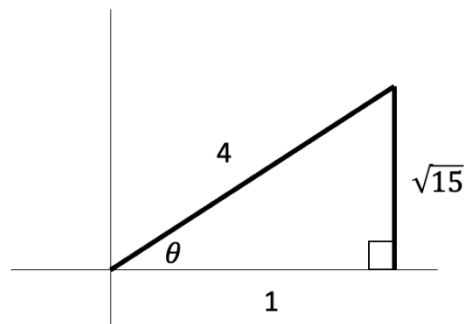
$$\text{Since } \arccos \left(\frac{1}{4} \right) = \theta, \text{ we have } \cos\theta = \frac{1}{4}$$

$$\text{So, } 2\cos^2\theta - 1 = 2 \left(\frac{1}{4} \right)^2 - 1 = \frac{2}{16} - \frac{16}{16} = -\frac{14}{16} = -\frac{7}{8}$$

$$\sin \left[2 \arccos \left(\frac{1}{4} \right) \right] = \text{we need to find } \sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{15}}{4}$$

$$\sin \left[2 \arccos \left(\frac{1}{4} \right) \right] = 2 \left(\frac{\sqrt{15}}{4} \right) \left(\frac{1}{4} \right) = \frac{1}{8} \sqrt{15}$$



***note: for the first part, you can use any of the identities for $\cos 2\theta$ and you will get the same answer**

Example 3.

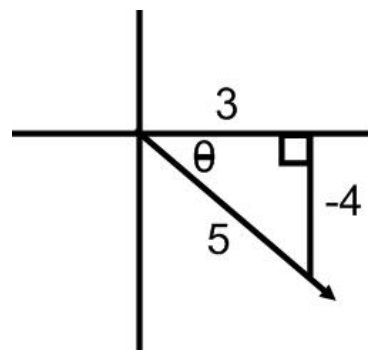
$$\cos \left[\arctan \left(-\frac{4}{3} \right) \right]$$

Let $\arctan \left(-\frac{4}{3} \right) = \theta$, so $\tan \theta = -\frac{4}{3} = \frac{\text{opp}}{\text{adj}}$ and since it was arctanx we

know it is in the fourth quadrant since it was negative (p.32 of booklet)

Find the other side using Pythagorean theorem and we get 5.

Now, we want $\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5}$.

**Example 4.**

$$\cos \left[2\arcsin \left(\frac{1}{5} \right) \right]$$

Let $\arcsin \left(\frac{1}{5} \right) = \theta$, we want $\cos 2\theta = 2\cos^2 \theta - 1$

From $\arcsin \left(\frac{1}{5} \right) = \theta$, we know $\sin \theta = \frac{1}{5} = \frac{\text{opp}}{\text{hyp}}$ Using Pythagorean

theorem, the other side would be $\sqrt{24}$ and then $\cos \theta = \frac{\sqrt{24}}{5}$. (In first quadrant, since it was arcsinx of a positive x)

$$\text{So, } \cos 2\theta = 2\cos^2 \theta - 1 = 2\left(\frac{\sqrt{24}}{5}\right)^2 - 1 = 2\left(\frac{24}{25}\right) - 1 = \frac{48}{25} - \frac{25}{25} = \frac{23}{25}$$

Example 5.

$$\sin(2 \cos^{-1}(x))$$

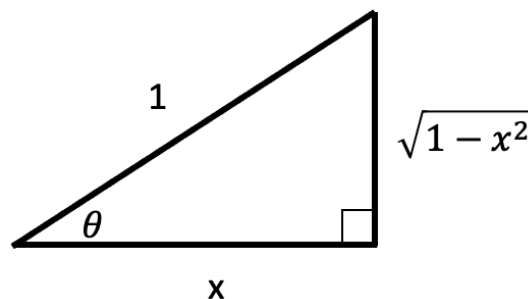
$$\text{Let } \theta = \cos^{-1}(x)$$

$$\therefore \cos \theta = \frac{x}{1} = \frac{\text{adj}}{\text{hyp}}$$

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{1-x^2}}{1}$$

$$\therefore \sin 2\theta = 2 \sin \theta \cos \theta$$

$$= 2 \left(\frac{\sqrt{1-x^2}}{1} \right) \left(\frac{x}{1} \right) = 2x\sqrt{1-x^2}$$



$$\text{D1. } \sin(\sin^{-1}(\frac{-1}{2})) = \frac{-1}{2} \quad \text{since } x \in [-1,1]$$

D2.

$\cos^{-1} x$ defined in 1st and 2nd quadrants since we want $\arccos\left(\frac{-1}{2}\right)$
negative number

\therefore looking at 2nd quadrant by CAST rule

$$\arccos\left(\frac{-1}{2}\right) = \theta \quad \cos \theta = \frac{-1}{2} \quad \text{adj hyp}$$

$$\therefore \sin(\arccos(\frac{-1}{2})) = \sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$$

$$D3. \quad \sin(\tan^{-1}\left(\frac{-1}{2}\right)) = \sin \theta \quad \tan^{-1}\left(\frac{-1}{2}\right) = \theta \quad \text{negative}$$

\therefore we are in 4th quadrant

$$2^2 + (-1)^2 = c^2$$

$$\tan \theta = \frac{-1 \text{ opp}}{2 \text{ adj}}$$

$$5 = c^2$$

$$c = \sqrt{5}$$

$$\therefore \sin(\tan^{-1}\left(\frac{-1}{2}\right)) = \sin \theta = \frac{-1}{\sqrt{5}}$$

$$D4. \quad \tan(\arctan 1) = 1 \quad \text{since } \arctan x \text{ is defined for all } x \in \mathbb{R}$$

$$D5. \quad \arccos(\cos \frac{\pi}{3}) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

**recall $\arccos(\cos x) = x$ for all $x \in [0, \pi]$*

$$D6. \quad \sin(\arccos \frac{1}{2}) = \sin \theta$$

$$\arccos \frac{1}{2} = \theta$$

$$\cos \theta = \frac{1 \text{ adj}}{2 \text{ hyp}} \quad \arccos x \text{ is defined in 1st and 2nd quadrant.}$$

Since $\frac{1}{2}$ is positive, we are in the 1st quadrant.

$$\sin(\arccos \frac{1}{2}) = \sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$$

D7. $\tan(\arccos \frac{1}{\sqrt{2}})$ *positive* \therefore *in 1st quadrant*

$$\arccos \frac{1}{\sqrt{2}} = \theta$$

$$\cos \theta = \frac{1}{\sqrt{2}} \frac{\text{adj}}{\text{hyp}}$$

$$\therefore \tan(\arccos \frac{1}{\sqrt{2}}) = \tan \theta = \frac{1}{1} = 1$$

D8. $\cos(\arcsin \frac{1}{2})$ *arc sinx is defined in 1st and 4th*

$$\text{Let } \arcsin \frac{1}{2} = \theta \quad \sin \theta = \frac{1}{2} \frac{\text{opp}}{\text{hyp}}$$

Since 1/2 is a positive number, we are in the 1st quadrant

$$\cos(\arcsin \frac{1}{2}) = \cos(\theta) = \frac{\sqrt{3}}{2}$$

D9. $\cos(\arccos \frac{\sqrt{3}}{\sqrt{2}}) = \frac{\sqrt{3}}{2}$ *since* $\frac{\sqrt{3}}{2} \in [-1, 1]$

D10. $\cos^{-1}(\tan \frac{\pi}{4}) = \cos^{-1}(1) = 0$

D11. $\cos(\sin^{-1}(\frac{-3}{5}))$

$\sin^{-1}(\frac{-3}{5}) = \theta$

$\sin \theta = \frac{-3}{5} \quad \text{negative} \therefore$

$\sin^{-1}x$ is in the 4th quadrant

$$(-3)^2 + x^2 = 5^2$$

$$x^2 = 25 - 9$$

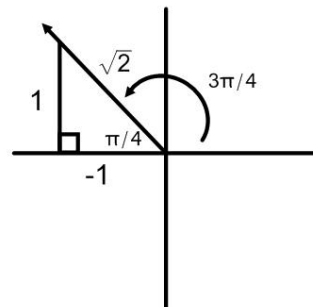
$$x^2 = 16$$

$$x = 4$$

$$\cos(\sin^{-1}(\frac{-3}{5})) = \cos \theta = \frac{4}{5}$$

$$D12. \arctan \left[\tan \left(\frac{3\pi}{4} \right) \right]$$

$\tan \left(\frac{3\pi}{4} \right) = \frac{\text{opp}}{\text{adj}} = \frac{1}{-1} = -1$ using special triangles

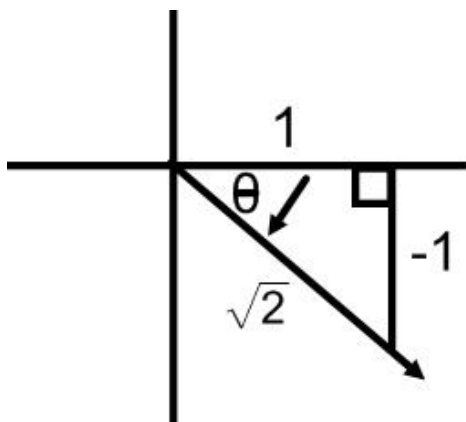


$$\arctan \left[\tan \left(\frac{3\pi}{4} \right) \right] = \arctan(-1)$$

$= \theta$ so we draw the angle in the 4th quadrant (see p. 36)

and $\tan \theta = -1$

$$\theta = -\frac{\pi}{4}$$



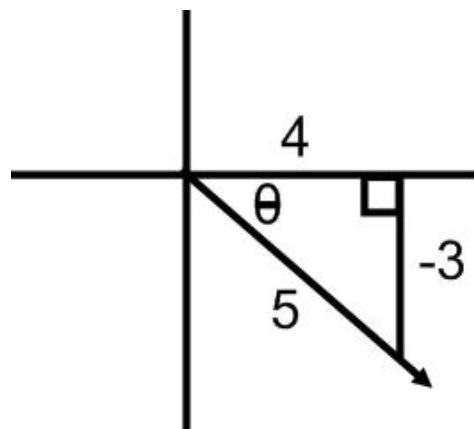
$$D13. \cos \left[\arcsin \left(-\frac{3}{5} \right) \right]$$

Let $\arcsin \left(-\frac{3}{5} \right) = \theta$, then $\sin \theta =$

$-\frac{3}{5}$ and it is in the 4th quadrant. Using Pythagorean Theorem, we get the other side is 4.

We want to find $\cos \left[\arcsin \left(-\frac{3}{5} \right) \right] = \cos \theta =$

$\frac{\text{adj}}{\text{hyp}} = \frac{4}{5}$ from our diagram.



$$D14. \cos(2 \tan^{-1}(x))$$

Let $\theta = \tan^{-1} x$

$$\tan \theta = \frac{x}{1}$$

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{1+x^2}}$$

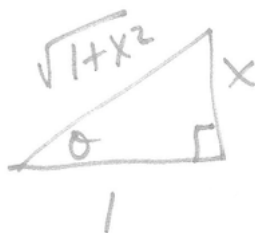
$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1+x^2}}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$= \left(\frac{1}{\sqrt{1+x^2}} \right)^2 - \left(\frac{x}{\sqrt{1+x^2}} \right)^2$$

$$= \frac{1}{1+x^2} - \frac{x^2}{1+x^2}$$

$$= \frac{1-x^2}{1+x^2}$$



E. Inverse

Example 1.

1. no
2. no
3. yes
4. yes

Visually Example

$$f(4) = 5 \text{ means } f^{-1}(5) = 4$$

$$f^{-1}(10) = 12 \therefore f(12) = 10$$

Example 2

$$g^{-1}(4) = 6$$

$$\therefore g(6) = 4$$

$$f(x) = \sqrt[3]{2x - 2}$$

$$f(-1) = \sqrt[3]{2(-1) - 2}$$

But

$$f^{-1}(-1)?$$

$$(-1)^3 = \left(\sqrt[3]{2(-1) - 2}\right)^3$$

$$-1 = 2x - 2$$

$$-1 + 2 = 2x$$

$$2x = 1$$

$$x = \frac{1}{2}$$

Point (7,2) is on g

$$\therefore g(7) = 2$$

$$g^{-1}(2) = 7$$

$$\text{a) } f^{-1}(-1) + g(6)$$

$$= \frac{1}{2} + \frac{4}{1}$$

$$= \frac{1}{2} + \frac{8}{2} = \frac{9}{2}$$

$$\text{b) } g^{-1}(f(5))$$

$$f(x) = \sqrt[3]{2x - 2}$$

$$\begin{aligned} f(5) &= \sqrt[3]{2(5) - 2} \\ &= \sqrt[3]{8} = 2 \end{aligned}$$

$$\therefore g^{-1}(f(5)) = g^{-1}(2) = 7$$

Example 4.

$$y = \frac{4}{x + 2}$$

$$x = \frac{4}{y + 2}$$

$$xy + 2x = 4$$

$$xy = 4 - 2x$$

$$y = \frac{4-2x}{x} \quad \therefore f^{-1}(x) = \frac{4-2x}{x}$$

Example 5. $y = \frac{3x+1}{x-1}$

$$x = \frac{3y + 1}{y - 1}$$

$$xy - x = 3y + 1$$

$$xy - 3y = 1 + x$$

$$y(x - 3) = 1 + x$$

$$y = \frac{1 + x}{x - 3} \quad \therefore f^{-1}(x) = \frac{1 + x}{x - 3}$$

Example 6.

$$y = \sqrt{1 - x^2}, 0 \leq x \leq 1$$

$$x = \sqrt{1 - y^2} \text{ square both sides}$$

$$x^2 = 1 - y^2$$

$$y^2 = -x^2 + 1$$

$$y = \sqrt{1 - x^2}, 0 \leq y \leq 1$$

$$\therefore f^{-1}(x) = \sqrt{1 - x^2}, 0 \leq y \leq 1$$

Example 7. $y = \tan^{-1}(\ln x)$

$$x = \tan^{-1}(\ln y)$$

$$\tan x = \ln y \text{ exponential form gives: } y = e^{\tan x} \quad \therefore f^{-1}(x) = e^{\tan x}$$

Example 8.

$$y = 20(1 - e^{-4x})$$

$$X \leftrightarrow Y$$

$$\frac{x}{20} = \frac{20}{20}(1 - e^{-4y})$$

$$\frac{x}{20} = (1 - e^{-4y})$$

$$\frac{x}{20} - 1 = -e^{-4y}$$

$$-\frac{x}{20} + 1 = e^{-4y}$$

$$\ln \left| -\frac{x}{20} + 1 \right| = \ln e^{-4y}$$

$$\ln \left| -\frac{x}{20} + 1 \right| = -4y \ln e$$

$$y = -\frac{1}{4} \ln \left| -\frac{x}{20} + 1 \right|$$

$$E1. y = 9e^{x+5} - 27$$

$$x = 9e^{y+5} - 27$$

$$x + 27 = 9e^{y+5}$$

$$\frac{x + 27}{9} = e^{y+5}$$

$$\ln\left(\frac{x + 27}{9}\right) = \ln e^{y+5}$$

$$y + 5 = \ln\left(\frac{x + 27}{9}\right)$$

$$y = \ln\left(\frac{x + 27}{9}\right) - 5$$

$$\therefore f^{-1}(x) = \ln\left(\frac{x+27}{9}\right) - 5$$

$$E2. y = \ln(7x + 2)$$

$$x = \ln(7y + 2)$$

$$e^x = e^{\ln(7y+2)}$$

$$e^x = 7y + 2$$

$$7y = e^x - 2$$

$$y = \frac{e^x - 2}{7}$$

$$f^{-1}(x) = \frac{e^x - 2}{7}$$

$$\text{E3. } y = e^{3x+5}$$

$$x = e^{3y+5}$$

$$\ln x = \ln e^{3y+5}$$

$$\ln x = (3y + 5) \ln e$$

$$\frac{\ln x - 5}{3} = y \quad \therefore f^{-1}(x) = \frac{\ln x - 5}{3}$$

$$\text{E4. } y = \frac{x}{3+x}$$

$$x = \frac{y}{3+y}$$

$$3x + xy = y$$

$$3x = y - xy$$

$$3x = y(1 - x)$$

$$y = \frac{3x}{1-x} \quad \therefore f^{-1}(x) = \frac{3x}{1-x}$$

$$\text{E5. } y = \ln(4x - 2)$$

$$x = \ln(4y - 2)$$

$$e^x = e^{\ln(4y-2)}$$

$$e^x = 4y - 2$$

$$4y = e^x + 2 \quad \therefore f^{-1}(x) = \frac{e^x + 2}{4}$$

$$E6. g(x) = \frac{1}{\sqrt{4-x^2}} \quad -2 < x \leq 0$$

$$y = \frac{1}{\sqrt{4-x^2}}$$

$$x \leftrightarrow y \quad x = \frac{1}{\sqrt{4-y^2}}$$

$$(x \sqrt{4-y^2})^2 = (1)^2$$

$$x^2(4-y^2) = 1$$

$$4x^2 - x^2y^2 = 1$$

$$4x^2 = 1 + x^2y^2$$

$$4x^2 - 1 = x^2y^2$$

$$y^2 = \frac{4x^2 - 1}{x^2}$$

$$y = -\sqrt{\frac{4x^2-1}{x^2}} \quad (-2 < y \leq 0)$$

$$f^{-1}(x) = -\sqrt{\frac{4x^2-1}{x^2}}$$

F. Logarithms

Example 1. Solve each of the following:

a) $\log_3 27 = 3$ since $3^3 = 27$

b) $\log_{25} 5 = \frac{1}{\log_5 25} = \frac{1}{2}$

c) $\log_2 32 = 5$ since $2^5 = 32$

Example 2. Simplify each of the following using logarithmic rules.

a) $6^{3\log_6 2} = 6^{\log_6 2^3}$

$$= 2^3$$

$$= 8$$

b) $\log_4 4^3 = 3$

c) $e^{\ln 3 + \ln 6}$

$$= e^{\ln 18} = 18$$

d) $e^{\ln 12 - \ln 4}$

$$= e^{\ln\left(\frac{12}{4}\right)} = e^{\ln 3} = 3$$

e) $e^{2\ln 4 + \ln 5}$

$$= e^{\ln 4^2 + \ln 5}$$

$$= e^{\ln 16 + \ln 5}$$

$$= e^{\ln(16 \times 5)} = 80$$

f) $\log_3 27 - \log_2 32$

$$= 3 - 5 = -2$$

Example 3. Solve each of the following:

b) $e^{3t} = 14$ take \ln of both sides $\ln e^{3t} = \ln 14$

$$3t \ln e = \ln 14$$

$$3t = \ln 14$$

$$t = \frac{\ln 14}{3}$$

c) $\log_3(4x + 7) - \log_3(x + 1) = 2$

$$\log_3 \left(\frac{4x+7}{x+1} \right) = 2$$

$$3^2 = \frac{4x+7}{x+1}$$

$$9x + 9 = 4x + 7$$

$$5x = -2$$

$x = -\frac{2}{5}$ substitute into both original logarithms and both are >0 ,
so it is a solution

d) $2(3^{x-5}) = 28$ divide both sides by 2

$$3^{x-5} = 14 \quad \text{take ln of both sides}$$

$$\ln 3^{x-5} = \ln 14$$

$$(x - 5) \ln 3 = \ln 14$$

$$x \ln 3 - 5 \ln 3 = \ln 14$$

$$x \ln 3 = \ln 14 + \ln 3^5$$

$$x = \frac{\ln 14 + \ln 3^5}{\ln 3}$$

$$*f) \frac{\log_3 5 + \log_3 25}{\log_3 125 - \log_3 5} = \frac{\log_3 125}{\log_3 \left(\frac{125}{5}\right)} = \frac{\log_3 125}{\log_3 25} = \frac{\log_5 125}{\log_5 25} = \frac{3}{2}$$

$$*g) \log_{25} 625 + \log_6 36 + \log_4 32 = \frac{\log 625}{\log 25} + 2 + \frac{\log 32}{\log 4}$$

$$= \frac{\log_5 625}{\log_5 25} + 2 + \frac{\log_2 32}{\log_2 4}$$

$$= \frac{4}{2} + \frac{4}{2} + \frac{5}{2}$$

$$= \frac{13}{2}$$

$$*h) \frac{\log_4 x + \log_4 y}{\log_4 x^2 + \log_4 y^2} = \frac{\log_4 x + \log_4 y}{2 \log_4 x + 2 \log_4 y} = \frac{\log_4 x + \log_4 y}{2(\log_4 x + \log_4 y)}$$

$$= \frac{1}{2}$$

$$*i) \frac{\log_4 32}{\log_{27} 9} = \frac{\frac{\log 32}{\log 4}}{\frac{\log 9}{\log 27}} = \frac{\frac{\log_2 32}{\log_2 4}}{\frac{\log_3 9}{\log_3 27}} = \frac{\frac{5}{2}}{\frac{2}{3}} = \frac{5}{2} \times \frac{3}{2} = \frac{15}{4}$$

$$J) \quad \log_9 x + \log_3 x = \frac{1}{2}$$

$$\frac{\log x}{\log 9} + \log_3 x = \frac{1}{2}$$

$$\frac{\log_3 x}{\log_3 9} + \log_3 x = \frac{1}{2}$$

$$\frac{\log_3 x}{2} + \log_3 x = \frac{1}{2}$$

$$\frac{1}{2} \log_3 x + 1 \log_3 x = \frac{1}{2}$$

$$\frac{3}{2} \log_3 x = \frac{1}{2}$$

$$\log_3 x = \frac{1}{2} \div \frac{3}{2}$$

$$\log_3 x = \frac{1}{2} \times \frac{2}{3}$$

$$\log_3 x = \frac{1}{3}$$

$$x = 3^{\frac{1}{3}} = \sqrt[3]{3}$$

Example 4.

$$\text{Substitute } \begin{pmatrix} 1, 2 \\ x, y \end{pmatrix} \quad 2 = \log_a 1 + b$$

$$2 = 0 + b$$

$$b = 2$$

$$\text{Substitute } B \begin{pmatrix} 4, 3 \\ x, y \end{pmatrix} \quad y = \log_a x + 2$$

$$3 = \log_a 4 + 2$$

$$1 = \log_a 4$$

$$a^1 = 4$$

$$a = 4$$

\therefore the equation is $y = \log_4 x + 2$

$$\text{F1. } e^{3x+5} = 6$$

$$\ln e^{3x+5} = \ln 6$$

$$(3x + 5) \ln e = \ln 6$$

$$\frac{3x}{3} = \frac{\ln 6 - 5}{3} \quad x = \frac{\ln 6 - 5}{3}$$

$$\text{F2. } \ln(x^2 + x) = 2 + \ln x$$

$$\ln(x^2 + x) - \ln x = 2$$

$$\ln\left(\frac{x^2 + x}{x}\right) = 2 \quad \textit{put into exp. form}$$

$$e^2 = \frac{x^2 + x}{x}$$

$$e^2 x = x^2 + x$$

$$x^2 + x - e^2 x = 0$$

$$x^2 + x(1 - e^2) = 0$$

$$x[x + (1 - e^2)] = 0$$

$$x = 0 \quad \textit{OR} \quad x + 1 - e^2 = 0 \quad x = e^2 - 1$$

$$x \neq 0 \textit{ since } \ln 0 \textit{ is undefined} \quad \therefore x = e^2 - 1$$

$$\text{F3. } \log_8 2 + \log_2 8 = \frac{1}{\log_2 8} + \log_2 8 = \frac{1}{3} + 3$$

$$= 3\frac{1}{3} \textit{ OR } \frac{10}{3}$$

$$F4. \log_2(x + 2) - \log_2(2x + 1) = 4$$

$$\log_2\left(\frac{x + 2}{2x + 1}\right) = 4$$

$$2^4 = \frac{x + 2}{2x + 1}$$

$$16 = \frac{x + 2}{2x + 1}$$

$$32x + 16 = x + 2$$

$$31x = 2 - 16$$

$$31x = -14$$

$$x = -\frac{14}{31} \quad \text{Check } \log_2\left(2\left(-\frac{14}{31}\right) + 1\right)$$

$$= \log_2\left(-\frac{28}{31} + \frac{31}{31}\right)$$

$$= \log_2\left(\frac{3}{31}\right) \text{ is defined} \quad \therefore x = -\frac{14}{31}$$

$$F5. \log_2 16 - \log_2 32 = \log_2\left(\frac{16}{32}\right) = \log_2\frac{1}{2} = -1$$

Therefore, answer is E).

$$F6. 2\log_2 4 + \log_2 8 = \log_2 4^2 + 3 = \log_2 16 + 3 = 4 + 3 = 7$$

Therefore, answer is A).

$$F7. \log_4 32 - \log_4 16 = \log_4 \left(\frac{32}{16} \right) = \log_4 2 = \frac{1}{\log_2 4} = \frac{1}{2}$$

Therefore, answer is C).

$$F8. e^{\ln 3 + \ln 9} = e^{\ln 27} = 27$$

Therefore, answer is B).

$$F9. \frac{\log_5 3 + \log_5 81}{\log_5 81} = \frac{\log_5 243}{\log_5 81} = \frac{\log_3 243}{\log_3 81} = \frac{5}{4}$$

Change both bases, so you can evaluate

$$F10. \frac{\log_4 25}{\log_4 125} = \frac{\log_5 25}{\log_5 125} = \frac{2}{3} \quad \text{Change of base}$$

Therefore, answer is C).

$$F11. \text{Simplify: } \frac{\log_9 3 - \log_2 16}{\log_4 64} = \frac{\frac{1}{\log_3 9} - 4}{3} = \frac{\frac{1}{2} - 4}{3} = \frac{\frac{1-8}{2}}{\frac{3}{1}} = \frac{-7}{2} \times \frac{1}{3} = -\frac{7}{6}$$

$$F12. \log_8 32 + \log_9 27 = \frac{\log 32}{\log 8} + \frac{\log 27}{\log 9} = \frac{\log_2 32}{\log_2 8} + \frac{\log_3 27}{\log_3 9}$$

$$= \frac{5}{3} + \frac{3}{2} = \frac{10}{6} + \frac{9}{6} = \frac{19}{6}$$

$$\begin{aligned} \text{F13. } \frac{\ln x^8 + \ln y^8}{\ln(x^4 y^4)} &= \frac{\ln(x^8 y^8)}{\ln(x^4 y^4)} = \frac{\ln(x^4 y^4)^2}{\ln(x^4 y^4)} = \frac{2\ln(x^4 y^4)}{\ln(x^4 y^4)} \\ &= 2 \end{aligned}$$

$$\text{F14. Solve: } \log_3 x + \log_3(x + 2) = \log_3(x + 20)$$

$$\log_3(x(x + 2)) = \log_3(x + 20)$$

$$x(x + 2) = x + 20$$

$$x^2 + x - 20 = 0 \text{ factor}$$

$$(x+5)(x-4)=0$$

$x = -5, 4$ but we can't take the log of a negative number (or 0), so we exclude $x = -5$ and the only solution is $x = 4$.

G. Limits

Example 1. a) $\lim_{x \rightarrow -4^-} f(x) = 3$ $\lim_{x \rightarrow -4^+} f(x) = 0$

$\therefore \lim_{x \rightarrow -4} f(x) = DNE$

b) $\lim_{x \rightarrow -2} f(x) = -4$

c) $\lim_{x \rightarrow 4^+} f(x) = 5$

d) $\lim_{x \rightarrow 4^-} f(x) = 5$

e) $\lim_{x \rightarrow 4} f(x) = 5$

f) $f(4) = 5$

g) $\lim_{x \rightarrow 0} f(x) = DNE$ since $\lim_{x \rightarrow 0^-} f(x) = 0$ and $\lim_{x \rightarrow 0^+} f(x) = 1$

h) $f(0) = 1$

H. Properties of Limits

Example 1. $\lim_{x \rightarrow 4} \sqrt{x} = \sqrt{4} = 2$

Example 2. $\lim_{x \rightarrow 1} (x^2 + 3x) = 1^2 + 3(1) = 4$

Example 3. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{5x} = \frac{2^2 - 4}{5(2)} = \frac{0}{10} = 0$

Example 5.

$$= \lim_{x \rightarrow 2} \frac{(x-2)(x-3)}{(x-2)}$$

$$= \lim_{x \rightarrow 2} (x - 3)$$

$$= 2 - 3$$

$$= -1$$

Example 6.

$$= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2}+1)}{(\sqrt{x-2}-1)(\sqrt{x-2}+1)}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2}+1)}{x-2-1}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2}+1)}{(x-3)}$$

$$= \lim_{x \rightarrow 3} (\sqrt{x-2} + 1)$$

$$= 1 + 1$$

$$= 2$$

Example 7. Since $x \rightarrow 3$ makes $|x - 3| = 0 \therefore$ we must do left and right hand limits

$$\begin{aligned} \therefore \lim_{x \rightarrow 3^-} \frac{(x-2)(x-3)}{-(x-3)} & \qquad \qquad \lim_{x \rightarrow 3^+} \frac{(x-2)(x-3)}{(x-3)} \\ \lim_{x \rightarrow 3^-} \frac{x-2}{-1} & \qquad \qquad \lim_{x \rightarrow 3^+} (x-2) \\ = \frac{3-2}{-1} & \qquad \qquad = 3-2 \\ = -1 & \qquad \qquad = 1 \end{aligned}$$

$$\therefore \lim_{x \rightarrow 3} \frac{x^2-5x+6}{|x-3|} = DNE$$

Since $\lim_{x \rightarrow 3^+} f(x) \neq \lim_{x \rightarrow 3^-} f(x)$

Example 8. $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 2x}$

$$= \lim_{x \rightarrow 0} \frac{5x \sin 5x / 5x}{2x \sin 2x / 2x}$$

$$= \frac{5}{2} \lim_{x \rightarrow 0} \frac{\sin 5x / 5x}{\sin 2x / 2x}$$

$$= \frac{5}{2} \frac{\lim_{x \rightarrow 0} \sin 5x / 5x}{\lim_{x \rightarrow 0} \sin 2x / 2x}$$

$$= \frac{5}{2} \text{ since } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (\text{p.79 of booklet})$$

Example 9. The limit $\lim_{x \rightarrow \pi} \frac{f(x)-4}{x-\pi}$ can only equal a number if $f(x)-4 \rightarrow 0$

a) So, therefore, $f(x) \rightarrow 4$ and the limit is 4 as well. Since the answer to the limit is just a number, it would have to be $0/0$ and you did something to make it a constant answer. Otherwise, since the bottom is 0, it would be $\#/0$ which would either be plus or minus infinity.

b) If $\lim_{x \rightarrow 3^+} \frac{f(x)-7}{x-3} = 2$, find $\lim_{x \rightarrow 3^+} f(x)$. Here again, in order to get a numerical answer of 2, it must be $0/0$ as otherwise it would be $\#/0$ which is $+/-$ infinity. So, $f(x) - 7 \rightarrow 0$ and $\lim_{x \rightarrow 3^+} f(x) = 7$

Example 10.

$$\begin{aligned}
& \lim_{x \rightarrow 4} \frac{\sqrt{8-x} - 2}{\sqrt{5-x} - 1} \quad (0/0) \\
&= \lim_{x \rightarrow 4} \frac{(\sqrt{8-x} - 2)(\sqrt{5-x} + 1)}{(\sqrt{5-x} - 1)(\sqrt{5-x} + 1)} \\
&= \lim_{x \rightarrow 4} \frac{(\sqrt{8-x} - 2)(\sqrt{5-x} + 1)}{(5-x-1)} \\
&= \lim_{x \rightarrow 4} \frac{(\sqrt{5-x} + 1)(\sqrt{8-x} - 2)(\sqrt{8-x} + 2)}{4-x(\sqrt{8-x} + 2)} \\
&= \lim_{x \rightarrow 4} \frac{(\sqrt{5-x} + 1)(8-x-4)}{(4-x)(\sqrt{8-x} + 2)} \\
&= \lim_{x \rightarrow 4} \frac{(\sqrt{5-x} + 1)(x-4)}{(4-x)(\sqrt{8-x} + 2)} \\
& \lim_{x \rightarrow 4} \frac{(\sqrt{5-x} + 1)}{(\sqrt{8-x} + 2)} \\
&= \frac{\sqrt{5-4} + 1}{\sqrt{8-4} + 2} \\
&= \frac{2}{4} = \frac{1}{2}
\end{aligned}$$

$$\textbf{Example 11} = \lim_{t \rightarrow 0} \left(\frac{1}{t} + \frac{1}{t(t-1)} \right)$$

$$= \lim_{t \rightarrow 0} \left(\frac{1(t-1)}{t(t-1)} + \frac{1}{t(t-1)} \right)$$

$$= \lim_{t \rightarrow 0} \frac{t-1+1}{t(t-1)}$$

$$= \lim_{t \rightarrow 0} \frac{t}{t(t-1)}$$

$$= \lim_{t \rightarrow 0} t = \frac{1}{0-1} = -1$$

H1. (%)

$$= \lim_{h \rightarrow 0} \frac{h(4h+1)}{h(1+h)}$$

$$= \lim_{h \rightarrow 0} \frac{4h+1}{1+h}$$

$$= \frac{1}{1}$$

$$= 1$$

$$\text{H2. } = \lim_{x \rightarrow 5} \frac{\sqrt{5-2}}{5-5} = \frac{\sqrt{3}}{0} = \infty$$

H3. (%)

$$= \lim_{x \rightarrow 4} \frac{(x-4)(x+4)}{(x-4)}$$

$$= \lim_{x \rightarrow 4} \frac{(x+4)}{1} = 4 + 4 = 8$$

$$\text{H4. } = \frac{0}{1+2(4)} = \frac{0}{9} = 0$$

H5. (%)

$$= \lim_{h \rightarrow 0} \frac{h(6+h)}{h(1+h^2)}$$

$$= \lim_{h \rightarrow 0} \frac{6+h}{1+h^2} = \frac{6}{1} = 6$$

H6. $|x| = x$ since $x \rightarrow 0^+$

$$\therefore \lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = 1$$

$$\begin{aligned} \text{H7.} &= \lim_{x \rightarrow 7} \frac{(x-7)(x+7)}{x-7} \\ &= \lim_{x \rightarrow 7} (x+7) = 7+7 = 14 \end{aligned}$$

H8. (%)

$$\begin{aligned} &= \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)(x-2)} \\ &= \lim_{x \rightarrow 2} \frac{x+2}{x-2} = \frac{2+2}{2-2} = \frac{4}{0} = \infty \end{aligned}$$

$$\begin{aligned} \text{H9.} &= \lim_{h \rightarrow 0} \left(\frac{1}{h} + \frac{1}{h(h-1)} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{1(h-1)}{h(h-1)} + \frac{1}{h(h-1)} \right) \\ &= \lim_{h \rightarrow 0} \frac{h-1+1}{h(h-1)} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(h-1)} \\ &= \lim_{h \rightarrow 0} \frac{1}{h-1} = \frac{1}{0-1} = -1 \end{aligned}$$

H10. (%)

$$\begin{aligned} &= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2}+1)}{(\sqrt{x-2}-1)(\sqrt{x-2}+1)} \\ &= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2}+1)}{x-2-1} \\ &= \lim_{x \rightarrow 3} (\sqrt{x-2}+1) \\ &= \sqrt{3-2}+1 = 2 \end{aligned}$$

H11. (%)

$$\lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}$$

$$= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x}+1)}{x-1}$$

$$\lim_{x \rightarrow 1} (\sqrt{x} + 1) = 1 + 1 = 2$$

$$\text{H12.} = \lim_{x \rightarrow 0} \frac{-(x-3)-(6x+3)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{-x+3-6x-3}{x}$$

$$= \lim_{x \rightarrow 0} \frac{-7x}{x} = -7$$

$$\text{H13.} = \lim_{x \rightarrow 0} \frac{-(x-5)-(x+5)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{-x+5-x-5}{x}$$

$$= \lim_{x \rightarrow 0} \frac{-2x}{x} = -2$$

$$\text{H14.} \quad \lim_{h \rightarrow 0} \frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)}$$

$$= \lim_{h \rightarrow 0} \frac{9+h-9}{h(\sqrt{9+h}+3)}$$

$$= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{9+h}+3)}$$

$$= \lim_{h \rightarrow 0} \frac{1}{(\sqrt{9+h}+3)} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

$$\text{H15. } \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 4x}$$

$$= \lim_{x \rightarrow 0} \frac{3x \sin 3x / 3x}{4x \sin 4x / 4x}$$

$$= \frac{3}{4} \lim_{x \rightarrow 0} \frac{\sin 3x / 3x}{\sin 4x / 4x}$$

$$= \frac{3}{4} \frac{\lim_{x \rightarrow 0} \sin 3x / 3x}{\lim_{x \rightarrow 0} \sin 4x / 4x}$$

$$= \frac{3}{4} \text{ since } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (\text{p. 70 of booklet})$$

H16. The limit $\lim_{x \rightarrow 5^+} \frac{f(x)+5}{x-5}$ can only equal a number if $f(x)+5 \rightarrow 0$

So, therefore, $f(x) \rightarrow -5$

$$\text{H17. } \lim_{x \rightarrow -\infty} \frac{5x-2}{\sqrt{5x^2-x^3-5x+1}}$$

$$\lim_{x \rightarrow -\infty} \frac{5x-2}{\sqrt{x^3\left(\frac{5}{x}-1-\frac{5}{x^2}+\frac{1}{x^3}\right)}} \quad \sqrt{x^3} = x^{\frac{3}{2}}$$

$$\lim_{x \rightarrow -\infty} \frac{5x-2}{x^{\frac{3}{2}} \sqrt{\left(\frac{5}{x}-1-\frac{5}{x^2}+\frac{1}{x^3}\right)}}$$

$$= \lim_{x \rightarrow -\infty} \frac{\frac{5x}{x^{\frac{3}{2}}} - \frac{2}{x^{\frac{3}{2}}}}{\frac{3}{x^{\frac{3}{2}}} \sqrt{\left(\frac{5}{x}-1-\frac{5}{x^2}+\frac{1}{x^3}\right)}}$$

$$= \lim_{x \rightarrow -\infty} \frac{\frac{5x^1}{x^{\frac{3}{2}}} - \frac{2}{x^{\frac{3}{2}}}}{\sqrt{\left(\frac{5}{x}-1-\frac{5}{x^2}+\frac{1}{x^3}\right)}} = \lim_{x \rightarrow -\infty} \frac{\frac{5}{x^{\frac{1}{2}}} - \frac{2}{x^{\frac{3}{2}}}}{\sqrt{\left(\frac{5}{x}-1-\frac{5}{x^2}+\frac{1}{x^3}\right)}} = \frac{0-0}{\sqrt{(0-1-0+0)}}$$

$$= \frac{0}{\sqrt{-1}} = 0$$

I. Continuity

Example 1. $x^2 - 25 = 0$

$$x^2 = 25$$

$$x = 5, -5$$

$\therefore f(x)$ is discontinuous at $x = 5, -5$

Example 2. a) $= 1^2 - 9 = -8$

b) $= \lim_{x \rightarrow 1^-} (x^2 - 9) = 1^2 - 9 = -8$

c) $= \lim_{x \rightarrow 1^+} (2x + 4) = 2(1) + 4 = 6$

d) No, since $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$

Example 3. Since the function is continuous everywhere,
 $f(x)$ is continuous at $x = 3$

$$\therefore k(3^2 - 3) = 1 + 3k^2(3)$$

$$6k = 9k^2 + 1$$

$$9k^2 - 6k + 1 = 0$$

$$(3k - 1)(3k - 1) = 0$$

$$k = \frac{1}{3}$$

Example 4.

$$\lim_{x \rightarrow 0^+} f(x) = f(0) \quad \therefore \lim_{x \rightarrow 0^+} \frac{\sin(3x)}{x} = A$$

$$H = \lim_{x \rightarrow 0^+} \frac{3 \cos(3x)}{1} = A$$

$$\frac{3(1)}{1} = A \quad \therefore A = 3$$

Or use the limit $\lim_{x \rightarrow 0^+} \frac{\sin(3x)}{x} = A$

$$\lim_{x \rightarrow 0^+} \frac{3 \sin(3x)}{3x} = A$$

$$3 \lim_{x \rightarrow 0^+} \frac{\sin(3x)}{3x} = A$$

A=3 since the limit is 1. (see p. 79 booklet)

$$\lim_{x \rightarrow 0^-} f(x) = f(0) = A = 3$$

$$\lim_{x \rightarrow 0^-} \frac{(3 - \sqrt{9-x})(3 + \sqrt{9-x})}{Bx(3 + \sqrt{9-x})} = 3$$

$$\lim_{x \rightarrow 0^-} \frac{(9 - (9-x))}{Bx(3 + \sqrt{9-x})} = 3$$

$$\lim_{x \rightarrow 0^-} \frac{x}{Bx(3 + \sqrt{9-x})} = 3$$

$$\frac{1}{B(3 + \sqrt{9})} = 3$$

$$\frac{1}{B(6)} = 3$$

$$18B = 1$$

$$B = \frac{1}{18}$$

$$11. a) = 1^2 + 4 = 5$$

$$b) = \lim_{x \rightarrow 1^-} (x^2 + 4) = 1^2 + 4 = 5$$

$$c) = \lim_{x \rightarrow 1^+} (2x + 3) = 2(1) + 3 = 5$$

$$d) \therefore \text{Yes, since } f(1) = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$$

$$12. x^2 - 100 = 0$$

$$x^2 = 100$$

$$x = 10, -10$$

$\therefore f(x)$ is discontinuous at $x = 10, -10$

$$13. a) f(2) = 2^2 - 4 = 0$$

$$b) \lim_{x \rightarrow 2^-} (x^2 - 4) = 2^2 - 4 = 0$$

$$c) \lim_{x \rightarrow 2^+} (4x + 3) = 4(2) + 3 = 11$$

$$d) \therefore \text{No, since } \lim_{x \rightarrow 2^+} f(x) \neq \lim_{x \rightarrow 2^-} f(x)$$

$$14. f(x) \text{ is continuous at } x = 3$$

$$\therefore k(3)^2 + 1 = 4(3) + 3$$

$$9k + 1 = 12 + 3$$

$$9k = 14$$

$$k = \frac{14}{9}$$

$$15. \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (\ln x) = \ln 1 = 0$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} e^x = e^1$$

\therefore No, since $e \neq 0$

$$16. \quad f(x) \text{ is continuous at } x = 2$$

$$\therefore 4(2)^2 + k(2) = 2(2)^3 - 5k$$

$$16 + 2k = 16 - 5k$$

$$7k = 0$$

$$k = 0$$

$$17. \quad \lim_{x \rightarrow 0^+} f(x) = f(0) \quad \therefore \lim_{x \rightarrow 0^+} \frac{\sin(2x)}{x} = A$$

$$H = \lim_{x \rightarrow 0^+} \frac{2 \cos(2x)}{1} = A$$

$$\frac{2(1)}{1} = A \quad \therefore A = 2$$

$$\lim_{x \rightarrow 0^-} f(x) = f(0) = A = 2$$

$$\lim_{x \rightarrow 0^-} \frac{(4 - \sqrt{16-x})(4 + \sqrt{16-x})}{Bx(4 + \sqrt{16-x})} = 2$$

$$\lim_{x \rightarrow 0^-} \frac{(16 - (16-x))}{Bx(4 + \sqrt{16-x})} = 2$$

$$\lim_{x \rightarrow 0^-} \frac{x}{Bx(4 + \sqrt{16-x})} = 2$$

$$\frac{1}{B(4 + \sqrt{16})} = 2$$

$$\frac{1}{B(8)} = 2$$

$$16B = 1 \quad B = \frac{1}{16}$$

J. Limits at Infinity

Example 2. $\lim_{x \rightarrow \infty} \frac{x^2(3 + \frac{4}{x} - \frac{5}{x^2})}{x^2(6 + \frac{2}{x} - \frac{1}{x^2})}$

$$= \lim_{x \rightarrow \infty} \frac{3 + \frac{4}{x} - \frac{5}{x^2}}{6 + \frac{2}{x} - \frac{1}{x^2}} = \frac{3}{6} = \frac{1}{2}$$

Example 3. $\lim_{t \rightarrow -\infty} \frac{\sqrt{16t^2 + t - 4}}{t - 4}$

$$= \lim_{t \rightarrow -\infty} \frac{\frac{\sqrt{16t^2 + t - 4}}{t}}{\frac{t - 4}{t}}$$

$$= \lim_{t \rightarrow -\infty} \frac{-\sqrt{\frac{16t^2 + t - 4}{t^2}}}{1 - \frac{4}{t}}$$

$$= \lim_{t \rightarrow -\infty} \frac{-\sqrt{16 + \frac{1}{t} - \frac{4}{t^2}}}{1 - \frac{4}{t}}$$

$$= -\frac{\sqrt{16 + 0 + 0}}{1 - 0}$$

$$= -4$$

Factor out the highest power in the denominator

$$\begin{aligned} \text{J1.} &= \lim_{x \rightarrow \infty} \frac{x(3x-4-\frac{6}{x})}{x(1-\frac{6}{x})} \\ &= \lim_{x \rightarrow \infty} \frac{3x-4-\frac{6}{x}}{1-\frac{6}{x}} = \frac{\infty-4-0}{1-0} = \infty \end{aligned}$$

$$\text{J2.} = \lim_{x \rightarrow \infty} \frac{3-\frac{6}{x}}{1+\frac{2}{x}-\frac{1}{x^2}} = \frac{3}{1} = 3$$

$$\begin{aligned} \text{J3.} &= \lim_{x \rightarrow \infty} \frac{x^2(1)}{x^2(6+\frac{1}{x^2})} \\ &= \lim_{x \rightarrow \infty} \frac{1}{6+\frac{1}{x^2}} = \frac{1}{6+0} = \frac{1}{6} \end{aligned}$$

$$\begin{aligned} \text{J4.} &\lim_{t \rightarrow -\infty} \frac{t(t^2-\frac{2}{t})}{t(2-\frac{4}{t})} \\ &= \lim_{t \rightarrow -\infty} \frac{t^2-\frac{2}{t}}{2-\frac{4}{t}} = \frac{(-\infty)^2-0}{2-0} = \infty \end{aligned}$$

J5.

$$\begin{aligned}
&= \lim_{t \rightarrow -\infty} \frac{\frac{\sqrt{9t^2+t-3}}{t}}{\frac{t-4}{t}} \\
&= \lim_{t \rightarrow -\infty} \frac{-\sqrt{\frac{9t^2+t-3}{t^2}}}{1 - \frac{4}{t}} \\
&= \lim_{t \rightarrow -\infty} \frac{-\sqrt{9 + \frac{1}{t} - \frac{3}{t^2}}}{1 - \frac{4}{t}} \\
&= \frac{-\sqrt{9+0+0}}{1-0} \\
&= -3
\end{aligned}$$

$$J6. \lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right) = DNE \text{ since } -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$J7. \lim_{x \rightarrow \infty} \frac{4x^4+3x^2+2x}{x^4+2x+1} = \lim_{x \rightarrow \infty} \frac{x^4(4+\frac{3}{x^2}+\frac{2}{x^3})}{x^4(1+\frac{2}{x^3}+\frac{1}{x^4})} = \lim_{x \rightarrow \infty} \frac{(4+\frac{3}{x^2}+\frac{2}{x^3})}{(1+\frac{2}{x^3}+\frac{1}{x^4})} = \frac{4}{1} = 4$$

K. Vertical and Horizontal Asymptotes

Example 1.

a) For VA, if you have a rational function

i.e. $y = \frac{p(x)}{q(x)}$, we check the values of $x = a$ where the denominator

$$q(a) = 0 \text{ and$$

a) if numerator $p(a) \neq 0$

→ we have a VA

$$f(x) = \frac{p(x)}{q(x)} = \frac{x + 8}{x^2 - 9}$$

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = 3, -3$$

$$p(3) = 3 + 8 = 11 \neq 0$$

$$p(-3) = -3 + 8 = 5 \neq 0$$

∴ we have 2 VA at $x = 3, -3$

$$HA \quad \lim_{x \rightarrow \infty} \frac{x + 3}{x^2 - 9} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{3}{x^2}}{1 - \frac{9}{x^2}} = \frac{0 + 0}{1 - 0} = \frac{0}{1} = 0$$

$$\therefore y = 0$$

NOTE: if we do the limit as x approaches $-\infty$, it is the same!

$$\text{b) } f(x) = \frac{x^2 - x - 12}{x - 4}$$

if $p(a) = 0$, then, we have to find $\lim_{x \rightarrow a} f(x)$ to see if it is $\pm \infty$

if it is $\pm \infty$, there is a VA

VA

$$f(x) = \frac{p(x)}{q(x)}$$

$$q(x) = 0 \quad x - 4 = 0 \quad x = 4$$

if $p(4) \neq 0$ we have a VA

$$p(x) = (x - 4)(x + 3)$$

$$\text{but here } p(4) = (4 - 4)(4 + 3) = 0$$

Since $p(4) = 0$ we have to do the limit

$$\lim_{x \rightarrow 4} \frac{\cancel{(x-4)}(x+3)}{\cancel{(x-4)}} = 4 + 3 = 7 \neq \infty$$

No VA at $x = 4$

$$\text{factor } f(x) = \frac{(x-4)(x+3)}{(x-4)} = x + 3 \text{ (cancel) This is really just a line}$$

with a hole at $x=4$!

$$\text{HA } \lim_{x \rightarrow \pm \infty} (x + 3) = \pm \infty \quad \therefore \text{none}$$

Example 2.

$$\text{a) } \lim_{x \rightarrow \infty} \frac{\frac{3x+1}{x} + \frac{1}{x}}{\frac{x}{x} + \frac{\sqrt{4x^2+5}}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{3 + \frac{1}{x}}{1 + \sqrt{\frac{4x^2+5}{x^2}}}$$

$$= \lim_{x \rightarrow \infty} \frac{3 + \frac{1}{x}}{1 + \sqrt{4 + \frac{5}{x^2}}}$$

$$= \frac{3+0}{1+\sqrt{4+0}}$$

$$= \frac{3}{1+2} = \frac{3}{3}$$

$$= 1$$

$$\lim_{x \rightarrow -\infty} \frac{\frac{3x+1}{x} + \frac{1}{x}}{\frac{x}{x} - \frac{\sqrt{4x^2+5x}}{x^2}}$$

$$= \lim_{x \rightarrow -\infty} \frac{3 + \frac{1}{x}}{1 - \sqrt{4 + \frac{5}{x^2}}}$$

$$= \frac{3+0}{1-\sqrt{4+0}}$$

$$= \frac{3}{1-2}$$

$$= \frac{3}{-1}$$

$$= -3$$

$y = 1$ and $y = -3$ are the horizontal asymptotes.

$$\begin{aligned}
\text{b) } & \lim_{x \rightarrow \infty} \frac{2^{2x} - 3^x}{3^x + 4^{x+1}} \\
&= \lim_{x \rightarrow \infty} \frac{(2^2)^x - 3^x}{3^x + 4^{x+1}} \\
&= \lim_{x \rightarrow \infty} \frac{4^x - 3^x}{3^x + 4^1 \cdot 4^x} \\
&= \lim_{x \rightarrow \infty} \frac{\frac{4^x}{4^x} - \frac{3^x}{4^x}}{\frac{3^x}{4^x} + \frac{4 \cdot 4^x}{4^x}} \\
&= \lim_{x \rightarrow \infty} \frac{1 - \left(\frac{3}{4}\right)^x}{\left(\frac{3}{4}\right)^x + 4} \\
&\left(\frac{3}{4}\right)^x \rightarrow 0 \text{ as } x \rightarrow \infty \text{ (fraction less than 1)}
\end{aligned}$$

Final answer

$$= \frac{1+0}{0+4} = 1/4$$

****NOTE: If it is a long answer, do the full solution we went through in prep!!!**

$$\text{K1. } f(x) = \frac{x-4}{x^2-5x-6} = \frac{x-4}{(x-6)(x+1)}$$

$$\text{VA } x = 6, -1$$

$$\text{HA } \lim_{x \rightarrow \infty} \frac{x-4}{x^2-5x-6} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} - \frac{4}{x^2}}{1 - \frac{5}{x} - \frac{6}{x^2}} = \frac{0-0}{1-0-0} = 0$$

$$y = 0$$

$$\text{K2. } f(x) = \frac{4-x}{2x+3}$$

$$\text{VA } 2x + 3 = 0$$

$$x = -\frac{3}{2}$$

$$\text{HA } \lim_{x \rightarrow \infty} \frac{4-x}{2x+3} = \lim_{x \rightarrow \infty} \frac{\frac{4}{x} - 1}{2 + \frac{3}{x}} = \frac{0-1}{2+0} = -\frac{1}{2} \quad \therefore y = -\frac{1}{2}$$

$$\text{K3. VA } 2e^x - 4 = 0$$

$$2e^x = 4$$

$$e^x = 2$$

$$\ln e^x = \ln 2$$

$$x \ln e = \ln 2$$

$$x = \ln 2$$

$$\text{HA } \lim_{x \rightarrow \infty} \frac{\sqrt{4e^{2x}+1}}{2e^x-4}$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{e^{2x}(4+\frac{1}{e^{2x}})}}{e^x(2-\frac{4}{e^x})}$$

$$= \lim_{x \rightarrow \infty} \frac{|e^x| \sqrt{4+\frac{1}{e^{2x}}}}{e^x(2-\frac{4}{e^x})}$$

NOTE: $|e^x| = e^x$ since $x \rightarrow \infty$

$$\therefore e^x > 0$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{4+e^{-2x}}}{2-4e^{-x}}$$

$$= \frac{\sqrt{4+0}}{2-0}$$

$$= \frac{\sqrt{4}}{2}$$

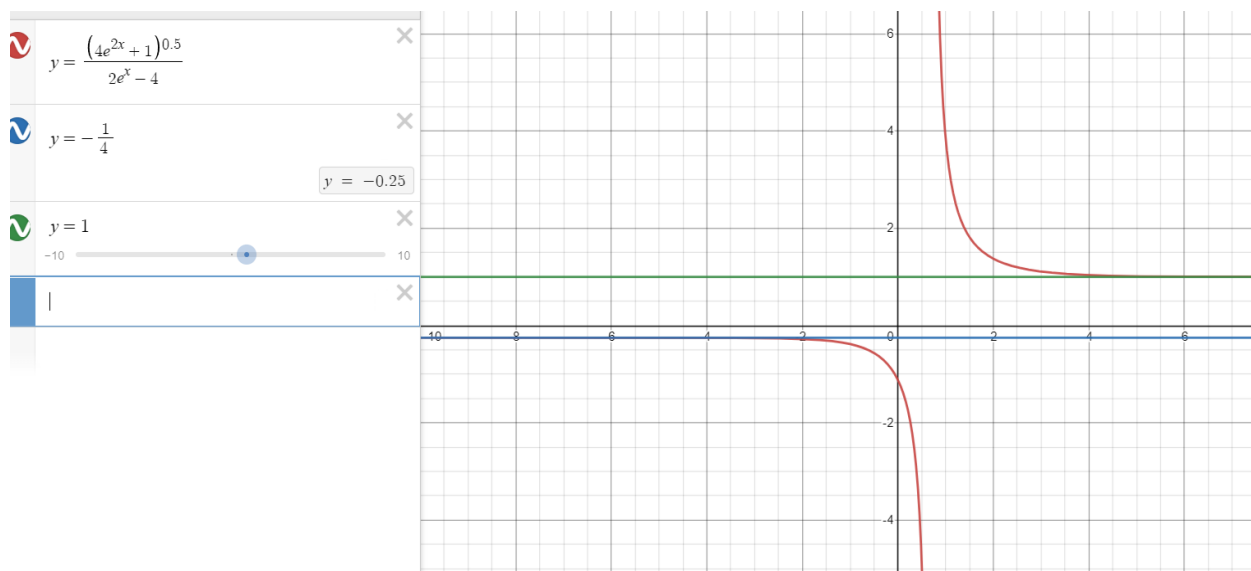
$$= 1$$

$$\therefore y = 1$$

$$HA \lim_{x \rightarrow -\infty} \frac{\sqrt{4e^{2x} + 1}}{2e^x - 4}$$

$$= \frac{\sqrt{4 \lim_{x \rightarrow -\infty} e^{2x} + 1}}{2 \lim_{x \rightarrow -\infty} e^x - 4}$$

$$= \frac{\sqrt{0+1}}{0-4} = -\frac{1}{4} \quad \therefore y = -\frac{1}{4}$$



$$\text{K4. a. } f(x) = \frac{e^{2x} + 5e^x + 2}{e^{2x} - 1}$$

VA

$$e^{2x} - 1 = 0$$

$$e^{2x} = 1$$

$$\ln e^{2x} = \ln 1$$

$$2x \ln e = \ln 1$$

$$2x = 0$$

$$x = 0$$

HA

$$\lim_{x \rightarrow \infty} \frac{e^{2x} + 5e^x + 2}{e^{2x} - 1}$$

$$= \lim_{x \rightarrow \infty} \frac{e^{2x} \left(1 + \frac{5}{e^x} + \frac{2}{e^{2x}}\right)}{e^{2x} \left(1 - \frac{1}{e^{2x}}\right)}$$

$$= \lim_{x \rightarrow \infty} \frac{1 + \frac{5}{e^x} + \frac{2}{e^{2x}}}{1 - \frac{1}{e^{2x}}}$$

$$= \frac{1+0+0}{1-0} = 1 \quad \therefore y = 1$$

$$\lim_{x \rightarrow -\infty} \frac{e^{2x} + 5e^x + 2}{e^{2x} - 1}$$

$$= \frac{\lim_{x \rightarrow -\infty} e^{2x} + \lim_{x \rightarrow -\infty} e^x + \lim_{x \rightarrow -\infty} 2}{\lim_{x \rightarrow -\infty} e^{2x} - \lim_{x \rightarrow -\infty} 1}$$

$$= \frac{0+0+2}{0-1} = \frac{2}{-1} = -2$$

$$\therefore y = -2$$

$$\therefore HA \text{ at } y = 1, y = -2$$

$$\text{b) } g(x) = \frac{2x}{\sqrt[4]{2x^4+4}}$$

VA None since $2x^4 > 0$ for all x

HA

$$\lim_{x \rightarrow \infty} \frac{2x}{\sqrt[4]{2x^4+4}}$$

$$= \lim_{x \rightarrow \infty} \frac{2x}{\sqrt[4]{x^4 \left(2 + \frac{4}{x^4}\right)}}$$

$$= \lim_{x \rightarrow \infty} \frac{2x}{x \sqrt[4]{2 + \frac{4}{x^4}}} \quad \text{since } x \rightarrow +\infty \quad \sqrt[4]{x^4} = |x| = x$$

$$= \lim_{x \rightarrow \infty} \frac{2}{\sqrt[4]{2 + \frac{4}{x^4}}} = \frac{2}{\sqrt[4]{2+0}} = \frac{2}{\sqrt{2}} = 2$$

$$\therefore y = \frac{2}{\sqrt{2}} \text{ is a HA or } \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\lim_{x \rightarrow -\infty} \frac{2x}{\sqrt[4]{2x^4+4}}$$

$$= \lim_{x \rightarrow -\infty} \frac{2x}{\sqrt[4]{x^4 \left(2 + \frac{4}{x^4}\right)}}$$

$$= \lim_{x \rightarrow -\infty} \frac{2x}{(-x) \sqrt[4]{\left(2 + \frac{4}{x^4}\right)}} \quad \text{since } |x| = -x \text{ as } x \rightarrow -\infty$$

$$= \lim_{x \rightarrow -\infty} \frac{-2}{\sqrt{2 + \frac{4}{x^4}}}$$

$$= \lim_{x \rightarrow -\infty} \frac{-2}{\sqrt{2+0}} = -\frac{2}{\sqrt{2}} = -\frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{-2\sqrt{2}}{2}$$

$$y = -\sqrt{2}$$

$$\therefore HA y = \pm\sqrt{2}$$

L. Squeeze Theorem

Example 2.

$$\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right) \qquad -1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

By squeeze theorem, $x^2(-1) \leq x^2 \cos\left(\frac{1}{x}\right) \leq 1(x^2)$

$$\lim_{x \rightarrow 0} -x^2 = 0 \quad \& \quad \lim_{x \rightarrow 0} x^2 = 0$$

↑

↑

as $x \rightarrow 0$ as $x \rightarrow 0$ $\lim \rightarrow 0$ $\lim \rightarrow 0$

$$\therefore \lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right) = 0$$

Example 3.

$$\lim_{x \rightarrow \infty} \frac{3 - \cos x}{x+3}$$

$$-1 \leq \cos x \leq 1 \quad \text{multiply by } -1$$

$$1 \geq -\cos x \geq -1 \quad \text{reverse}$$

inequalities

$$\therefore -1 \leq -\cos x \leq 1$$

Add 3 to each component

$$-1 + 3 \leq 3 - \cos x \leq 1 + 3$$

$$\therefore 2 \leq 3 - \cos x \leq 4$$

Divide by $x + 3$ since $x \rightarrow \infty, x + 3 > 0$

$$\frac{2}{x+3} \leq \frac{3 - \cos x}{x+3} \leq \frac{4}{x+3}$$

$$\lim_{x \rightarrow \infty} \frac{2}{x+3} = 0 = \lim_{x \rightarrow \infty} \frac{4}{x+3}$$

\(\therefore\) from squeeze theorem

$$\lim_{x \rightarrow \infty} \frac{3 - \cos x}{x+3} = 0$$

Example 4.

$$-1 \leq \sin\left(\frac{2020}{x}\right) \leq 1 \quad \text{multiply by } e^{\frac{1}{x}}$$

$$-e^{\frac{1}{x}} \leq e^{\frac{1}{x}} \sin\left(\frac{2020}{x}\right) \leq e^{\frac{1}{x}}$$

$$\lim_{x \rightarrow 0^-} \left(-e^{\frac{1}{x}}\right) = -e^{-\infty} = -\frac{1}{e^{\infty}} = 0$$

$$\text{And } \lim_{x \rightarrow 0^-} \left(e^{\frac{1}{x}}\right) = e^{-\infty} = \frac{1}{e^{\infty}} = 0$$

↓ below 0 ∴ a negative number

$$\therefore \lim_{x \rightarrow 0^-} \sin\left(\frac{2020}{x}\right)e^{\frac{1}{x}} = 0 \quad \text{by Squeeze Theorem}$$

Example 5.

$$\lim_{x \rightarrow -\infty} g(x)$$

$$= \lim_{x \rightarrow -\infty} \frac{5x^2 + 10x + 15}{x^2 + 1}$$

$$= \lim_{x \rightarrow -\infty} \frac{5 + \frac{10}{x} + \frac{15}{x^2}}{1 + \frac{1}{x^2}}$$

$$= \frac{5}{1} = 5$$

$$\lim_{x \rightarrow -\infty} h(x)$$

$$= \lim_{x \rightarrow -\infty} (5 - 3^x)$$

$$= 5 - 3^{-\infty}$$

$$= 5 - \frac{1}{\infty}$$

$$= 5 - 0 = 5$$

\therefore since $g(x) \leq f(x) \leq h(x)$

$$\text{And } \lim_{x \rightarrow -\infty} g(x) = \lim_{x \rightarrow -\infty} h(x) = 5$$

$\therefore \lim_{x \rightarrow -\infty} f(x) = 5$ by the squeeze theorem

$$\text{L1. } \lim_{x \rightarrow 0} x^2 e^{\sin(\frac{1}{x})}$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$e^{-1} \leq e^{\sin(\frac{1}{x})} \leq e^1$$

$$\frac{1}{e} \leq e^{\sin(\frac{1}{x})} \leq e$$

$$\frac{x^2}{e} \leq x^2 e^{\sin(\frac{1}{x})} \leq ex^2$$

$$\lim_{x \rightarrow 0} \frac{x^2}{e} = 0 \quad \& \quad \lim_{x \rightarrow 0} ex^2 = 0$$

$$\therefore \text{By squeeze Theorem, } \lim_{x \rightarrow 0} x^2 e^{\sin(\frac{1}{x})} = 0$$

$$\text{L2. } \lim_{x \rightarrow \infty} \frac{\cos^2(2x)}{5-2x}$$

$$-1 \leq \cos(2x) \leq 1 \quad \therefore \text{think about squaring this graph, it}$$

would only have values from 0 to 1

divide by $5 - 2x$ and $x \rightarrow \infty$, assume $5 - 2x < 0$

$$\therefore \frac{0}{5-2x} \geq \frac{\cos^2(2x)}{5-2x} \geq \frac{1}{5-2x}$$

(inequalities switch because we're dividing by a negative value)

$$\therefore \frac{1}{5-2x} \leq \frac{\cos^2(2x)}{5-2x} \leq 0$$

$$\text{Now, } \lim_{x \rightarrow \infty} \frac{1}{5-2x} = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} 0 = 0$$

$$\therefore \text{by squeeze theorem } \lim_{x \rightarrow \infty} \frac{\cos^2(2x)}{5-2x} = 0$$

$$\text{L3. } \lim_{x \rightarrow \infty} \frac{6x^2 - \sin(2x)}{x^2 + 10}$$

$-1 \leq \sin(2x) \leq 1$ since the sine curve oscillates

between -1 and 1.

multiply by -1

$1 \geq -\sin(2x) \geq -1$ same as

$-1 \leq -\sin(2x) \leq 1$

add $6x^2$ to each term

$$6x^2 - 1 \leq 6x^2 - \sin(2x) \leq 6x^2 + 1$$

$$\frac{6x^2 - 1}{x^2 + 10} \leq \frac{6x^2 - \sin(2x)}{x^2 + 10} \leq \frac{6x^2 + 1}{x^2 + 10}$$

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{6x^2 - 1}{x^2 + 10} &= \lim_{x \rightarrow -\infty} \frac{x^2 \left(6 - \frac{1}{x^2}\right)}{x^2 \left(1 + \frac{10}{x^2}\right)} = \\ &= \frac{6 - 0}{1 + 0} = 6 \end{aligned}$$

$$\lim_{x \rightarrow -\infty} \frac{6x^2 + 1}{x^2 + 10} = 6$$

$$\lim_{x \rightarrow -\infty} \frac{6x^2 - \sin(2x)}{x^2 + 10} = 6 \text{ by the Squeeze Theorem}$$

M. Intermediate Value Theorem

Example 2. $f(x) = 2x^3 - 3x^2 - 3$

First, always state that $f(x)$ is continuous on $[a,b]$ or they take off marks:(

$$f(1) = 2 - 3 - 3 = -4 < 0$$

$$f(3) = 2(3)^3 - 3(3)^2 - 3$$

$$= 54 - 27 - 3 > 0$$

$$f(1) < f(c) < f(3)$$

$$= 24 > 0$$

There is at least one c in $(1,3)$ such that $f(c) =$

0 by IVT and c is a root of

$$2c^3 - 3c^2 - 3 = 0$$

Example 3.

$$f(x) = \frac{1}{3}e^{\cos(2\pi x)} - 2\sin\left(\frac{\pi}{6}x\right)$$

$f(x)$ is continuous on $(-\infty, \infty)$

$$f(0) = \frac{1}{3}e^{\cos 0} - 2\sin 0 = \frac{1}{3}e^1 - 0 = \frac{1}{3}e > 0$$

$$f(1) = \frac{1}{3}e^{\cos 2\pi} - 2\sin\frac{\pi}{6} = \frac{1}{3}e^1 - 1 < 0$$

\therefore by IVT there is at least one c in $[0, 1]$ such that $f(c) = 0$

$$\therefore C \text{ is a root of } \frac{1}{3}e^{\cos(2\pi c)} = 2\sin\left(\frac{\pi}{6}c\right)$$

Example 4. IVT

$$f(x) = \sqrt[n]{x} - 1 + 2x$$

$$\text{Check } f(0) = \sqrt[n]{0} - 1 + 2(0)$$

$$= 0 - 1 + 0 = -1 < 0$$

$$f(1) = \sqrt[n]{1} - 1 + 2(1) \quad n \geq 4$$

$$= 1^{\frac{1}{n}} - 1 + 2$$

$$= 1^{\frac{1}{n}} + 1$$

$$= 1 + 1 = 2 > 0 \quad \text{but } 1^{\frac{1}{n}} = 1 \text{ for } n \geq 4$$

$$\therefore f(0) < f(c) < f(1)$$

\therefore is at least one c in $(0,2)$ such that $f(c) = 0$.

\therefore there is at least one root in $(0,2)$. A root means there is at least one solution

$\therefore c$ is a root of $f(c) = \sqrt[n]{c} - 1 + 2c$ by IVT

$$M1. f(x) = x^4 + x^3 - x - 2$$

First, always state that $f(x)$ is continuous on $[a,b]$ or they take off marks:(

$$f(0) = 0 + 0 - 0 - 2 = -2 < 0$$

$$f(2) = 16 + 8 - 2 - 2 = 24 - 4 = 20 > 0$$

$$f(0) < f(c) < f(2)$$

\therefore there is at least one c in $(0,2)$ such that $f(c) = 0$ and by IVT c is a root of

$$c^4 + c^3 - c - 2 = 0$$

M2. $f(x) = \cos x$ $[0,2\pi]$ First, always state that $f(x)$ is continuous on $[a,b]$ or they take off marks:(Note: if you used the endpoints, you would get both are positive

$$f(0) = \cos 0 = 1 > 0$$

$$f(\pi) = \cos \pi = -1 < 0$$

$$f(\pi) < f(c) < f(0)$$

\therefore there is at least one c in $(0,\pi)$ such that $f(c) = 0$ and by IVT, c is a root of

$$\cos c = 0$$

$$M3. f(x) = \sin^2 x - x^3 + 2 \quad \left(0, \frac{3\pi}{2}\right)$$

$$f(0) = (\sin 0)^2 - 0^3 + 2 = 2 > 0$$

$$f\left(\frac{3\pi}{2}\right) = \sin^2\left(\frac{3\pi}{2}\right) - \left(\frac{3\pi}{2}\right)^3 + 1 = (-1)^2 - \left(\frac{27\pi^3}{8}\right) + 2$$

= pretty difficult without a calculator

$$\text{try } f(\pi) = \sin^2 \pi - \pi^3 + 2$$

$$= 0 - \pi^3 + 2 = 2 - (3.14)^3 < 0$$

$$f(\pi) < f(c) < f(0)$$

∴ there is at least one c in $\left(0, \frac{3\pi}{2}\right)$ such that $f(c) = 0$

and by IVT c is a root of the equation

$$\sin^2 c - c^3 + 2 = 0$$

N. Definition of the Derivative

Example 1.

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$f'(3) = \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3} \frac{x^3 - 3^3}{x - 3} = \lim_{x \rightarrow 3} \frac{x^3 - 27}{x - 3} = H =$$

$$= \lim_{x \rightarrow 3} \frac{3x^2}{1} = 3(3)^2 = 27$$

*here you can factor as a difference of cubes, or use L'Hospitals Rule

$$\dots \lim_{x \rightarrow 3} \frac{x^3 - 27}{x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x^2 + 3x + 9)}{x - 3} = \lim_{x \rightarrow 3} (x^2 + 3x + 9) =$$

$$= 3^2 + 3(3) + 9 = 27$$

Example 2.

$$f(x) = \sqrt{x + 1}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h+1} - \sqrt{x+1})(\sqrt{x+h+1} + \sqrt{x+1})}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \lim_{h \rightarrow 0} \frac{x+h+1 - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \lim_{h \rightarrow 0} \frac{x+h+1 - x - 1}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \lim_{h \rightarrow 0} \frac{1}{(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \frac{1}{\sqrt{x+1} + \sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

Example 3.

$$f(x) = \sin x \quad f(x+h) = \sin(x+h)$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Now, from identities $\sin(x+h) = \sin x \cosh + \cos x \sinh$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sin x \cosh + \cos x \sinh - \sin x}{h} \text{ rearrange}$$

$$= \lim_{h \rightarrow 0} \frac{\sin x \cosh - \sin x + \cos x \sinh}{h} \text{ common factor of } \sin x \text{ from first two terms}$$

$$= \lim_{h \rightarrow 0} \frac{\sin x (\cosh - 1) + \cos x \sinh}{h} \text{ split up limits}$$

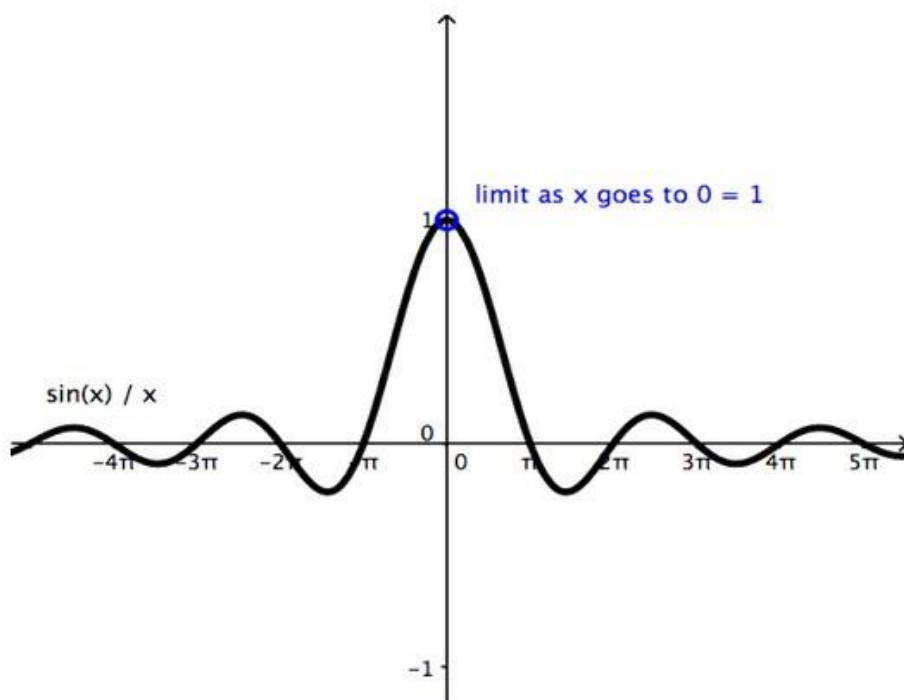
$$= \lim_{h \rightarrow 0} \sin x \left(\frac{\cosh - 1}{h} \right) + \lim_{h \rightarrow 0} \cos x \left(\frac{\sinh}{h} \right)$$

$$= \sin x (0) + \cos x (1) \text{ from limits you've memorized! (I hope!!) p.56}$$

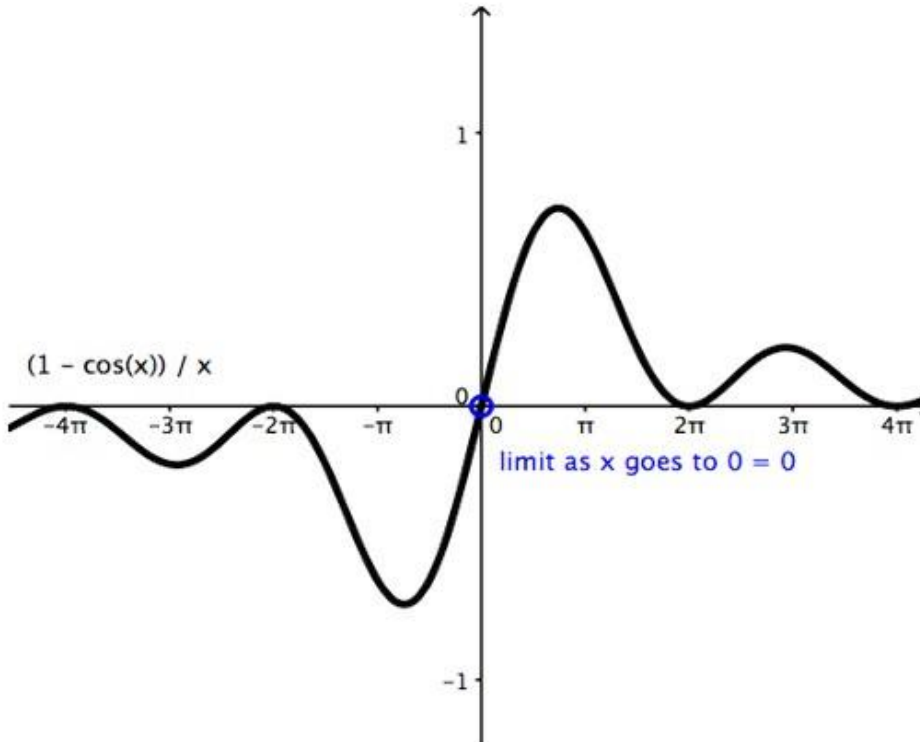
booklet

$$= \cos x$$

We can see that the first limit converges to 1



and the second limit converges to 0.



We can plug in 1 and 0 for the limits and get $\cos(x)$

$$\frac{d}{dx}\sin(x) = \cos(x)(1) - \sin(x)(0)$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

Example 4.

$$f(x) = (x + 2)^2 \text{ at } x = 2.$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

$$f'(2) = \lim_{h \rightarrow 0} \frac{(2+h+2)^2 - (2+2)^2}{h}$$

$$f'(2) = \lim_{h \rightarrow 0} \frac{(4+h)^2 - 16}{h}$$

$$f'(2) = \lim_{h \rightarrow 0} \frac{16 + 8h + h^2 - 16}{h}$$

$$f'(2) = \lim_{h \rightarrow 0} \frac{h^2 + 8h}{h}$$

$$f'(2) = \lim_{h \rightarrow 0} \frac{h(h+8)}{h}$$

$$m=8$$

$$f(x) = (x + 2)^2 \text{ at } x = 2.$$

Subst. $x=2$ into $f(x)$ to find y

$$f(2) = (2 + 2)^2 = 16$$

Point $(2,16)$

$$y - 16 = 8(x - 2)$$

$$y - 16 = 8x - 16$$

$$y=8x$$

Example 5.

Subst. $x=3$ into $f(x)$ and get $f(3) = \sqrt{3+1} = 2$ Point $(3,2)$

$$m = f'(3) = \lim_{h \rightarrow 0} \frac{(\sqrt{3+h+1} - 2)(\sqrt{3+h+1} + 2)}{h(\sqrt{3+h+1} + 2)}$$

$$m = \lim_{h \rightarrow 0} \frac{(3+h+1-4)}{h(\sqrt{3+h+1} + 2)}$$

$$m = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{3+h+1} + 2)}$$

$$m = \lim_{h \rightarrow 0} \frac{1}{(\sqrt{3+h+1} + 2)}$$

$$m = \frac{1}{4}$$

$$y - 2 = \frac{1}{4}(x - 3) \text{ or } y = \frac{1}{4}x - \frac{3}{4} + 2$$

$$y = \frac{1}{4}x + \frac{5}{4}$$

Example 6.

a) Yes, $f(x)$ is defined at $x = 0$

$$f(0) = 4(0) = 0$$

b) Check differentiability

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$$

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{4x - 0}{x - 0} = \lim_{x \rightarrow 0^+} \frac{4x}{x} = 4$$

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{2x \sin\left(\frac{1}{x^2}\right) - 0}{x - 0}$$

$$= \lim_{x \rightarrow 0^-} \frac{2 \sin\left(\frac{1}{x^2}\right)}{1} = DNE$$

$\therefore f'(0)$ is undefined

c)

$$\lim_{x \rightarrow 0^+} 4x = 4(0) = 0$$

$$\lim_{x \rightarrow 0^+} 2 \sin\left(\frac{1}{x^2}\right) = 0 \text{ by Squeeze Theorem}$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x)$$

$\therefore f(x)$ is continuous at $x = 0$

$$N1. f(x) = \sqrt{x}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \lim_{h \rightarrow 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}} \end{aligned}$$

$$N2. f(x) = (x+1)^2 = x^2 + 2x + 1$$

$$\begin{aligned} f(x+h) &= (x+h+1)^2 = (x+h+1)(x+h+1) \\ &= x^2 + xh + x + xh + h^2 + h + x + h + 1 \end{aligned}$$

$$= x^2 + 2xh + 2x + 2h + h^2 + 1$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + 2x + 2h + h^2 + 1 - (x^2 + 2x + 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + 2h + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2x + 2 + h)}{h} \\ &= \lim_{h \rightarrow 0} (2x + 2 + h) \\ &= 2x + 2 \end{aligned}$$

$$N3. f(x) = x^2 - x$$

$$f(x+h) = (x+h)^2 - (x+h) = x^2 + 2xh + h^2 - x - h$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x - h - (x^2 - x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2xh + h^2 - h}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(2x + h - 1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(2x + h - 1)}{1}$$

$$= 2x + 0 - 1 = 2x - 1$$

$$N4. f(x) = \frac{1}{x+3} \quad f(x+h) = \frac{1}{x+h+3}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} \quad \text{get a common denominator}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1(x+3)}{(x+3)(x+h+3)} - \frac{1(x+h+3)}{(x+3)(x+h+3)}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{x+3-x-h-3}{(x+3)(x+h+3)}}{h}$$

$$\lim_{h \rightarrow 0} \frac{-h}{(x+3)(x+h+3)} \left(\frac{1}{h} \right) = \frac{-1}{(x+3)(x+0+3)} = \frac{-1}{(x+3)^2}$$

$$N5. \quad f(x) = \frac{1}{\sqrt{x}} \quad f(x+h) = \frac{1}{\sqrt{x+h}}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{\sqrt{x}}{\sqrt{x}\sqrt{x+h}} - \frac{\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{(\sqrt{x} - \sqrt{x+h})(\sqrt{x} + \sqrt{x+h})}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{x - (x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{x - x - h}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-h}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} \left(\frac{1}{h}\right) = \frac{-1}{\sqrt{x}\sqrt{x+0}(\sqrt{x} + \sqrt{x+0})} = \frac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})} =$$

$$\frac{-1}{x(2\sqrt{x})} = \frac{-1}{2x^1 x^{\frac{1}{2}}}$$

$$= \frac{-1}{2x^{\frac{3}{2}}} = \frac{-1x^{-\frac{3}{2}}}{2}$$

N6. A) Find $f'(x)$ using the definition of the derivative if

$$f(x) = (2x - 1)^2.$$

$$f(x + h) = [2(x + h) - 1]^2 = (2x + 2h - 1)(2x + 2h - 1)$$

$$f(x + h) = 4x^2 + 4xh - 2x + 4xh + 4h^2 - 2h - 2x - 2h + 1$$

$$f(x + h) = 4x^2 + 8xh - 4x - 4h + 4h^2 + 1$$

And

$$f(x) = (2x - 1)^2 = (2x - 1)(2x - 1) = 4x^2 - 4x + 1$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{4x^2 + 8xh - 4x - 4h + 4h^2 + 1 - (4x^2 - 4x + 1)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{4x^2 + 8xh - 4x - 4h + 4h^2 + 1 - 4x^2 + 4x - 1}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{8xh - 4h + 4h^2}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{h(8x - 4 + 4h)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} (8x - 4 + 4h)$$

$$f'(x) = 8x - 4$$

B) Find the equation of the tangent to $f(x)$ at $x=3$.

$$f'(x) = 8x - 4$$

$$f'(3) = 8(3) - 4 = 20$$

Subst. $x=3$ into $f(x) = (2x - 3)^2$ to find y

$f(3)=9$, so $y=9$. The point is $(3,9)$

So, the equation is

$$y - 9 = 20(x - 3) \text{ or } y = 20x - 60 + 9 \text{ or } y = 20x - 51$$

*N7. A) Find $f'(x)$ using the definition of the derivative if $f(x) = \sqrt{2x - 4}$.

A) Find $f'(x)$ using the definition of the derivative if $f(x) = \sqrt{2x - 4}$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{2x+2h-4} - \sqrt{2x-4}}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(\sqrt{2x+2h-4} - \sqrt{2x-4})(\sqrt{2x+2h-4} + \sqrt{2x-4})}{h(\sqrt{2x+2h-4} + \sqrt{2x-4})}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2x+2h-4 - (2x-4)}{h(\sqrt{2x+2h-4} + \sqrt{2x-4})}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2x+2h-4-2x+4}{h(\sqrt{2x+2h-4} + \sqrt{2x-4})}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{2x+2h-4} + \sqrt{2x-4})}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2}{(\sqrt{2x+2h-4} + \sqrt{2x-4})}$$

$$f'(x) = \frac{2}{\sqrt{2x-4} + \sqrt{2x-4}}$$

$$f'(x) = \frac{2}{2\sqrt{2x-4}}$$

$$f'(x) = \frac{1}{\sqrt{2x-4}}$$

The slope, m at $x=7$ would be $m = \frac{1}{\sqrt{2(7)-4}} = \frac{1}{\sqrt{10}}$

B). Find the equation of the tangent to $f(x)$ at $x=7$.

$$\text{At } x=7, f(7) = \sqrt{2(7) - 5} = \sqrt{9}=3$$

The point is $(7,3)$

So, the equation is

$$y - 3 = \frac{1}{3}(x - 7) \text{ or } y = \frac{1}{3}x - \frac{7}{3} + 3$$

$$\text{or } y = \frac{1}{3}x + \frac{2}{3}$$

N8. See example 3. Repeat for you to try it again!

N9. Find the derivative of $y = \cos x$ using the definition of the derivative.

Using the definition of a derivative:

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \text{ where } h = \delta x$$

We substitute in our function to get:

$$\lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h}$$

Using the Trig identity:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b,$$

we get:

$$\lim_{h \rightarrow 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$$

Factoring out the $\cos x$ term, we get:

$$\lim_{h \rightarrow 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

This can be split into 2 fractions:

$$\lim_{h \rightarrow 0} \frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \sin h}{h}$$

Now comes the more difficult part: recognizing known formulas.

The 2 which will be useful here are:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \text{ and } \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$$

Since those identities rely on the variable inside the functions being the same as the one used in the \lim portion, we can only use these identities on terms using h , since that's what our \lim uses. To work these into our equation, we first need to split our function up a bit more:

$$\lim_{h \rightarrow 0} \frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \sin h}{h}$$

becomes:

$$\lim_{h \rightarrow 0} \cos x \left(\frac{\cos h - 1}{h} \right) - \sin x \left(\frac{\sin h}{h} \right)$$

Using the previously recognized formulas, we now have:

$$\lim_{h \rightarrow 0} \cos x(0) - \sin x(1)$$

which equals:

$$\lim_{h \rightarrow 0} (-\sin x)$$

Since there are no more h variables, we can just drop the $\lim_{h \rightarrow 0}$, giving us a final answer of: $-\sin x$.

O. Evaluating Limits by recognizing them as derivatives

Example 2.

$a=x=4$ to find $f(x)$ replace $4+h$ with an x

$$f(x)=\ln x$$

$$f'(x) = \frac{1}{x}$$

$$f'(x) = \frac{1}{4}$$

Example 3.

$a=x=3$ to find $f(x)$ look at the first part of the limit

$$f(x) = e^{3x}$$

$$f'(x) = 3e^{3x}$$

$$f'(x) = 3e^9$$

Example 4.

$$\lim_{h \rightarrow 0} \frac{(1+h)^3 - 2(1+h)^2 + 3(1+h) - 2}{h} \quad (\%)$$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \quad \therefore a = x = 1$$

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

Replace $1+h \rightarrow x$

$$\therefore f(x) = x^3 - 2x^2 + 3x - 2 \text{ note that } f(1)=0$$

$$f'(x) = 3x^2 - 4x + 3$$

$$\begin{aligned} f'(1) &= 3(1)^2 - 4(1) + 3 \\ &= 2 \end{aligned}$$

$$01. \quad a+h = 4+h \quad \therefore a = 4$$

To find $f(x)$ replace $4+h$ with an x

$$\therefore f(x) = 2^x$$

$$\therefore f'(x) = 2^x \ln 2$$

$$f'(4) = 2^4 \ln 2 = 16 \ln 2$$

$$02. \quad x \rightarrow a \quad \therefore a = \frac{\pi}{3}$$

$$f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$f'\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$03. \quad a + h = 5 + h \quad \therefore a = 5$$

To find $f(x)$ replace $5 + h$ with an $x \quad \therefore f(x) = x^3$

$$f'(x) = 3x^2$$

$$f'(5) = 3(5)^2 = 3(25) = 75$$

$$04. \quad \lim_{x \rightarrow \pi/3} \frac{e^{\sin x} - \frac{1}{2}e^{\sqrt{3}/2}}{x - \pi/3}$$

$$\text{Definition} \quad f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$\therefore a = \frac{\pi}{3} \quad f(a) = f\left(\frac{\pi}{3}\right) = \frac{1}{2}e^{\sqrt{3}/2} \quad f(x) = e^{\sin x}$$

$$f'(x) = e^{\sin x}(\cos x)$$

$$f'\left(\frac{\pi}{3}\right) = e^{\sin\left(\frac{\pi}{3}\right)}\left(\cos\left(\frac{\pi}{3}\right)\right) = \frac{1}{2}e^{\sqrt{3}/2}$$

O5.

$$\lim_{x \rightarrow \frac{1}{2}} \frac{\arcsin x - \frac{\pi}{6}}{x - \frac{1}{2}}$$

$$a = \frac{1}{2}$$

$$f(x) = \arcsin x$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f'\left(\frac{1}{2}\right) = \frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{1}{\sqrt{\frac{3}{4}}}$$

$$\therefore f'\left(\frac{1}{2}\right) = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

$$\text{Definition } f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$f\left(\frac{1}{2}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$O6. \quad \lim_{h \rightarrow 0} \frac{3^{2h} - 1}{h} \quad f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$a = 0 \quad f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

$$f(x) = 3^{2x}$$

$$f'(x) = 3^{2x}(\ln 3)(2)$$

$$f'(0) = 3^0(\ln 3)(2) = 2 \ln 3 = \ln 3^2 = \ln 9$$

$$O7. \lim_{h \rightarrow 0} \frac{\sin\left(\frac{\pi}{2}+h\right)-1}{h}$$

$$a = \pi/2$$

$$f(\pi/2) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

$$f'(\pi/2) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}$$

P. Average vs. Instantaneous Velocity and Acceleration

P1. a) $s(1) = 1^3 + 2(1)^2 + 1$

$$= 1 + 2 + 1 = 4m$$

b) $\frac{\Delta s}{\Delta t} = \frac{(2^3 + 2(2)^2 + 1) - 4}{2 - 1} = \frac{8 + 8 + 1 - 4}{1} = 13 \text{ m/s}$

c) $s'(t) = 0$ $s'(t) = 3t^2 + 4t$

d) $s'(3) = 3(3)^2 + 4(3) = 3(9) + 12$
 $= 27 + 12 = 39 \text{ m/s}$

e) $s''(t) = 6t + 4$

$$s''(3) = 6(3) + 4 = 22 \text{ m/s}^2$$

P2. a) $\frac{\Delta s}{\Delta t} = \frac{(50 - 2(2)^2) - (50 - 2(1)^2)}{2 - 1}$
 $= \frac{(50 - 8) - (50 - 2)}{1} = \frac{42 - 48}{1} = -6 \text{ m/s}$

b) $s'(t) = 0$

$$s'(2) = -4(2) = -8 \text{ m/s}$$

c) $s(t) = 0$ $50 - 2t^2 = 0$

$$50 = 2t^2$$

$$25 = t^2$$

$$t = 5s$$

d) $s''(t) = -4 \text{ m/s}^2$

$$s''(3) = -4 \text{ m/s}^2 \quad (\text{constant acceleration})$$

Q. Derivative Rules

Example 1. (a) $f(x) = x^{\frac{1}{2}}(x^4 + 3x)$

$$f(x) = x^{\frac{9}{2}} + 3x^{\frac{3}{2}}$$

$$f'(x) = \frac{9}{2}x^{\frac{7}{2}} + \frac{9}{2}x^{\frac{1}{2}}$$

(b) $g'(x) = \frac{3(2-x) - (-1)(1+3x)}{(2-x)^2}$

$$= \frac{6-3x+1+3x}{(2-x)^2}$$

$$= \frac{7}{(2-x)^2}$$

(c) $f'(x) = \frac{(2x+2)(x^3+4) - 3x^2(x^2+2x+2)}{(x^3+4)^2}$

$$= \frac{2x^4+8x+2x^3+8-3x^4-6x^3-6x^2}{(x^3+4)^2}$$

$$= \frac{-x^4-4x^3-6x^2+8x+8}{(x^3+4)^2}$$

(d) $f(x) = x^2 - \sqrt{x}x + \sqrt{x} \cdot x - x$

$$= x^2 - x$$

$$f'(x) = 2x - 1$$

Example 2. Find the derivative of each of the following:

a) $f(x) = 3^x$

$$f'(x) = 3^x \ln 3$$

b) $f(x) = 7^x + 4x^7$

$$f'(x) = 7^x \ln 7 + 28x^6$$

$$c) f(x) = 4e^x + 3x^9 + 6$$

$$f'(x) = 4e^x + 27x^8$$

$$(d) f'(x) = 5^x \ln 5 (3e^x + 5x^2 + 6x + 12) + 5^x (3e^x + 10x + 6)$$

$$= 5^x [\ln 5 (3e^x + 5x^2 + 6x + 12) + (3e^x + 10x + 6)]$$

$$(e) f'(x) = \frac{3^x \ln 3 (x^3 + 4x^2) - 3^x (3x^2 + 8x)}{(x^3 + 4x^2)^2}$$

$$(f) (x) = \sqrt{4e^x - 9x^3} = (4e^x - 9x^3)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} (4e^x - 9x^3)^{-\frac{1}{2}} (4e^x - 27x^2)$$

$$f'(x) = \frac{4e^x - 27x^2}{2(4e^x - 9x^3)^{\frac{1}{2}}} = \frac{4e^x - 27x^2}{2\sqrt{4e^x - 9x^3}}$$

Example 3. Find the derivative of each of the following using the Product Rule.

$$a) f(x) = 6x^7 e^x \quad f'(x) = 42x^6 e^x + e^x (6x^7)$$

$$f'(x) = 6x^6 e^x (7 + x)$$

$$b) y = 8^x \log_5 x \quad y' = 8^x \ln 8(x) + \frac{1}{x \ln 5} (8^x)$$

Example 4. Find the derivative of each of the following using the Quotient Rule.

$$f(x) = \frac{\ln x}{6x^5} \quad f'(x) = \frac{\frac{1}{x}(6x^5) - 30x^4(\ln \ln x)}{(6x^5)^2} = \frac{6x^4 - 30x^4 \ln x}{36x^{10}} = \frac{6x^4(1 - 5 \ln x)}{36x^{10}}$$

$$= \frac{(1 - 5 \ln x)}{6x^6}$$

Example 6. Find the derivative of each of the following using the Chain Rule.

$$a) y = (3x + 6)^4 \quad y' = 4(3x + 6)^3(3) = 12(3x + 6)^3$$

$$b) f(x) = 6^{8x-1} \quad f'(x) = 6^{8x-1}(\ln 6)(8) \\ = (8 \ln 6)6^{8x-1}$$

$$c) y = e^{9x-2} \quad y' = 9e^{9x-2}$$

$$d) f(x) = 4 \ln(8x - 7) \quad f'(x) = 4 \left(\frac{8}{8x-7} \right) = \frac{32}{8x-7}$$

$$e) y = \log_3(2x^2 - x + 5) \quad y' = \frac{(4x-1)}{(2x^2-x+5)\ln 3}$$

$$f) f(x) = [\ln(2x - 5)]^4 \quad f'(x) = 4[\ln(2x - 5)]^3 \left(\frac{2}{2x-5} \right) \\ = \frac{8}{2x-5} [\ln(2x - 5)]^3$$

$$g) f(x) = \ln(2x - 5)^4 \quad f(x) = 4 \ln(2x - 5) \text{ log rules} \\ f'(x) = 4 \left(\frac{2}{2x-5} \right) = \frac{8}{2x-5}$$

***Example 7.** Find the derivative of each of the following:

$$a) f(x) = \ln(\ln x) \text{ at } x = e^2$$

$$f'(x) = \frac{\frac{1}{x}}{\ln x} = \frac{1}{x \ln x}$$

$$f'(e^2) = \frac{1}{e^2 \ln e^2} = \frac{1}{e^2(2 \ln e)} = \frac{1}{2e^2}$$

$$b) y = e^x \ln x \text{ at } x = e \quad y' = e^x \ln x + \frac{1}{x} e^x = e^x \left(\ln x + \frac{1}{x} \right)$$

$$y'(e) = e^e \left(\ln e + \frac{1}{e} \right) = e^e \left(1 + \frac{1}{e} \right)$$

***Example 8.** Find the derivative of $y = \ln(2x^2 + 5e)^{3e}$

$$y = 3e \ln(2x^2 + 5e)$$

$$y' = 3e \left(\frac{4x}{2x^2 + 5e} \right) = \frac{12ex}{2x^2 + 5e}$$

***Example 9.** Find the derivative of $y = [\ln(2x^2 + 5e)]^{3e}$

$$\begin{aligned} y' &= 3e [\ln(2x^2 + 5e)]^{3e-1} \left(\frac{4x}{2x^2 + 5e} \right) \\ &= \frac{12ex}{2x^2 + 5e} [\ln(2x^2 + 5e)]^{3e-1} \end{aligned}$$

Q1. a) $f'(x) = 4e^{4x}$

b) $f'(x) = 3^{-4x} (\ln 3)(-4)$

c) $f'(x) = 3x^2(e^{7x}) + 7e^{7x}(x^3) = x^2(e^{7x})(3 + x)$

d) $f'(x) = \frac{1}{(x-5)\ln 3}$

e) $f'(x) = 12x^2 + 4e^{3x}(3) = 12x^2 + 12e^{3x}$

f) $f'(x) = \frac{4e^{4x}(x^2) - 2x(e^{4x})}{(x^2)^2} = \frac{2xe^{4x}(2x-1)}{x^4} = \frac{2e^{4x}(2x-1)}{x^3}$

g) $f'(x) = e^{\sqrt{x}} \left(\frac{1}{2} x^{-\frac{1}{2}} \right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

h) $f'(x) = 6^{x^2-7x} (\ln 6)(2x - 7)$

$$\text{i) } f'(x) = \frac{6}{6x-3} = \frac{2}{2x-1}$$

$$\text{j) } f'(x) = \frac{e^{x+5}(x+e^x) - (1+e^x)(e^{x+5})}{(x+e^x)^2}$$

$$\text{k) } f(x) = 10^x (x^2 + 3x)^{\frac{1}{2}}$$

$$f'(x) = 10^x \ln 10 \sqrt{x^2 + 3x} + \frac{1}{2} (x^2 + 3x)^{-\frac{1}{2}} (2x + 3)(10^x)$$

$$\text{l) } f(x) = \ln(x - 5) - \ln(x - 2) \quad \dots \text{using log rules}$$

$$f'(x) = \frac{1}{x-5} - \frac{1}{x-2}$$

$$\text{m) } \quad \text{USE LOG RULES } y = 3 \ln(x^2 - 2x)$$

$$y' = \frac{3(2x-2)}{x^2-2x}$$

$$\text{Q2. } f'(x) e^{x^4+2x} (4x^3 + 2)$$

$$f'(1) = e^3 (4 + 2) = 6e^3$$

Therefore, answer is B).

$$\text{Q3. } f'(x) = \frac{1}{x \ln 5} \quad f'(2) = \frac{1}{2 \ln 5}$$

Therefore, answer is B).

Q4.

$$\frac{d}{dx} (4^{2 \log_4 x}) = \frac{d}{dx} (4^{\log_4 x^2}) = \frac{d}{dx} (x^2) = 2x$$

$$\begin{aligned} \text{Q5. } f'(x) &= \frac{2x}{(x^2+2)\ln 3} \\ f'(1) &= \frac{2(1)}{3\ln 3} = \frac{2}{3\ln 3} \end{aligned}$$

Therefore, answer is E).

$$\begin{aligned} \text{Q6. } f'(x) &= 2x \ln x + \frac{1}{x}(x^2) = 2x \ln x + x \\ f'(e^2) &= 2e^2 \ln e^2 + e^2 = 2e^2(2) + e^2 = 5e^2 \end{aligned}$$

Therefore, answer is A).

$$\begin{aligned} \text{Q7. } f'(x) &= \frac{\frac{1}{x}(x^2) - 2x \ln x}{x^4} = \frac{x - 2e \ln x}{x^4} \\ f'(e) &= \frac{e - 2e \ln e}{e^4} = \frac{-e}{e^4} = -\frac{1}{e^3} \end{aligned}$$

Therefore, answer is C).

$$\begin{aligned} \text{Q8. } f(x) &= 6e^x - 4 \ln x + \frac{1}{4}x^{-1} \\ f'(x) &= 6e^x - 4\left(\frac{1}{x}\right) + \frac{1}{4}(-x^{-2}) = 6e^x - \frac{4}{x} - \frac{1}{4x^2} \end{aligned}$$

$$\text{Q9. } f'(x) = 4e[\ln(3x + e)]^{4e-1} \left(\frac{3}{3x+e}\right)$$

$$\text{Q10. } f'(x) = 6^{2x-1}(\ln 6)(2)$$

$$\text{Q11. } f(x) = \frac{e^x}{3} - x^2$$

$$f'(x) = \frac{1}{3}e^x - 2x$$

$$f'(\ln 2) = \frac{1}{3}e^{\ln 2} - 2\ln 2 = \frac{2}{3} - \ln 4$$

$$\text{Q12. } \frac{d}{dx}(\log_4 x^8) = \frac{8x^7}{x^8 \ln 4} = \frac{8}{x \ln 4}$$

$$\text{Q13.a) } f(x) = 2e^{x^{\frac{1}{2}}} \text{ at } x=16$$

$$f'(x) = 2e^{x^{\frac{1}{2}}} \left(\frac{1}{2} x^{-\frac{1}{2}} \right) = \frac{2e^{\sqrt{x}}}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{\sqrt{x}}$$

$$f'(16) = \frac{e^{\sqrt{16}}}{\sqrt{16}} = \frac{e^4}{4}$$

$$\text{b) } y = \log_3 x \text{ at } x = 27.$$

$$y' = \frac{1}{x \ln 3}$$

$$y'(27) = \frac{1}{27 \ln 3}$$

$$\text{c) } f(x) = xe^{2x}$$

$$f'(x) = (1)e^{2x} + x(2e^{2x}) = e^{2x}(1 + 2x)$$

$$f'(1) = e^2(1 + 2(1)) = 3e^2$$

$$\text{d) } f(x) = \ln(\ln x)$$

$$f'(x) = \frac{1}{\ln x} \left(\frac{1}{x} \right)$$

$$f'(e^2) = \frac{1}{e^2 \ln e^2} = \frac{1}{2e^2}$$

$$\text{Q14. } F'(x) = f'(f(x)) \cdot f'(x)$$

$$F'(4) = f'(f(4)) \cdot f'(4)$$

$$= f'(3) \cdot 12$$

$$= 5 \cdot 12 = 60$$

$$\text{Q15. } g'(x) = 5[f(x)]^4 \cdot f'(x)$$

$$\therefore g'(4) = 5[f(4)]^4 \cdot f'(4)$$

$$= 5(3)^4 \cdot (2)$$

$$= 5(81)(2)$$

$$= 5(162)$$

$$= 810$$

Q16.

$$\text{a) } f'(x) = \frac{2x(2x+3) - 2(x^2-4)}{(2x+3)^2}$$

$$= \frac{4x^2 + 6x - 2x^2 + 8}{(2x+3)^2}$$

$$= \frac{2x^2 + 6x + 8}{(2x+3)^2}$$

$$\text{b) } f'(x) = \frac{1(x+3) - 1(x+2)}{(x+3)^2}$$

$$= \frac{x+3-x-2}{(x+3)^2}$$

$$= \frac{1}{(x+3)^2}$$

$$\text{c) } f(x) = \frac{1}{2}e^x + 18x + 6$$

$$f'(x) = \frac{1}{2}e^x + 18$$

$$\text{d) } f(x) = 3x^{2\ln 3}$$

$$f(x) = 3x^{\ln 9}$$

$$f'(x) = 3[(\ln 9)x^{\ln 9 - 1}]$$

$$\text{e) } f(x) = x^3 \ln x, \text{ find } f'(e^2).$$

$$f'(x) = 3x^2 \ln x + \frac{1}{x}(x^3) = 3x^2 \ln x + x^2$$

$$f'(x) = x^2(3 \ln x + 1)$$

$$f'(e^2) = (e^2)^2(3 \ln e^2 + 1) = e^4(6 \ln e + 1) = 7e^4$$

$$\text{f) } f(x) = 3^{2x} \quad f'(x) = 3^{2x}(\ln 3)(2) \text{ OR } 2 \ln 3(3)^{2x}$$

$$\text{g) find } \frac{d}{dx} \ln \left(x + 3 + \frac{1}{x}\right) = \frac{1+0-1x^{-2}}{x+3+\frac{1}{x}} = \frac{1-\frac{1}{x^2}}{x+3+\frac{1}{x}}$$

$$\text{Q17. a) } f(x) = x^4 e^x \quad f'(x) = 4x^3(e^x) + e^x(x^4)$$

$$= x^3 e^x(4 + x)$$

$$\text{b) } y = (3x^3 - 5)\ln x \quad y' = 9x^2 \ln x + \frac{1}{x}(3x^3 - 5)$$

Q18.

$$\text{a) } f(x) = \frac{e^x}{x^8} \quad f'(x) = \frac{e^x(x^8) - 8x^7(e^x)}{(x^8)^2} = \frac{x^7 e^x(x-7)}{x^{16}} = \frac{e^x(x-8)}{x^9}$$

$$\begin{aligned} \text{b) } y &= \frac{(4x+5)}{x^4} & y' &= \frac{4(x^4) - 4x^3(4x+5)}{(x^4)^2} = \frac{4x^4 - 16x^4 - 20x^3}{x^8} = \frac{-12x^4 - 20x^3}{x^8} \\ & & &= \frac{-4x^3(3x+5)}{x^8} = \frac{-4(3x+5)}{x^5} \end{aligned}$$

Q19. Find $\frac{d}{dx} (\ln (e^{-3x} + 2x))$ at $x = 1$

$$\frac{dy}{dx} = \frac{(-3e^{-3x} + 2)}{e^{-3x} + 2x} \quad \text{at } x = 1, \quad \frac{dy}{dx} = \frac{-3e^{-3} + 2}{e^{-3} + 2}$$

R. Trig Derivatives

Example 1. Find the derivative of each of the following. Don't forget to use the Product Rule, the Quotient Rule and the Chain Rule!!

a) $y = 2\cos x \quad y' = -2 \sin x$

b) $f(x) = 2\sin 8x \quad f'(x) = 2(8 \cos 8x) = 16 \cos(8x)$

c) $f(x) = \sin^2 x + \cos x + e^{8x} \quad f(x) = (\sin x)^2 + \cos x + e^{8x}$

$$f'(x) = 2(\sin x)^1(\cos x) + (-\sin x) + 8e^{8x}$$

$$f'(x) = 2 \sin x \cos x - \sin x + 8e^{8x}$$

$$f'(x) = \sin 2x - \sin x + 8e^{8x} \text{ trig identity}$$

d) $f(x) = \cot(3x)$ chain rule

$$f'(x) = -3 \csc^2(3x)$$

e) $f(x) = \sin 3x + e^{\sqrt{x}}$ chain rule

$$f'(x) = 3 \cos 3x + e^{\sqrt{x}} \left(\frac{1}{2} x^{-\frac{1}{2}}\right) = 3 \cos 3x + \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

f) $f(x) = (\cos x + \tan x)^6$ chain rule

$$f'(x) = 6(\cos x + \tan x)^5(-\sin x + \sec^2 x)$$

g) $f(x) = \frac{\tan x}{(1+\cos x)^5}$ quotient rule

$$f'(x) = \frac{(\sec^2 x)(1+\cos x)^5 - 5(1+\cos x)^4(-\sin x)(\tan x)}{[(1+\cos x)^5]^2}$$

$$f'(x) = \frac{(1+\cos x)^4 [\sec^2 x(1+\cos x) + 5\sin x \tan x]}{(1+\cos x)^{10}}$$

$$f'(x) = \frac{[\sec^2 x(1+\cos x) + 5\sin x \tan x]}{(1+\cos x)^6}$$

Example 2. If $f(x) = \sin^2 x$, find $f''\left(\frac{\pi}{4}\right)$. chain rule

$$f(x) = (\sin x)^2$$

$f'(x) = 2(\sin x)(\cos x) = \sin 2x$ (easier than doing a product rule for the second derivative)

$$f''(x) = 2\cos 2x$$

$$f''\left(\frac{\pi}{4}\right) = 2\cos\left(\frac{2\pi}{4}\right) = 2\cos\frac{\pi}{2} = 0$$

Example 3. Find the equation of the tangent of $y = x^3 e^{\sin x}$ at $x = \pi$.

$$y' = 3x^2 e^{\sin x} + e^{\sin x}(\cos x)(x^3)$$

$$y'(\pi) = 3\pi^2 e^{\sin \pi} + e^{\sin \pi}(\cos \pi)(\pi)^3$$

$$= 3\pi^2(e)^0 + e^0(-1)(\pi)^3$$

$$= 3\pi^2 - \pi^3 = \pi^2(3 - \pi)$$

$$\text{at } x = \pi, y = x^3 e^{\sin x} = \pi^3 e^0 = \pi^3$$

$$y - y_1 = m(x - x_1) \text{ becomes } y - \pi^3 = \pi^2(3 - \pi)(x - \pi)$$

Practice Exam Question on Trigonometric Derivatives

R1.a) $y = 2\sec x$

$$y' = 2\sec x \tan x$$

b) $f(x) = 3\cos 7x$

$$f'(x) = 3(-4\sin 7x) = -12\sin 7x$$

c) $f(x) = (\sin x)^2$

$$f'(x) = 2(\sin x)(\cos x)$$

d) $f(x) = \sec(2x)$

$$f'(x) = (2x) \tan(2x)$$

e) $f(x) = 3e^x \sin x$

$$f'(x) = (3e^x)(\sin x) + (\cos x)(3e^x) = 3e^x(\sin x + \cos x)$$

f) $f(x) = \ln(\sin x)$

$$f'(x) = \frac{1}{\sin x}(\cos x) = \cot x$$

g) $f(x) = (x + \sin x)^4$

$$f'(x) = 4(x + \sin x)^3(1 + \cos x)$$

h) $f(x) = \frac{\sin x}{\sec x} = \frac{\sin x}{1/\cos x} = \sin x(\cos x)$

$$\begin{aligned} f'(x) &= (\cos x)(\cos x) + (-\sin x)(\sin x) \\ &= \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) \\ &= -1 + 2\cos^2 x \end{aligned}$$

i) $f(x) = \sin(x^{\frac{1}{2}})$

$$f'(x) = \cos(x^{\frac{1}{2}}) \frac{1}{2} x^{-1/2}$$

$$f'(x) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$$

j) $f(x) = \sin(e^x)$

$$f'(x) = \cos(e^x) \cdot e^x = e^x \cos(e^x)$$

k) $y = \cot x \sin x = \frac{\cos x}{\sin x} \sin x = \cos x$

$$y' = -\sin x$$

l) $y = \csc(2x)$

$$y' = -\csc(2x) \cot(2x) \cdot 2 = -2 \csc(2x) \cot(2x)$$

R2. $f(x) = x^2 \sin x$

$$f'(x) = 2x \sin x + x^2 \cos x$$

$$f'(\pi) = 2\pi \sin \pi + \pi^2 \cos \pi = -\pi^2$$

R3. $f(x) = (\sin x)^2$

$$f'(x) = 2 \sin x (\cos x)$$

$$f''(x) = (2 \cos x)(\cos x) + (-\sin x)(2 \sin x) = 2 \cos^2 x - 2 \sin^2 x$$

$$f''\left(\frac{\pi}{6}\right) = 2 \left(\cos \frac{\pi}{6}\right)^2 - 2 \left(\sin \frac{\pi}{6}\right)^2$$

$$f''\left(\frac{\pi}{6}\right) = 2 \left(\frac{\sqrt{3}}{2}\right)^2 - 2 \left(\frac{1}{2}\right)^2 = 2 \left(\frac{3}{4}\right) - 2 \left(\frac{1}{4}\right) = \frac{4}{4} = 1$$

R4. a)

$$f(x) = \sin(5x)$$

$$f'(x) = 5 \cos(5x)$$

b)

$$f(x) = \cos \sqrt{x} = \cos(x^{\frac{1}{2}})$$

$$f'(x) = -\sin\left(x^{\frac{1}{2}}\right) \left(\frac{1}{2} x^{-\frac{1}{2}}\right)$$

$$f'(x) = \frac{-1 \sin \sqrt{x}}{2\sqrt{x}}$$

c)

$$f(x) = e^{\sin x}$$

$$f'(x) = e^{\sin x} (\cos x)$$

d)

$$f(x) = \ln x \sec x$$

$$f'(x) = \frac{1}{x} \sec x + (\sec x \tan x) \ln x = \sec x \left(\frac{1}{x} + \tan x \ln x \right)$$

e)

$$f(x) = \sec^2(2x + 1) = [\sec(2x + 1)]^2$$

$$\begin{aligned} f'(x) &= 2[\sec(2x + 1)]^1 \sec(2x + 1) \tan(2x + 1)(2) \\ &= 4\sec^2(2x + 1) \tan(2x + 1) \end{aligned}$$

$$f) \quad f(x) = e^{\tan x} \ln(\cos x)$$

$$\begin{aligned} f'(x) &= e^{\tan x} (x) \ln(\cos x) + \frac{1(\sin x)}{\cos x} e^{\tan x} \\ &= e^{\tan x} (x) \ln(\sin x) + e^{\tan x} \\ &= e^{\tan x} [(x) \ln(\sin x) + \tan x] \end{aligned}$$

$$g) \quad f(x) = \frac{\sin x}{1 + \cos x}$$

$$f'(x) = \frac{\cos x(1 + \cos x) - (-\sin x)(\sin x)}{(1 + \cos x)^2} = \frac{\cos x + \cos^2 x + \sin x}{(1 + \cos x)^2}$$

$$= \frac{\cos x + 1}{(1 + \cos x)^2} \quad \text{since } x + x = 1$$

$$= \frac{1}{1 + \cos x}$$

$$\begin{aligned} \text{R5. } f(x) &= (\cos x)^2 \\ f'(x) &= 2\cos x (-\sin x) = -2\sin x \cos x \end{aligned}$$

Therefore, answer is C).

$$\begin{aligned} \text{R6. } f'(x) &= \cos(4x)(4) \\ f'\left(\frac{\pi}{4}\right) &= 4\cos\left(\frac{4\pi}{4}\right) = -4 \end{aligned}$$

Therefore, answer is C).

$$\begin{aligned} \text{R7. } f'(x) &= (1)\sin x + (\cos x)(x) \\ f'(\pi) &= \sin \pi + (\cos \pi)(\pi) = 0 - \pi = -\pi \\ \text{At } x=0, y=0, \text{ so the equation is } y=0 &= -\pi(x-0) \text{ or } y = -\pi \end{aligned}$$

$$\begin{aligned} \text{R8. } f'(x) &= (2\sec 2x \tan 2x)(\tan 2x) + 2\sec^2(2x)(\sec 2x) \\ &= 2\sec(2x)\tan^2(2x) + 2\sec^3(2x) \quad \text{Product rule} \end{aligned}$$

$$\begin{aligned} \text{R9. } f(x) &= [\csc(2x)]^2 \\ f'(x) &= 2\csc(2x)(-\csc(2x)\cot(2x)(2)) \\ &= -4\cot(2x)\csc^2(2x) \quad \text{chain rule} \end{aligned}$$

$$\begin{aligned} \text{R10. } f'(x) &= 5\cos x(\cos(2x)) + (-\sin(2x))(2)(5\sin x) \quad \text{product} \\ &\text{rule} \\ f'(0) &= 5(1)(1) - 0 = 5 \end{aligned}$$

R11. Find the slope of the tangent of $f(x) = \sin(2x)$ at $x = \frac{\pi}{4}$.

$$\begin{aligned} f'(x) &= 2 \cos(2x) \\ f'\left(\frac{\pi}{4}\right) &= 2 \cos \frac{2\pi}{4} = 2\left(-\cos \frac{\pi}{2}\right) = 0 \end{aligned}$$

R12.

a) $f(x) = 3e^{2x} \cos(6x)$ product rule

$$\begin{aligned} f'(x) &= 3e^{2x}(2) \cos(6x) + (-6 \sin(6x))(3e^{2x}) \\ &= 6e^{2x} \cos(6x) - 18e^{2x} \sin(6x) \\ &= 6e^{2x}(\cos(6x) - 3 \sin(6x)) \end{aligned}$$

b) $f(x) = \sin\sqrt{\cos x}$ chain rule

$$\begin{aligned} f'(x) &= \cos(\cos x)^{\frac{1}{2}} \cdot \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x) \\ f'(x) &= \frac{-\sin x \cos \sqrt{\cos x}}{2\sqrt{\cos x}} \end{aligned}$$

c) $y = \cos(e^x)$ chain rule

$$y' = -\sin(e^x) \cdot e^x = -e^x \sin(e^x)$$

d) $y = \cot x \csc x$ product rule

$$y' = -\csc^2 x (\csc x) + \cot x (-\csc x \cot x) = -\csc^3 x - \cot^2 x \csc x$$

If the answers involve $\cos x/\sin x$:

$$\begin{aligned} y &= \frac{\cos x}{\sin x} \left(\frac{1}{\sin x} \right) = \frac{\cos x}{(\sin x)^2} \\ y' &= \frac{-\sin x (\sin^2 x) - 2 \sin x (\cos x) (\cos x)}{\sin^4 x} \\ &= \frac{-\sin^3 x - 2 \sin x \cos^2 x}{\sin^4 x} \end{aligned}$$

R13. If $f(x) = x^2 \sin x$, find $f' \left(\frac{\pi}{2} \right)$. product rule

$$f'(x) = 2x(\sin x) + (\cos x)(x^2)$$

$$\begin{aligned} f' \left(\frac{\pi}{2} \right) &= \frac{2\pi}{2} \sin \frac{\pi}{2} + \left(\cos \frac{\pi}{2} \right) \left(\frac{\pi}{2} \right)^2 \\ &= \pi(1) + 0 = \pi \end{aligned}$$

R14. Find the slope of the tangent of $y = x - \cos x$ at $x = \frac{\pi}{6}$.

$$y' = 1 + \sin x$$

$$y' \left(\frac{\pi}{6} \right) = 1 + \sin \frac{\pi}{6} = 1 + \frac{1}{2} = \frac{3}{2}$$

S. Derivatives of Inverse Trig

Example 1. $f'(x) = (1)(\arctan x) + \left(\frac{1}{1+x^2}\right)(x)$
 $= \arctan x + \frac{x}{1+x^2}$

Example 2. $f'(x) = \frac{1}{1+x^2}(e^x) + (\arctan x)(e^x)$
 $= e^x \left[\frac{1}{1+x^2} + \arctan x \right]$ (product rule)

vs. $f'(x) = \frac{e^x}{1+(e^x)^2} = \frac{e^x}{1+e^{2x}}$ (chain rule)

Example 3. $f'(x) = \frac{1}{3x\sqrt{(3x)^2-1}} = \frac{1}{3x\sqrt{9x^2-1}}$

Example 4. $f(x) = e^x \cos^{-1} x$

$$f'(x) = e^x \cos^{-1} x + \left(\frac{-1}{\sqrt{1-x^2}}\right)e^x$$

$$= e^x \left[\cos^{-1} x - \frac{1}{\sqrt{1-x^2}} \right]$$

Example 5. $f'(x) = \frac{1}{\sqrt{1-x^2}}(e^x) + e^x \sin^{-1} x$
 $= e^x \left(\frac{1}{\sqrt{1-x^2}} + \sin^{-1} x \right)$

Example 6. $f'(x) = \frac{e^x(\arctan x) - \frac{1(e^x)}{1+x^2}}{(\arctan x)^2}$

$$\begin{aligned} \text{S1. } f'(x) &= e^x \sin^{-1} x + \left(\frac{1}{\sqrt{1-x^2}} \right) (e^x) \\ &= e^x \left[\sin^{-1} x + \frac{1}{\sqrt{1-x^2}} \right] \end{aligned}$$

$$\begin{aligned} \text{S2. } f'(x) &= 3x^2 \sin^{-1} x + \frac{1}{\sqrt{1-x^2}} (x^3) \\ &= x^2 \left[3 \sin^{-1} x + \frac{x}{\sqrt{1-x^2}} \right] \end{aligned}$$

$$\text{S3. } f'(x) = 0 \text{ since } e^{10} \text{ is a constant}$$

$$\text{S4. } f'(x) = \frac{-1}{\sqrt{1-x^2}} + \cos x$$

$$\text{S5. } f'(x) = \frac{1}{1+x^2} + \sec^2 x$$

$$\text{S6. } f'(x) = \sec^2 x (\arccos x) + \frac{-1}{\sqrt{1-x^2}} (\tan x)$$

$$\begin{aligned} \text{S7. } f'(x) &= e^x (\operatorname{arc} \sec x) + \frac{1}{x\sqrt{x^2-1}} (e^x) \\ &= e^x \left[\operatorname{arc} \sec x + \frac{1}{x\sqrt{x^2-1}} \right] \end{aligned}$$

$$\begin{aligned} \text{S8. } f'(x) &= 3x^2 \tan^{-1} x + \frac{1}{1+x^2} (x^3) \\ &= x^2 \left(3 \tan^{-1} x + \frac{x}{1+x^2} \right) \end{aligned}$$

$$\begin{aligned} \text{S9. } f'(x) &= \frac{1}{\sqrt{1-(e^x)^2}} (e^x) \\ &= \frac{e^x}{\sqrt{1-e^{2x}}} \end{aligned}$$

$$S10. f'(x) = \frac{3e^{3x}}{1+(e^{3x})^2} = \frac{3e^{3x}}{1+e^{6x}}$$

$$S11. f'(x) = \sec^2(\arcsin(e^x)) \frac{1}{\sqrt{1-(e^x)^2}} (e^x)$$

$$f'(x) = \frac{e^x \sec^2(\arcsin(e^x))}{\sqrt{1-e^{2x}}}$$

S12.

$$h'(1)$$

$$h(x) = \tan^{-1}(f(x))^3$$

$$h'(x) = \frac{3[f(x)]^2 \cdot f'(x)}{1+(f(x)^3)^2}$$

$$h'(1) = \frac{3[f(1)]^2 \cdot f'(1)}{1+(f(1))^6}$$

$$h'(1) = \frac{3(-1)^2(2)}{1+(-1)^6} = \frac{6}{2} = 3$$

T. Equation of a Tangent Line

Example 1. Find the slope of the tangent line to the curve $f(x) = 2x^3 + 5x + 4$ at $x=1$.

$$\text{slope} = f'(x) = 6x^2 + 5$$

$$\text{at } x = 1, f'(1) = 6(1)^2 + 5 = 11$$

Example 2. Find the slope of the tangent line to the curve $y = 4^{2x+1}$ at $x=2$.

$$y' = 4^{2x+1}(\ln 4)(2)$$

$$\text{at } x = 2, y' = 4^5(\ln 4)(2)$$

Example 3. Find the points on the curve $f(x) = 3x^3 + 9x$ where the slope of the tangent is parallel to the line $-18x + y - 6 = 0$.

parallel ∴ same slope or derivative

$$-18x + y - 6 = 0.$$

$$y = 18x + 6 \quad \therefore y' = 18$$

$$f(x) = 3x^3 + 9x$$

$$f'(x) = 9x^2 + 9 \quad \therefore 9x^2 + 9 = 18$$

$$9x^2 = 9$$

$$x^2 = 1 \quad x = 1, -1$$

$$x = 1 \quad f(1) = 12 \quad \therefore (1, 12) \text{ is one point}$$

$$x = -1 \quad f(-1) = -12 \quad \therefore (-1, -12) \text{ is the other point}$$

Example 4. Find the equation of a tangent line to $f(x) = \sin 2x$ at

$$x = \frac{\pi}{6}. \quad f'(x) = 2\cos 2x$$

$$f' \left(\frac{\pi}{6} \right) = 2 \cos \left(\frac{\pi}{3} \right) = 2 \left(\frac{1}{2} \right) = 1$$

Find the value of y , when $x = \frac{\pi}{6}$

$$f(x) = \sin 2x$$

$$f \left(\frac{\pi}{6} \right) = \sin 2 \left(\frac{\pi}{6} \right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$y - y_1 = m(x - x_1) \text{ becomes } y - \frac{\sqrt{3}}{2} = 1 \left(x - \frac{\pi}{6} \right)$$

$$y = x - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

Example 5. $g(3) = 2(3)^2 - 5(3) + 1 = 18 - 15 + 1 = 4, \quad x_1 = 3$

$$\therefore y_1 = 4$$

$$g'(x) = 4x - 5 \quad g'(3) = 4(3) - 5 = 7$$

$$\therefore M_{\text{normal}} = \frac{-1}{7}$$

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -\frac{1}{7}(x - 3)$$

$$y - 4 = -\frac{1}{7}x + \frac{3}{7}$$

$$y = -\frac{1}{7}x + \frac{3}{7} + \frac{28}{7}$$

$$y = -\frac{1}{7}x + \frac{31}{7}$$

$$T1. y = x^2 e^{3x} \text{ at } x=1$$

$$y' = 2xe^{3x} + 3e^{3x}(x^2)$$

$$y'(1) = 2e^3 + 3e^3 = 5e^3$$

Find y when $x = 1$

$$y = (1)^2 e^{3(1)} = e^3$$

$$y - e^3 = 5e^3(x - 1)$$

$$y = 5e^3x - 5e^3 + e^3 = 5e^3x - 4e^3$$

T2.

$$y = 12(e^{3x} + 1) \text{ parallel to } y = 4x - 1$$

$$y' = 12(3e^{3x}) = 36e^{3x} \text{ parallel to } y' = 4 \dots \text{same slopes, so we get:}$$

$$36e^{3x} = \frac{4}{1}$$

$$e^{3x} = \frac{1}{9}$$

$$\ln e^{3x} = \ln\left(\frac{1}{9}\right)$$

$$3x = \ln 1 - \ln 9$$

$$3x = -\ln 9$$

$$x = -\frac{\ln 9}{3}$$

T3. Horizontal tangent $\therefore \text{slope} = 0$

$$f(x) = 4x^2 - 8x + 3 - 4x + 17$$

$$f(x) = 4x^2 - 12x + 20$$

$$f'(x) = 8x - 12$$

$$\therefore 0 = 8x - 12$$

$$x = \frac{12}{8} = \frac{3}{2}$$

$$f\left(\frac{3}{2}\right) = 4\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) + 20 = 4\left(\frac{9}{4}\right) - 18 + 20 = 11$$

So, the point is $(3/2, 11)$.

T4. $f(x) = 3x^3 - 6x$ at $x = 2$

$$f(2) = 3(2)^3 - 6(2) = 12 \text{ So } y = 12$$

$$f'(x) = 9x^2 - 6$$

$$f'(2) = 9(2)^2 - 6 = 30 \dots \text{so the slope is } m = 30$$

$$y - y_1 = m(x - x_1)$$

$$y - 12 = 30(x - 2) \quad y = 30x - 48$$

T5. $y = (2x - 1)^4$ at $x = 0$

$$y' = 4(2x - 1)^3(2)$$

$$y'_{x=0} = 4(-1)^3(2) = -8 \dots \text{so slope} = m = -8$$

$$\text{at } x = 0 \quad y = (2(0) - 1)^4 = (-1)^4 = 1$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -8(x - 0) \quad y = -8x + 1$$

$$T6. f'(x) = 9x^2 e^x + e^x (3x^3)$$

$$f'(2) = 9(2)^2 e^2 + e^2 (3(2)^3) = 36e^2 + 24e^2 = 60e^2$$

U. Second Derivatives

Example 1.

$$\text{a) } f(x) = e^{2x}$$

$$f'(x) = 2e^{2x}$$

$$f''(x) = 4e^{2x}$$

$$\text{b) } f(x) = xe^{2x}$$

$$f'(x) = (1)(e^{2x}) + (2e^{2x})(x) = e^{2x}(1 + 2x)$$

$$f''(x) = 2e^{2x}(1 + 2x) + (2)(e^{2x}) = 4xe^{2x} + 4e^{2x}$$

$$= 4e^{2x}(x + 1)$$

Example 2.

$$f(x) = (2x + 1)^3$$

$$f'(x) = 3(2x + 1)^2(2) = 6(2x + 1)^2$$

$$f''(x) = 12(2x + 1)^1(2) = 24(2x + 1)^1 = 48x + 24$$

$$f'''(x) = 48$$

$$U1. y' = 3(2x - 7)^2(2) = 6(2x - 7)^2$$

$$y'' = 12(2x - 7)^1(2) = 24(2x - 7)$$

$$y''' = 24(2) = 48$$

$$U2. f'(x) = \frac{1}{2}(4x - 1)^{-1/2}(4) = 2(4x - 1)^{-1/2}$$

$$f''(x) = -1(4x - 1)^{-3/2}(4) = -4(4x - 1)^{-3/2}$$

$$f'''(x) = 6(4x - 1)^{-5/2}(4) = 24(4x - 1)^{-5/2}$$

U3. If $f(x) = \cos 2x$, find $f^6(x)$.

$$f'(x) = -2 \sin 2x$$

$$f''(x) = -2(\cos 2x \cdot 2) = -4 \cos 2x$$

$$f^3(x) = -4(-\sin 2x \cdot 2) = 8 \sin 2x$$

$$f^4(x) = 16 \cos 2x$$

$$f^5(x) = -32 \sin 2x$$

$$f^6(x) = -64 \cos 2x$$

U4. If $f(x) = (2x + 1)^4$, find $f^6(x)$.

$$f'(x) = 4(2x + 1)^3(2) = 8(2x + 1)^3$$

$$f''(x) = 24(2x + 1)^2(2) = 48(2x + 1)^2$$

$$f'''(x) = 96(2x + 1)(2) = 192(2x + 1)$$

$$f^4(x) = 192(2) = 384$$

$$f^5(x) = f^6(x) = 0$$

U5. $f(x) = \sin x$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f^3(x) = -\cos x$$

$$f^4(x) = \sin x \dots \text{keep going in this pattern} \dots f^{10}(x) = -\sin x$$

V. Tricky Domain Questions

Example 1. $x \neq \pm 2$

$$x^2 - 4 \neq 0 \quad D = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$$

$$x^2 \neq 4$$

Example 2. $x - 5 \geq 0$

$$x \geq 5 \quad \therefore D = [5, \infty)$$

Example 3. $x^2 - 4 \neq 0$ $x^2 - 4 \geq 0$

$$x \neq \pm 2 \quad x^2 \geq 4$$

$$x \geq 2 \text{ or } x \leq -2$$

$$\therefore D = (-\infty, -2) \cup (2, \infty)$$

Example 4. $x - 2 > 0$

$$x > 2 \quad \therefore D = (2, \infty)$$

Example 5. D of $\arccos x$ is $[-1, 1]$

D of e^x is R

So, D of $f(x)$ is $[-1, 1]$

Example 6. Domain of $\cos^{-1}(x)$ is $[-1, 1]$

Solve $-1 \leq e^{3x} \leq 1$ but $e^{3x} > 0$ for all x , so

$$0 < e^{3x} \leq 1$$

$$\ln 0 < \ln e^{3x} \leq \ln 1$$

$$-\infty < 3x \leq 0$$

$$-\infty < x \leq 0$$

$$\therefore D = (-\infty, 0]$$

V1. $x + 2 \neq 0$

$$x \neq -2 \quad \therefore D = (-\infty, -2) \cup (-2, \infty)$$

V2. $D = [-1, 1]$

V3. Domain of $\cos^{-1}(x)$ is $[-1, 1]$

Solve $-1 \leq e^x \leq 1$ but $e^x > 0$ for all x , so

$$0 < e^x \leq 1$$

$$\ln 0 < \ln e^x \leq \ln 1$$

$$-\infty < x \leq 0$$

$$\therefore D = (-\infty, 0]$$

V4. $x+6 \geq 0$

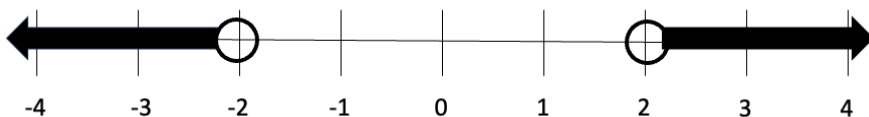
$$x \geq -6 \quad \therefore D = [-6, \infty)$$

V5.

$$y = \ln(x^2 - 4)$$

$$x^2 - 4 > 0$$

$$x^2 > 4$$



$$\therefore D = (-\infty, -2) \cup (2, \infty)$$

V6.

$$\text{I } f(x) = \sqrt{\frac{x}{x+4}} \quad \text{II } g(x) = \frac{\sqrt{x}}{\sqrt{x+4}}$$

No, they do NOT have the same domain

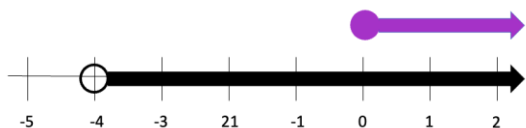
For I:

$$\frac{x}{x+4} \geq 0 \quad \text{and } x+4 \neq 0$$

$$\text{ie. } x \neq -4$$

Case 1

$$x \geq 0 \text{ and } x + 4 > 0, \text{ ie) } x > -4$$

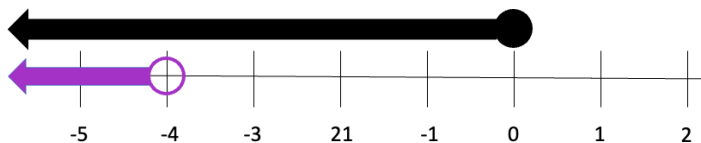


$$\therefore x \geq 0$$

OR

Case 2

$$x \leq 0 \text{ and } x + 4 < 0, \text{ ie) } x < -4$$



$$\therefore x < -4$$

\therefore solution $x \geq 0$ or $x < -4$ $(-\infty, -4) \cup [0, \infty)$ which is not the answer for II below.

For II:

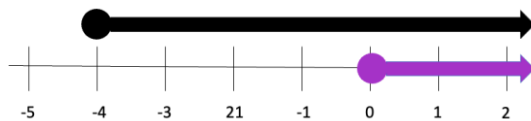
D of \sqrt{x}

D of $\sqrt{x+4}$

$$x \geq 0$$

$$x + 4 \geq 0$$

$$x \geq -4$$



\therefore combined we get:

ie.) $x \geq 0$ or $[0, \infty)$

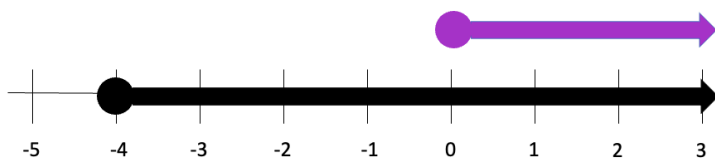
V7.

$$h(x) = \sqrt{x(x+4)}$$

$$x(x+4) \geq 0$$

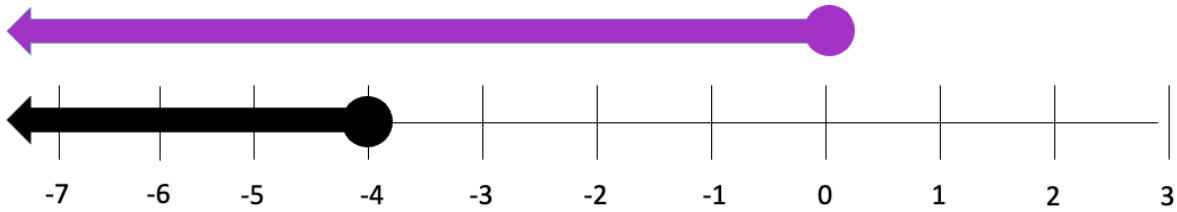
Case 1 $x \geq 0$ and $x + 4 \geq 0$

ie). $x \geq -4$



\therefore both are coloured in for $x \geq 0$ ie. $[0, \infty)$

Case 2 $x \leq 0$ and $x + 4 \leq 0$, ie.) $x \leq -4$



\therefore both are coloured in for $x \leq -4$ ie. $(-\infty, -4]$

\therefore final answer is $(-\infty, -4] \cup [0, \infty)$

W. Inverse Function Theorem

Example 1.

Use the inverse *function* theorem to find derivative of $g(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

First, find the inverse of $y = x^{\frac{1}{3}}$

Switch x and y: $x = y^{\frac{1}{3}}$ cube both sides and get $x^3 = y$ and $f(x) = x^3$

The *function* $g(x) = x^{\frac{1}{3}}$ is the inverse of $f(x) = x^3$

$$f'(x) = 3x^2$$

$$f'(g(x)) = f'\left(x^{\frac{1}{3}}\right) = 3\left(x^{\frac{1}{3}}\right)^2 = 3x^{\frac{2}{3}}$$

$$\therefore g'(x) = \frac{1}{f'(g(x))} = \frac{1}{3x^{\frac{2}{3}}}$$

Example 2. $f(x) = x + \sqrt{x}$ Find $(f^{-1})'(2)$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Step 1. Find $f'(x)$

$$f'(x) = 1 + \frac{1}{2}x^{-\frac{1}{2}} = 1 + \frac{1}{2\sqrt{x}}$$

Step 2. Find $f^{-1}(2) \leftarrow f(x)$ is 2

$$2 = x + \sqrt{x}$$

$$f(x) = x + \sqrt{x} \leftarrow \text{true if } x = 1$$

$$\therefore f^{-1}(2) = 1$$

Step 3. $(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}$

$$= \frac{1}{f'(1)} \leftarrow f'(x) = 1 + \frac{1}{2\sqrt{x}}$$

$$f'(1) = 1 + \frac{1}{2\sqrt{1}} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$= \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}}$$

W1.

Use the inverse *function* theorem to find derivative of $g(x) = \frac{x+2}{x}$

$$y = \frac{x+2}{x}$$

$$x \leftrightarrow y \quad x = \frac{y+2}{y}$$

$$xy = y + 2$$

$$xy - y = 2$$

$$y(x - 1) = 2$$

$$y = \frac{2}{x-1} \quad \therefore f(x) = \frac{2}{x-1} = 2(x-1)^{-1}$$

Need $f'(g(x))$

$$f'(x) = -2(x-1)^{-2}$$

$$\begin{aligned} f'(g(x)) &= f'\left(\frac{x+2}{x}\right) = -2\left[\frac{x+2}{x} - 1\right]^{-2} \\ &= -2\left[1 + \frac{2}{x} - 1\right]^{-2} \\ &= -2\left[\frac{2}{x}\right]^{-2} = -2\left[\frac{x}{2}\right]^2 \\ &= \frac{-2x^2}{4} = \frac{-x^2}{2} \end{aligned}$$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\frac{-x^2}{2}} = \boxed{\frac{-2}{x^2}}$$

W2.

$$f(x) = 2x^3 + 3x^2 + 7x + 4$$

Find $(f^{-1})'(4)$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Step 1. find $f'(x)$

$$f'(x) = 6x^2 + 6x + 7$$

Step 2. find $f^{-1}(4)$ 4 is $f(x)$ not x

$$\therefore f(x) = 2x^3 + 3x^2 + 7x + 4$$

$$4 = 2x^3 + 3x^2 + 7x + 4 \leftarrow \text{true if } x = 0$$

$$\therefore f^{-1}(4) = 0$$

$$\text{Step 3. } (f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))}$$

$$= \frac{1}{f'(0)}$$

$$= \frac{1}{6(0)^2 + 6(0) + 7} \leftarrow f'(x) = 6x^2 + 6x + 7$$

$$\therefore f'(0) = 7$$

$$= \boxed{\frac{1}{7}}$$

X. Implicit Differentiation

Example 2.

$$y = x^x$$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y} \frac{dy}{dx} \left[(1) \ln x + \frac{1}{x} (x) \right]$$

$$\frac{dy}{dx} = y[\ln x + 1] = x^x(\ln x + 1)$$

Example 3. If $y = (4x + 3)^x$, find $\frac{dy}{dx}$ at $x=1$.

$$\ln y = \ln(4x + 3)^x$$

$$\ln y = x \cdot \ln(4x + 3)$$

$$\frac{1}{y} \frac{dy}{dx} = (1) \ln(4x + 3) + \frac{4x}{4x+3}$$

$$\frac{dy}{dx} = (4x + 3)^x \left[\ln(4x + 3) + \frac{4x}{4x+3} \right]$$

$$\frac{dy}{dx} \Big|_{x=1} = 7^1 \left[\ln 7 + \frac{4}{7} \right]$$

Example 4. If $f(x) = \ln(x^x)$, find $f'(x)$.

$$y = \ln x^x \quad \text{use log rules}$$

$$y = x \ln x \quad \text{not log diff}$$

$$y' = (1) \ln x + \frac{1}{x} (x)$$

$$y' = \ln x + 1$$

Example 5. If $y = (\ln x)^x$, find $f'(e)$.

$$\ln y = \ln(\ln x)^x$$

$$\ln y = x \cdot \ln(\ln x)$$

$$\frac{1}{y} \frac{dy}{dx} = (1) \ln(\ln x) + \frac{1 \left(\frac{1}{x}\right)(x)}{\ln x}$$

$$\frac{dy}{dx} = (\ln x)^x \left[\ln(\ln x) + \frac{x}{x \ln x} \right]$$

$$\frac{dy}{dx} = (\ln x)^x \left[\ln(\ln x) + \frac{1}{\ln x} \right]$$

$$\text{at } x = e \quad \frac{dy}{dx} = (\ln e)^e \left[\ln(\ln e) + \frac{1}{\ln e} \right]$$

$$(1)^e \left[\ln 1 + \frac{1}{1} \right]$$

$$= \ln 1 + 1 = 1$$

Example 6. If $f(x) = (x + 5)^x x^{x+6}$, find $\frac{dy}{dx}$.

$$y = (x + 5)^x x^{x+6}$$

$$\ln y = \ln[(x + 5)^x \cdot x^{x+6}]$$

$$\ln y = x \ln(x + 5) + (x + 6) \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = (1) \ln(x + 5) + \frac{x}{x+5} + (1) \ln x + \frac{1}{x}(x + 6)$$

$$\frac{dy}{dx} = y \left[\ln(x + 5) + \frac{x}{x+5} + \ln x + \frac{x+6}{x} \right]$$

$$\frac{dy}{dx} = (x + 5)^x x^{x+6} \left[\ln(x + 5) + \frac{x}{x+5} + \ln x + \frac{x+6}{x} \right]$$

Example 7. Find the derivative of $y(x) = \frac{x^2(x-6)^3}{\sqrt[3]{3-x}}$

$$y = x^2(x-6)^3(3-x)^{-\frac{1}{3}}$$

$$\ln y = \ln \left[x^2(x-6)^3(3-x)^{-\frac{1}{3}} \right]$$

$$\ln y = 2 \ln x + 3 \ln(x-6) - \frac{1}{3} \ln(3-x)$$

$$\frac{1}{y} \frac{dy}{dx} = 2 \left(\frac{1}{x} \right) + 3 \left(\frac{1}{x-6} \right) - \frac{1}{3} \left(\frac{-1}{3-x} \right)$$

$$\frac{dy}{dx} = y \left[\frac{2}{x} + \frac{3}{x-6} + \frac{1}{9-3x} \right]$$

$$\frac{dy}{dx} = \frac{x^2(x-6)^3}{\sqrt[3]{3-x}} \left[\frac{2}{x} + \frac{3}{x-6} + \frac{1}{9-3x} \right]$$

Implicit Differentiation

Example 1.

$$x^3 + y^2 = 2$$

$$3x^2 + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{3x^2}{2y}$$

Example 3. Find the derivative $x^4 + 6y^3 = 12x + 6$.

$$4x^3 + 18y^2 \frac{dy}{dx} = 12$$

$$18y^2 \frac{dy}{dx} = 12 - 4x^3$$

$$\frac{dy}{dx} = \frac{12 - 4x^3}{18y^2}$$

$$dy/dx \text{ at } (0,1) = \frac{12 - 4(0)^3}{18(1)^2} = \frac{12}{18} = \frac{2}{3}$$

Example 4. Differentiate: $x^3y^2 - 4x = 3 - 2y$

product rule

$$3x^2(y^2) + 2y \cdot \frac{dy}{dx}(x^3) - 4 = 0 - 2 \frac{dy}{dx}$$

$$2x^3y \frac{dy}{dx} + 2 \frac{dy}{dx} = 4 - 3x^2y^2$$

$$\text{Factor } \frac{dy}{dx}(2x^3y + 2) = 4 - 3x^2y^2$$

$$\frac{dy}{dx} = \frac{4 - 3x^2y^2}{2x^3y + 2}$$

Example 5. Differentiate: $e^{xy} + 4x = 5 - \ln x + y$

$$e^{xy} \left[(1)(y) + \frac{dy}{dx}(x) \right] + 4 = 0 - \frac{1}{x} + (1) \frac{dy}{dx}$$

$$ye^{xy} + xe^{xy} \frac{dy}{dx} + 4 = -\frac{1}{x} + \frac{dy}{dx}$$

$$xe^{xy} \frac{dy}{dx} - \frac{dy}{dx} = -\frac{1}{x} - 4 - ye^{xy}$$

factor

$$\frac{dy}{dx}(xe^{xy} - 1) = -\frac{1}{x} - 4 - ye^{xy}$$

$$\frac{dy}{dx} = \frac{-\frac{1}{x} - 4 - ye^{xy}}{xe^{xy} - 1} \text{ OR } \frac{dy}{dx} = \frac{\frac{1}{x} + 4 + ye^{xy}}{-xe^{xy} + 1} \text{ (if you brought all$$

the terms to the opposite sides)

Example 6. $x^3 - 4y^3 + 2y^2 = -1$ Find $y'(1,1)$ and $y''(1,1)$.

$$3x^2 - 12y^2y' + 4yy' = 0$$

$$y'(-12y^2 + 4y) = -3x^2$$

$$y' = \frac{-3x^2}{-12y^2 + 4y}$$

$$y'(1,1) = \frac{-3}{-12 + 4} = \frac{-3}{-8} = \frac{3}{8}$$

$$\frac{d^2y}{dx^2} = \frac{(-6x)(-12y^2 + 4y) - (-24yy' + 4y')(-3x^2)}{(-12y^2 + 4y)^2}$$

$$\frac{d^2y}{dx^2}(1,1) = \frac{(-6)(-12 + 4) - \left(-24(1)\left(\frac{3}{8}\right) + 4\left(\frac{3}{8}\right)\right)(-3)}{(-12 + 4)^2}$$

$$\begin{aligned} \frac{d^2y}{dx^2}(1,1) &= \frac{48 - \left(-9 + \frac{3}{2}\right)(-3)}{64} = \frac{48 - \left(-\frac{18}{2} + \frac{3}{2}\right)(-3)}{64} \\ &= \frac{48 - \left(-\frac{15}{2}\right)(-3)}{64} \end{aligned}$$

$$\frac{d^2y}{dx^2}(1,1) = \frac{\frac{96}{2} - \frac{45}{2}}{64} = \frac{51}{2} \left(\frac{1}{64}\right) = \frac{51}{128}$$

Example 7.

$$2x^2 + xy + 2y^2 = 4 \quad \boxed{1}$$

Horizontal tangent $y' = 0$

Implicit

$$4x + 1(y) + xy' + 4yy' = 0$$

$$y'(x + 4y) = -4x - y$$

$$y' = -\frac{4x - y}{x + 4y}$$

$y' = 0$, so $-4x - y = 0$ and make sure $(x + 4y \neq 0)$

$$y = -4x$$

Substitute into $\boxed{1}$

$$2x^2 + xy + 2y^2 = 4$$

$$2x^2 + x(-4x) + 2(-4x)^2 = 4$$

$$2x^2 - 4x^2 + 2(16x^2) = 4$$

$$-2x^2 + 32x^2 = 4$$

$$30x^2 = 4$$

$$x^2 = \frac{4}{30}$$

$$x = \pm \sqrt{\frac{4}{30}} = \pm \frac{2}{\sqrt{30}}$$

$$x = \frac{2}{\sqrt{30}}$$

$$y = -4x$$

$$y = -4\left(\frac{2}{\sqrt{30}}\right)$$

$$y = \frac{-8}{\sqrt{30}}$$

$$x = \frac{-2}{\sqrt{30}}$$

$$y = -4x$$

$$y = -4\left(\frac{-2}{\sqrt{30}}\right)$$

$$y = \frac{8}{\sqrt{30}}$$

∴ points are $\left(\frac{2}{\sqrt{30}}, \frac{-8}{\sqrt{30}}\right)$ and $\left(\frac{-2}{\sqrt{30}}, \frac{8}{\sqrt{30}}\right)$

Practice Exam Questions on Implicit Differentiation

X1. $y = (x + 2)^x$

$$\ln y = \ln(x + 2)^x$$

$$\ln y = x \ln(x + 2)$$

$$\frac{1}{y} \frac{dy}{dx} = \left[(1) \ln(x + 2) + \frac{1}{x+2} (x) \right]$$

$$\frac{dy}{dx} = (x + 2)^x \left[\ln(x + 2) + \frac{x}{x+2} \right]$$

$$\frac{dy}{dx} (x = 1) = 3^1 \left[\ln 3 + \frac{1}{3} \right]$$

Therefore, answer is B).

X2. $\ln y = \ln(x + 2)^{\ln x}$

$$\ln y = \ln x \ln(x + 2)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \ln(x + 2) + \frac{1}{x+2} (\ln x)$$

$$\frac{dy}{dx} = y \left[\frac{\ln(x+2)}{x} + \frac{\ln x}{x+2} \right]$$

$$\frac{dy}{dx} = (x + 2)^{\ln x} \left[\frac{\ln(x+2)}{x} + \frac{\ln x}{x+2} \right]$$

$$\frac{dy}{dx} (x = 1) = 3^{\ln 1} \left[\frac{\ln 3}{1} + \frac{\ln 1}{3} \right] = 3^0 (\ln 3) = \ln 3$$

$$\text{X3. } y = (5x + 1)^x$$

$$\ln y = \ln(5x + 1)^x$$

$$\ln y = x \ln(5x + 1)$$

$$\frac{1}{y} \frac{dy}{dx} = (1) \ln(5x + 1) + \frac{5}{5x+1} (x)$$

$$\frac{dy}{dx} = y \left[\ln(5x + 1) + \frac{5x}{5x + 1} \right]$$

$$\frac{dy}{dx} = (5x + 1)^x \left[\ln(5x + 1) + \frac{5x}{5x + 1} \right]$$

$$\frac{dy}{dx} (x = 1) = 6^1 \left[\ln 6 + \frac{5}{6} \right] = 6 \ln 6 + 5$$

$$\text{X4. } \ln y = \ln \left[\frac{x^{1/3} (3x-1)^3}{(3-x)^{1/3}} \right]$$

$$\ln y = \ln x^{1/3} + \ln(3x - 1)^3 - \ln(3 - x)^{1/3}$$

$$\ln y = \frac{1}{3} \ln x + 3 \ln(3x - 1) - \frac{1}{3} \ln(3 - x)$$

$$\frac{1}{y} y' = \frac{1}{3} \left(\frac{1}{x} \right) + 3 \left(\frac{3}{3x - 1} \right) - \frac{1}{3} \left(\frac{-1}{3 - x} \right)$$

$$y' = \frac{x^{1/3} (3x - 1)^3}{(3 - x)^{1/3}} \left[\frac{1}{3x} + \frac{9}{3x - 1} + \frac{1}{9 - 3x} \right]$$

X5.

$$y = (x + e^x)^{\ln x}$$

$$\ln y = \ln(x + e^x)^{\ln x}$$

$$\ln y = \ln x \ln(x + e^x)$$

$$\frac{1}{y} y' = \frac{1}{x} \ln(x + e^x) + \frac{(1+e^x) \ln x}{x+e^x}$$

$$\frac{1}{y} y' = \frac{1}{x} \ln(1 + e^x) + \frac{(1+e^x) \ln x}{x+e^x}$$

$$y' = (x + e^x)^{\ln x} \left[\frac{\ln(1+e^x)}{x} + \frac{(1+e^x) \ln x}{x+e^x} \right]$$

X6.

$$y = (\ln x)^{\ln x}$$

$$\ln y = \ln (\ln x)^{\ln x}$$

$$\ln y = \ln x \ln(\ln x)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} (\ln(\ln x)) + \frac{1}{\ln x} \left(\frac{1}{x} \right) (\ln x)$$

$$\frac{dy}{dx} = y \left[\frac{\ln(\ln x)}{x} + \frac{\frac{\ln x}{x}}{\ln x} \right]$$

$$\frac{dy}{dx} = (\ln x)^{\ln x} \left[\frac{\ln(\ln x)}{x} + \frac{\ln x}{x} \left(\frac{1}{\ln x} \right) \right] = (\ln x)^{\ln x} \left[\frac{\ln(\ln x)}{x} + \frac{1}{x} \right]$$

$$\text{At } x=e, \frac{dy}{dx} = (\ln e)^{\ln e} \left[\frac{\ln(\ln e)}{e} + \frac{1}{e} \right] = 1 \left(\frac{\ln 1}{e} + \frac{1}{e} \right) = \frac{1}{e}$$

$$\text{X7. } y = x^{(2^x)}$$

$$\ln y = \ln x^{(2^x)}$$

$$\ln y = 2^x \ln x$$

$$\frac{1}{y} \cdot y' = 2^x \ln 2 (\ln x) + \frac{1}{x} (2^x)$$

$$y' = y \left[2^x \ln 2 (\ln x) + \frac{2^x}{x} \right]$$

$$y' = x^{(2^x)} \left[2^x \ln 2 (\ln x) + \frac{2^x}{x} \right]$$

$$\text{X8. } x^2 + y^2 = 2$$

$$2x + 2yy' = 0$$

$$2yy' = -2x$$

$$y' = \frac{-2x}{2y} = \frac{-x}{y} \quad \text{subst } x = y = 1$$

$$y' = -\frac{1}{1} = -1$$

$$\text{X9. } x^2 + y^2 + 3x - y = 5x^2$$

$$2x + 2y y' + 3 - 1y' = 10x$$

$$y'(2y - 1) = 10x - 2x - 3$$

$$y'(2y - 1) = 8x - 3$$

$$y' = \frac{8x-3}{2y-1}$$

$$\text{X10. } \sin(x + y) = x \cos y$$

$$\cos(x + y) (1 + y') = (1) \cos y + (x)(-\sin y)y'$$

$$\cos(x + y) + y' \cos(x + y) = \cos y - x \sin y y'$$

$$y'(\cos(x + y) + x \sin y) = \cos y - \cos(x + y)$$

$$y' = \frac{\cos y - \cos(x + y)}{(\cos(x + y) + x \sin y)}$$

X11.

$$x^3 + y^3 = 27$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = -\frac{x^2}{y^2}$$

$$\text{X12. } x^2 + y^2 = 5$$

$$2x + 2yy' = 0$$

$$2yy' = -2x$$

$$y' = \frac{-2x}{2y} = -\frac{x}{y} = -xy^{-1}$$

$$y'' = (-1)(y^{-1}) + (-y^{-2})y'(-x) = \textit{substitute } y' = -xy^{-1}$$

$$y'' = (-1)(y^{-1}) + (-y^{-2})(-xy^{-1})(-x) \dots \textit{you could leave your answer like this!!}$$

$$y'' = -y^{-1} - x^2y^{-3} = -y^{-3}(y^2 + x^2) \dots \textit{factor}$$

Now, the tricky step is to substitute the original equation

$$\text{in } x^2 + y^2 = 5$$

Crazy, eh?

$$y'' = -y^{-3}(5) = \frac{-5}{y^3}$$

$$\text{X13. } x^3 + y^2 = 6$$

$$3x^2 + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-3x^2}{2y}$$

$$X14. x^2y + x^3 = 12$$

$$2x(y) + (1)\frac{dy}{dx}x^2 + 3x^2 = 0$$

$$2xy + x^2\frac{dy}{dx} = -3x^2$$

$$x^2\frac{dy}{dx} = -3x^2 - 2xy$$

$$\frac{dy}{dx} = \frac{-3x^2 - 2xy}{x^2} = \frac{-(3x + 2y)}{x}$$

$$\frac{dy}{dx}(2,1) = \frac{-(3x+2y)}{x} = -\frac{6+2}{2} = -4$$

$$X15. e^{xy} = 4x + 7y$$

$$e^{xy}[(1)(y) + (1)y'(x)] = 4 + 7y'$$

$$ye^{xy} + xe^{xy}y' = 4 + 7y'$$

$$y'(xe^{xy} - 7) = 4 - ye^{xy}$$

$$y' = \frac{4 - ye^{xy}}{xe^{xy} - 7}$$

$$X16. x^2y + x^3 = 8 + y^3$$

$$(2x)(y) + (1)\frac{dy}{dx}(x^2) + 3x^2 = 0 + 3y^2\frac{dy}{dx}$$

$$\frac{dy}{dx}(x^2 - 3y^2) = -3x^2 - 2xy$$

$$\frac{dy}{dx} = \frac{-3x^2 - 2xy}{x^2 - 3y^2}$$

X17. Find y' if $\sin(xy) = x^2 + \cos y + 3$.

$$\cos(xy) \left[1(y) + \frac{dy}{dx}(x) \right] = 2x - \sin y \cdot \frac{dy}{dx}$$

$$y \cos(xy) + x \cos(xy) \frac{dy}{dx} = 2x - \sin y \frac{dy}{dx}$$

$$x \cos(xy) \frac{dy}{dx} + \sin y \frac{dy}{dx} = 2x - y \cos(xy)$$

$$\frac{dy}{dx} [x \cos(xy) + \sin y] = 2x - y \cos(xy)$$

$$\frac{dy}{dx} = \frac{2x - y \cos(xy)}{x \cos(xy) + \sin y}$$

X18. Find y'

$$3x^2 - 4x(y) + y'(-2x^2) + 6yy' = 0$$

$$y'(6y - 2x^2) = -3x^2 + 4xy$$

$$y' = \frac{-3x^2 + 4xy}{6y - 2x^2} \quad y'(1,0) = \frac{-3(1)^2 + 0}{0 - 2(1)^2} = \frac{3}{2}$$

$$\frac{d^2y}{dx^2} = y'' = \frac{[-6x + 4y + y'(4x)][6y - 2x^2] - [(6y' - 4x)(-3x^2 + 4xy)]}{(6y - 2x^2)^2}$$

At (1,0)

$$y'' = \frac{[-6 + 0 + \frac{3}{2}(4)(1)][0 - 2(1)^2] - [6(\frac{3}{2}) - 4(1)][-3(1)^2 + 0]}{(0 - 2)^2}$$

$$= \frac{0(-2) - (9 - 4)(-3)}{4}$$

$$= \frac{-(5)(-3)}{4}$$

$$= \frac{15}{4}$$

Y. Related Rates

Example 1. given $\frac{dx}{dt} = 2 \text{ ft/s}$

find $\frac{dy}{dt}$ when $x = 5 \text{ ft}$ $8^2 = y^2 + 5^2$

$$64 = y^2 + 25$$

$$y = \sqrt{39}$$

$$x^2 + y^2 = 8^2$$

$$x^2 + y^2 = 64$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

$$5(2) + \sqrt{39} \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-10}{\sqrt{39}} \left(\frac{\sqrt{39}}{\sqrt{39}} \right) = \frac{-10\sqrt{39}}{39} \text{ ft/s}$$

Example 2. given $\frac{dv}{dt} = 120 \text{ cm}^3/\text{s}$

find $\frac{dr}{dt}$ when $V = 36\pi$

$$V = \frac{4}{3}\pi r^3 \text{ sphere}$$

find r

$$V = \frac{4}{3}\pi r^3$$

$$36\pi = \frac{4}{3}\pi r^3$$

$$\frac{36(3)}{4} = r^3$$

$$r^3 = 27$$

$$r = 3$$

$$\frac{dV}{dt} = \frac{4\pi}{3} (3r^2) \frac{dr}{dt}$$

$$120 = 4\pi(3)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{120}{4\pi(9)} = \frac{120}{36\pi} = \frac{10}{3\pi} \text{ cm/s}$$

Example 3. given $\frac{dx}{dt} = -50$, $\frac{dy}{dt} = -60 \text{ km/h}$

find $\frac{dr}{dt}$ when $x = 9$ and $y = 12$

$$r^2 = 9^2 + 12^2 = 81 + 144 = 225$$

$$r = 15$$

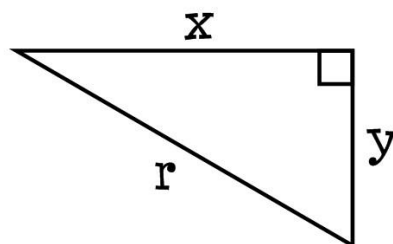
$$x^2 + y^2 = r^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2r \frac{dr}{dt} \quad (\text{divide by 2})$$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = r \frac{dr}{dt}$$

$$9(-50) + 12(-60) = 15 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-450 - 720}{15} = \frac{-1170}{15} \text{ km/h or } -78 \text{ km/h}$$



Example 4

$$6 \text{ km/hr} \cdot 1 \text{ hr} = 6$$

$$3 \text{ km/hr} \cdot 1 \text{ hr} = 3$$

$$r^2 = 9^2 + 12^2$$

$$r^2 = 81 + 144$$

$$r^2 = 225$$

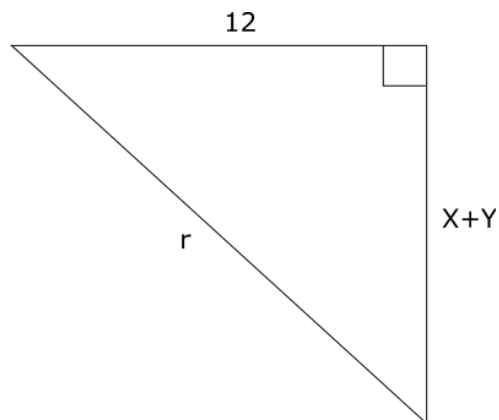
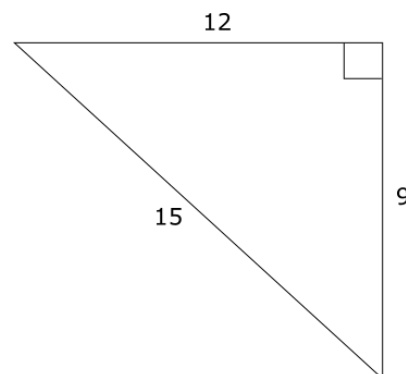
$$r = \sqrt{225} = 15$$

$$r^2 = 12^2 + (x + y)^2$$

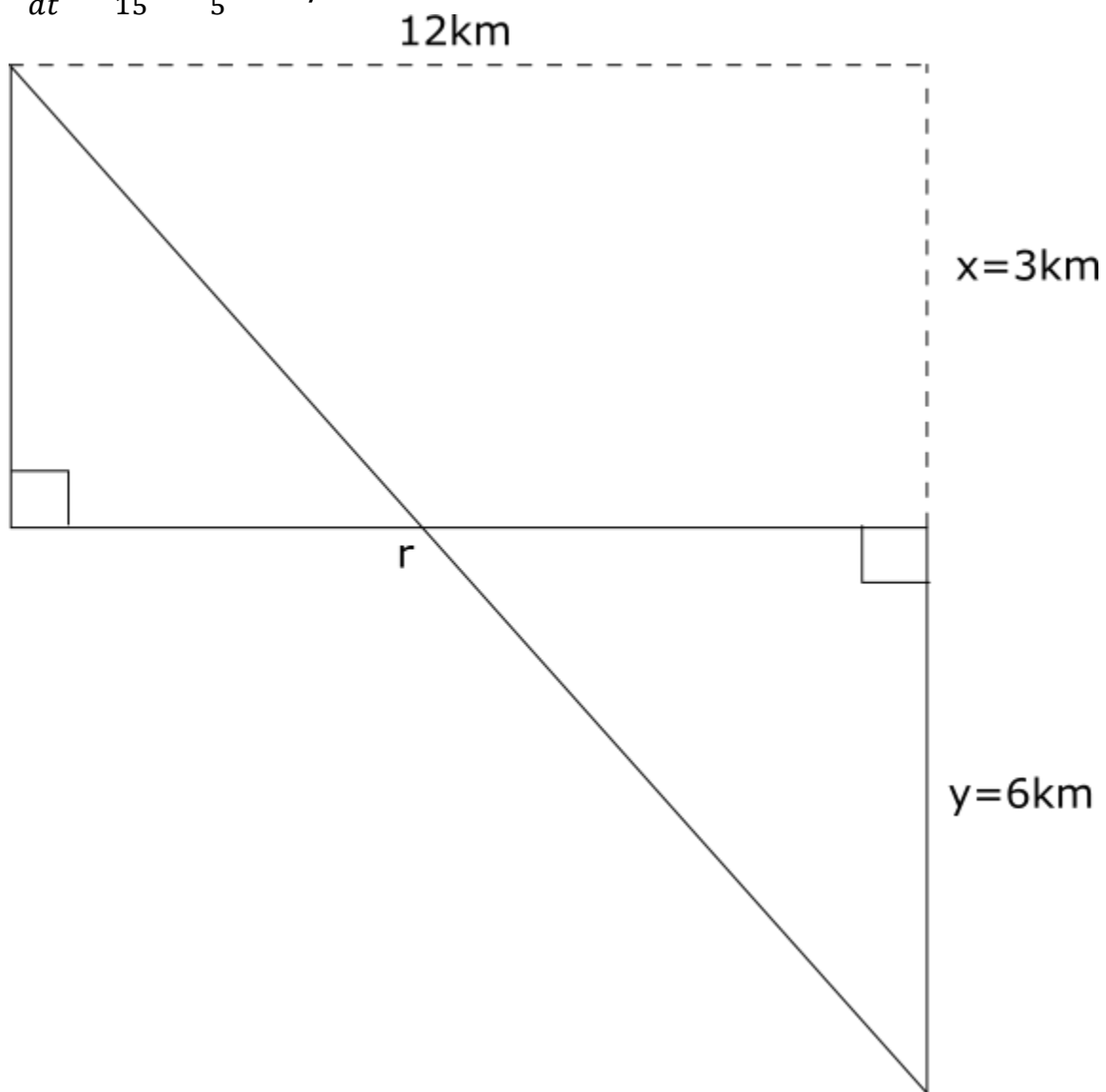
$$2r \frac{dr}{dt} = 0 + 2(x + y) \left(\frac{dx}{dt} + \frac{dy}{dt} \right)$$

$$15 \frac{dr}{dt} = (3 + 6)(3 + 6)$$

$$15 \frac{dr}{dt} = 81$$



$$\frac{dr}{dt} = \frac{81}{15} = \frac{27}{5} \text{ km/hr}$$



Solve each of the following:

Y1. given $\frac{dA}{dt} = -4 \text{ cm}^2/\text{mm}$ (melting \therefore decreasing)

find $\frac{dr}{dt}$ when $d = 1\text{m} = 100\text{cm}$ $r = 50 \text{ cm}$

$$A = 4\pi r^2 \text{ sphere}$$

$$\frac{dA}{dt} = 4\pi(2r) \frac{dr}{dt}$$

$$-4 = 4\pi(2)(50) \frac{dr}{dt}$$

$$-1 = \pi(100) \frac{dr}{dt}$$

$$\frac{-1}{100\pi} = \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-1}{100\pi} \text{ cm/min}$$

Y2. given $\frac{dy}{dt} = 3\text{m/s}$

find $\frac{dx}{dt}$ when $y = 10$

using similar Δ 's $\frac{2}{x} = \frac{20}{x+y}$

$$2x + 2y = 20x$$

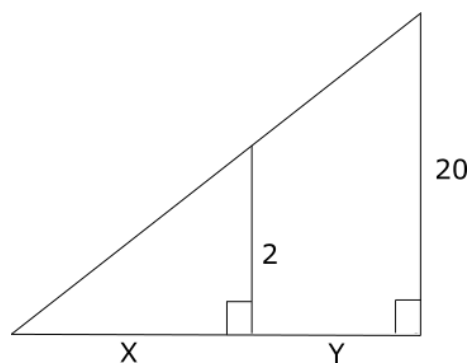
$$2y = 18x \quad y = 9x$$

$$\frac{dy}{dt} = 9 \frac{dx}{dt}$$

$$3 = 9 \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{3} \text{ m/s}$$

the shadow moves at $\frac{1}{3} \text{ m/s}$. The tip of the shadow moves at $3 + \frac{1}{3} \text{ m/s}$.



Y3. given $\frac{dV}{dt} = 5 \text{ cm}^3/\text{s}$

find $\frac{dr}{dt}$ when $V = 36\pi$

$V = \frac{4}{3}\pi r^3$ sphere

$$36\pi = \frac{4}{3}\pi r^3$$

$$\frac{36}{4} \times 3 = r^3 \quad r = 3$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2) \frac{dr}{dt} \quad \text{substitute } 5 = 4\pi(3)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = 5/36\pi \text{ cm/s}$$

Y4. given $\frac{dx}{dt} = 10 \text{ ft/s}$

find $\frac{dr}{dt}$ when $r = 100$ $100^2 = 80^2 + x^2$

$$x^2 = 3600 \quad x = 60 \text{ ft}$$

$$x^2 + 80^2 = r^2$$

$$2x \frac{dx}{dt} + 0 = 2r \frac{dr}{dt}$$

$$60(10) = 100 \frac{dr}{dt}$$

$$\frac{dr}{dt} = 6 \text{ ft/s}$$

Y5. given $\frac{dV}{dt} = 3 \text{ ft}^3/\text{min}$

find $\frac{dh}{dt}$ when $h = 6 \text{ ft}$...note: $h=2r$ and

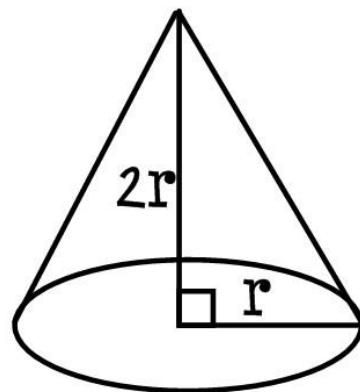
$r=1/2 h$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{1}{2}h\right)^2 h = \frac{1}{3}\pi \left(\frac{1}{4}h^3\right)$$

$$\frac{dV}{dt} = \frac{1}{12}\pi(3h^2) \frac{dh}{dt}$$

$$3 = \frac{1}{12}\pi(3(6)^2) \frac{dh}{dt}$$

$$3 = 9\pi \frac{dh}{dt} \quad \therefore \frac{dh}{dt} = \frac{1}{3\pi} \text{ ft/min}$$



Y6.

$$\text{given } \frac{dx}{dt} = 1.5 \text{ km/h} \quad \frac{dy}{dt} = -1.0 \text{ km/h}$$

$$\text{find } \frac{dr}{dt} \text{ when } x = 1.5 \text{ km/h} (2 \text{ hr}) = 3$$

$$y = (1.0 \text{ km/h})(2) = 2$$

6-2=4 inside the triangle

$$x^2 + y^2 = r^2$$

$$(3)^2 + (4)^2 = r^2$$

$$r^2 = 25 \quad r = 5$$

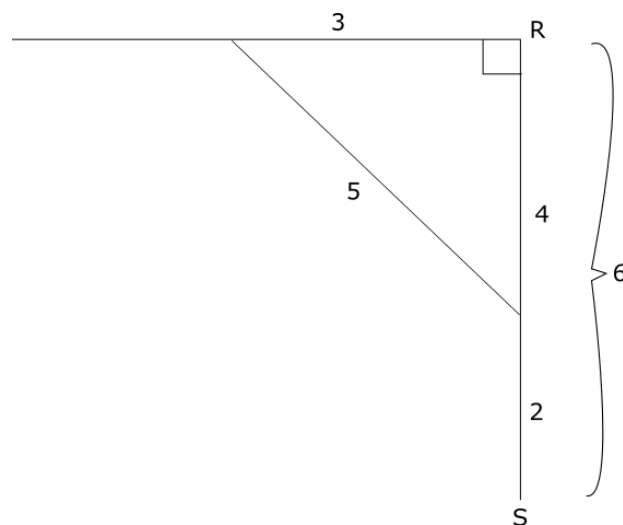
$$x^2 + y^2 = r^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2r \frac{dr}{dt}$$

$$3(1.5) + (4)(-1.0) = 5 \frac{dr}{dt}$$

$$4.5 - 4 = 5 \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = -\frac{0.5}{5} = -0.1 \text{ km/h}$$

Y7. let $\frac{dx}{dt}$ be Cara's speed let $\frac{dy}{dt}$ be Jocelyn's speed

$$\text{given } \frac{dx}{dt} = 4 \text{ m/s} \quad \frac{dy}{dt} = 3 \text{ m/s}$$

find $\frac{dr}{dt}$ after 5s

$$y = 4 \text{ m/s}(5 \text{ s}) = 20 \text{ m} \quad x = 3 \text{ m/s}(5 \text{ s}) = 15 \text{ m}$$

$$x^2 + y^2 = r^2$$

$$20^2 + 15^2 = r^2$$

$$r^2 = 625$$

$$r = 25$$

$$x^2 + y^2 = r^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2r \frac{dr}{dt}$$

$$15(3) + 20(4) = 25 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{45+80}{25} = \frac{125}{5} = 5 \text{ m/s}$$

Y8. $A = 4\pi r^2$ sphere

given $\frac{dr}{dt} = -3 \text{ m/s}$ find $\frac{dA}{dt}$ when $r = 4 \text{ m}$

Formula $A = 4\pi r^2$

derivative $\frac{dA}{dt} = 4\pi(2r) \frac{dr}{dt}$

substitute $\frac{dA}{dt} = 4\pi(2)(4)(-3) \therefore \frac{dA}{dt} = -96\pi \text{ m}^2/\text{s}$

Y9. Sphere $V = \frac{4}{3}\pi r^3$

given $\frac{dv}{dt} = -4 \text{ cm}^3/\text{min}$

find $\frac{dr}{dt}$ when $V = 36\pi \text{ cm}^3$ find r $V = \frac{4}{3}\pi r^3$

$$36\pi = \frac{4}{3}\pi(r)^3$$

cancel π by dividing both sides by π

$$36 = \frac{4}{3}r^3 \quad \text{divide by 4}$$

$$9 = \frac{1}{3}r^3 \quad \text{multiply 9 by 3}$$

$$27 = r^3 \quad r = 3$$

formula $V = \frac{4}{3}\pi r^3$

derivative $\frac{dv}{dt} = \frac{4}{3}\pi(3r^2) \frac{dr}{dt}$

substitute $-4 = 4\pi(3)^2 \frac{dr}{dt} \therefore -4 = 36\pi \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{-1}{9\pi} \text{ cm/min}$$

Y10. $V = \pi r^2 h$ (cylinder)

given $\frac{dv}{dt} = -36 \text{ cm}^3/\text{s}$ (being drained)

find $\frac{dh}{dt}$ when $d = 60 \text{ cm}$ ($r = 30 \text{ cm}$)

formula $V = \pi r^2 h$ (radius is constant)

$$V = \pi(30)^2 h = 900\pi h$$

derivative $\frac{dv}{dt} = 900\pi \left[\frac{dh}{dt} \right]$

$$-36 = 900\pi \frac{dh}{dt}$$

$$\frac{-36}{900\pi} = \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{-36}{900\pi} = \frac{-1}{25\pi} \text{ cm/s}$$

Y11. given $h = 2r$, $\frac{dv}{dt} = 12 \text{ ft}^3/\text{min}$

find $\frac{dr}{dt}$ when $h = 6 \text{ ft}$ ($r = 3 \text{ ft}$)

formula cone $V = \frac{1}{3}\pi r^2 h$ substitute $h = 2r$

$$\text{we get } V = \frac{1}{3}\pi r^2(2r)$$

$$\therefore V = \frac{2}{3}\pi r^3$$

derivative $\frac{dv}{dt} = \frac{2}{3}\pi(3r^2) \frac{dr}{dt}$

$$\frac{dv}{dt} = 2\pi r^2 \frac{dr}{dt}$$

substitute $12 = 2\pi(3)^2 \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{12}{18\pi} = \frac{2}{3\pi} \text{ ft/min}$$

Y12. *given:* $x = 6$

$$\frac{dx}{dt} = 2 \text{ ft/s}, \quad x = 6 \text{ ft}, \quad r = 9 \text{ ft} \quad \text{Find } \frac{dy}{dt}$$

$$x^2 + y^2 = r^2$$

$$y^2 = r^2 - x^2$$

$$y^2 = 81 - 36 = 45$$

$$y = \sqrt{45} = 3\sqrt{5}$$

derivative $x^2 + y^2 = 9^2$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

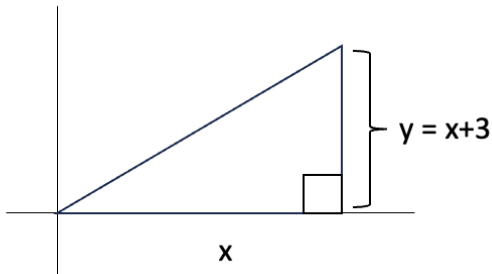
$$6(2) + 3\sqrt{5} \frac{dy}{dt} = 0$$

$$3\sqrt{5} \frac{dy}{dt} = -12$$

$$\frac{dy}{dt} = \frac{-12}{3\sqrt{5}}$$

$$\frac{dy}{dt} = -\frac{4}{\sqrt{5}} \text{ or } -\frac{4\sqrt{5}}{5} \text{ ft/s}$$

Y13.



Given: $\frac{dx}{dt} = 4 \text{ m/s}$

Find: $\frac{dA}{dt}$ at (3,6) at $x=3$

$$\text{Area} = \frac{\text{base} \times \text{height}}{2}$$

$$A = \frac{x(x+3)}{2} = \frac{1}{2}(x^2 + 3x)$$

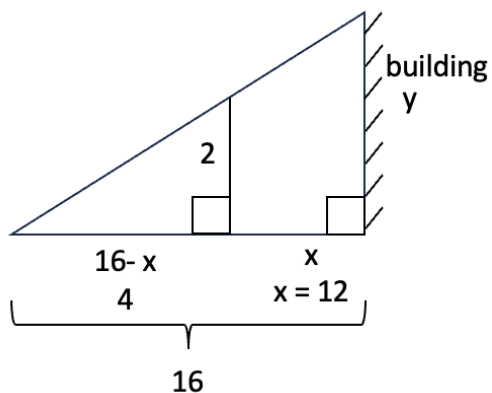
$$A(x) = \frac{1}{2}x^2 + \frac{3}{2}x$$

$$\frac{dA}{dt} = \frac{1}{2}(2x) \left(\frac{dx}{dt}\right) + \frac{3}{2} \left(\frac{dx}{dt}\right) \text{ Substitute } \frac{dx}{dt} = 4 \text{ m/s and } x=3:$$

$$\frac{dA}{dt} = \frac{1}{2}(2)(3)(4) + \frac{3}{2}(4)$$

$$= 12 + 6 = 18 \text{ m}^2/\text{second}$$

Y14.



Given: $\frac{dx}{dt} = 1.4 \text{ m/s}$

Let y be the height of the building

Find $\frac{dy}{dt}$ when $x = 12$

Similar Triangle's

$$\frac{2}{4} = \frac{y}{16}$$

$$4y = 32$$

$$y = 8$$

In general $\rightarrow \frac{2}{16-x} = \frac{y}{16}$

Flip the fractions: $\frac{16-x}{2} = \frac{16}{y}$

$$8 - \frac{1}{2}x = 16y^{-1}$$

Do derivative

$$-\frac{1}{2}(1.4) = -\frac{16}{(8)^2} \frac{dy}{dt}$$

$$-0.7 = -\frac{16}{64} \frac{dy}{dt}$$

$$-0.7 = -\frac{1}{4} \frac{dy}{dt}$$

$$-0.7(4) = -\frac{dy}{dt}$$

$$\frac{dy}{dt} = 2.8 \text{ m/s}$$

Y15. Given: $\frac{dr}{dt} = 6 \text{ cm/s}$

Find: $\frac{dV}{dt}$ when $r = 3$

Formula: $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{4}{3}\pi (3r^2) \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substitute: $\frac{dr}{dt} = 6, r = 3$

$$\frac{dV}{dt} = 4\pi(3)^2(6)$$

$$\frac{dV}{dt} = 4\pi(9)(6)$$

$$\frac{dV}{dt} = 216 \text{ cm}^3/\text{s}$$

Best of luck on your exam!!

NEW MATERIAL
SINCE THE MIDTERM

A. L'Hospital's Rule

Example 3. $\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \frac{\infty}{\infty}$

$$H = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \frac{\infty}{\infty} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$$

Example 4. $\lim_{x \rightarrow 0} \frac{\sin(x^3)}{2x}$

$$H = \lim_{x \rightarrow 0} \frac{\cos(x^3)(3x^2)}{2} = \frac{(\cos 0)(3(0))}{2} = 0$$

Example 5. $\lim_{x \rightarrow \frac{\pi}{2}} (\sec x - \frac{1}{\cot x})$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) = \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{1 - \sin x}{\cos x} \right) = \frac{0}{0}$$

$$H = \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{-\cos x}{-\sin x} \right) = \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\cos x}{\sin x} \right) = \frac{\cos \frac{\pi}{2}}{\sin \frac{\pi}{2}} = \frac{0}{1} = 0$$

Example 6. $\lim_{x \rightarrow \infty} \frac{x^3 + 2x}{x^4 - 2x} = \frac{\infty}{\infty}$

$$H = \lim_{x \rightarrow \infty} \frac{3x^2 + 2}{4x^3 - 2} = \frac{\infty}{\infty}$$

$$H = \lim_{x \rightarrow \infty} \frac{6x}{12x^2} = \frac{\infty}{\infty}$$

$$H = \lim_{x \rightarrow \infty} \frac{6}{24x} = \frac{6}{24(\infty)} = 0$$

Example 7. $\lim_{x \rightarrow 0^+} x^x$ indeterminate 0^0

Let $y = x^x$ take the ln of both sides NOTE: $x^x = e^{x \ln x}$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x \text{ log rules}$$

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$$

$$H = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \rightarrow 0^+} \frac{1}{x} \left(\frac{-x^2}{1} \right) = \lim_{x \rightarrow 0^+} (-x) = -(0) = 0$$

$$\text{So, } \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} x^x = e^0 = 1$$

$$\therefore \lim_{x \rightarrow 0^+} x^x = 1$$

Example 8.

Let $y = (x^4 + e^{3x})^{\frac{1}{x}}$...take the ln of both sides

$$\ln y = \ln(x^4 + e^{3x})^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x} \ln(x^4 + e^{3x})$$

$$\lim_{x \rightarrow \infty} \ln y$$

$$= \lim_{x \rightarrow \infty} \ln(x^4 + e^{3x})^{\frac{1}{x}}$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln(x^4 + e^{3x})}{x} \quad \frac{\infty}{\infty}$$

$$(H) = \lim_{x \rightarrow \infty} \frac{(4x^3 + 3e^{3x})}{(x^4 + e^{3x})} \quad \frac{\infty}{\infty}$$

$$(H) = \lim_{x \rightarrow \infty} \frac{12x^2 + 9e^{3x}}{(4x^3 + 3e^{3x})} \quad \frac{\infty}{\infty}$$

$$(H) = \lim_{x \rightarrow \infty} \frac{24x + 27e^{3x}}{12x^2 + 9e^{3x}} \quad \frac{\infty}{\infty}$$

$$(H) = \lim_{x \rightarrow \infty} \frac{24 + 81e^{3x}}{24x + 27e^{3x}} \quad \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{243e^{3x}}{24 + 81e^{3x}} \quad \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{729e^{3x}}{243e^{3x}} = \frac{729}{243} = 3$$

$$\therefore \lim_{x \rightarrow \infty} y = e^3$$

Practice Exam Questions on L'Hospital's Rule

$$A1. \lim_{x \rightarrow \infty} \frac{x^3 + x}{x^4 - x} \quad \frac{\infty}{\infty}$$

$$H = \lim_{x \rightarrow \infty} \frac{3x^2 + 1}{4x^3 - 1} \quad \frac{\infty}{\infty}$$

$$H = \lim_{x \rightarrow \infty} \frac{6x}{12x^2} \quad \frac{\infty}{\infty}$$

$$H = \lim_{x \rightarrow \infty} \frac{6}{24x} = 0$$

$$A2. \lim_{x \rightarrow 0} \frac{\tan x - x}{3x^2}$$

$$H = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{6x} \quad \frac{0}{0} \quad \sec^2 0 = (\sec 0)^2 = \left(\frac{1}{\cos 0}\right)^2 = \left(\frac{1}{1}\right)^2 = 1$$

$$H = \lim_{x \rightarrow 0} \frac{2 \sec x (\sec x \tan x)}{6} = \frac{2 \sec^2 0 \tan 0}{6} = \frac{0}{6} = 0$$

$$\text{A3. } \lim_{x \rightarrow 0} \frac{2 \cos x \tan x}{4 \sin x} = \lim_{x \rightarrow 0} \frac{2 \cos x \left(\frac{\sin x}{\cos x}\right)}{4 \sin x} = \lim_{x \rightarrow 0} \frac{2 \sin x}{4 \sin x} = \lim_{x \rightarrow 0} \left(\frac{2}{4}\right) = \frac{1}{2}$$

$$\text{A4. } \lim_{x \rightarrow \pi} \frac{\sin x}{4 + \cos x} = \frac{\sin \pi}{4 + \cos \pi} = \frac{0}{4 + (-1)} = \frac{0}{3} = 0$$

$$\text{A5. } \lim_{x \rightarrow 0^+} 9^{\ln x} = 9^{\ln 0^+} = 9^{-\infty} = \frac{1}{9^\infty} = 0$$

$$\text{A6. } \lim_{x \rightarrow 0} \frac{e^x - 1 - 2x - \frac{x^3}{6}}{x^3} = \frac{0}{0}$$

$$H = \lim_{x \rightarrow 0} \frac{e^{x+0} - 2 - \frac{3x^2}{6}}{3x^2} = -\frac{1}{0} = -\infty$$

$$\text{A7. } \lim_{x \rightarrow \infty} \frac{5x^2 - x}{5x + 15}$$

$$H = \lim_{x \rightarrow \infty} \frac{10x - 1}{5} = \frac{10(\infty) - 1}{5} = \infty$$

$$\text{A8. } \lim_{x \rightarrow 2\pi} \frac{\sin x}{1 - \cos x} = \frac{0}{0}$$

$$H = \lim_{x \rightarrow 2\pi} \frac{\cos x}{\sin x} = \frac{1}{0} = \infty$$

$$\text{A9. } \lim_{x \rightarrow 2\pi} \frac{\sin x}{x - 2\pi} = \frac{\sin \pi}{2\pi - 2\pi} = \frac{0}{0}$$

$$H = \lim_{x \rightarrow 2\pi} \frac{\cos x}{1} = \frac{\cos 2\pi}{1} = 1$$

$$\text{A10. } \lim_{x \rightarrow 0} \frac{5^x - e^x}{2x} = \frac{1 - 1}{0} = \frac{0}{0}$$

$$H = \lim_{x \rightarrow 0} \frac{5^x \ln 5 - e^x}{2} = \frac{5^0 \ln 5 - e^0}{2} = \frac{\ln 5 - 1}{2}$$

A11.

$$\lim_{x \rightarrow 0^+} x^{\sqrt{x}}$$

indeterminate 0^0 Let $y = x^{\sqrt{x}}$ take the ln of both sides NOTE: $x^{\sqrt{x}} = e^{\ln x^{\sqrt{x}}} = e^{\sqrt{x} \ln x}$

$$\ln y = \ln x^{\sqrt{x}} \text{ log rules}$$

$$\ln y = \sqrt{x} \ln x$$

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \sqrt{x} \ln x$$

$$= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}}$$

$$= \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-\frac{1}{2}}}$$

$$H = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-1}{2} x^{-\frac{3}{2}}}$$

$$= \lim_{x \rightarrow 0^+} \frac{x^{-1}}{\frac{-1}{2} x^{-\frac{3}{2}}} = \lim_{x \rightarrow 0^+} -2x^{\frac{1}{2}} = -2(0)^{\frac{1}{2}} = 0$$

$$\therefore e^0 = 1 \quad \therefore \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} x^{\sqrt{x}} = 1$$

$$A12. \quad \lim_{x \rightarrow 4^-} \frac{1}{e^{x-4}} = \frac{1}{e^{4-4}} = \frac{1}{e^0} = \frac{1}{1} = 1$$

$$A13. \quad \lim_{x \rightarrow 0} \frac{e^{2x}-1}{\cos x} = \frac{e^0-1}{\cos 0} = \frac{1-1}{1} = \frac{0}{1} = 0$$

$$A14. \quad \lim_{x \rightarrow 0^+} \ln \left(\frac{1}{\cot x} \right) = \lim_{x \rightarrow 0^+} \ln (\tan x) = \ln(\tan 0^+) = \ln(0^+) = -\infty \text{ from the}$$

y=lnx graph

$$\begin{aligned}
 \text{A15. } & \lim_{x \rightarrow \infty} x^2 \sin\left(\frac{1}{x^2}\right) \\
 &= \lim_{x \rightarrow \infty} \frac{\sin\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}} \quad \frac{0}{0} \\
 \text{(H)} &= \lim_{x \rightarrow \infty} \frac{\cos\left(\frac{1}{x^2}\right)(-2x^{-3})}{(-2x^{-3})} \\
 &= \lim_{x \rightarrow \infty} \cos\left(\frac{1}{x^2}\right) \\
 &= \cos 0 \\
 &= 1
 \end{aligned}$$

$$\text{A16. } \lim_{x \rightarrow 0^+} (\cos x)^{\frac{1}{x}} \quad \text{let } y = (\cos x)^{\frac{1}{x}}$$

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \ln (\cos x)^{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\ln (\cos x)}{x} \quad \frac{0}{0}$$

$$\text{(H)} = \lim_{x \rightarrow 0^+} \frac{-\frac{1}{\cos x} \sin x}{1}$$

$$= \lim_{x \rightarrow 0^+} \frac{-\tan x}{1}$$

$$= 0$$

$$\lim_{u \rightarrow 0^+} y = e^0 = 1$$

B. Critical Numbers

Example 1. $D_f = (-\infty, \infty)$

$$f'(x) = 3x^2 - 3$$

$$f'(x) = 0 \qquad 3x^2 - 3 = 0$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = 1, -1$$

Example 2. $D_f = (-\infty, \infty)$

$$f'(x) = \frac{2}{3}(x-2)^{-\frac{1}{3}} = 0$$

$$\frac{2}{3(x-2)^{\frac{1}{3}}} = 0 \qquad \text{no solution}$$

$f'(x)$ is undefined at $x = 2$ and $x = 2$ is in DF

$\therefore x = 2$ is a critical number

Example 3. $f'(x) = \cos x - \sin x = 0$

$$\frac{\cos x}{\cos x} = \frac{\sin x}{\cos x}$$

$$1 = \tan x$$

$$x = \frac{\pi}{4}$$

By CAST, $\tan x$ is positive in T.

$$\therefore x = \pi + \frac{\pi}{4} = \frac{4\pi}{4} + \frac{\pi}{4}$$

$$x = \frac{5\pi}{4} \quad \therefore \text{critical number's are } \frac{\pi}{4}, \frac{5\pi}{4}$$

Practice Exam Questions on Critical Numbers

B1. $D_f = (-\infty, 3) \cup (3, \infty)$

$$\begin{aligned} f'(x) &= \frac{2x(x-3) - (1)(x^2-5)}{(x-3)^2} \\ &= \frac{2x^2 - 6x - x^2 + 5}{(x-3)^2} \\ &= \frac{x^2 - 6x + 5}{(x-3)^2} \end{aligned}$$

$$f'(x) = 0$$

$$x^2 - 6x + 5 = 0$$

$$(x - 5)(x - 1) = 0$$

$$x = 5, 1 \quad \therefore \text{critical numbers are } 5, 1$$

NOTE: the derivative is undefined at $x=3$, but so is $f(x)$ so since $x=3$ is NOT in the domain of $f(x)$, it is not a critical number

B2. $D_f = (-\infty, \infty)$

$$f'(x) = \frac{4}{3}x^{-\frac{1}{3}}(2-x) + (-1)\left(2x^{\frac{2}{3}}\right) = 0$$

$$x^{-\frac{1}{3}} \left[\frac{4}{3}(2-x) - 2x \right] = 0$$

$$x^{-\frac{1}{3}} = 0 \quad \text{no solution}$$

$$\frac{8}{3} - \frac{4}{3}x - 2x = 0$$

$$\frac{8}{3} = \frac{6}{3}x + \frac{4}{3}x$$

$$8 = 10x$$

$$x = \frac{8}{10} = \frac{4}{5} \quad \text{Also, } f'(x) \text{ is undefined at } x=0 \text{ and } x=0 \text{ is in } D_f$$

$\therefore 0$ is also a critical number. The critical numbers are $4/5$ and 0 .

$$\text{B3. } f'(x) = -4 \sin x + 4 \sin x(-\cos x) = 0$$

$$-4 \sin x(1 + \cos x) = 0$$

$$\sin x = 0$$

$$x = 0, \pi, 2\pi$$

$$1 + \cos x = 0$$

$$\cos x = -1$$

$$x = \pi$$

\therefore critical numbers are $0, \pi$ and 2π

C. Maximum and Minimum

Example 1. $D_f = (-\infty, 2) \cup (2, \infty)$

$$f(x) = (x - 2)^{-1}$$

$$f'(x) = -1(x - 2)^{-2}$$

$$f'(x) = 0 \quad \frac{-1}{(x-2)^2} = 0 \quad \therefore \text{no solution}$$

$f'(x)$ is undefined at $x = 2$ but $x = 2$ is undefined in f

\therefore it is not a critical number

$\therefore x = 2$ is a vertical asymptote

2

Test point 0	Test point 3
<input type="checkbox"/>	<input type="checkbox"/>

Put $x = 2$ in chart (critical and VA)

\therefore decreasing $(-\infty, 2) \cup (2, \infty)$

Example 2. $f(x) = 4x^3 - 6x^2 - 72x$

$$f'(x) = 12x^2 - 12x - 72 = 0$$

$$12(x^2 - x - 6) = 0$$

$$12(x - 3)(x + 2) = 0$$

$$x = 3, -2$$

-2

3

Test point -3	Test point 0	Test point 4
+(-)(-)	+(-)(+)	+ (+) (+)
+	-	+

Increasing $(-\infty, -2) \cup (3, \infty)$

Decreasing $(-2, 3)$

Example 3.

$$f(x) = x^3 - 3x^2 + 2$$

$$f'(x) = 3x^2 - 6x = 0$$

$$3x(x - 2) = 0$$

$$x = 0, 2 \quad (\text{both in } [-1, 4])$$

$$f(x) = x^3 - 3x^2 + 2$$

$$f(0) = 2$$

$$f(2) = 2^3 - 3(2)^2 + 2 = 8 - 12 + 2 = -2 \quad \text{min}$$

$$f(-1) = (-1)^3 - 3(-1)^2 + 2 = -1 - 3 + 2 = -2 \quad \text{min}$$

$$f(4) = 4^3 - 3(4)^2 + 2 = 64 - 48 + 2 = 18 \quad \text{max}$$

\therefore absolute minimum is -2 and absolute maximum is 18

Practice Exam Questions on Maximum and Minimum

C1. $f'(x) = 8x^3 - 64x = 0$

$$8x(x^2 - 8) = 0$$

$$8x(x - \sqrt{8})(x + \sqrt{8}) = 0$$

$$x = 0, \quad x = \sqrt{8}, -\sqrt{8}$$

$$-\sqrt{8} \quad 0 \quad \sqrt{8}$$

-3	-1	1 test point	3
(-)(-)(-)	(-)(-)(+)	+(-)(+)	+(+)(+)
<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
min	max	min	

$$f(0) = 0$$

$$f(\sqrt{8}) = 2(\sqrt{8})^4 - 32(\sqrt{8})^2 = 2(8)(8) - 32(8) = 128 - 256 = -128$$

$$f(-\sqrt{8}) = 2(-\sqrt{8})^4 - 32(-\sqrt{8})^2 = 2(8)(8) - 32(8) = -128$$

$$-\sqrt{8} = -2\sqrt{2} \text{ and } \sqrt{8} = 2\sqrt{2}$$

\therefore local min $(-2\sqrt{2}, -128)$ and $(2\sqrt{2}, -128)$, local max $(0,0)$

Increasing $(-2\sqrt{2}, 0) \cup (2\sqrt{2}, \infty)$, Decreasing $(-\infty, -2\sqrt{2}) \cup (0, 2\sqrt{2})$

C2. $f'(x) = 3x^2 - 12x = 0$

$$3x(x - 4) = 0$$

$$x = 0, 4 \quad \therefore 4 \text{ is not in } [-3, 2] \text{ so don't check it}$$

$$f(x) = x^3 - 6x^2 + 5$$

$$f(0) = 5$$

$$f(2) = 2^3 - 6(2)^2 + 5 = 8 - 24 + 5 = -11$$

$$f(-3) = (-3)^3 - 6(-3)^2 + 5 = -27 - 54 + 5 = -76$$

$$\text{absolute maximum} = 5 \quad \text{absolute minimum} = -76$$

$$\text{C3. a) } a = -1 \quad b = 3$$

$$f'(x) = 3x^2 - 6x$$

$$f'(x) = 0 \quad 3x^2 - 6x = 0$$

$$\text{factor} \quad 3x(x - 2) = 0$$

$x = 0, 2$ [2 is not in the interval \therefore not tested in $f(x)$] check endpoints and critical values in f

$$f(x) = x^3 - 3x^2 + 4$$

$$f(-1) = (-1)^3 - 3(-1)^2 + 4 = 0$$

$$f(0) = 4$$

$$f(3) = 3^3 - 3(3)^2 + 4 = 4 \quad \therefore \text{abs max} = 4$$

$$\text{abs min} = 0$$

$$\text{b) } a = 0 \quad b = 4$$

$$f'(x) = 3(x^2 - 1)^2(2x)$$

$$f'(x) = 0 \quad 3(x^2 - 1)^2(2x) = 0$$

$$3(x - 1)(x + 1)(2x) = 0$$

$$x = 0, \pm 1 \quad -1 \text{ is not in } [0, 4]$$

$$f(x) = (x^2 - 1)^3$$

$$f(0) = -1 \quad \leftarrow \text{abs min} = -1$$

$$f(1) = 0$$

$$f(4) = 15^3 \quad \leftarrow \text{abs max} = 15^3$$

$$\text{c) } f'(x) = -8 \sin x + 8 \cos 2x = 0$$

$$-8 \sin x + 8(1 - 2 \sin^2 x) = 0$$

$$-8 \sin x + 8 - 16 \sin^2 x = 0$$

$$0 = 16 \sin^2 x + 8 \sin x - 8$$

$$0 = 8(2 \sin^2 x + \sin x - 1)$$

$$0 = 2(2 \sin x - 1)(\sin x + 1)$$

$$\sin x = -1$$

$$\sin x = \frac{1}{2}$$

$$\text{No solution in } \left[0, \frac{\pi}{2}\right]$$

$$x = \frac{\pi}{6}$$

$$f(0) = 8$$

$$f\left(\frac{\pi}{2}\right) = 8 \cos \frac{\pi}{2} + 4 \sin\left(\frac{2\pi}{2}\right) = 0 \quad \text{abs min}$$

$$f\left(\frac{\pi}{6}\right) = 8\cos\frac{\pi}{6} + 4\sin\left(\frac{\pi}{6}\right) = \frac{8\sqrt{3}}{2} + \frac{4\sqrt{3}}{2} = 6\sqrt{3} \text{ abs max}$$

d) $f(x) = x - \ln x$

$$f'(x) = 1 - \frac{1}{x} = 0$$

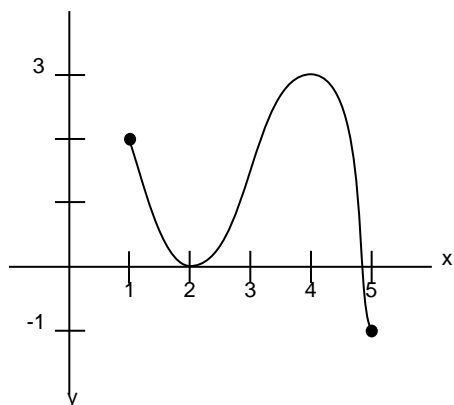
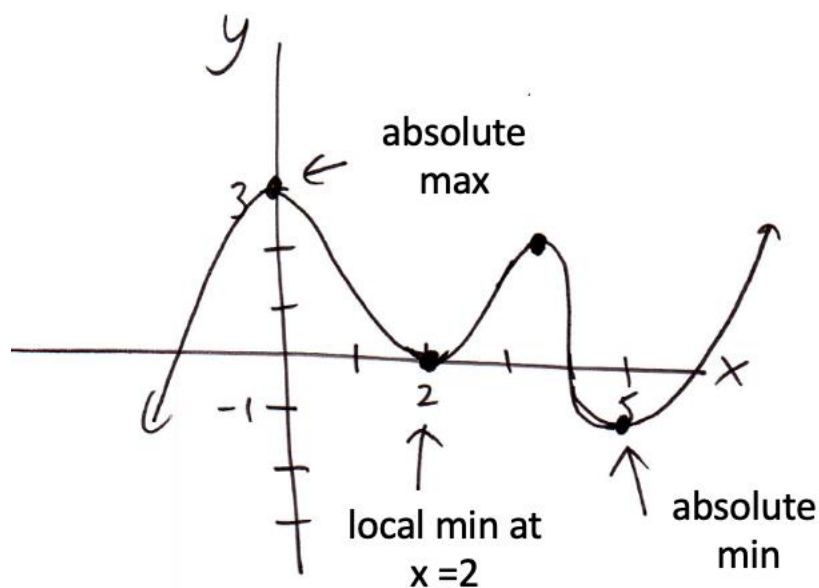
$$1 = \frac{1}{x} \quad x = 1 \text{ critical value}$$

$$f(1) = 1 - \ln 1 = 1$$

$$f(6) = 6 - \ln 6 = 4.2$$

$$\text{abs max } (6 - \ln 6) \quad \text{abs min} = 1$$

C4. Answers vary. Two examples are given below:



D. Concavity and Points of Inflection and The Second Derivative Test

Example 1. $f'(x) = 6x^2 - 24x + 18$

$$6(x^2 - 4x + 3) = 0$$

$$6(x - 3)(x - 1) = 0$$

$$x = 3, 1$$

1		3	
Test point 0		Test point 2	
+(-)(-)		+(-)(+)	
$\boxed{+}$		$\boxed{-}$	
max		min	

$$\text{local max}(1,12) \quad f(1) = 2 - 12 + 18 + 4 = 12$$

$$\text{local min}(3,4) \quad f(3) = 2(3)^3 - 12(3)^2 + 18(3) + 4 = 4$$

$$f''(x) = 12x - 24 = 0$$

$$12x = 24$$

$$x = 2$$

2	
Test point 0	Test point 3
-	+

\therefore POI at (2, 8)

$$f(2) = 2(2)^3 - 12(2)^2 + 18(2) + 4 = 8$$

concave up $(2, \infty)$ concave down $(-\infty, 2)$

Example 2. $f'(x) = \cos x - \sin x$

$$f''(x) = -\sin x - \cos x$$

$$f''(x) = 0$$

$$\frac{-\sin x}{\cos x} = \frac{\cos x}{\cos x}$$

$$-\tan x = 1$$

$$\tan x = -1$$

From CAST, $\tan x$ is negative in C & S

From special Δ 's, $\tan x = 1$ gives $x = \frac{\pi}{4}$

$$\therefore x = 2\pi - \frac{\pi}{4} = \frac{8\pi}{4} - \frac{\pi}{4} = \frac{7\pi}{4} \quad (C \text{ quadrant})$$

$$\therefore x = \pi - \frac{\pi}{4} = \frac{4\pi}{4} - \frac{\pi}{4} = \frac{3\pi}{4} \quad (S \text{ quadrant})$$

$$\frac{3\pi}{4}$$

$$\frac{7\pi}{4}$$

Test point 0	π	2π
$-\sin x - \cos x$ $= -\sin 0 - \cos 0$ -1 <div style="text-align: center;">$\boxed{-}$</div>	$-\sin \pi - \cos \pi$ $= 1$ <div style="text-align: center;">$\boxed{+}$</div>	$-\sin 2\pi - \cos 2\pi$ $= -1$ <div style="text-align: center;">$\boxed{-}$</div>

There is a point of inflection at $x = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$

concave up $(\frac{3\pi}{4}, \frac{7\pi}{4})$ concave down $[0, \frac{3\pi}{4})$ or $(\frac{7\pi}{4}, 2\pi]$

Example 3. $f'(x) = 15x^4 - 15x^2 = 0$

$$15x^2(x^2 - 1) = 0$$

$$x = 0, 1, -1$$

$$f''(x) = 60x^3 - 30x$$

$$f''(0) = 0 \text{ inconclusive}$$

$$f''(1) = 60 - 30 = 30 > 0 \quad \therefore \text{local min at } x = 1 \text{ sub } x=1 \text{ into } f(x) \text{ and}$$

get (1, -2)

$f''(-1) = 60(-1) + 30 = -30 < 0 \quad \therefore \text{local max at } x = -1 \text{ sub } x=-1$
into $f(x)$ and get (-1,2)

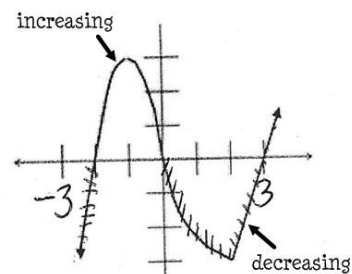
Making Conclusions about Graphs Based on First or Second Derivatives

Example 4. Part (1)

local max at $x=0$, local min at $x=3, -2$

Increasing $(-2, 0) \cup (3, \infty)$

Decreasing $(-\infty, -2) \cup (0, 3)$



We can also tell by the slope of the tangent where it is concave up and down. It is concave up if the slope is positive, ie. up and to the right. It is concave down if the slope is down and to the right (negative). So, here it is concave up from $(-\infty, -1) \cup (0, \infty)$ and down for $(-1, 2)$.

Example 5. $f''(x) < 0$ concave down

$$f''(x) > 0 \text{ concave up}$$

Points of Inflection \rightarrow

occur when graph switches from concave

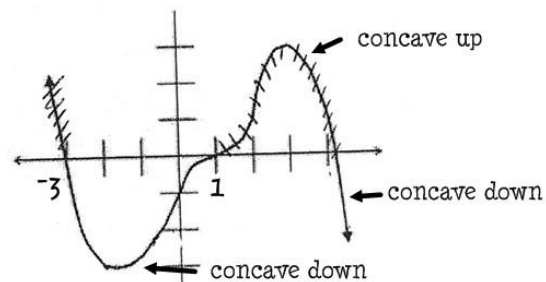
a)

up to down or concave down to concave up

(3) concave up $(-\infty, -3) \cup (1, 4)$

concave down $(-3, 1) \cup (4, \infty)$

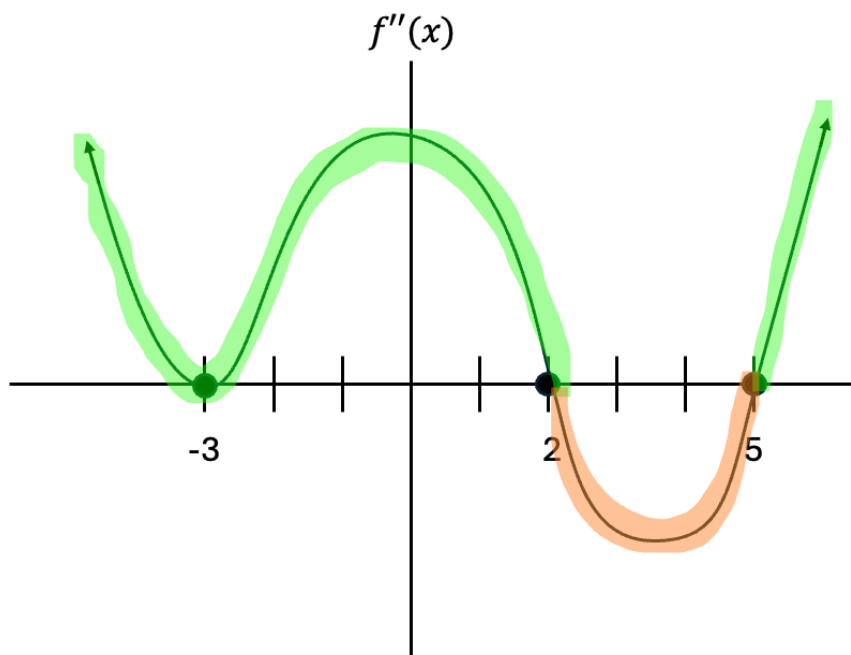
POI at $x = -3, x = 1, x = 4$



b) concave up $(-\infty, -3) \cup (-3, 2) \cup (5, \infty)$

concave down $(2, 5)$

Points of inflection at $x = -3, 2$ and



Practice Exam Questions on Concavity and Points of Inflection and The Second Derivative Test

D1. $f(x) = 4x^3 + 6x^2 - 72x$

$$f'(x) = 12x^2 + 12x - 72$$

$$12(x^2 + x - 6) = 0$$

$$12(x + 3)(x - 2) = 0$$

$$x = -3, 2$$

	-3	2	
Test point	-4	0	3
Sign	$+(-)(-)$	$+(+)(-)$	$+(+)(+)$
Result	$\boxed{+}$	$\boxed{-}$	$\boxed{+}$

Increasing $(-\infty, -3) \cup (2, \infty)$

Decreasing $(-3, 2)$

$$f''(x) = 24x + 12 = 0$$

$$24x = -12$$

$$x = \frac{-1}{2}$$

POI at $(\frac{-1}{2}, 37)$

$\frac{-1}{2}$	
-1	1
$\boxed{-}$	$\boxed{+}$

$$f\left(\frac{-1}{2}\right) = 4\left(-\frac{1}{2}\right)^3 + 6\left(\frac{-1}{2}\right)^2 - 72\left(-\frac{1}{2}\right)$$

$$= 4\left(-\frac{1}{8}\right) + 6\left(\frac{1}{4}\right) + \frac{72}{2}$$

$$= \frac{-1}{2} + \frac{3}{2} + \frac{72}{2}$$

$$= \frac{74}{2} = 37$$

D2. $f(x) = 3x^4 - 6x^2 + 9$

$$f'(x) = 12x^3 - 12x$$

$$12x(x^2 - 1) = 0$$

$$12x(x - 1)(x + 1) = 0$$

$$x = 0, 1, -1$$

-1	0	1	
Test point -2	$\frac{-1}{2}$	$\frac{1}{2}$	2
$\boxed{-}$	$\boxed{+}$	$\boxed{-}$	$\boxed{+}$

Increasing $(-1, 0) \cup (1, \infty)$

Decreasing $(-\infty, -1) \cup (0, 1)$

$$f''(x) = 36x^2 - 12 = 0$$

$$36x^2 = 12$$

$$x^2 = \frac{1}{3}$$

$$x = \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$$

	$\frac{-1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	
-2	0	2	
$\boxed{+}$	$\boxed{-}$	$\boxed{+}$	

concave up $\left(-\infty, \frac{-1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$

concave down $\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$


D3. $f(x) = 3x + 6\sin x$ on $[0, 2\pi]$

$$f'(x) = 3 + 6\cos x = 0$$

$$6\cos x = -3$$

$$\cos x = \frac{-1}{2}$$

$\cos x$ is negative in S & T quadrants

From Special 

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}$$

$$\therefore x = \pi - \frac{\pi}{3} = \frac{3\pi}{3} - \frac{\pi}{3} = \frac{2\pi}{3} \quad (\text{S quadrant})$$

$$x = \pi + \frac{\pi}{3} = \frac{3\pi}{3} + \frac{\pi}{3} = \frac{4\pi}{3} \quad (T \text{ quadrant})$$

	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	
Test point	0	π	2π
	$3 + 6 \cos 0$	$3 + 6\pi$	$3 + 6 \cos x = 3 + 6 \cos 2\pi$
	$\boxed{+}$	$\boxed{-}$	$\boxed{+}$
	local max		local min

Increasing $\left(0, \frac{2\pi}{3}\right) \cup \left(\frac{4\pi}{3}, 2\pi\right)$

Decreasing $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

Local maximum $\left(\frac{2\pi}{3}, 2\pi + 3\sqrt{3}\right)$

Local minimum $\left(\frac{4\pi}{3}, 4\pi - 3\sqrt{3}\right)$

$$f(x) = 3x + 6 \sin x$$

$$f\left(\frac{2\pi}{3}\right) = 3\left(\frac{2\pi}{3}\right) + 6 \sin \frac{2\pi}{3}$$

$$= 2\pi + 6\left(\frac{\sqrt{3}}{2}\right)$$

$$= 2\pi + 3\sqrt{3}$$

$$f\left(\frac{4\pi}{3}\right) = 3\left(\frac{4\pi}{3}\right) + 6 \sin \frac{4\pi}{3}$$

$$= 4\pi + 6\left(-\frac{\sqrt{3}}{2}\right)$$

$$= 4\pi - 3\sqrt{3}$$

$$D4. \quad f(x) = 6x^5 - 40x^3$$

$$f'(x) = 30x^4 - 120x^2$$

$$f''(x) = 120x^3 - 240x = 0$$

$$120x(x^2 - 2) = 0$$

$$120x(x - \sqrt{2})(x + \sqrt{2}) = 0$$

$$x = 0, \sqrt{2}, -\sqrt{2}$$

$-\sqrt{2}$		0		$\sqrt{2}$	
-2		-1		1	
(-)(-)(-)		(-)(-)(+)		(+)(-)(+)	
$\boxed{-}$		$\boxed{+}$		$\boxed{-}$	
$\boxed{+}$		$\boxed{-}$		$\boxed{+}$	

POI at $-\sqrt{2}, 0$ and $\sqrt{2}$

$$D5. \quad f''(x) = 12x^2 - 48 = 0$$

$$12x^2 = 48$$

$$x^2 = 4$$

$$x = 2, -2$$

-2		2	
-3		3	
$\boxed{+}$		$\boxed{-}$	
$\boxed{-}$		$\boxed{+}$	

POI at $x = 2, -2$

Concave up $(-\infty, -2) \cup (2, \infty)$

Concave down $(-2, 2)$

$$D6. a) f(x) = 9x^5 - 15x^3 + 9$$

$$f'(x) = 45x^4 - 45x^2 = 0$$

$$45x^2(x^2 - 1) = 0$$

$$x = 0, 1 - 1$$

$$f''(x) = 180x^3 - 90x$$

$$f''(0) = 0 \text{ inconclusive}$$

$f''(1) = 90 > 0 \therefore$ local min at $x = 1$ $f(1)=3$, so the local minimum is (1,3)

$$f''(-1) = 180(-1) - 90(-1) = -90 < 0 \therefore \text{local max at } x = -1$$

$f(-1)=-15$, so the local maximum is (-1, 15)

$$b) f(x) = 12x^3 - 32x$$

$$f'(x) = 36x^2 - 32 = 0$$

$$36x^2 = 32$$

$$x^2 = \frac{32}{26} \text{ so, } x = \pm \sqrt{\frac{32}{36}} = \pm \frac{\sqrt{16\sqrt{2}}}{6} = \pm \frac{2\sqrt{2}}{3}$$

$$f''(x) = 16x$$

$$f''\left(\frac{2\sqrt{2}}{3}\right) = 16\left(\frac{2\sqrt{2}}{3}\right) > 0 \therefore \text{local min at } x = \frac{2\sqrt{2}}{3} \text{ sub. into } f(x) \text{ to find } y$$

$$f''\left(-\frac{2\sqrt{2}}{3}\right) = 16\left(-\frac{2\sqrt{2}}{3}\right) < 0 \therefore \text{local max at } x = -\frac{2\sqrt{2}}{3}$$

D7. NOTE D_f = doesn't include 0 $D_f(-\infty, 0) \cup (0, \infty)$

$$\begin{aligned} \text{a) } f'(x) &= (1)e^{1/x} + e^{1/x}(-x^{-2})(x) = e^{1/x} - e^{1/x}x^{-1} \\ e^{1/x} \left(1 - \frac{1}{x}\right) &= 0 \\ 1 - \frac{1}{x} &= 0 \\ 1 &= \frac{1}{x} \\ x &= 1 \end{aligned}$$

Include where it is undefined

	0	1
-1	0.5	2
$e^{1/x} \left(1 - \frac{1}{x}\right)$	$e^{1/x} \left(1 - \frac{1}{x}\right)$	$e^{1/x} \left(1 - \frac{1}{x}\right)$
$e^{1/-1} \left(1 - \frac{1}{-1}\right)$	$e^{1/0.5} \left(1 - \frac{1}{0.5}\right)$	$e^{1/2} \left(1 - \frac{1}{2}\right)$
$e^{-1}(1 + 1)$	$e^2(1 - 2)$	$\boxed{+}$
$= \frac{2}{e} \quad \boxed{+}$	$= \boxed{-}$	

Decreasing (0, 1) and local minimum (1,e)

$$c) \quad f'(x) = e^{\frac{1}{x}}(1 - x^{-1})$$

$$\begin{aligned} f''(x) &= e^{\frac{1}{x}}(-x^{-2})(1 - x^{-1}) + (x^{-2})e^{\frac{1}{x}} \\ &= e^{\frac{1}{x}}(-x^{-2} + x^{-3}) + x^{-2}e^{\frac{1}{x}} \\ &= e^{\frac{1}{x}}\left(-\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^2}\right) \\ &= e^{\frac{1}{x}}\left(\frac{1}{x^3}\right) \end{aligned}$$

$f''(x) = 0$ test points around $x = 0$ where the graph is undefined

$$e^{\frac{1}{x}}\left(\frac{1}{x^3}\right) = 0$$

↓ ↘

No solution $\frac{1}{x^3} = 0$

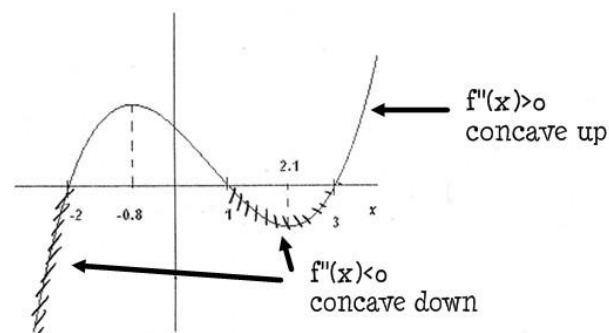
no solution \therefore no POI

0

-1	1
$e^{\frac{1}{x}}\left(\frac{1}{x^3}\right)$	$e^{\frac{1}{x}}\left(\frac{1}{x^3}\right)$
$e^{\frac{1}{-1}}\left(\frac{1}{(-1)^3}\right)$	$e^{\frac{1}{1}}\left(\frac{1}{1}\right)$
$= \frac{-1}{e} \quad \boxed{-}$	$= \boxed{+}$
Concave down $(-\infty, 0)$	Concave up $(0, \infty)$

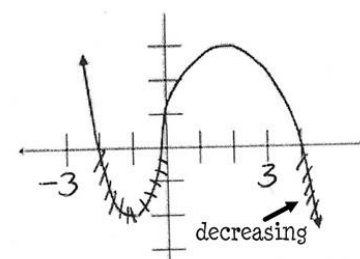
D8. $f''(x) > 0$ concave up $f''(x) < 0$ concave down

concave up $(-2, 1) \cup (3, \infty)$
 concave down $(-\infty, -2) \cup (1, 3)$
 POI at $x = -2, 1, 3$



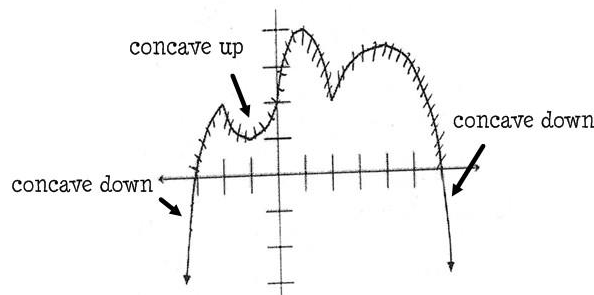
D9. Part (2)

local max at $x = -2, 4$ local min at $x = -0.25$
 If $f'(x) < 0 \rightarrow$ decreasing If $f'(x) > 0 \rightarrow$ increasing
 local max \Rightarrow increasing to decreasing
 local min \Rightarrow decreasing to increasing



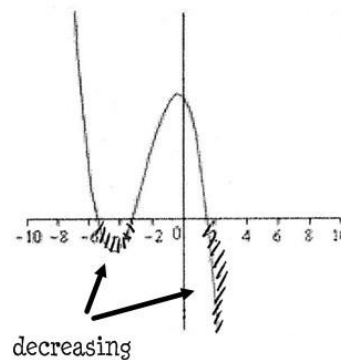
Increasing $(-\infty, -2) \cup (-0.25, 4)$
 Decreasing $(-2, -0.25) \cup (4, \infty)$

D10. (4) concave up $(-3, 6)$
 concave down $(-\infty, -3) \cup (6, \infty)$
 POI at $x = -3$ and $x = 6$



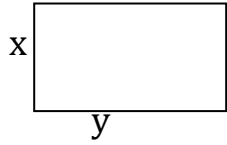
D11.

Plot 2 $f'(x) > 0$ $f(x)$ is increasing
 $f'(x) < 0$ $f(x)$ is decreasing
 from increasing to decreasing \rightarrow local max
 from decreasing to increasing \rightarrow local min



local max at $x = -5$ and 1.8 (approx.)
 local min at $x = -3$

E. Optimization

Example 1.

$$2x + 2y = 128$$

$$x + y = 64$$

$$y = 64 - x$$

FIND $\max A = xy$

$$A = x(64 - x)$$

$$A = 64x - x^2$$

$$A'(x) = 64 - 2x = 0$$

$$64 = 2x$$

$$x = 32$$

$$\therefore y = 64 - x$$

$$= 64 - 32$$

$$= 32$$

\therefore length and width are both 32 and Area is $32 \times 32 = 1024 \text{ m}^2$.

Prove it is a maximum

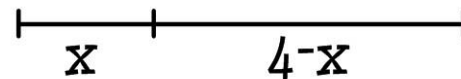
$A''(x) = -2 < 0$, so it is a maximum

Example 2. Let x be the number of feet that goes toward the circle
 Let $4 - x$ be the number of feet that goes toward the square

$$C = 2\pi r$$

$$x = 2\pi r$$

$$r = \frac{x}{2\pi}$$



Square $\frac{4-x}{4}$ goes to each side ($4 - x$ total for 4 sides)

$$\text{Area} = A_{\text{circle}} + A_{\text{square}}$$

$$A = \pi r^2 + x^2 = \pi \left(\frac{x}{2\pi}\right)^2 + \left(\frac{4-x}{4}\right)\left(\frac{4-x}{4}\right)$$

$$A = \frac{x^2}{4\pi} + \frac{16-8x+x^2}{16}$$

$$A' = \frac{1}{4\pi}(2x) + \frac{1}{16}(-8 + 2x) = 0$$

$$\frac{-8+2x}{16} = \frac{-2x}{4\pi}$$

$$\frac{-8+2x}{4} = \frac{-2x}{\pi}$$

$$\frac{-8+2x}{4} = \frac{-2x}{\pi}$$

$$-8\pi + 2\pi x = -8x$$

$$-8\pi = -8x - 2\pi x$$

$$8\pi = 8x + 2\pi x$$

$$8\pi = x(8 + 2\pi) \quad x = \frac{8\pi}{8+2\pi} \text{ to circle or } \frac{4\pi}{4+\pi}$$

for a minimum, $\frac{4\pi}{4+\pi}$ ft to the circle, $4 - x = 4 - \frac{4\pi}{4+\pi} = \frac{4(4+\pi)}{4+\pi} - \frac{4\pi}{4+\pi} = \frac{16}{4+\pi}$ to the square

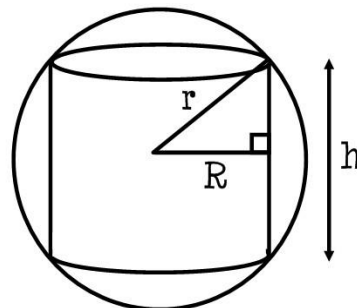
Prove it is a minimum $A'' = \frac{1}{2\pi} + \frac{1}{16}(2) > 0$, so it is a minimum

Example 3.

Let R = radius of cylinder
 h = height of cylinder

From Pythagorean Theorem:

$$r^2 = R^2 + \left(\frac{1}{2}h\right)^2 = R^2 + \frac{h^2}{4} \quad \therefore R^2 = r^2 - \frac{h^2}{4}$$



$$\text{Volume of cylinder } V = \pi R^2 h = \pi \left(r^2 - \frac{h^2}{4} \right) h$$

$= \pi \left(r^2 h - \frac{h^3}{4} \right)$ a function of h alone

$$\frac{dV}{dh} = \pi \left(r^2 - \frac{3h^2}{4} \right) = 0 \quad \text{when } r^2 = \frac{3h^2}{4} \quad h^2 = \frac{4r^2}{3}$$

$$h = \sqrt{\frac{4}{3}r^2} = \frac{2r}{\sqrt{3}}$$

$$R^2 = r^2 - \frac{h^2}{4} = r^2 - \frac{\left(\frac{2r}{\sqrt{3}}\right)^2}{4} = r^2 - \frac{1}{3}r^2 = \frac{2}{3}r^2$$

$$\therefore \max \text{Volume } V = \pi R^2 h = \pi \left(\frac{2}{3}r^2 \right) \left(\frac{2r}{\sqrt{3}} \right) = \frac{4\pi r^3}{3\sqrt{3}}$$

Example 4. $V=90$

let x be width $3x$ be length y be height

$$V = 3x(x)(y) = 3x^2y$$

$$90 = 3x^2y \quad y = \frac{90}{3x^2} = \frac{30}{x^2} \quad \boxed{1}$$

$$A = 2(3x^2) + 2(xy) + 2(3xy) = 6x^2 + 8xy$$

$$\text{Cost } C = 10(6x^2) + 5(8xy) = 60x^2 + 40xy \quad \text{sub } \boxed{1}$$

$$C = 60x^2 + 40x \left(\frac{30}{x^2} \right) = 60x^2 + \frac{1200}{x} \\ = 60x^2 + 1200x^{-1}$$

$$C' = 120x - 1200x^{-2}$$

$$C' = 0 \quad 120x - 1200x^{-2} = 0$$

$$120x = \frac{1200}{x^2}$$

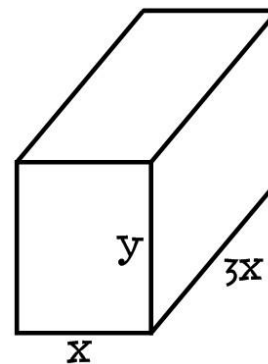
$$120x^3 - 1200 = 0$$

$$120(x^3 - 10) = 0 \quad x = \sqrt[3]{10}$$

$$y = \frac{30}{(\sqrt[3]{10})^2}$$

dimensions are $3\sqrt[3]{10} \times \sqrt[3]{10} \times \frac{30}{(\sqrt[3]{10})^2}$ feet

$$C''(x) = 120 + \frac{2400}{x^2} > 0, \text{ since } x > 0, \text{ so it is a minimum}$$

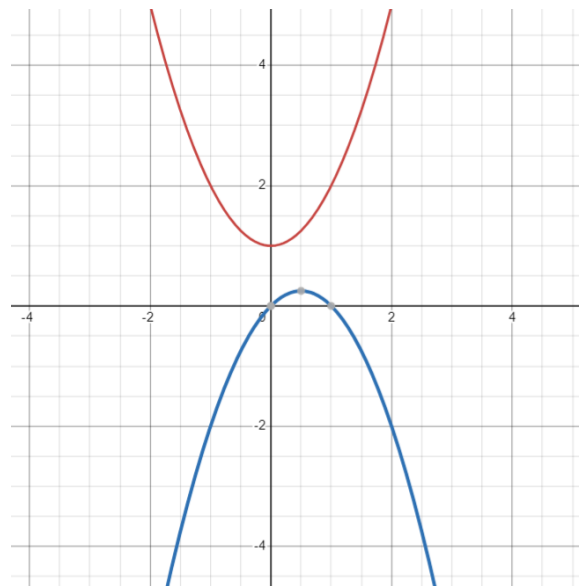
**Example 5.**

$$\frac{df}{dx} = 4x - 1$$

$$\frac{df}{dx} = 0 \text{ when } x = \frac{1}{4} \text{ for minimum } f(x)$$

$$f\left(\frac{1}{4}\right) = 2\left(\frac{1}{4}\right)^2 - \frac{1}{4} + 1 = \frac{2}{16} - \frac{4}{16} + \frac{16}{16} = \frac{14}{16} \\ = \frac{7}{8}$$

Prove Minimum $f''(x) = 4 > 0$, so it is a minimum.



Practice Exam Questions on Optimization

E1. *let x, y be the numbers*

$$x + 2y = 24 \quad x = 24 - 2y$$

$$\max = xy = (24 - 2y)y = 24y - 2y^2$$

$$\max f(y) = 24y - 2y^2$$

$$f'(y) = 24 - 4y = 0$$

$$24 = 4y \quad y = \frac{24}{4} = 6$$

$$\therefore x = 24 - 2(6) \quad x = 12$$

The numbers are 12 and 6.

E2. *Max $A = xy$*

$$A(x) = x(150 - 2x) = 150x - 2x^2 \quad y + 2x = 150$$

$$A'(x) = 150 - 4x = 0 \quad y = 150 - 2x$$

$$150 = 4x \quad x = 37.5$$

$$\therefore y = 150 - 2(37.5) = 75$$

$$\therefore \text{Max } A = (75)(37.5) = 2812.50 \text{ m}^2$$

E3.

$$V = \pi r^2 h$$

$$1000 = \pi r^2 h$$

$$h = \frac{1000}{\pi r^2}$$

$$\text{Min}A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2} \right)$$

$$A = 2\pi r^2 + 2000r^{-1}$$

$$A'(r) = 4\pi r - 2000r^{-2}$$

$$\text{let } A'(r) = 0 \quad 4\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{2000}{4\pi} \quad r = \sqrt[3]{\frac{2000}{4\pi}} = \sqrt[3]{\frac{500}{\pi}}$$

E4. Perimeter is 100...

$$2x + 2y = 100$$

$$x + y = 50 \quad \text{substitute } y = 50 - x \text{ into equation for area}$$

$$A = xy = x(50 - x)$$

$$A(x) = 50x - x^2$$

$$A'(x) = 50 - 2x = 0$$

$$\therefore 50 = 2x \quad x = 25$$

$$y = 50 - 25 \quad y = 25$$

$$\text{max}A = 25(25) = 625\text{cm}^2$$

E5. Solve for h $h^2 + y^2 = (2y)^2$
 $h^2 = 3y^2 \quad h = \sqrt{3}y$

$\text{minArea} = 2 - 4x = \text{left for } \Delta$

$$2 - 4x = 6y$$

$$y = \frac{2-4x}{6} = \frac{1-2x}{3}$$

$$A = x^2 + \frac{1}{2}(\text{base})(\text{height})$$

$$A = x^2 + \frac{1}{2}(2y)(\sqrt{3}y)$$

$$A = x^2 + \sqrt{3}y^2$$

$$\therefore A = x^2 + \sqrt{3} \left(\frac{1-2x}{3} \right)^2 = x^2 + \frac{\sqrt{3}}{9} (1 - 4x + 4x^2)$$

$$A' = 2x + \frac{\sqrt{3}}{9} (-4 + 8x) = 0$$

$$2x = \frac{-\sqrt{3}}{9} (-4 + 8x)$$

$$18x = 4\sqrt{3} - 8\sqrt{3}x$$

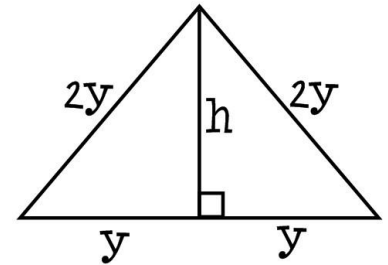
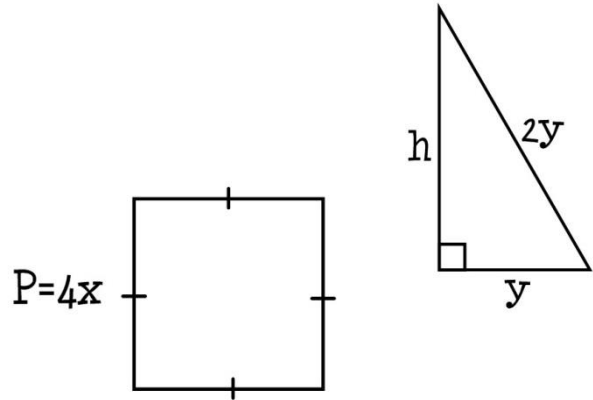
$$18x + 8\sqrt{3}x = 4\sqrt{3}$$

$$x(18 + 8\sqrt{3}) = 4\sqrt{3} \quad x = \frac{4\sqrt{3}}{18+8\sqrt{3}} =$$

0.217 sides of square

$$y = \frac{1-2(0.217)}{3} = 0.19$$

$$\text{sides of } \Delta = 2y = 2(0.19) = 0.38\text{m}$$



E6. $4x + 3y = 120 \quad y = \frac{120-4x}{3}$

$$A = 2xy = 2x \left(\frac{120-4x}{3} \right) = \frac{1}{3} (240x - 8x^2)$$

$$A' = \frac{1}{3} (240 - 16x) = 0$$

$$\therefore 240 = 16x \quad x = 15$$

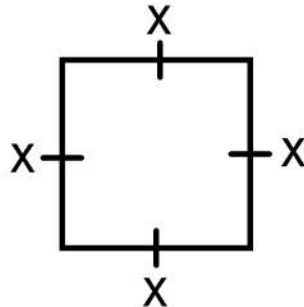
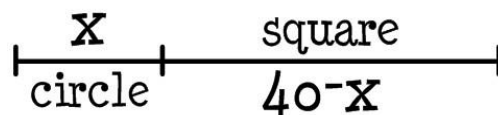
$$\therefore y = \frac{120-4(15)}{3} = 20$$

$$\text{max area} = 2xy = 2(15)(20) = 600\text{m}^2$$

E7. square (each side $\frac{40-x}{4}$)

$$2\pi r = x \quad r = \frac{x}{2\pi}$$

$$A = \left(\frac{40-x}{4}\right)\left(\frac{40-x}{4}\right) + \pi\left(\frac{x}{2\pi}\right)^2 = \frac{1600-80x+x^2}{16} + \frac{\pi x^2}{4\pi^2}$$



$$A' = 0 \quad \frac{1}{16}(-80 + 2x) + \frac{1}{4\pi}(2x) = 0$$

$$\frac{-40+x}{8} = \frac{-x}{2\pi}$$

$$-80\pi + 2\pi x = -8x$$

$$80\pi = x(8 + 2\pi) \quad x = \frac{80\pi}{8+2\pi} = 17.59 \text{ to circle}$$

$(40 - 17.59)$ to square

E8. Solve for h $h^2 + y^2 = (2y)^2$

$$h^2 + y^2 = 4y^2 \quad h^2 = 3y^2 \quad h = \sqrt{3}y$$

Min area $20 - 4x =$ left for equilateral Δ

$$20 - 4x = 6y \quad y = \frac{20-4x}{6} = \frac{10-2x}{3}$$

$$A = x^2 + \frac{1}{2}(\text{base})(\text{height}) = x^2 + \frac{1}{2}(2y)(\sqrt{3}y)$$

$$= x^2 + \sqrt{3}y^2$$

$$A = x^2 + \sqrt{3}\left(\frac{10-2x}{3}\right)\left(\frac{10-2x}{3}\right) = x^2 + \frac{\sqrt{3}}{9}(100 - 40x + 4x^2)$$

$$A'(x) = 2x + \frac{\sqrt{3}}{9}(-40 + 8x) = 0$$

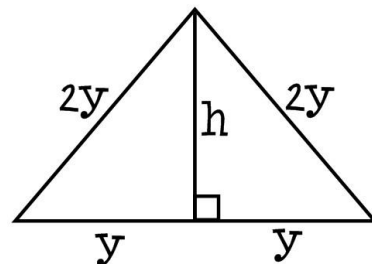
$$\frac{\sqrt{3}}{9}(-40 + 8x) = -2x$$

$$-18x = -40\sqrt{3} + 8\sqrt{3}x$$

$$40\sqrt{3} = 18x + 8\sqrt{3}x$$

$$40\sqrt{3} = x(18 + 8\sqrt{3})$$

$$\therefore x = \frac{40\sqrt{3}}{18+8\sqrt{3}} = 2.17 \text{ sides of square}$$



E9. $4x + 2\pi r = 12$ *perimeter*

$$\frac{2\pi r}{2\pi} = \frac{12-4x}{2\pi} \quad \therefore r = \frac{12-4x}{2\pi}$$

$$\min A = x^2 + \pi r^2 = x^2 + \pi \left(\frac{12-4x}{2\pi} \right) \left(\frac{12-4x}{2\pi} \right)$$

$$A = x^2 + \frac{1}{4\pi} (144 - 96x + 16x^2)$$

$$A'(x) = 2x + \frac{1}{4\pi} (-96 + 32x) = 0$$

$$\frac{-96+32x}{4\pi} = -2x$$

$$-96 + 32x = -8\pi x$$

$$-96 = -8\pi x - 32x$$

$$x = \frac{-96}{-8\pi-32} = \frac{-4(24)}{-4(2\pi+8)} = \frac{12}{\pi+4}$$

$$\therefore \text{sides of square} = \frac{12}{\pi+4}$$

$$r = \frac{12-4\left(\frac{12}{\pi+4}\right)}{2\pi} \rightarrow \text{circle}$$

E10. $A = 120 = x^2 + 4xy$

$$\therefore y = \frac{120-x^2}{4x}$$

$$\text{Max } V = x^2 y = x^2 \left(\frac{120-x^2}{4x} \right) = \frac{1}{4} (120x - x^3)$$

$$V'(x) = \frac{1}{4} (120 - 3x^2) = 0$$

$$120 = 3x^2 \quad x^2 = 40 \quad x = \sqrt{40} = 2\sqrt{10}$$

$$\therefore y = \frac{120-\sqrt{40}^2}{4\sqrt{40}} = \frac{120-40}{4\sqrt{40}} = \frac{20}{\sqrt{40}} = \frac{20}{2\sqrt{10}} = \frac{10}{\sqrt{10}} = \frac{10\sqrt{10}}{\sqrt{10}} = 10$$

$$\therefore \text{dimensions are } 2\sqrt{10} \times 2\sqrt{10} \times 10 \text{ inches}$$

E11.

let x, y be the two numbers

$$xy = 300 \quad \boxed{1} \quad \therefore y = \frac{300}{x}$$

$$\text{min} = x + 3y$$

$$M = x + 3\left(\frac{300}{x}\right) = x + 900x^{-1}$$

$$M' = 1 - 900x^{-2} = 0$$

$$1 = 900x^{-2}$$

$$1 = \frac{900}{x^2} \quad \therefore x^2 = 900 \quad x = 30$$

$$\therefore y = \frac{300}{x} = \frac{300}{30} = 10 \quad \therefore y = 10$$

The two numbers are 30 and 10.

E12. $V=120$ *let x be width $4x$ be length y be height*

$$V = 4x(x)(y) = 4x^2y$$

$$120 = 4x^2y \quad y = \frac{120}{4x^2} = \frac{30}{x^2} \quad \boxed{1}$$

$$A = 2(4x^2) + 2(xy) + 2(4xy) = 8x^2 + 10xy$$

$$\text{Cost } C = 5(8x^2) + 3(10xy) = 40x^2 + 30xy \quad \text{sub } \boxed{1}$$

$$C = 40x^2 + 30x\left(\frac{30}{x^2}\right) = 40x^2 + \frac{900}{x} \\ = 40x^2 + 900x^{-1}$$

$$C' = 80x - 900x^{-2}$$

$$C' = 0 \quad 80x - 900x^{-2} = 0$$

$$80x = \frac{900}{x^2}$$

$$80x^3 - 900 = 0$$

$$20(4x^3 - 45) = 0 \quad x = \sqrt[3]{11.25}$$

$$y = \frac{30}{(\sqrt[3]{11.25})^2}$$

dimensions are $4\sqrt[3]{11.25} \times \sqrt[3]{11.25} \times \frac{30}{(\sqrt[3]{11.25})^2}$ feet

F. Sums and Sigma Notation

Example 1.
$$= \frac{4}{n} \sum_{i=1}^n \left(\frac{1}{n}\right)^i = \frac{4}{n^2} (\sum_{i=1}^n i) = \frac{4}{n^2} \left(\frac{n(n+1)}{2}\right)$$

$$= \frac{2n(n+1)}{n^2} = \frac{2(n+1)}{n} = 2 + \frac{2}{n}$$

Example 3.

$$S_n = \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (i^3 + 4i^2 + 6i)$$

$$= \frac{1}{n^4} \left[\sum_{i=1}^n i^3 + 4 \sum_{i=1}^n i^2 + 6 \sum_{i=1}^n i \right]$$

$$= \left[\frac{n^2(n+1)^2}{4n^4} + \frac{4n(n+1)(2n+1)}{6n^4} + \frac{6n(n+1)}{2n^4} \right]$$

$$= \frac{(n+1)(n+1)}{4n^2} + \frac{4(2n^2+3n+1)}{6n^3} + \frac{6n+6}{2n^3}$$

$$= \frac{3n(n^2+2n+1)}{12n^3} + \frac{2(8n^2+12n+4)}{12n^3} + \frac{6(6n+6)}{12n^3}$$

Get common denominator = $\frac{3n^3+6n^2+3n}{12n^3} + \frac{16n^2+24n+8}{12n^3} + \frac{36n+36}{12n^3}$

$$S_n = \frac{3n^3+22n^2+63n+44}{12n^3}$$

$$\lim_{n \rightarrow \infty} \left(\frac{3n^3}{12n^3} + \frac{22n^2}{12n^3} + \frac{63n}{12n^3} + \frac{44}{12n^3} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{4} + \frac{22}{12n} + \frac{63}{12n^2} + \frac{44}{12n^3} \right)$$

$$= \frac{1}{4}$$

Example 4. $a = 1$ $r = 3$ $n = 9$

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$S_9 = \frac{1(3^9-1)}{3-1} = \frac{(3^9-1)}{2}$$

Example 5.
$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 -$$

$$\frac{1}{n+1} = \frac{n+1}{n+1} - \frac{1}{n+1} = \frac{n}{n+1}$$

Example 6.

$$\begin{aligned}
 \text{a) } \sum_{i=3}^{25} \sqrt{3i+1} \\
 \downarrow \quad \downarrow \\
 m=3 \qquad \qquad j=i-m+1 \\
 \downarrow \qquad \qquad j=i-3+1 \\
 \qquad \qquad \qquad j=i-2 \quad \text{solve for } i \\
 \qquad \qquad \qquad i=j+2 \\
 \text{for } i=25 \quad j=i-2 \\
 \qquad \qquad \qquad j=25-2 \\
 \qquad \qquad \qquad j=23 \\
 \therefore \sum_{j=1}^{23} \sqrt{3(j+2)+1} \\
 = \sum_{j=1}^{23} \sqrt{3j+7}
 \end{aligned}$$

Example 7. Suppose $\sum_{k=1}^{125} a_k = -8$ and $\sum_{k=1}^{125} b_k = 5$

Find $\sum_{k=1}^{125} (2a_k + 3b_k + 10)$

$$\begin{aligned}
 &= 2 \sum_{k=1}^{125} a_k + 3 \sum_{k=1}^{125} b_k + \sum_{k=1}^{125} 10 \\
 &= 2(-8) + 3(5) + 125(10) = -16 + 15 + 1250 = 1249
 \end{aligned}$$

Example 8. a) $A_n = \sum_{k=1}^n \left(2 + \frac{2k}{n} + \frac{k^2}{n^2} \right) = \sum_{k=1}^n 2 + \frac{2}{n} \sum_{k=1}^n k + \frac{1}{n^2} \sum_{k=1}^n k^2$

$$\begin{aligned}
 &= 2n + \frac{2}{n} \frac{(n)(n+1)}{2} + \frac{1}{n^2} \frac{n(n+1)(2n+1)}{6} = \frac{2n}{1} + \frac{n+1}{1} + \frac{2n^2+3n+1}{6n} \\
 &= \frac{12n^2+6n^2+6n+2n^2+3n+1}{6n} = \frac{20n^2+9n+1}{6n}
 \end{aligned}$$

b) $\lim_{n \rightarrow \infty} \frac{20n^2+9n+1}{6n} = \lim_{n \rightarrow \infty} \left(\frac{20}{6}n + \frac{9}{6} + \frac{1}{6n} \right) = \infty$

Practice Exam Questions on Sums and Sigma Notation

$$\begin{aligned}
 \text{F1.} &= \frac{1}{n^2} \sum_{i=1}^n (4i^2 + 3i) \\
 &= \frac{1}{n^2} \left[(4 \sum_{i=1}^n i^2) + (3 \sum_{i=1}^n i) \right] = \frac{1}{n^2} \left[\frac{4n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} \right] \\
 &= \frac{4n(n+1)(2n+1)}{6n^2} + \frac{3n(n+1)}{2n^2} = \frac{4(2n^2+3n+1)}{6n} + \frac{3n+3}{2n} \\
 &= \frac{8n^2+12n+4+9n+9}{6n} = \frac{8n^2+21n+13}{6n}
 \end{aligned}$$

$$\begin{aligned}
 \text{F2.} &= \sum_{k=1}^{10} a_k + \sum_{k=1}^{10} b_k + \sum_{k=1}^{10} 10 \\
 &= -20 + 30 + 100 = 110
 \end{aligned}$$

$$\begin{aligned}
 \text{F3.} &= 4 \sum_{i=1}^{20} a_i - 2 \sum_{i=1}^{20} b_i \\
 &= 4(10) - 2(15) = 40 - 30 = 10
 \end{aligned}$$

$$\begin{aligned}
 \text{F4.} &1^2 + 2^2 + 3^2 + \dots + 120^2 \\
 &= \sum_{i=1}^{120} i^2
 \end{aligned}$$

$$\begin{aligned}
 \text{F5.} &= \sum_{i=1}^n \frac{2}{n} (i^2 + 4i + 4) = \frac{1}{n} \sum_{i=1}^n (i^2 + 4i + 4) \\
 &= \frac{2}{n} \left(\sum_{i=1}^n i^2 + 4 \sum_{i=1}^n i + \sum_{i=1}^n 4 \right) = \frac{2}{n} \left[\frac{n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} + 4n \right] \\
 &= \frac{n(n+1)(2n+1)}{3n} + \frac{4n(n+1)}{n} + 8 = \frac{(n+1)(2n+1)}{3} + \frac{4(n+1)}{1} + 8 \\
 &= \frac{2n^2+3n+1}{3} + \frac{12n+12}{3} + \frac{24}{3} = \frac{2n^2+15n+37}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{F6.} &= \frac{2}{n} \sum_{i=1}^n \left(i^2 + \frac{6i}{n} + \frac{9}{n^2} \right) = \frac{2}{n} \left[\sum_{i=1}^n i^2 + \frac{6}{n} \sum_{i=1}^n i + \frac{9}{n^2} \sum_{i=1}^n 1 \right] \\
 &= \frac{2}{n} \left[\frac{n(n+1)(2n+1)}{6} + \frac{6}{n} \frac{(n)(n+1)}{2} + \frac{9}{n^2} (n) \right] = \frac{(n+1)(2n+1)}{3} + 6 \left(\frac{n+1}{n} \right) + \frac{18}{n^2} \\
 &= \frac{n^2(2n^2+3n+1)}{3n^2} + \frac{18n^2+18n}{3n^2} + \frac{54}{3n^2} = \frac{2n^4+3n^3+n^2+18n^2+18n+54}{3n^2} \\
 &= \frac{2n^4+3n^3+19n^2+18n+54}{3n^2} = \frac{2n^2}{3} + n + \frac{19}{3} + \frac{6}{n} + \frac{18}{n^2}
 \end{aligned}$$

$$\text{F7.} = 100(0.6) = 60$$

$$\begin{aligned}
 \text{F8. a) } & \frac{1}{n} \left(\sum_{i=1}^n 4 + \frac{1}{n} \sum_{i=1}^n i + \frac{1}{n^2} \sum_{i=1}^n i^2 \right) \\
 &= \frac{1}{n} \left[4n + \frac{1}{n} \frac{(n)(n+1)}{2} + \frac{1}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\
 &= \frac{1}{n} \left[4n + \frac{n+1}{2} + \frac{2n^2+3n+1}{6n} \right] = \left[4 + \frac{n+1}{2n} + \frac{2n^2+3n+1}{6n^2} \right] \\
 &= \frac{24n^2+3n^2+3n+2n^2+3n+1}{6n^2} = \frac{29n^2+6n+1}{6n^2}
 \end{aligned}$$

$$\text{c) } \lim_{n \rightarrow \infty} \frac{29n^2+6n+1}{6n^2} = \lim_{n \rightarrow \infty} \left(\frac{29}{6} + \frac{1}{n} + \frac{1}{6n^2} \right) = \frac{29}{6}$$

$$\begin{aligned}
 \text{F9. } &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sum_{i=1}^n \left(\frac{i^2}{n^2} - 2 \right) \right] = \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sum_{i=1}^n \frac{i^2}{n^2} - \sum_{i=1}^n 2 \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{n^2} \frac{n(n+1)(2n+1)}{6} - 2n \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{(n+1)(2n+1)}{6n} - \frac{12n^2}{6n} \right] = \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{2n^2+3n+1-12n^2}{6n} \right] \\
 &= \lim_{n \rightarrow \infty} \left(\frac{-10n^2+3n+1}{6n^2} \right) = \lim_{n \rightarrow \infty} \left(-\frac{10}{6} + \frac{1}{2n} + \frac{1}{6n^2} \right) = -\frac{5}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{F10. } &= \lim_{n \rightarrow \infty} \frac{1}{n^7} \left[\sum_{i=1}^n i^2 i^3 \right] = \lim_{n \rightarrow \infty} \frac{1}{n^7} \left[\sum_{i=1}^n i^2 \sum_{i=1}^n i^3 \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n^7} \left[\left(\frac{n(n+1)(2n+1)}{6} \right) \left(\frac{n^2(n+1)^2}{4} \right) \right] = \lim_{n \rightarrow \infty} \left[\frac{n^3(n+1)(2n+1)(n+1)^2}{24n^7} \right] \\
 &= \lim_{n \rightarrow \infty} \left[\left(\frac{2n^2+3n+1}{24n^4} \right) \left(\frac{n^2+2n+1}{1} \right) \right] \\
 &= \frac{1}{24} \lim_{n \rightarrow \infty} \frac{(2n^4+4n^3+2n^2+3n^3+6n^2+3n+n^2+2n+1)}{n^4}
 \end{aligned}$$

$$= \frac{1}{24} \lim_{n \rightarrow \infty} \frac{(2n^4 + 7n^3 + 9n^2 + 5n + 1)}{n^4}$$

$$= \frac{1}{24} \lim_{n \rightarrow \infty} \left(2 + \frac{7}{n} + \frac{9}{n^2} + \frac{5}{n^3} + \frac{1}{n^4} \right) = \frac{2}{24} = \frac{1}{12}$$

F11. Write the sum starting at 1: $\sum_{i=7}^{30} \sqrt{4i+5}$

$$m = 7 \qquad j = i - m + 1$$

$$j = i - 7 + 1$$

$$j = i - 6 \quad \text{solve for } i$$

$$i = j + 6$$

$$\text{for } i = 30 \quad j = i - 6$$

$$j = 30 - 6 = 24$$

$$j = 24$$

$$\therefore \sum_{j=1}^{24} \sqrt{4(j+6)+5}$$

$$= \sum_{j=1}^{24} \sqrt{4j+29}$$

F12. $\sum_{i=4}^{27} i^2$

$$m=4$$

$$j = i - m + 1$$

$$j = i - 4 + 1$$

$$j = i - 3 \quad \text{solve for } i$$

$$i = j + 3$$

$$\text{for } i = 27 \quad j = i - 3$$

$$j = 27 - 3$$

$$j = 24$$

$$\sum_{i=4}^{27} i^2 = \sum_{j=1}^{24} (j+3)^2$$

G. Riemann Sums

Example 2.

$$\Delta x = \frac{b-a}{n} = \frac{12-0}{3} = 4$$

$$\begin{aligned} \text{Sum} &= \sum \Delta x f(x_i) = 4f(2) + 4f(6) + 4f(10) \\ &= 4(7) + 4(39) + 4(103) = 596 \end{aligned}$$

Example 4.

$$\int_0^4 x^2 dx$$

$$f(x)=x^2$$

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(a + \Delta x i) \Delta x \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(0 + \frac{4i}{n}\right) \left(\frac{4}{n}\right)$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{4i}{n}\right) \left(\frac{4}{n}\right)$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{4i}{n}\right)^2 \left(\frac{4}{n}\right)$$

$$= \frac{4}{n} \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{4i}{n}\right)^2$$

$$= \frac{4}{n} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{16i^2}{n^2}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{64i^2}{n^3}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{64n(n+1)(2n+1)}{6n^3}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{64(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{(64n+64)(2n+1)}{6n^2}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{128n^2 + 128n + 64n + 64}{6n^2}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{128}{6} + \frac{192}{6n} + \frac{64}{6n^2}\right) = \frac{128}{6} = \frac{64}{3}$$

Example 5.

$$a) \frac{b-a}{n} = \frac{5}{n} \quad \therefore b - a = 5$$

$$xi = a + \Delta xi = a + \frac{5}{n}i \quad \therefore a = 3 \quad b = 8$$

$$\therefore \int_3^8 (x^3) dx$$

$$b) \frac{b-a}{n} = \frac{7}{n} \quad \therefore b - a = 7$$

$$xi = a + \Delta xi = a + \frac{7}{n}i \quad \therefore a = 0, b = 7$$

$$\text{let } f(x) = \cos(x^2)$$

$$\int_0^7 \cos(x^2) dx$$

Example 6. Estimate the area under the graph of the function

$f(x) = \ln(x^2 + 1)$ from $x=0$ to $x=8$ using four approximating rectangles and right endpoints. Repeat using left end points and midpoints.

$$\Delta x = \frac{8-0}{4} = 2$$

$$R4 = \Delta x [f(2) + f(4) + f(6) + f(8)] \\ = 2[\ln 5 + \ln 17 + \ln 37 + \ln 65]$$

$$L4 = \Delta x [f(0) + f(2) + f(4) + f(6)] \\ = 2[\ln 1 + \ln 5 + \ln 17 + \ln 37] \\ = 2[\ln 5 + \ln 17 + \ln 37]$$

$$M4 = \Delta x [f(1) + f(3) + f(5) + f(7)] \\ = 2[\ln 2 + \ln 10 + \ln 26 + \ln 50]$$

Practice Exam Questions on Reimann Sums

$$\text{G1. } \Delta x = \frac{b-a}{n} = \frac{5}{n} \quad b - a = 5$$

$$f(x) = e^x$$

$$xi = a + \Delta xi$$

$$xi = a + \frac{5}{n}i \quad a = 0, b = 5$$

$$\therefore \int_0^5 e^x dx$$

$$\text{G2. } f(xi) = f(a + i\Delta x) = f\left(a + \frac{\pi i}{n}\right)$$

$$\frac{b-a}{n} = \frac{2\pi}{n} \quad \therefore b - a = 2\pi$$

$$a = 0 \quad b = 2\pi \quad f(x) = \sin(x^2)$$

$$\therefore \int_0^{2\pi} \sin(x^2) dx$$

$$\text{G3. } \Delta x = \frac{b-a}{n} = \frac{1}{n} \quad b - a = 1$$

$$f(xi) = f\left(a + \frac{1}{n}i\right)$$

$$= \sqrt{\sec\left(\frac{i}{n}\right)} \quad \therefore a = 0 \quad b = 1$$

$$\text{let } f(x) = \sqrt{\sec x}$$

$$\int_0^1 \sqrt{\sec x} dx$$

$$\text{G4. } \frac{b-a}{n} = \frac{6}{n} \quad b - a = 6$$

$$f(a + i\Delta x)$$

$$= f\left(a + \frac{6}{n}i\right) \quad \therefore a = 1, b = 7$$

$$\int_1^7 \ln(x) dx \quad \text{or} \quad \text{use } \int_0^6 \ln(1+x) dx$$

$$\text{G5. } \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(xi) \Delta x$$

$$\Delta x = \frac{b-a}{n} = \frac{4}{n} \quad a = 1, b = 5$$

$$f(x) = \frac{1}{x}$$

$$f(xi) = f\left(1 + i\left(\frac{4}{n}\right)\right)$$

$$= \lim_{n \rightarrow \infty} \left[\frac{4}{n} \sum_{i=1}^n \frac{1}{1 + i\left(\frac{4}{n}\right)} \right]$$

$$\text{G6. } \Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

$$\begin{aligned}
 xi &= a + i\Delta x = 0 + i\left(\frac{3}{n}\right) = \frac{3}{n}i \\
 \therefore \lim_{n \rightarrow \infty} \sum_{i=1}^n f(xi)\Delta x & \\
 \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[\left(\frac{3i}{n}\right)^3 - 3\left(\frac{3i}{n}\right) \right] &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[\sum_{i=1}^n \frac{27i^3}{n^3} - \frac{9i}{n} \right] \\
 &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[\frac{27n^2(n+1)^2}{4n^3} - \frac{9n(n+1)}{2n} \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{3(27)(n^2+2n+1)}{4n^2} - \frac{27n+27}{2n} \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{81n^2+162n+81}{4n^2} - \left(\frac{27n+27}{2n}\right) \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{81}{4} + \frac{162}{4n} + \frac{81}{4n^2} - \left(\frac{27}{2} + \frac{27}{2n}\right) \right] = \frac{81}{4} - \frac{27}{2} = \frac{81}{4} - \frac{54}{4} = \frac{27}{4}
 \end{aligned}$$

G7.

$$\Delta x = \frac{b-a}{n} = \frac{8-0}{4} = 2$$

$$\begin{aligned}
 Sum &= \sum \Delta x f(x_i) = 2f(1) + 2f(3) + 2f(5) + 2f(7) \\
 &= 2((1)^3 + 1) + 2((3)^3 + 1) + 2((5)^3 + 1) + \\
 &\quad 2((7)^3 + 1)
 \end{aligned}$$

$$= 2(2) + 2(28) + 2(126) + 2(344) = 1000$$

G8.

$$\int_0^2 x^3 dx$$

$$f(x)=x^3$$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(a + \Delta x i) \Delta x$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(0 + \frac{2i}{n}\right) \left(\frac{2}{n}\right)$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right)$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2i}{n}\right)^3 \left(\frac{2}{n}\right)$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{8i^3}{n^3} \left(\frac{2}{n}\right)$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{16i^3}{n^4}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{16}{n^4} \left[\frac{n(n+1)}{2}\right]^2$$

$$= \lim_{n \rightarrow \infty} \frac{16}{n^4} \left[\frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n \rightarrow \infty} \frac{16}{n^2} \left[\frac{(n^2+2n+1)}{4}\right]$$

$$= \lim_{n \rightarrow \infty} \frac{16}{n^2} \left(\frac{16n^2+32n+16}{4n^2}\right)$$

$$= \lim_{n \rightarrow \infty} \left(4 + \frac{8}{n} + \frac{4}{n^2}\right)$$

$$\text{G9. } \Delta x = \frac{6-2}{4} = \frac{4}{4} = 1$$

$$R4 = \Delta x [f(3) + f(4) + f(5) + f(6)]$$

$$= 1[\ln 3 + \ln 4 + \ln 5 + \ln 6]$$

$$= \ln 3 + \ln 4 + \ln 5 + \ln 6$$

H. Indefinite Integrals or Antiderivatives

Example 1. Find the indefinite integral.

$$\begin{aligned} \text{a) } \int x^{12} dx \\ = \frac{x^{13}}{13} + c \end{aligned}$$

$$\begin{aligned} \text{b) } \int \frac{1}{2} \sqrt{x^3} dx \\ = \frac{1}{2} \int x^{\frac{3}{2}} dx \\ = \frac{1}{2} \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c \\ = \frac{1}{5} x^{\frac{5}{2}} + c \end{aligned}$$

$$\begin{aligned} \text{c) } \int \frac{1}{4\sqrt{x}} dx \\ = \frac{1}{4} \int x^{-\frac{1}{2}} dx \\ = \frac{1}{4} \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ = \frac{1}{2} \sqrt{x} + c \end{aligned}$$

$$\begin{aligned} \text{d) } \int 6e^{2x} dx \\ = \frac{6e^{2x}}{2} + c \\ = 3e^{2x} + c \end{aligned}$$

Example 2. Integrate each of the following:

$$\begin{aligned} \text{a) } \int 2\sin 3x dx \\ = -\frac{2 \cos 3x}{3} + c \\ = -\frac{2}{3} \cos 3x + c \end{aligned}$$

$$\begin{aligned} \text{b) } \int \frac{\sec^2 4x}{2} dx \\ = \frac{1}{2} \int \sec^2 4x \\ = \frac{1}{2} \frac{\tan 4x}{4} + c \\ = \frac{\tan 4x}{8} + c \end{aligned}$$

$$\text{c) } \int \frac{1}{5x} dx$$

$$= \frac{1}{5} \int \frac{1}{x} dx$$

$$= \frac{1}{5} \ln|x| + c$$

$$\text{d) } \int e^{7x} dx$$

$$= \frac{e^{7x}}{7} + c$$

$$\text{e) } \int \frac{x^5}{3} dx = \frac{1}{3} \int x^5 dx = \frac{1}{3} \frac{x^6}{6} + c = \frac{x^6}{18} + c$$

$$\text{f) } \int (2x - 1)(3x + 2) dx$$

$$= \int (6x^2 + x - 2) dx = \frac{6x^3}{3} + \frac{x^2}{2} - 2x + c = 2x^3 + \frac{x^2}{2} - 2x + c$$

$$\text{g) } \int \frac{x^7 + 4x^6}{2x^2} dx = \int \left(\frac{1}{2}x^5 + 2x^4 \right) dx = \frac{\frac{1}{2}x^6}{6} + \frac{2x^5}{5} + c = \frac{1}{12}x^6 + \frac{2}{5}x^5 + c$$

Practice Exam Questions on Indefinite Integrals or Antiderivatives

$$\text{H1. } \int \frac{1}{2x^4} dx = \frac{1}{2} \int x^{-4} dx$$

$$= \frac{1}{2} \frac{x^{-3}}{-3} + c$$

$$= -\frac{1}{6x^3} + c$$

$$\text{H2. } \int 2e^{7x} dx$$

$$= 2 \frac{e^{7x}}{7} + c$$

Integrate each of the following:

$$\text{H3. } \int \sin 3x dx$$

$$= \frac{-\cos 3x}{3} + c$$

$$\text{H4. } \int \frac{\sec^2 2x}{2} dx$$

$$= \frac{1}{2} \frac{\tan 2x}{2} + c$$

$$= \frac{1}{4} \tan 2x + c$$

$$\text{H5. } \int \frac{1}{8x} dx$$

$$= \frac{1}{8} \ln|x| + c$$

$$\text{H6. } \int e^{6x} dx$$

$$= \frac{e^{6x}}{6} + c$$

$$\text{H7. } \int \frac{x^5}{3} dx$$

$$= \frac{1}{3} \frac{x^6}{6} + c \text{ or } \frac{x^6}{18} + c$$

$$\text{H8. Integrate: } \int \sec x \tan x dx = \sec x + c$$

$$\text{H9. Integrate: } \int -5 \sin x dx = 5 \cos x + c$$

$$\text{H10. Integrate: } \int \left(2 \sec^2 x + \frac{3x^6 + 2\sqrt{x}}{x} \right) dx$$

$$= 2\tan x + \int \frac{3x^6}{x} dx + \int 2 \frac{\sqrt{x}}{x} dx$$

$$= 2\tan x + 3 \int x^5 dx + 2 \int x^{-\frac{1}{2}} dx$$

$$= 2\tan x + \frac{3x^6}{6} + \frac{2x^{\frac{1}{2}}}{1/2} + c$$

$$= 2\tan x + \frac{x^6}{2} + 4\sqrt{x} + c$$

H11. Integrate: $\int \frac{1}{5} \sec^2 x dx = \frac{1}{5} \tan x + c$

H12. Integrate: $\int \frac{1}{2x} dx = \frac{1}{2} \ln|x| + c$

H13. Integrate: $\int \frac{1}{2} \sec 4x \tan 4x dx = \frac{1}{2} \frac{\sec 4x}{4} + c = \frac{1}{8} \sec 4x + c$

H14. Integrate: $\int \cos 8x dx = \frac{\sin 8x}{8} + c$

H15. $= 6 \tan^{-1} x + c$

H16. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$
 $2 \int \frac{1}{\sqrt{4^2-x^2}} dx = 2 \sin^{-1} \left(\frac{x}{4} \right) + c$

H17. "a"=5 $\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x + c$
 $\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + c$
 $= \sin^{-1} \left(\frac{x}{5} \right) + c$

I. Definite Integrals and The Fundamental Theorem of Calculus

Example 4. a) $\int_1^2 x^3 dx$

$$= \left[\frac{x^4}{4} \right]_1^2 = \frac{2^4}{4} - \frac{1^4}{4} = \frac{16-1}{4} = \frac{15}{4}$$

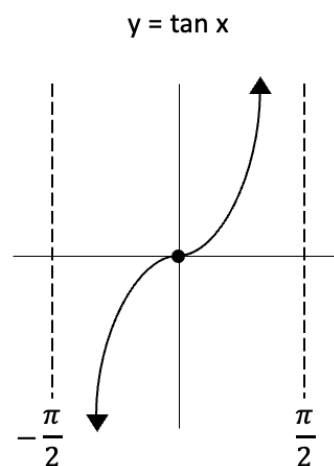
b) $\int_0^1 3e^x dx = 3[e^x]_0^1 = 3[e^1 - e^0] = 3(e - 1)$

c) $\int_0^1 (x^3 - 4x) dx$

$$\begin{aligned} \left[\frac{x^4}{4} - \frac{4x^2}{2} \right]_0^1 &= \left[\frac{x^4}{4} - 2x^2 \right]_0^1 \\ &= \left(\frac{1}{4} - 2 \right) - (0 - 0) = \frac{1}{4} - \frac{8}{4} = -\frac{7}{4} \end{aligned}$$

d)

$$\begin{aligned} \int_0^1 \frac{1}{1+x^2} dx &= [\tan^{-1} x]_0^1 \\ &= \tan^{-1} 1 - \tan^{-1} 0 \\ &= \frac{\pi}{4} - 0 = \frac{\pi}{4} \end{aligned}$$



To solve $\tan^{-1} 1$:

Ask yourself $\tan ? = 1 ? = \frac{\pi}{4}$ (special triangle's)

To solve $\tan^{-1} 0$:

Ask yourself $\tan ? = 0 ? = 0$ (special triangle's)

Recall, $\arctan x$ is defined in $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$



Example 5. Evaluate each of the following:

$$a) \frac{d}{dx} \left(\int_0^{2/\pi} 3\cos^2 x dx \right) = 0 \text{ since the derivative of a constant is } 0$$

$$b) \frac{d}{dx} \int_2^x \sqrt{t^2 + 4} dt = \sqrt{x^2 + 4}$$

$$c) \frac{d}{dx} \int_x^5 \sin t dt = -\frac{d}{dx} \int_5^x \sin t dt = -\sin x$$

Example 6.

$$a) = (3x^2) \tan(x^3) - \tan(x)(1) \\ = 3x^2 \tan(x^3) - \tan x$$

$$b) = e^{x^2} (2x) - e^{3x} (3) \\ = 2xe^{x^2} - 3e^{3x}$$

Example 7.

$$f'(x) = \int (1 + \cos x) dx = x + \sin x + c \text{ substitute } f'(0) = 5 \text{ and get}$$

$$5 = 0 + \sin 0 + c$$

$$c=5$$

$$f'(x) = x + \sin x + 5$$

$$f(x) = \int (x + \sin x + 5) dx = \frac{x^2}{2} - \cos x + 5x + k$$

$$\text{Substitute } f(0)=0$$

$$0=0 - \cos 0 + 0 + k$$

$$k=1$$

$$f(x) = \frac{x^2}{2} - \cos x + 5x + 1$$

Example 8.

$$a) 0$$

$$b) [e^x]_2^4 = e^4 - e^2$$

$$c) [\sin x]_{\pi/2}^{\pi} = \sin \pi - \sin \left(\frac{\pi}{2} \right) = 0 - 1 = -1$$

Example 9.

$$\cos x > 0 \text{ for } 0 < x < \pi$$

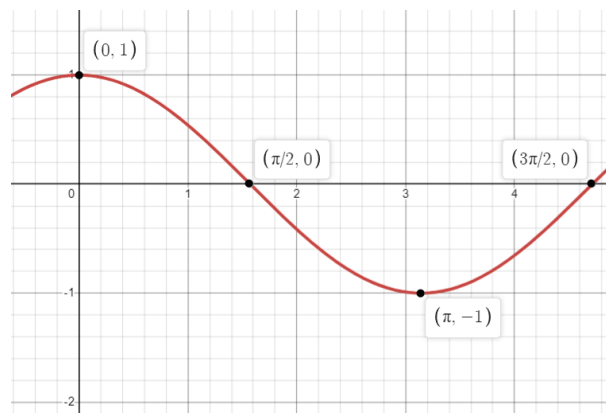
$$\cos x < 0 \text{ for } \pi < x < \frac{3\pi}{2}$$

$$|\cos x| = \cos x \quad 0 < x < \pi$$

$$|\cos x| = -\cos x \quad \pi < x < \frac{3\pi}{2}$$

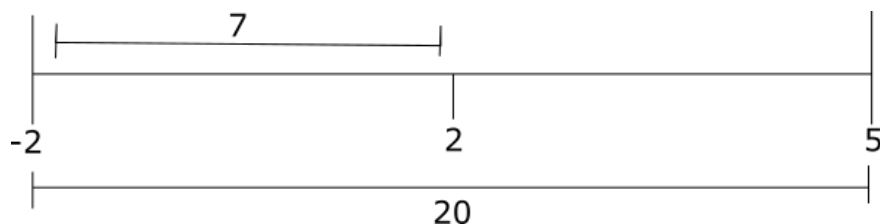
$$= \int_0^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\cos x dx = [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \left(\sin \frac{\pi}{2} - \sin 0\right) - \left(\sin \frac{3\pi}{2} - \sin \frac{\pi}{2}\right) = (1 - 0) - (-1 - 1) = 3$$

**Example 10.**

$$\int_{-2}^2 3f(x)dx = 21, \quad \int_{-2}^5 f(x)dx = 20, \quad \text{find } \int_2^5 f(x)dx$$

$$\int_{-2}^2 f(x)dx = \frac{21}{3} = 7$$



$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

$$\therefore \int_{-2}^5 f(x)dx = \int_{-2}^2 f(x)dx + \int_2^5 f(x)dx$$

$$20 = 7 + \int_2^5 f(x)dx$$

$$\int_2^5 f(x)dx = 20 - 7 = 13$$

Example 11.

$$= \lim_{x \rightarrow 0} \frac{\int_0^x e^{t^2} dt}{3x} \quad \left(\frac{0}{0}\right)$$

$$H = \lim_{x \rightarrow 0} \frac{e^{x^2}}{3} = \frac{1}{3}$$

Example 12.

$$\text{a) } F'(x) = \frac{d}{dx} \int_0^x (5t^5 + t^4 - 5t^3 - t^2 - 3t - 5) dt$$

$$F'(x) = 5x^5 + x^4 - 5x^3 - x^2 - 3x - 5$$

$$F''(x) = 25x^4 + 4x^3 - 15x^2 - 2x - 3$$

$$\text{b) } F'(x) \frac{d}{dx} \int_{x^4}^{x^5} (-4t - 6)^2 dt$$

$$= (-4x^5 - 6)^2 (5x^4) - (-4x^4 - 6)^2 (4x^3)$$

Example 13.

$$\text{Net Area} = \int_{-4}^4 |2x| dx = A1 - A2$$

$$= \int_{-4}^4 2x dx$$

$$= [x^2]_{-4}^4$$

$$= 16 - (-4)^2$$

$$= 16 - 16 = 0 \quad \text{OR}$$

$$A1 = \frac{bh}{2} = \frac{4(8)}{2} = 16 \quad A2 = \frac{bh}{2} = \frac{4(8)}{2} = 16$$

$$\therefore \text{Net area} = A1 - A2 = 16 - 16 = 0$$

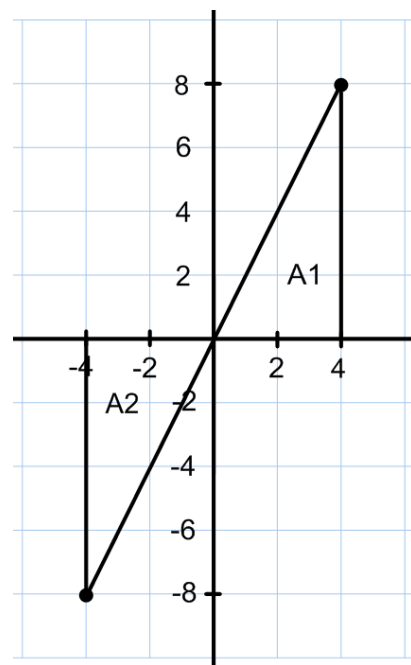
$$\text{Total Area} = \int_{-4}^0 -2x dx + \int_0^4 2x dx$$

$$A1 + A2 = 16 + 16 = 32 \quad \text{or DO}$$

$$2 \int_0^4 2x dx = 2[x^2]_0^4$$

$$= 2[4^2 - 0]$$

$$= 32$$



Example 14.

$$f(x) = 2x - x^2$$

$$g(x) = x^2$$

$$f(x) \geq g(x) \text{ for } 0 \leq x \leq 1$$

$$\therefore \int_0^1 (2x - x^2) dx \geq \int_0^1 x^2 dx$$

Practice Exam Questions on Definite Integrals and The Fundamental Theorem of Calculus

$$11. = \sqrt{x^8 + 1}(4x^3) - \sqrt{\cos^2 x + 1}(-\sin x)$$

$$12. = [(x^2)^2 + x^2](2x) - (x^2 + x)(1) \\ = 2x(x^4 + x^2) - (x^2 + x) \text{ or } 2x^5 + 2x^3 - x^2 - x$$

$$13. = \frac{d}{dx}(\text{constant}) = 0$$

$$14. \text{ FTC } = \arccos x$$

$$15. \text{ FTC } = 7 \tan^{-1} \sqrt{7x} - 2 \tan^{-1} \sqrt{2x}$$

$$16. \cos x > 0 \text{ for } 0 < x < \pi$$

$$\cos x < 0 \text{ for } \pi < x < \frac{3\pi}{2}$$

$$|\cos x| = \cos x \quad 0 < x < \pi$$

$$|\cos x| = -\cos x \quad \pi < x < \frac{3\pi}{2}$$

$$= \int_0^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\cos x dx = [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \left(\sin \frac{\pi}{2} - \sin 0\right) - \left(\sin \frac{3\pi}{2} - \sin \frac{\pi}{2}\right) = (1 - 0) - (-1 - 1) = 3$$

$$17. = \left[\frac{3^x}{\ln 3 \ln 3} \right]_0^1 = \left[\frac{3^x}{(\ln 3)^2} \right]_0^1 = \frac{3^1}{(\ln 3)^2} - \frac{3^0}{(\ln 3)^2} = \frac{2}{(\ln 3)^2}$$

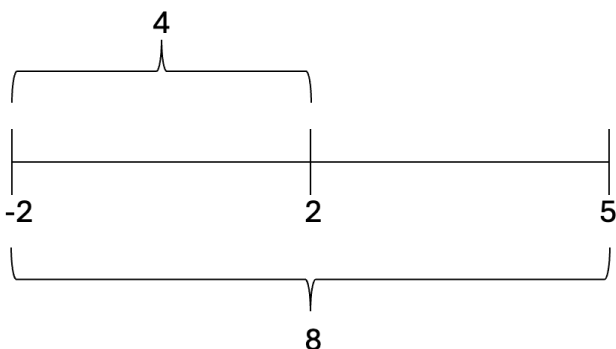
$$18. \int_{-2}^2 f(x) dx = \frac{12}{3} = 4$$

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$$

$$\int_{-2}^5 f(x) dx = \int_{-2}^2 f(x) dx + \int_2^5 f(x) dx$$

$$\therefore \int_2^5 f(x) dx = \int_{-2}^5 f(x) dx - \int_{-2}^2 f(x) dx$$

$$= 8 - 4 = 4$$



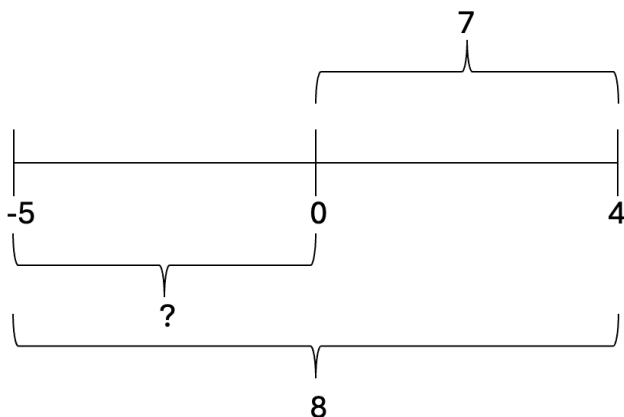
$$19. [4\tan^{-1}(x)]_0^1 = 4\tan^{-1}(1) - 4\tan^{-1}(0) = 4\left(\frac{\pi}{4}\right) - 0 = \pi$$

$$110. \text{ Evaluate } \int_0^1 10^x dx = \left[\frac{10^x}{\ln 10}\right]_0^1 = \frac{10^1}{\ln 10} - \frac{10^0}{\ln 10} = \frac{9}{\ln 10}$$

$$111. \text{ Integrate } \int_0^\pi 3\cos x dx = [3\sin x]_0^\pi = 3\sin\pi - 3\sin 0 = 0 - 0 = 0$$

$$112. \text{ Suppose } f \text{ is continuous on the interval } [-5, 4], \int_0^4 f(x) dx = 7 \text{ and } \int_{-5}^4 2f(x) dx = 16. \text{ Find } \int_{-5}^0 3f(x) dx.$$

$$\int_{-5}^4 f(x) dx = \frac{16}{2} = 8$$



$$\therefore 3 \int_{-5}^0 f(x) dx$$

$$= 3 \left[\int_{-5}^4 f(x) dx - \int_0^4 f(x) dx \right]$$

$$= 3 [8 - 7] = 3$$

113. Find $f' \left(\frac{\pi}{2} \right)$ where $f(t) = \int_{-\pi}^{\pi} \sqrt{\sin t} dt$

$$f'(t) = \frac{d}{dt} \left(\int_{-\pi}^{\pi} \sqrt{\sin t} dt \right) = 0 \text{ by FTC}$$

↓ a constant

The answer is C).

114. Evaluate $\int_1^3 \frac{1}{1+x} dx$

$$[\ln(1+x)]_1^3 = \ln(1+3) - \ln(1+1)$$

$$= \ln 4 - \ln 2 = \ln\left(\frac{4}{2}\right) = \ln 2$$

The answer is C).

115. Evaluate $\int_{-2}^1 |x| dx$

$$|x| = x \text{ if } x > 0$$

$$= -x \text{ if } x < 0$$

$$\begin{aligned} \int_{-2}^1 |x| dx &= \int_{-2}^0 -x dx + \int_0^1 x dx \\ &= \left[\frac{-x^2}{2} \right]_{-2}^0 + \left[\frac{x^2}{2} \right]_0^1 \\ &= \left(0 + \frac{(-2)^2}{2} \right) + \left(\frac{1}{2} - 0 \right) = 2 + \frac{1}{2} = \frac{5}{2} \end{aligned}$$

The answer is B)

116. Integrate $\int_0^{\pi/6} \sec x \tan x dx = [\sec x]_0^{\pi/6} = \sec\left(\frac{\pi}{6}\right) - \sec 0$

$$= \frac{2}{\sqrt{3}} - 1$$

Using Special Triangles

$$\text{Recall, } \sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

117. $|x - 5| = x - 5$ if $x - 5 \geq 0$

$$= -(x - 5) \text{ if } x < 5$$

$$= \int_0^5 -(x - 5) dx + \int_5^6 (x - 5) dx$$

$$= \left[\frac{-x^2}{2} + 5x \right]_0^5 + \left[\frac{x^2}{2} - 5x \right]_5^6$$

$$= \left(-\frac{5^2}{2} + 5(5) \right) - (0) + \left(\frac{6^2}{2} - 5(6) \right) - \left(\frac{5^2}{2} - 5(5) \right)$$

$$= -\frac{25}{2} + 25 + 18 - 30 - \frac{25}{2} + 25$$

$$= 13$$

$$118. \text{ Integrate } \int_1^2 x^2 dx = \left[\frac{x^3}{3} \right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$$119. \text{ Integrate } \int_0^{\pi/4} \sin 8x dx = \left[\frac{-\cos 8x}{8} \right]_0^{\pi/4} = -\frac{\cos 2\pi}{8} + \frac{\cos 0}{8} = -\frac{1}{8} + \frac{1}{8} = 0$$

$$120. \text{ Integrate } \int x^2 \left(x + \frac{1}{x^3} \right) dx = \int \left(x^3 + \frac{1}{x} \right) dx = \frac{x^4}{4} + \ln|x| + c$$

$$121. \text{ If } F(x) = \int_x^5 \sin t dt = -\int_5^x \sin t dt, \quad \text{find } F' \left(-\frac{\pi}{4} \right)$$

$$F'(x) = -\sin x$$

$$F' \left(-\frac{\pi}{4} \right) = -\sin \left(-\frac{\pi}{4} \right) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$122. \frac{d}{dx} \int_7^x \tan \sqrt{2t} dt = \tan \sqrt{2x}$$

$$123. \text{ Integrate } \int_0^{\frac{3\pi}{2}} |\sin x| dx$$

$$|\sin x| = \sin x \quad \text{if } 0 < x < \pi$$

$$= -\sin x \quad \text{if } \pi < x < \frac{3\pi}{2}$$

$$= \int_0^{\pi} \sin x dx + \int_{\pi}^{\frac{3\pi}{2}} -\sin x dx = [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{\frac{3\pi}{2}}$$

$$= (-\cos \pi + \cos 0) + (\cos \frac{3\pi}{2} - \cos \pi)$$

$$= (-(-1) + 1) + (0 - (-1)) = 3$$

124. If $f'(x) = 3e^x + 2$, and $f(0) = 6$, find $f(2)$.

$$f(x) = \int (3e^x + 2) dx$$

$$f(x) = 3e^x + 2x + c \text{ substitute } x=0 \text{ and } f(x)=6$$

$$6 = 3e^0 + 2(0) + c$$

$$c = 6 - 3 = 3$$

$$f(x) = 3e^x + 2x + 3$$

$$f(2) = 3e^2 + 2(2) + 3 = 3e^2 + 7$$

125. $F(x) = \int (3x^2 - 6x + 1) dx = \frac{3x^3}{3} - \frac{6x^2}{2} + x + c$

$$F(x) = x^3 - 3x^2 + x + c$$

$$-4 = c$$

$$F(x) = x^3 - 3x^2 + x - 4$$

$$F(1) = 1 - 3 + 1 - 4 = -5$$

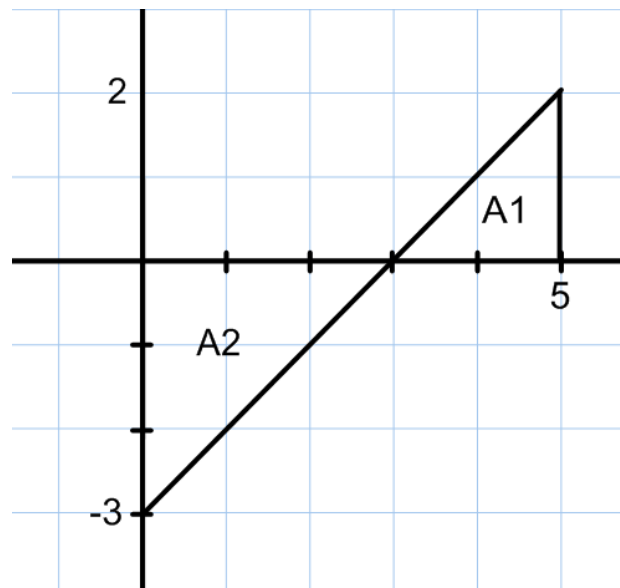
$$126. \text{ Total Area} = \int_0^3 -(x-3)dx + \int_3^5 (x-3)dx$$

$$\text{OR DO } A = \frac{bh}{2}$$

$$\begin{aligned} \text{Total Area} = A_1 + A_2 &= \frac{3(3)}{2} + \frac{2(2)}{2} \\ &= \frac{9}{2} + \frac{4}{2} \\ &= \frac{13}{2} \end{aligned}$$

$$\text{Net Signed Area} = A_1 - A_2$$

$$\begin{aligned} &= \frac{2(2)}{2} - \frac{3(3)}{2} \\ &= \frac{4}{2} - \frac{9}{2} \\ &= -\frac{5}{2} \end{aligned}$$



$$127. \text{ Net Area} = \int_{-2}^2 3xdx$$

$$\begin{aligned} &= \left[\frac{3x^2}{2} \right]_{-2}^2 \\ &= \frac{3}{2}(2^2) - \frac{3}{2}(-2)^2 \\ &= 6 - 6 = 0 \end{aligned}$$

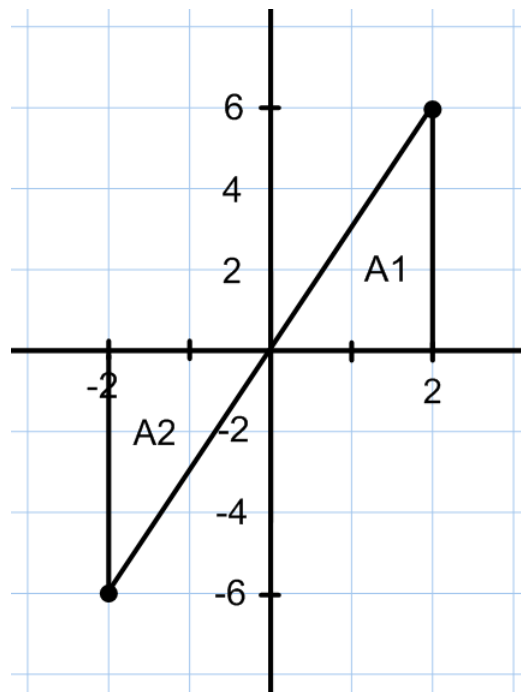
$$\text{Total Area} = 2 \int_0^2 3xdx \quad \text{or} \quad \int_{-2}^0 (-3x)dx + \int_0^2 3xdx$$

$$\begin{aligned} &= 2 \left[\frac{3x^2}{2} \right]_0^2 \\ &= 2 \left[\frac{3}{2}(2)^2 - 0 \right] = 12 \end{aligned}$$

$$t^2 + 3t - 4 = 0$$

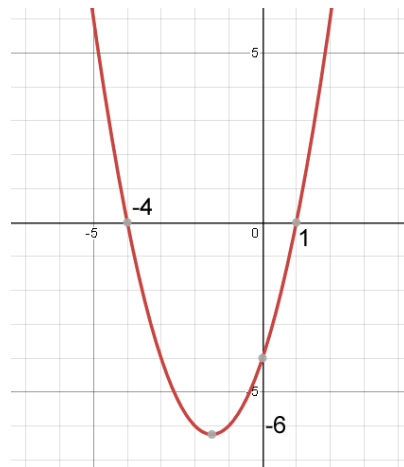
$$(t+4)(t-1) = 0$$

$$t = 1, -4$$



128.

$$\begin{aligned}
 \text{a) } s(2) - s(0) &= \int_0^2 (t^2 + 3t - 4) dt \\
 &= \left[\frac{t^3}{3} + \frac{3t^2}{2} - 4t \right]_0^2 \\
 &= \frac{2^3}{3} + \frac{3(2)^2}{2} - 4(2) - 0 \\
 &= \frac{8}{3} + 6 - 8 \\
 &= \frac{8}{3} - \frac{6}{3} \\
 &= \frac{2}{3}
 \end{aligned}$$



$$\begin{aligned}
 \text{b) } \int_0^2 |v(t)| dt &= \int_0^1 -v(t) dt + \int_1^2 v(t) dt \\
 &= \int_0^1 (-t^2 - 3t + 4) dt + \int_1^2 (t^2 + 3t - 4) dt \\
 &= \left[-\frac{t^3}{3} - \frac{3t^2}{2} + 4t \right]_0^1 + \left[\frac{t^3}{3} + \frac{3t^2}{2} - 4t \right]_1^2 \\
 &= \left(-\frac{1}{3} - \frac{3}{2} + 4 \right) - 0 + \left[\left(\frac{2^3}{3} + \frac{3(2)^2}{2} - 4(2) \right) - \left(\frac{1^3}{3} + \frac{3(1)^2}{2} - 4(1) \right) \right] \\
 &= -\frac{1}{3} - \frac{3}{2} + 4 + \frac{8}{3} + 6 - 8 - \frac{1}{3} - \frac{3}{2} + 4 \\
 &= -\frac{1}{3} + \frac{8}{3} - \frac{1}{3} - \frac{3}{2} - \frac{3}{2} + 4 + 6 - 8 + 4 \\
 &= \frac{6}{3} - \frac{6}{2} + 6 \\
 &= 2 - 3 + 6 = 5
 \end{aligned}$$

129. If $f''(x) = \frac{2}{x} + 3$ where $f'(1) = 2 + f$, find $f'(3)$

$$f'(x) = \int f''(x) dx = \int \left(\frac{2}{x} + 3 \right) dx$$

$$\therefore f'(x) = 2 \ln|x| + 3x + c$$

Substitution (1,2)

$$x, f'(x)$$

$$2 = 2 \ln 1 + 3(1) + c$$

$$2 = 3 + c$$

$$c = -1$$

$$\therefore f'(x) = 2 \ln|x| + 3x - 1$$

$$f'(3) = 2 \ln|3| + 3(3) - 1$$

$$= 2 \ln 3 + 8$$

130. If $f''(x) = 2e^{2x} + 5 \sin x + 1$, $f'(0) = 1$, $f(0) = 2$, find $f(x)$

$$f'(x) = \int f''(x) dx = \int (2e^{2x} + 5 \sin x + 1) dx$$

$$\therefore f'(x) = \frac{2e^{2x}}{2} - 5 \cos x + x + c = e^{2x} - 5 \cos x + x + c$$

Substitution $f'(0) = 1$

$$\therefore 1 = e^0 - 5 \cos 0 + 0 + c$$

$$1 = 1 - 5 + c$$

$$c = 5$$

$$\therefore f'(x) = e^{2x} - 5 \cos x + x + 5$$

$$f(x) = \int f'(x) dx = \int (e^{2x} - 5 \cos x + x + 5) dx$$

$$f(x) = \frac{e^{2x}}{2} - 5 \sin x + \frac{x^2}{2} + 5x + k$$

Substitution $f(0) = 2$

$$2 = \frac{e^0}{2} - 5 \sin 0 + \frac{0^2}{2} + 5(0) + k$$

$$2 = \frac{1}{2} - 0 + 0 + 0 + k$$

$$k = 2 - \frac{1}{2} = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

$$\therefore f(x) = \frac{e^{2x}}{2} - 5 \sin x + \frac{x^2}{2} + 5x + \frac{3}{2}$$

131. a)

$$\int_{-1}^0 \frac{1}{\sqrt{1-x^2}} dx = [\sin^{-1} x]_{-1}^0$$

$$= \sin^{-1} 0 - \sin^{-1}(-1)$$

$$= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Recall, arcsin x is defined in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

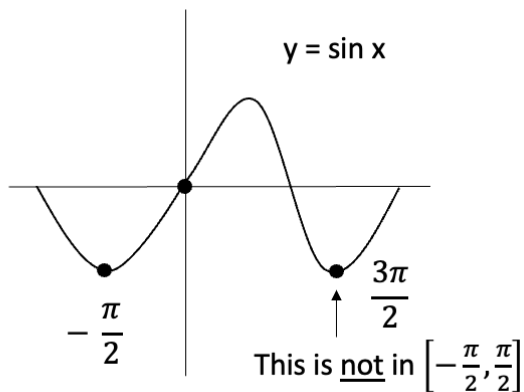
$$\frac{A}{c}$$

To solve $\sin^{-1}(-1)$

Ask yourself $\sin ? = -1$

$$? = -\frac{\pi}{2}$$

$\sin^{-1}(0) = 0$ since $\sin 0 = 0$



$$\begin{aligned} \text{b) } \int_0^1 \frac{-1}{\sqrt{1-x^2}} dx &= [\cos^{-1} x]_0^1 \\ &= \cos^{-1} 1 - \cos^{-1} 0 = 0 - \frac{\pi}{2} = -\frac{\pi}{2} \end{aligned}$$

To solve $\cos^{-1} 1$

Ask yourself $\cos ? = 1$

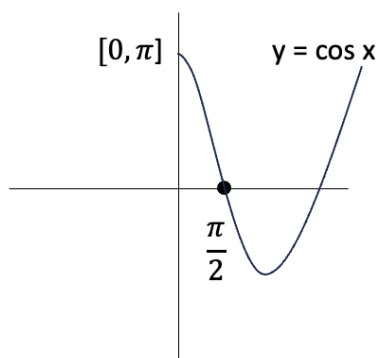
$$? = 1$$

To solve $\cos^{-1} 0$

Ask yourself $\cos ? = 0$

$$? = \frac{\pi}{2}$$

Recall, we draw $\arccos x$ in A,S



132. If $f'(x) = 3x^2 + 2$ and $f(0) = 3$, then $f(1) =$ _____.

$$f(x) = \int (3x^2 + 2) dx$$

$$f(x) = x^3 + 2x + c \quad \text{sub } x = 0 \quad f(0) = 3$$

$$3 = 0 + 0 + c$$

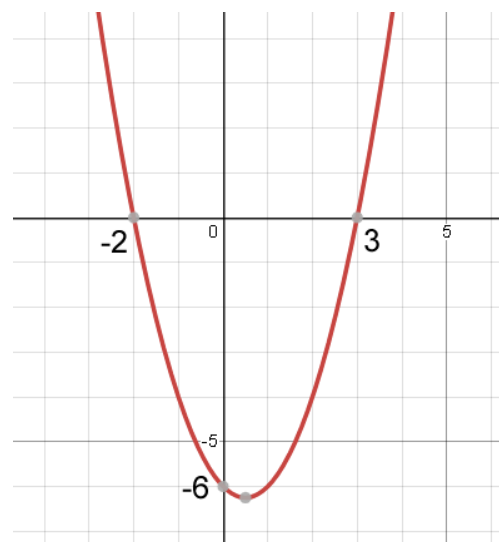
$$\therefore c = 3$$

$$f(x) = x^3 + 2x + 3 \quad \therefore f(1) = 1^3 + 2(1) + 3 = 6$$

133. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ in metres per second.

a) Find the net displacement of the particle from 0 to 4 seconds

$$\begin{aligned} s(4) - s(0) &= \int_0^4 (t^2 - t - 6) dt \\ &= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_0^4 \\ &= \frac{4^3}{3} - \frac{4^2}{2} - 6(4) - 0 \\ &= \frac{64}{3} - 8 - 24 \\ &= \frac{64}{3} - \frac{32}{3} \\ &= \frac{64}{3} - \frac{96}{3} \\ &= -\frac{32}{3} \end{aligned}$$



b) Find the distance traveled during 0 to 4 seconds

$$t^2 - t - 6 = 0$$

$$(t - 3)(t + 2) = 0$$

$$t = 3, -2$$

$$\begin{aligned}\int_0^4 |v(t)| dt &= \int_0^3 -v(t) + \int_3^4 v(t) dt \\ &= \int_0^3 (-t^2 + t + 6) dt + \int_3^4 (t^2 - t - 6) dt \\ &= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t \right]_0^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_3^4 \\ &= \left[\frac{-3^3}{3} + \frac{3^2}{2} + 6(3) - 0 \right] + \left[\left(\frac{4^3}{3} - \frac{4^2}{2} - 6(4) \right) - \left(\frac{3^3}{3} - \frac{3^2}{2} - 6(3) \right) \right] \\ &= \frac{-27}{3} + \frac{9}{2} + 18 + \frac{64}{3} - \frac{16}{2} - 24 - \frac{27}{3} + \frac{9}{2} + 18 \\ &= 16\frac{1}{3}\end{aligned}$$

J. Substitution

Example 1. $\int \frac{(3x^2+1)}{(x^3+x)^4} dx$ *subst let* $u = x^3 + x$ $du = (3x^2 + 1)dx$

$$= \int \frac{1}{u^4} du = \int u^{-4} du = \frac{u^{-3}}{-3} + c = \frac{(x^3+x)^{-3}}{-3} + c = \frac{1}{-3(x^3+x)^3} + c$$

b) $\int (40x - 16)(5x^2 - 4x)^6 dx$

$$= 4 \int u^6 du \quad \text{subst let } u = 5x^2 - 4x$$

$$= \frac{4u^7}{7} + c \quad du = (10x - 4)dx$$

$$= \frac{4(5x^2-4x)^7}{7} + c \quad 4du = (40x - 16)dx$$

c) $\int \frac{dx}{(2x+3)^8}$ *subst* $u = 2x + 3$ $du = 2dx$ $\frac{du}{2} = dx$

$$= \int (2x + 3)^{-8} dx$$

$$= \frac{1}{2} \int u^{-8} du$$

$$= \frac{1}{2} \left(\frac{u^{-7}}{-7} \right) + c$$

$$= \frac{-1}{14u^7} + c = \frac{-1}{14(2x+3)^7} + c$$

$$\text{d) } \int \frac{-2}{x(1+\ln x)^2} dx \quad \text{subst } u = 1 + \ln x \quad du = \frac{1}{x} dx$$

$$= -2 \int \frac{1}{u^2} du$$

$$= -2 \int u^{-2} du$$

$$= -2 \left[\frac{u^{-1}}{-1} \right] + c$$

$$= \left[\frac{2}{u} \right] + c$$

$$= \frac{2}{1+\ln x} + c$$

$$\text{e) } \int 4xe^{x^2} dx \quad \text{subst } u = x^2 \quad du = 2x dx \quad 2du=4x dx$$

$$= 2 \int e^u du$$

$$= 2e^u + c$$

$$= 2e^{x^2} + c$$

Example 2. Integrate $\int \frac{e^x}{\sqrt{e^x+4}} dx$

$$u = e^x + 4 \quad du = e^x dx$$

$$= \int \frac{1}{\sqrt{u}} du$$

$$= \int u^{-\frac{1}{2}} du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{u} + c = 2\sqrt{e^x + 4} + c$$

Example 3. Integrate $\int 2\sin x(1 + \cos x)^5 dx$

$$u = 1 + \cos x$$

$$du = -\sin x dx$$

$$-du = \sin x dx$$

$$= -2 \int u^5 du$$

$$= -2 \left[\frac{u^6}{6} \right] + c$$

$$= \left[\frac{-1}{3} u^6 \right] + c = \frac{-1}{3} (1 + \cos x)^6 + c$$

Example 4. Integrate $\int 2\tan^6 x \sec^2 x dx$

$$u = \tan x \quad du = \sec^2 x dx$$

$$= 2 \int u^6 du$$

$$= \frac{2u^7}{7} + c$$

$$= \frac{2}{7} \tan^7 x + c$$

Example 5. Integrate $\int \frac{8}{(\arctan 2x)(1+4x^2)} dx$

$$u = \arctan 2x \quad du = \frac{1}{1+(2x)^2} (2) dx = \frac{2}{1+4x^2} dx$$

$$4du = \frac{1}{1+(2x)^2} (2) dx = \frac{8}{1+4x^2} dx$$

$$= 4 \int \frac{1}{u} du = [4\ln|u|] + c = 4\ln |\arctan 2x| + c$$

Example 6. $\int \sin^3 x \cos^3 x dx$ Let $u = \sin x$ and $du = \cos x dx$

$$= \int \sin^3 x \cos^3 x dx = \int \sin^3 x \cos^2 x \cos x dx \text{ use the identity } \sin^2 x + \cos^2 x = 1$$

$$= \int \sin^3 x (1 - \sin^2 x) \cos x dx = \int u^3 (1 - u^2) du = \int (u^3 - u^5) du$$

$$= \frac{u^4}{4} - \frac{u^6}{6} + c = \frac{(\sin x)^4}{4} - \frac{(\sin x)^6}{6} + c = \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + c$$

Example 7. Evaluate: $\int_1^2 \frac{x}{1+x^2} dx$

$$u = 1 + x^2 \quad du = 2x dx$$

$$\frac{du}{2} = x dx$$

$$x = 1 \quad u = 1 + 1^2 = 2$$

$$x = 2 \quad u = 1 + 2^2 = 5$$

$$= \frac{1}{2} \int_2^5 \frac{1}{u} du$$

$$= \frac{1}{2} [\ln|u|]_2^5$$

$$= \frac{1}{2} [\ln 5 - \ln 2]$$

$$= \frac{1}{2} \ln\left(\frac{5}{2}\right)$$

Example 8. Evaluate: $\int_e^{e^3} \frac{\ln x}{x} dx$

$$\text{subst let } u = \ln x \quad du = \frac{1}{x} dx$$

$$x = e \quad u = \ln e = 1$$

$$x = e^3 \quad u = 3$$

$$= \int_1^3 u du$$

$$= \left[\frac{u^2}{2} \right]_1^3 = \frac{3^2}{2} - \frac{1^2}{2} = \frac{9}{2} - \frac{1}{2} = 4$$

Example 9. Evaluate: $\int_0^{\ln 2} \frac{e^x}{e^x+1} dx$

$$\text{subst } u = e^x + 1$$

$$du = e^x dx$$

$$x = 0 \quad u = e^0 + 1 = 1 + 1 = 2$$

$$x = \ln 2 \quad u = e^{\ln 2} + 1 = 2 + 1 = 3$$

$$= \int_2^3 \frac{1}{u} du$$

$$= [\ln u]_2^3 = \ln 3 - \ln 2 = \ln\left(\frac{3}{2}\right)$$

Example 10.

$$\int_0^1 2x(x-1)^4 dx$$

$$u = x - 1 \quad x = 0$$

$$x = u + 1 \quad u = 0 - 1 = -1$$

$$x = 1$$

$$u = 1 - 1 = 0$$

$$2 \int_{-1}^0 (u+1)u^4 du$$

$$= 2 \int_{-1}^0 (u^5 + u^4) du$$

$$= 2 \left[\frac{u^6}{6} + \frac{u^5}{5} \right]_{-1}^0$$

$$\begin{aligned}
&= 2 \left(0 - \left[\frac{(-1)^6}{6} + \frac{(-1)^5}{5} \right] \right) \\
&= 2 \left(- \left[\frac{1}{6} - \frac{5}{5} \right] \right) \\
&= -2 \left(\frac{5}{30} - \frac{6}{30} \right) \\
&= -2 \left(-\frac{1}{30} \right) \\
&= \frac{2}{30} \text{ or } \frac{1}{15}
\end{aligned}$$

Practice Exam Questions on Substitution

J1. $u = \arctan x \quad du = \frac{1}{1+x^2} dx$

$$= \int u^4 du = \frac{u^5}{5} + c \quad \text{or} \quad \frac{(\arctan x)^5}{5} + c$$

J2. Integrate: $\int \tan x dx \quad \tan x = \frac{\sin x}{\cos x} \quad \text{subst let } u = \cos x$

$$du = -\sin x dx$$

$$-du = \sin x dx$$

$$= \int \frac{\sin x}{\cos x} dx = - \int \frac{1}{u} du = -\ln|u| + c = -\ln|\cos x| + c$$

J3.

$$= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \quad \text{trick multiply by } \frac{\sec x + \tan x}{\sec x + \tan x}$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \quad \text{let } u = \sec x + \tan x$$

$$du = (\sec x \tan x + \sec^2 x) dx$$

$$= \int \frac{1}{u} du = \ln|u| + c \quad \text{or} \quad \ln|\sec x + \tan x| + c$$

J4. Evaluate $\int_1^e \frac{e^{(\ln x)^2}}{x} dx$

$$u = \ln x \quad du = \frac{1}{x} dx$$

$$x = 1 \quad u = \ln 1 = 0$$

$$x = e \quad u = \ln e = 1$$

$$= \int_0^1 u^2 du = \left[\frac{u^3}{3} \right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

J5. Integrate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$$u = \sqrt{x} \quad du = \frac{1}{2} x^{-\frac{1}{2}} dx \quad 2du = \frac{1}{\sqrt{x}} dx$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^u du = 2e^u + c = 2e^{\sqrt{x}} + c$$

J6. Integrate $\int 6x^2(x^3 + 1)^4 dx$

$$u = x^3 + 1 \quad du = 3x^2 dx \quad 2du = 6x^2 dx$$

$$= 2 \int u^4 du = 2 \frac{u^5}{5} + c = 2 \frac{u^5}{5} + c = 2 \frac{(x^3+1)^5}{5} + c$$

J7. Evaluate $\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x dx$

$$u = \tan x \quad du = \sec^2 x dx$$

$$x = 0 \quad u = \tan 0 = 0$$

$$x = \frac{\pi}{4} \quad u = \tan \frac{\pi}{4} = 1$$

$$= \int_0^1 u^3 du = \left[\frac{u^4}{4} \right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$$

J8. Integrate $\int_0^1 \frac{x}{(x^2+1)^4} dx$

$$u = x^2 + 1 \quad du = 2x dx \quad \frac{du}{2} = x dx$$

$$x = 0, u = 1$$

$$x = 1, u = 2$$

$$\begin{aligned} \int_0^1 \frac{x}{(x^2+1)^4} dx &= \frac{1}{2} \int_1^2 u^{-4} du = \frac{1}{2} \left[\frac{u^{-3}}{-3} \right]_1^2 = \frac{1}{2} \left[-\frac{1}{3u^3} \right]_1^2 \\ &= -\frac{1}{6} \left[\frac{1}{u^3} \right]_1^2 = -\frac{1}{6} \left(\frac{1}{8} - \frac{1}{1} \right) = -\frac{1}{6} \left(-\frac{7}{8} \right) = \frac{7}{48} \end{aligned}$$

J9. Integrate $\int_0^1 \frac{2x^3}{x^4+1} dx$

$$u = x^4 + 1 \quad du = 4x^3 dx \quad \frac{du}{2} = 2x^3 dx$$

$$x = 0, u = 0 + 1 = 1$$

$$x = 1, u = 1 + 1 = 2$$

$$\begin{aligned} \int_0^1 \frac{x^3}{x^4+1} dx &= \frac{1}{2} \int_1^2 \frac{1}{u} du = \frac{1}{2} [\ln|u|]_1^2 = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} (\ln 2 - 0) \\ &= \frac{1}{2} \ln 2 = \ln 2^{1/2} = \ln \sqrt{2} \end{aligned}$$

J10. Integrate $\int_0^2 \frac{x}{x^2+1} dx$

$$u = x^2 + 1 \quad du = 2x dx \quad \therefore \frac{du}{2} = x dx$$

$$x = 0 \quad u = 0^2 + 1 = 1$$

$$x = 2 \quad u = 2^2 + 1 = 5$$

$$\begin{aligned} &= \frac{1}{2} \int_1^5 \frac{du}{u} = \frac{1}{2} \int_1^5 \frac{1}{u} du = \frac{1}{2} [\ln u]_1^5 = \frac{1}{2} (\ln 5 - \ln 1) = \frac{1}{2} (\ln 5 - 0) \\ &= \frac{1}{2} \ln 5 \text{ or } \frac{\ln 5}{2} \end{aligned}$$

J11. $u = 1 + \cos x \quad du = -\sin x dx$

$$x = 0 \quad u = 1 + \cos 0 = 2$$

$$x = \frac{\pi}{3} \quad u = 1 + \cos \frac{\pi}{3} \quad u = 1 + \frac{1}{2} = \frac{3}{2}$$

$$= \int_2^{\frac{3}{2}} \frac{du}{u} = [\ln u]_2^{\frac{3}{2}} = \left[\ln \frac{3}{2} - \ln 2 \right]$$

$$= \ln \left(\frac{3/2}{2} \right) = \ln \left(\frac{3}{4} \right)$$

J12. Integrate $\int_1^2 \frac{3x^2}{(x^3+1)[\ln(x^3+1)]^4} dx$

$$u = \ln(x^3 + 1) \quad du = \frac{1}{x^3 + 1} (3x^2) dx$$

$$x = 1 \rightarrow u = \ln 2 \quad x = 2 \rightarrow u = \ln 9$$

$$= \int_{\ln 2}^{\ln 9} \frac{1}{u^4} du = \int_{\ln 2}^{\ln 9} u^{-4} du = \left[\frac{u^{-3}}{-3} \right]_{\ln 2}^{\ln 9} = \left[\frac{-1}{3u^3} \right]_{\ln 2}^{\ln 9}$$

$$= \left[\frac{-1}{3(\ln 9)^3} + \frac{1}{3(\ln 2)^3} \right]$$

J13. Integrate $\int_0^1 \frac{6x}{x^2+1} dx$

$$u = x^2 + 1 \quad x=0 \quad u=0^2 + 1 = 1$$

$$du = 2x dx \quad x=1 \quad u=1^2 + 1 = 2$$

$$3du = 6x dx$$

$$= 3 \int_1^2 \frac{du}{u} = 3[\ln u]_1^2 = 3(\ln 2 - \ln 1) = 3\ln 2 - 0 = 3\ln 2 = \ln 2^3 = \ln 8$$

J14. Integrate $\int_2^3 \frac{1}{x\sqrt{\ln x}} dx$

$$u = \ln x \quad du = \frac{1}{x} dx \quad du = \frac{1}{x} dx$$

$$x = 2 \quad u = \ln 2$$

$$x = 3 \quad u = \ln 3$$

$$= \int_{\ln 2}^{\ln 3} \frac{1}{\sqrt{u}} du = \int_{\ln 2}^{\ln 3} u^{-\frac{1}{2}} du = \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_{\ln 2}^{\ln 3}$$

$$= [2\sqrt{u}]_{\ln 2}^{\ln 3} = [2\sqrt{\ln 3} - 2\sqrt{\ln 2}]$$

J15. Find $\int_1^9 \frac{(1+\sqrt{x})^3}{\sqrt{x}} dx$

$$u = 1 + \sqrt{x} \quad du = \frac{1}{2} x^{-\frac{1}{2}} dx \quad 2du = \frac{1}{\sqrt{x}} dx$$

$$x = 1, u = 2$$

$$x = 9, u = 1 + \sqrt{9} = 1 + 3 = 4$$

$$\int_1^9 \frac{(1+\sqrt{x})^3}{\sqrt{x}} dx = 2 \int_2^4 u^3 du = 2 \left[\frac{u^4}{4} \right]_2^4 = 2 \left[\frac{4^4}{4} - \frac{2^4}{4} \right]$$

$$J16. \int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

$$u = \sqrt{x} \quad du = \frac{1}{2}x^{-\frac{1}{2}}dx \quad 2du = \frac{1}{\sqrt{x}}dx$$

$$x = 1, u = \sqrt{1} = 1$$

$$x = 9, u = \sqrt{9} = 3$$

$$\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int_1^3 e^u du = 2[e^u]_1^3 = 2(e^3 - e)$$

$$J17. \text{ Evaluate: } \int_0^{\pi/4} \frac{\sec^2 x}{\sqrt{1+\tan x}} dx$$

$$\text{subst } u = 1 + \tan x$$

$$du = \sec^2 x dx$$

$$x = 0 \quad u = 1 + \tan 0 = 1$$

$$x = \frac{\pi}{4} \quad u = 1 + \tan \frac{\pi}{4} = 1 + 1 = 2$$

$$= \int_1^2 \frac{1}{\sqrt{u}} du = \int_1^2 u^{-\frac{1}{2}} du = \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^2 = [2\sqrt{u}]_1^2$$

$$= 2\sqrt{2} - 2\sqrt{1} = 2\sqrt{2} - 2$$

$$J18. \text{ Evaluate } \int \tan^3 x \sec^2 x dx$$

$$u = \tan x \quad du = \sec^2 x dx$$

$$\int u^3 du = \frac{u^4}{4} + c = \frac{\tan^4 x}{4} + c$$

J19. Integrate $\int \frac{x}{(x^2+1)^4} dx$

$$u = x^2 + 1 \quad du = 2x dx \quad \frac{du}{2} = x dx$$

$$\begin{aligned} \int \frac{x}{(x^2+1)^4} dx &= \frac{1}{2} \int u^{-4} du = \frac{1}{2} \left(\frac{u^{-3}}{-3} \right) + c = -\frac{1}{6u^3} + c \\ &= -\frac{1}{6(x^2+1)^3} + c \end{aligned}$$

J20. Integrate $\int \frac{2x^3}{x^4+1} dx$

$$u = x^4 + 1 \quad du = 4x^3 dx \quad \frac{du}{2} = 2x^3 dx$$

$$\int \frac{x^3}{x^4+1} dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + c = \frac{1}{2} \ln|x^4+1| + c$$

J21. Integrate $\int \frac{8x}{x^2+1} dx$

$$u = x^2 + 1 \quad du = 2x dx \quad \therefore 4du = 8x dx$$

$$= 4 \int \frac{1}{u} du = 4 \ln|u| + c = 4 \ln|x^2+1| + c$$

J22. Integrate $\int 8x^3\sqrt{x^4 + 1}dx$

$$\text{subst } u = x^4 + 1 \quad du = 4x^3 dx$$

$$2du = 8x^3 dx$$

$$= 2 \int \sqrt{u} du$$

$$= 2 \int u^{\frac{1}{2}} du = 2 \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = 2 \left(\frac{2}{3}\right) u^{\frac{3}{2}} + c = \frac{4}{3} (x^4 + 1)^{\frac{3}{2}} + c$$

J23. Integrate $\int \frac{\sec^2(\ln x)}{2x} dx$

$$\text{subst } u = \ln x \quad du = \frac{1}{x} dx \quad \frac{1}{2} du = \frac{1}{2x} dx$$

$$= \frac{1}{2} \int \sec^2 u du$$

$$= \frac{1}{2} \tan u + c$$

$$= \frac{1}{2} \tan(\ln x) + c$$

J24. Integrate $\int_0^{\pi/2} \cos^3 x \, dx$

Substitution

$$u = \sin x$$

$$du = \cos x \, dx$$

$$x = 0 \quad u = \sin 0 = 0$$

$$x = \frac{\pi}{2} \quad u = \sin \frac{\pi}{2} = 1$$

Identity

$$\sin^2 x + \cos^2 x = 1$$

$$\therefore \cos^2 x = 1 - \sin^2 x$$

$$\int_0^{\pi/2} \cos^3 x \, dx$$

$$= \int_0^{\pi/2} \cos^2 x \cos x \, dx$$

$$= \int_0^{\pi/2} (1 - \sin^2 x) \cos x \, dx$$

$$= \int_0^1 (1 - u^2) du$$

$$= \left[u - \frac{u^3}{3} \right]_0^1$$

$$= \left(1 - \frac{1}{3} \right) - (0 - 0)$$

$$= \frac{2}{3}$$

J25. Evaluate: $\int_0^2 6x^2 e^{x^3} dx$

$$\text{subst } u = x^3 \quad du = 3x^2 dx \quad 2du = 6x^2 dx$$

$$x = 0 \quad u = 0^3 = 0$$

$$x = 2 \quad u = 2^3 = 8$$

$$= 2 \int_0^8 e^u du$$

$$= 2[e^u]_0^8 = 2(e^8 - e^0) = 2(e^8 - 1)$$

K. Other Types of Integrals

Example 1. Integrate $\int_{-5}^5 \sqrt{25 - x^2} dx$

If we write $y = \sqrt{25 - x^2}$ and square both sides, the square root will disappear and we get:

$y^2 = 25 - x^2$ simplifying, we get...

$x^2 + y^2 = 25$...which we recognize as half of a circle, since originally, we only had the positive square root of the function

This circle has centre at (0,0) and radius=5

$$A = \frac{\pi r^2}{2} = \frac{\pi(5^2)}{2} = 12.5\pi$$

Example 2. Determine whether each of the following functions are odd, even, or neither.

a) $f(x) = x^3 - x$

$$f(-x) = (-x)^3 - (-x)$$

$$= -x^3 + x$$

$$= -(x^3 - x)$$

$$= -f(x)$$

since in this case $f(-x) = -f(x)$, the function is odd

b) $f(x) = x^2 + 1$

$$f(-x) = (-x)^2 + 1$$

$$= x^2 + 1$$

$$= f(x)$$

since $f(-x) = f(x)$, the function is even

c) $f(x) = 3x - x^2$

$$f(-x) = 3(-x) - (-x)^2$$

$$= -3x - x^2$$

since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, this function is neither odd or even.

Example 3. Determine whether the following functions are odd or even:

a) $f(x) = x^3 \sin x$

x^3 is an odd function, since $(-x)^3 = -x^3$

$\sin x$ is an odd function, since $\sin(-x) = -\sin x$

Thus, $f(x) = x^3 \sin x$

$= \text{odd} \times \text{odd} = \text{even}$

b) $f(x) = x^3 \cos x$

x^3 is once again an odd function, however, this time $\cos x$ is an even function, since $\cos(-x) = \cos x$.

thus, $f(x) = x^3 \cos x$

$\text{odd} \times \text{even} = \text{odd}$

L. Practice With All Types of Integration

$$L1. \quad \Delta x = \frac{b-a}{n} = \frac{2\pi}{n} \quad \therefore b - a = 2\pi$$

$$f(xi) = f\left(a + i\Delta x\right) = f\left(a + \frac{2\pi}{n}i\right) \quad \therefore a = 0, b = 2\pi$$

$$\therefore \int_0^{2\pi} \cos x dx$$

$$L2. \quad \frac{b-a}{n} = \frac{1}{n} \quad \therefore b - a = 1 \quad a = 0 \quad b = 1$$

$$f(xi) = f\left(0 + \frac{1}{n}i\right) = f\left(\frac{1}{n}i\right) \quad \text{let } f(x) = \sqrt{\csc x}$$

The answer is B.

$$L3. \quad [\ln(1+x)]_1^4 = \ln(1+4) - \ln(1+1)$$

$$= \ln 5 - \ln 2 = \ln\left(\frac{5}{2}\right)$$

The answer is E.

$$L4. \quad |x| = x \text{ if } x > 0$$

$$= -x \text{ if } x < 0$$

$$= \int_{-2}^0 -x dx + \int_0^2 x dx$$

$$= \left[\frac{-x^2}{2}\right]_{-2}^0 + \left[\frac{x^2}{2}\right]_0^2$$

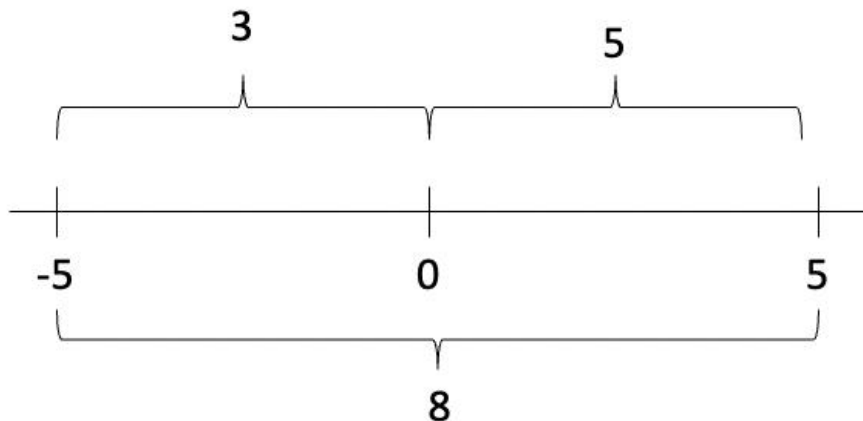
$$= \left(0 + \frac{(-2)^2}{2}\right) + (2 - 0) = 2 + 2 = 4$$

$$L5. \quad f'(t) = \frac{d}{dt} \left(\int_{\frac{\pi}{2}}^{\pi} \sqrt{1 + \cos t} dt \right) = 0 \quad \text{FTC}$$

↓ a constant

The answer is C.

$$\begin{aligned} \text{L6. } \int_{-5}^5 f(x) &= \frac{24}{3} = 8 \\ &= 5 \int_{-5}^0 f(x) dx = 5(8 - 5) = 5(3) = 15 \end{aligned}$$



$$\begin{aligned} \text{L7. } &= \sin^{-1} \sqrt{(x^2 + 1)^2 + 1} (2x) - \sin^{-1} \sqrt{x^2 + 1} (1) \\ &= 2x \sin^{-1} \sqrt{(x^2 + 1)^2 + 1} - \sin^{-1} \sqrt{x^2 + 1} \end{aligned}$$

$$\begin{aligned} \text{L8. } |x - 5| &= x - 5 \text{ if } x - 5 \geq 0 \\ &= -(x - 5) \text{ if } x < 5 \\ &= \int_{-1}^5 -(x - 5) dx + \int_5^6 (x - 5) dx \\ &= \left[\frac{-x^2}{2} + 5x \right]_{-1}^5 + \left[\frac{x^2}{2} - 5x \right]_5^6 \\ &= \left(-\frac{5^2}{2} + 5(5) \right) - \left(-\frac{(-1)^2}{2} + 5(-1) \right) + \left(\frac{6^2}{2} - 5(6) \right) - \left(\frac{5^2}{2} - 5(5) \right) \\ &= -\frac{25}{2} + 25 + \frac{1}{2} + 5 + 18 - 30 - \frac{25}{2} + 25 \\ &= 43 - \frac{25}{2} - \frac{25}{2} + \frac{1}{2} = 43 - \frac{49}{2} = \frac{86}{2} - \frac{49}{2} = \frac{37}{2} \end{aligned}$$

$$\begin{aligned} \text{L9. } u &= \tan^{-1} x \quad du = \frac{1}{1+x^2} dx \\ x = 0 \quad u &= \tan^{-1} 0 = 0 \\ x = 1 \quad u &= \tan^{-1} 1 = \frac{\pi}{4} \\ &= 2 \int_0^{\frac{\pi}{4}} u du = 2 \left[\frac{u^2}{2} \right]_0^{\frac{\pi}{4}} = 2 \left[\frac{\left(\frac{\pi}{4}\right)^2}{2} - 0 \right] = \frac{2\pi^2}{32} = \frac{\pi^2}{16} \end{aligned}$$

$$\begin{aligned} \text{L10.} \quad &= \int_0^5 (\cos^2 \sqrt{2x} + \sin^2 \sqrt{2x}) dx = \int_0^5 1 dx = [x]_0^5 \\ &= 5 - (0) = 5 \end{aligned}$$

$$\text{L11.} \quad \sqrt{49 - x^2} \quad \frac{1}{2} \text{ circle with radius 7}$$

$$A = \frac{1}{2} \pi r^2 = \frac{1}{2} \pi (7)^2 = \frac{49}{2} \pi$$

$$\begin{aligned} \text{L12.} \quad u = \tan^{-1}(x^4) \quad du &= \frac{(4x^3)}{1+(x^4)^2} = \frac{(4x^3)}{1+x^8} \\ x = 0 \quad u &= \tan^{-1} 0 = 0 \\ x = 1 \quad u &= \tan^{-1} 1 = \frac{\pi}{4} \\ &= \int_0^{\frac{\pi}{4}} u du = \left[\frac{u^2}{2} \right]_0^{\frac{\pi}{4}} = \frac{\left(\frac{\pi}{4}\right)^2}{2} - 0 = \frac{\pi^2}{32} \end{aligned}$$

$$\begin{aligned} \text{L13.} \quad u = 1 + x^2 \quad du &= 2x dx \quad \frac{du}{2} = x dx \\ x = 1 \quad u &= 1 + 1^2 = 2 \\ x = 2 \quad u &= 1 + 2^2 = 5 \\ &= \frac{1}{2} \int_2^5 \frac{1}{u} du = \frac{1}{2} [\ln u]_2^5 = \frac{1}{2} [\ln 5 - \ln 2] = \frac{1}{2} \ln\left(\frac{5}{2}\right) \end{aligned}$$

$$\text{L14.} \quad = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\text{L15.} \quad F'(x) = \sin(0)(0) - \sin x(1)$$

$$F'(x) = -\sin x$$

$$F'\left(-\frac{\pi}{4}\right) = -\sin\left(-\frac{\pi}{4}\right) = -(-\sin \frac{\pi}{4}) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\text{L16.} \quad u = \tan x \quad du = \sec^2 x dx$$

$$x = 0 \quad u = \tan 0 = 0$$

$$x = \frac{\pi}{4} \quad u = \tan \frac{\pi}{4} = 1$$

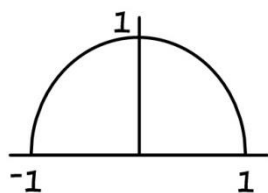
$$= 7 \int_0^1 u^2 du = 7 \left[\frac{u^3}{3} \right]_0^1 = 7 \left(\frac{1}{3} - 0 \right) = \frac{7}{3}$$

$$\begin{aligned} \text{L17. } u &= \ln x & du &= \frac{1}{x} dx \\ x &= e & u &= \ln e = 1 \\ x &= e^3 & u &= \ln e^3 = 3 \ln e = 3 \end{aligned}$$

$$= \int_1^3 u du = \left[\frac{u^2}{2} \right]_1^3 = \frac{3^2}{2} - \frac{1^2}{2} = \frac{8}{2} = 4$$

$$\text{L18. } = \cos^{-1} \sqrt{4x}(4) - \cos^{-1} \sqrt{x}(1)$$

$$\text{L19. } \sqrt{1-x^2} \quad \frac{1}{2} \text{ circle (from } -1 \text{ to } 1) \quad y = \sqrt{1-x^2}$$



$$\begin{aligned} y^2 &= 1 - x^2 \\ x^2 + y^2 &= 1 & r &= 1 \end{aligned}$$

Between -1 and 0 is $\frac{1}{4}$ of the circle $A = \frac{\pi r^2}{4} = \frac{\pi(1)^2}{4} = \frac{\pi}{4}$

$$\begin{aligned} \text{L20. } u &= \ln x & du &= \frac{1}{x} dx \\ x &= 1 & u &= \ln 1 = 0 \\ x &= 4 & u &= \ln 4 \end{aligned}$$

$$= \int_0^{\ln 4} u du = \left[\frac{u^2}{2} \right]_0^{\ln 4} = \frac{(\ln 4)^2}{2} - 0 = \frac{(\ln 4)^2}{2}$$

$$\text{L21. } u = \cos^{-1} x \quad du = \frac{-1}{\sqrt{1-x^2}} dx \quad -du = \frac{1}{\sqrt{1-x^2}} dx$$

$$x = -1 \quad u = \cos^{-1}(-1) = \pi$$

$$x = 0 \quad u = \cos^{-1} 0 = \frac{\pi}{2}$$

$$= - \int_{\frac{\pi}{2}}^{\pi} u du = \int_{\frac{\pi}{2}}^{\pi} u du = \left[\frac{u^2}{2} \right]_{\frac{\pi}{2}}^{\pi} = \frac{(\pi)^2}{2} - \frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{\pi^2}{2} - \frac{\pi^2}{8} = \frac{4\pi^2 - \pi^2}{8} = \frac{3\pi^2}{8}$$

$$\text{L22. } u = x^2 \quad du = 2x dx \quad 3du = 6x dx$$

$$x = 0 \quad u = 0^3 = 0$$

$$x = 2 \quad u = 2^2 = 4$$

$$= 3 \int_0^4 e^u du = [3e^u]_0^4 = 3e^4 - 3e^0 = 3e^4 - 3$$

$$\begin{aligned} \text{L23. } u &= \sin x \quad du = \cos x dx \\ &= \int \frac{1}{1+u^2} du = \tan^{-1} u + c = \tan^{-1}(\sin x) + c \end{aligned}$$

$$\begin{aligned} \text{L24. } u &= \sin x \quad du = \cos x dx \\ x &= \frac{\pi}{6} \quad u = \sin \frac{\pi}{6} = \frac{1}{2} \\ x &= \frac{\pi}{2} \quad u = \sin \frac{\pi}{2} = 1 \\ &= \int_{\frac{1}{2}}^1 u^4 du = \left[\frac{u^5}{5} \right]_{\frac{1}{2}}^1 = \frac{1}{5} - \frac{(0.5)^5}{5} = \frac{1}{5} - \frac{1}{160} = \frac{32}{160} - \frac{1}{160} = \frac{31}{160} \end{aligned}$$

$$\begin{aligned} \text{L25. } u &= \ln x \quad du = \frac{1}{x} dx \\ x &= 1 \quad u = \ln 1 = 0 \\ x &= e \quad u = \ln e = 1 \\ &= \int_0^1 u^5 du = \left[\frac{u^6}{6} \right]_0^1 = \frac{1}{6} - 0 = \frac{1}{6} \end{aligned}$$

$$\begin{aligned} \text{L26. } u &= 4 + e^x \quad du = e^x dx \\ &= \int \sqrt{u} du = \int u^{\frac{1}{2}} du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \quad \text{or} \quad \frac{2u^{\frac{3}{2}}}{3} + c \\ &= \frac{2(4+e^x)^{\frac{3}{2}}}{3} + c \\ \text{L27. } &= \left[\frac{\tan 5x}{5} \right]_0^{\frac{\pi}{4}} = \frac{\tan \frac{5\pi}{4}}{5} - \frac{\tan 0}{5} = \frac{\tan \frac{5\pi}{4}}{5} = \frac{\tan \frac{\pi}{4}}{5} = \frac{1}{5} \end{aligned}$$

$$\begin{aligned} \text{L28. } u &= \sec x \quad du = \sec x \tan x dx \\ &= \int \sec^4 x \sec x \tan x dx = \int u^4 du = \frac{u^5}{5} + c \quad \text{or} \quad \frac{\sec^5 x}{5} + c \end{aligned}$$

$$\begin{aligned} \text{L29. } &= \frac{1}{n} \lim_{n \rightarrow \infty} \left[4n + \frac{n(n+1)}{2n} + \frac{n(n+1)(2n+1)}{6n} \right] \\ &= \frac{1}{n} \lim_{n \rightarrow \infty} \left[\frac{4n}{1} + \frac{n+1}{2} + \frac{2n^2+3n+1}{6} \right] \\ &= \frac{1}{n} \lim_{n \rightarrow \infty} \left[\frac{24n+3n+3+2n^2+3n+1}{6} \right] = \lim_{n \rightarrow \infty} \left[\frac{2n^2+30n+4}{6n} \right] \\ &= \lim_{n \rightarrow \infty} \left[2n + 5 + \frac{4}{6n} \right] = 2(\infty) + 5 + 0 = \infty \end{aligned}$$

$$\text{L30. } \Delta x = \frac{b-a}{n} = \frac{6}{n} \quad \therefore b - a = 6$$

$$f(xi) = f(a + i\Delta x) = f\left(a + \frac{6i}{n}\right)$$

$$\text{let } a = 1 \quad b - 1 = 6 \quad b = 7$$

$$\therefore \int_1^7 \sqrt{x} dx$$

$$\text{L31. } \int \frac{6x^2}{1+x^3} dx$$

$$u = 1 + x^3$$

$$du = 3x^2 dx$$

$$2du = 6x^2 dx$$

$$2 \int \frac{1}{u} du = 2 \ln|u| + c$$

$$\text{or } 2 \ln|1 + x^3| + c$$

$$\text{L32. } \int \frac{\sin \sqrt{x}}{3\sqrt{x}} dx$$

$$u = \sqrt{x} = x^{\frac{1}{2}}$$

$$du = \frac{1}{2} x^{-\frac{1}{2}} dx$$

$$2du = \frac{1}{\sqrt{x}} dx$$

$$= \frac{1}{3} \int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

$$= \frac{2}{3} \int \sin u du$$

$$= \frac{2}{3} [-\cos u] + c$$

$$= -\frac{2}{3} \cos \sqrt{x} + c$$

L33.

$$\int x^2 e^{x^3} dx$$

$$u = x^3$$

$$du = 3x^2 dx$$

$$\frac{1}{3} du = x^2 dx$$

$$= \frac{1}{3} \int e^u du$$

$$= \frac{1}{3} e^u + c$$

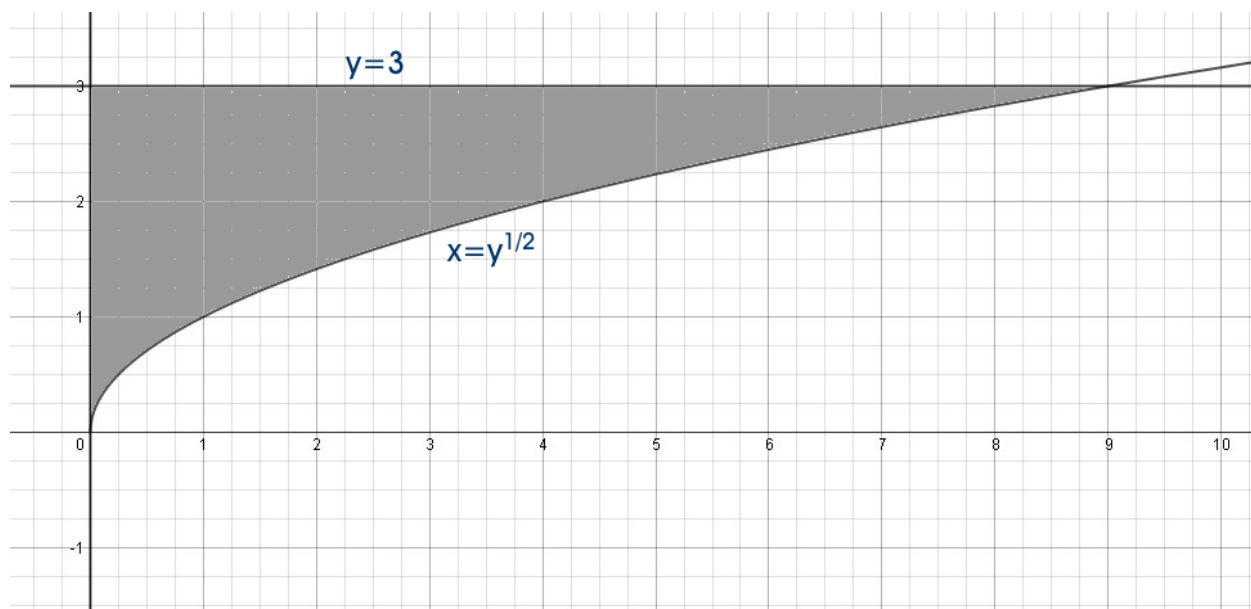
$$= \frac{1}{3} e^{x^3} + c$$

M. Area Between Curves

Example 1.

$$A = \int_a^b (\text{top} - \text{bottom}) dx$$

$$A = \int_0^9 (3 - \sqrt{x}) dx = \left[3x - \frac{2}{3} x^{3/2} \right]_0^9 = \left(27 - \frac{2}{3} (9^{3/2}) \right) - (0 - 0) =$$
$$27 - \frac{2}{3} (27) = \frac{81}{3} - \frac{54}{3} = 9$$



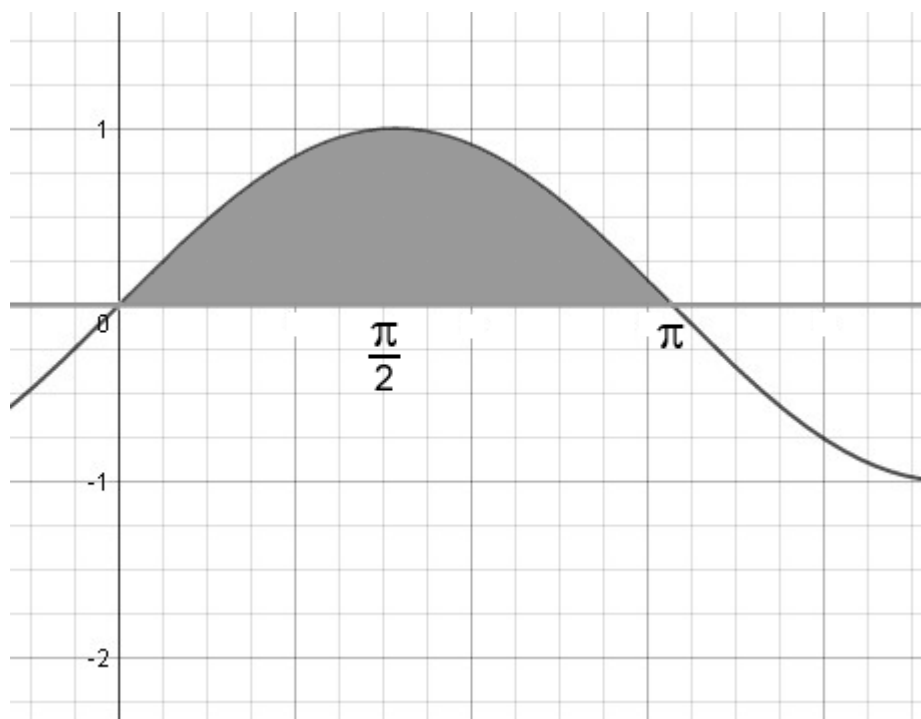
or use top and bottom with 0 to 9 for x-values

Example 2.

$$A = \int_a^b (\text{top} - \text{bottom}) dx$$

$$A = \int_0^{\pi} (\sin x - 0) dx = [-\cos x]_0^{\pi}$$

$$= -\cos \pi + \cos 0 = 1 + 1 = 2$$



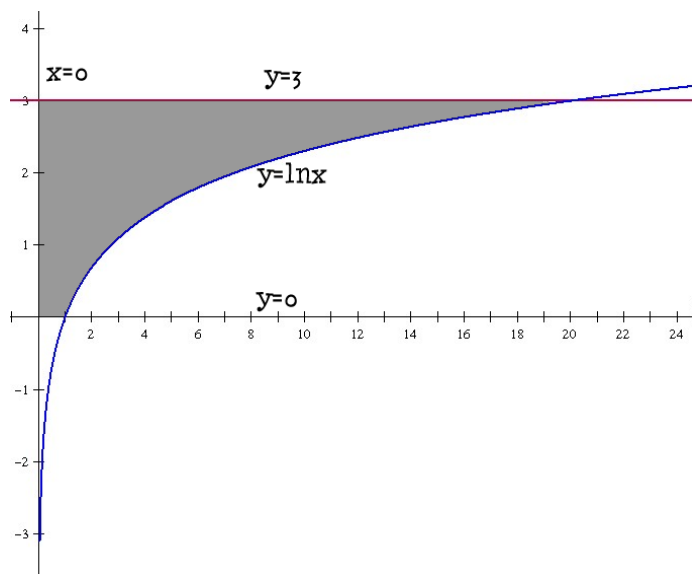
Example 3. Find the area of the region bounded by $y = \ln x$, $y = 3$, the x -axis and the y -axis.

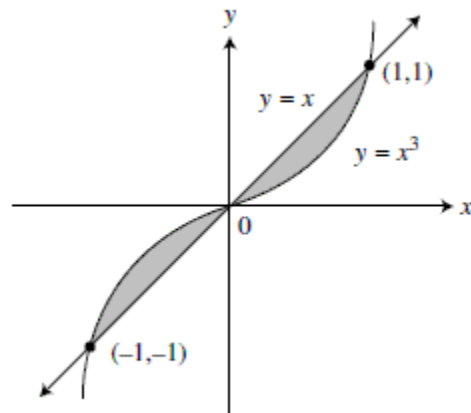
$$\text{Point of Intersection } 3 = \ln x \quad x = e^3$$

$$A = \int_0^3 (e^y - 0) dy$$

$$= \int_0^3 e^y dy$$

$$A = [e^y]_0^3 = e^3 - e^0 = e^3 - 1$$



Example 4.**Figure 12.6-2**

$$A = 2 \int_0^1 (x - x^3) dx \quad \text{using symmetry}$$

$$= 2 \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = 2 \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{1}{2}$$

Example 5.

$$x = 2y^2 - 2$$

x	y
-2	0
0	1
0	-1

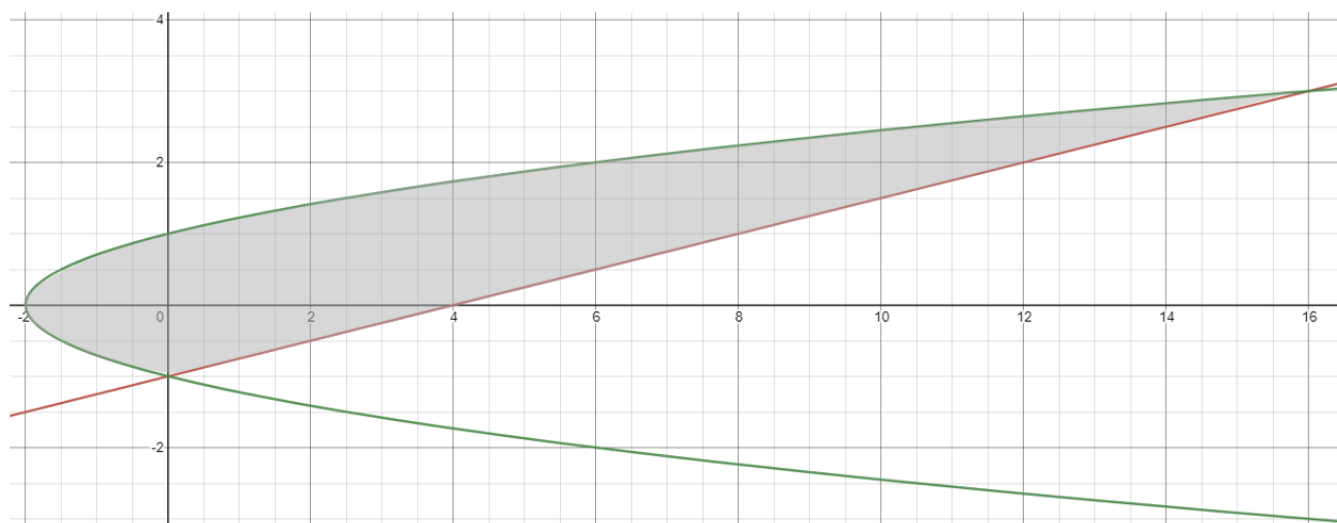
Point of intersection (Set $x=x$)

$$4y + 4 = 2y^2 - 2$$

$$2(y^2 - 2y - 3) = 0 \text{ factor}$$

$$(y - 3)(y + 1) = 0$$

$$y = 3, -1$$



$$A = \int_{-1}^3 (4y + 4) - (2y^2 - 2) dy = \int_{-1}^3 (-2y^2 + 4y + 6) dy$$

Practice Exam Questions on Area between Curves

M1. $0.5y^2 \quad \therefore$ use right and left

$$0.5y^2 - 1 = y + 3$$

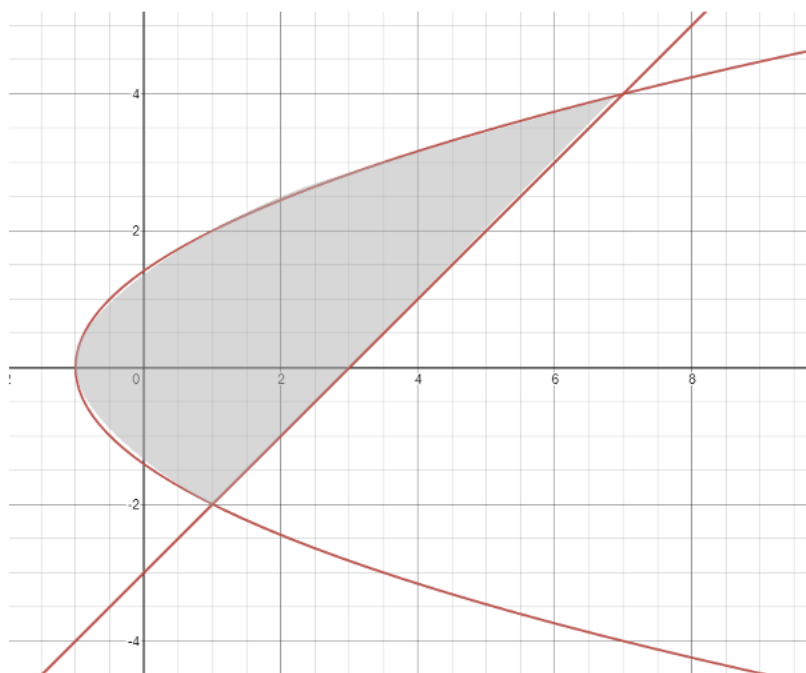
$$y^2 - 2y - 8 = 0$$

$$(y-4)(y+2) = 0$$

$$y = 4, -2$$

$$A = \int_{-2}^4 \left[\left(\frac{1}{2}y^2 - 1 \right) - (y + 3) \right] dy$$

$$= \int_{-2}^4 \left[\left(\frac{1}{2}y^2 - y - 3 \right) \right] dy$$



M2.

Point of intersection

$$(y - 1)^2 = -y^2 + 1$$

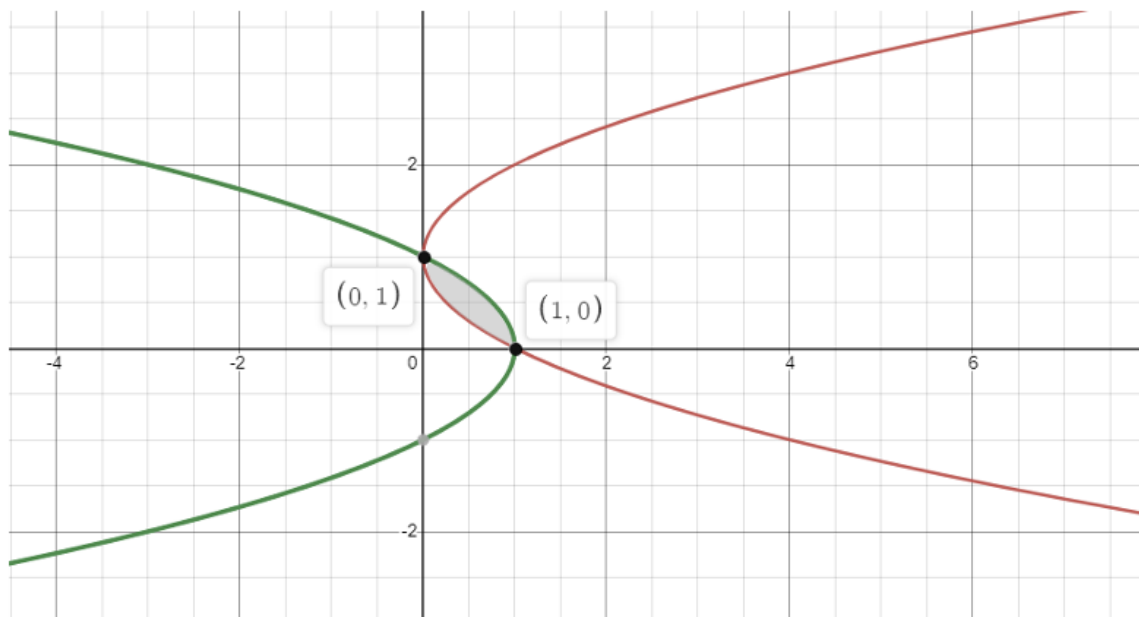
$$y^2 - 2y + 1 = -y^2 + 1$$

$$2y^2 - 2y = 0$$

$$2y(y - 1) = 0$$

$$y = 0, 1$$

$$A = \int_0^1 [(y - 1)^2 - (-y^2 + 1)] dy = \int_0^1 (y^2 - 2y + 1 + y^2 - 1) dy = \int_0^1 (2y^2 - 2y) dy$$



$$A = (-2/3(3^3 + 2(3^2) + 6(3))) - (-2/3(-1)^3 + 2(-1)^2 + 6(-1))$$

$$= -\frac{2}{3}(27) + 18 + 18 - \left(\frac{2}{3} + 2 - 6\right) = -\frac{54}{3} + 36 - \frac{2}{3} + 4 = -\frac{56}{3} + 40$$

$$= -\frac{56}{3} + \frac{120}{3} = \frac{64}{3}$$

M3. Intersection $x^2 = 2x - x^2$

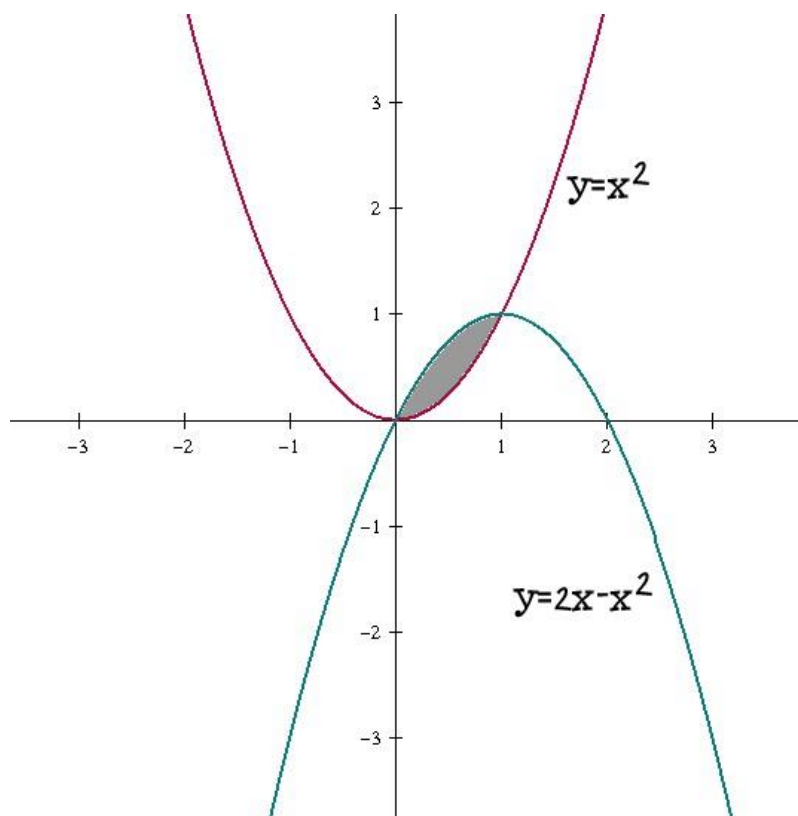
$$2x^2 - 2x = 0$$

$$2x(x - 1) = 0 \quad x = 0, 1$$

$$A = \int_0^1 [(2x - x^2) - (x^2)] dx$$

$$= \left[x^2 - \frac{x^3}{3} - \frac{x^3}{3} \right]_0^1 = \left[x^2 - \frac{2x^3}{3} \right]_0^1$$

$$= \left(1 - \frac{2}{3} \right) - (0 - 0) = \frac{1}{3}$$



$$\text{M4. } y = \sec^2 x \quad y = 2$$

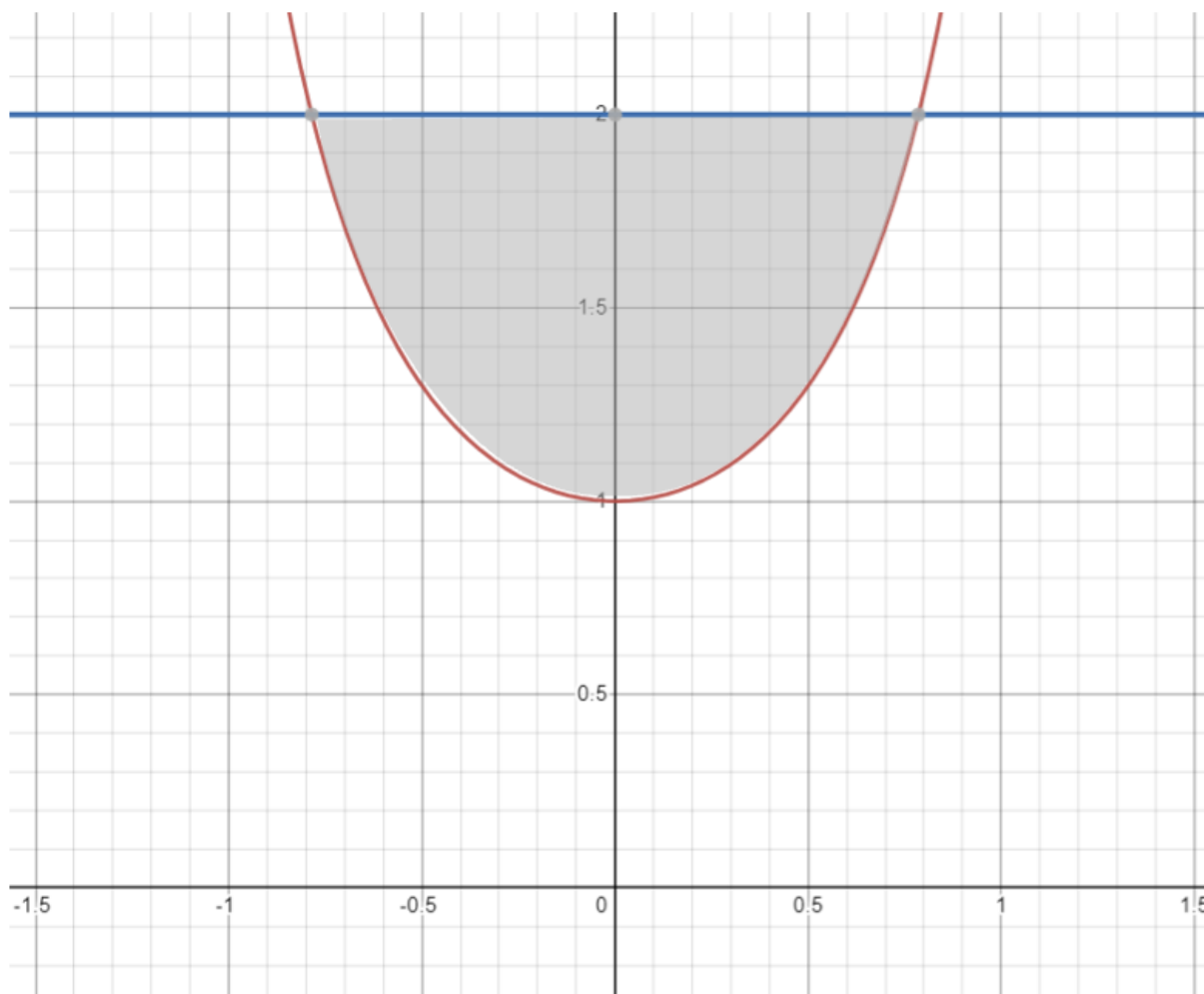
$$x = -\frac{\pi}{4} \quad \text{to} \quad \frac{\pi}{4}$$

$$A = \int_{-\pi/4}^{\pi/4} (2 - \sec^2 x) dx \quad \text{or} \quad 2 \int_0^{\pi/4} (2 - \sec^2 x) dx$$

$$= 2[2x - \tan x]_0^{\pi/4}$$

$$= 2 \left[\left(\frac{2\pi}{4} - \tan \frac{\pi}{4} \right) - (2(0) - \tan 0) \right]$$

$$= \pi - 2$$



M5. point of intersection $\sqrt{x} = \frac{x}{2}$

$$(2\sqrt{x})^2 = (x)^2$$

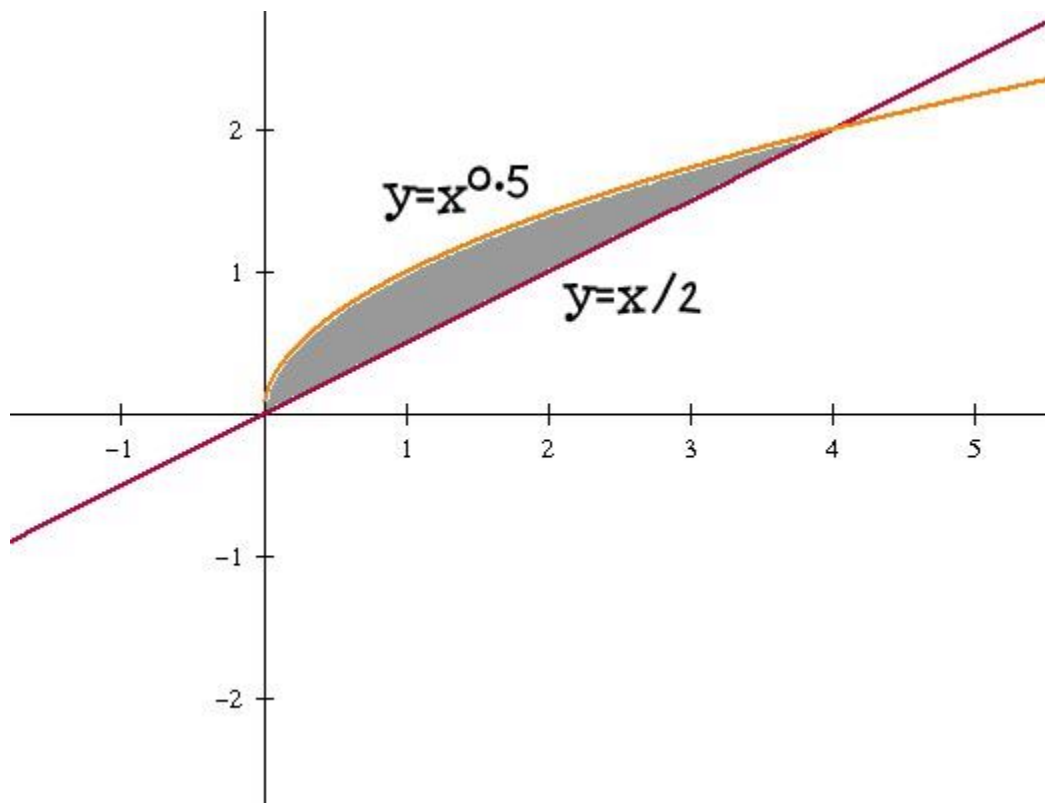
$$4x = x^2$$

$$0 = x^2 - 4x$$

$$0 = x(x - 4) \quad x = 0, 4$$

$$A = \int_0^4 \left(\sqrt{x} - \frac{x}{2} \right) dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{4} \right]_0^4 = \left(\frac{2}{3} (4)^{\frac{3}{2}} - \frac{4^2}{4} - (0) \right) = \frac{16}{3} - \frac{4}{1} = \frac{16-12}{3} = \frac{4}{3}$$



M6. point of intersection

$$e^x = e^{3x}$$

$$0 = e^{3x} - e^x$$

$$0 = e^x(e^{2x} - 1)$$

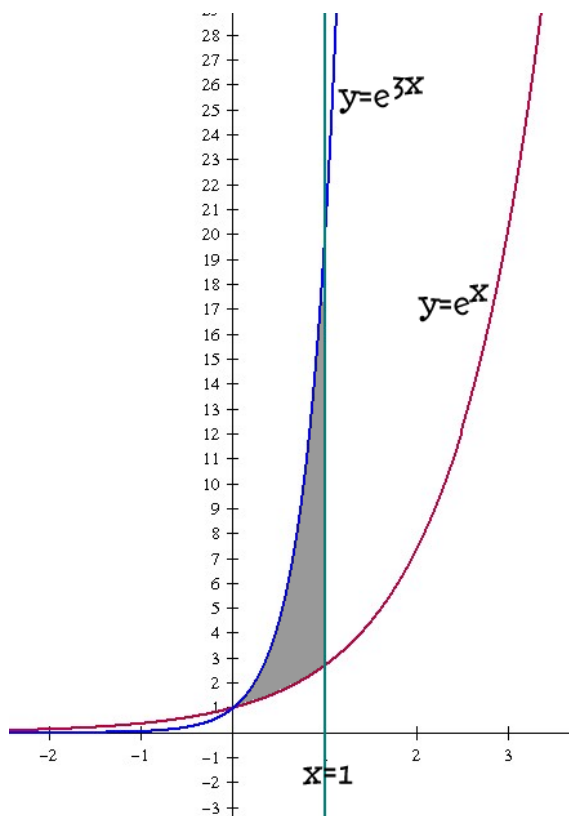
$$e^x = 0 \text{ no solution}$$

$$e^{2x} = 1$$

$$\ln e^{2x} = \ln 1$$

$$x = 0$$

$$\begin{aligned} A &= \int_0^1 (e^{3x} - e^x) dx = \left[\frac{e^{3x}}{3} - e^x \right]_0^1 = \left(\frac{e^3}{3} - e^1 \right) - \left(\frac{e^0}{3} - e^0 \right) \\ &= \frac{e^3}{3} - e - \frac{1}{3} + 1 = \frac{e^3}{3} - e + \frac{2}{3} \end{aligned}$$



M7. point of intersection $8 = e^x$

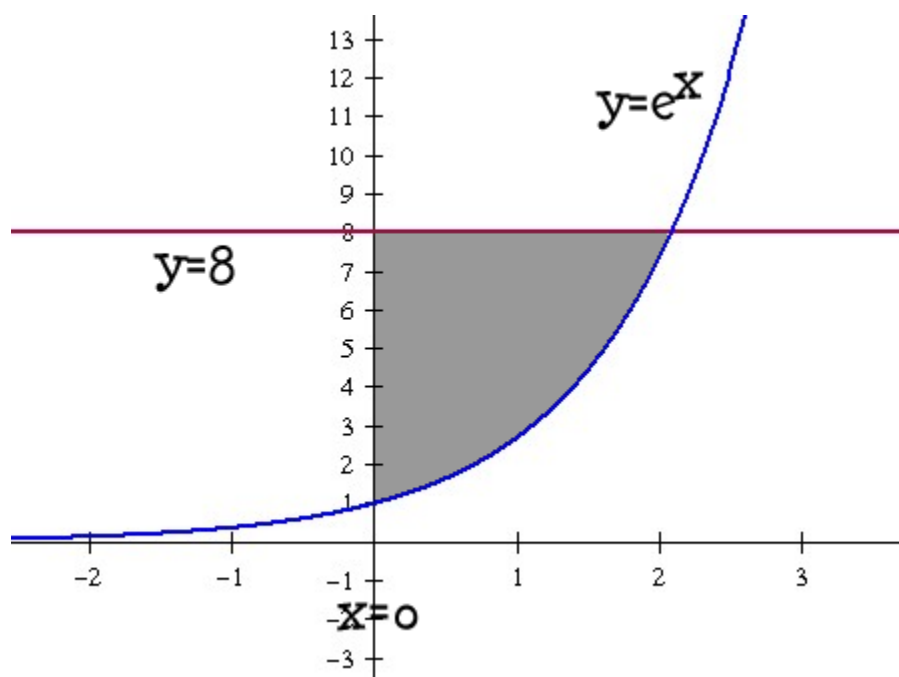
$$\ln 8 = \ln e^x$$

$$x = \ln 8$$

$$A = \int_0^{\ln 8} (8 - e^x) dx$$

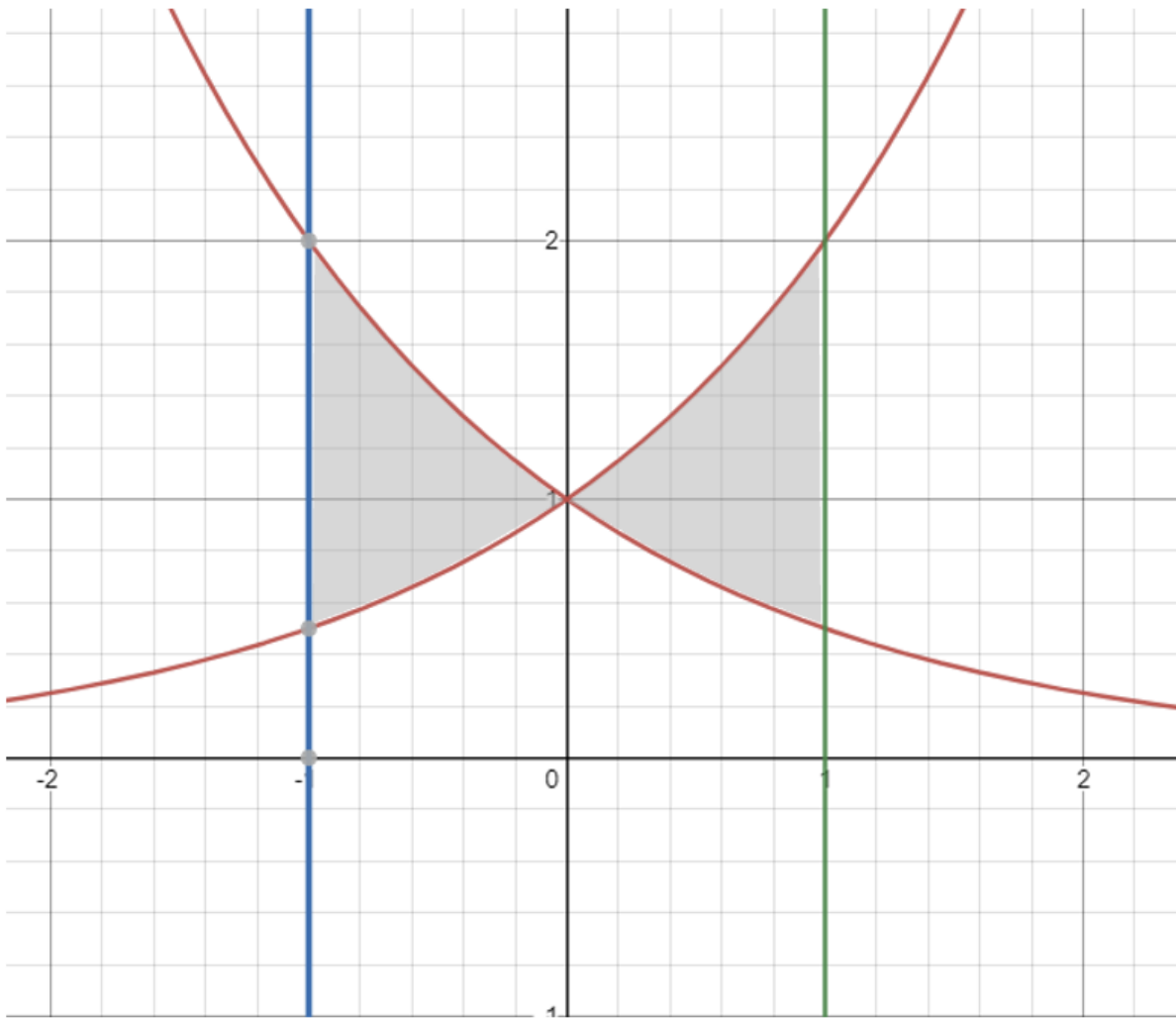
$$= [8x - e^x]_0^{\ln 8}$$

$$= (8 \ln 8 - e^{\ln 8}) - (0 - e^0) = 8 \ln 8 - 8 + 1 = 8 \ln 8 - 7$$

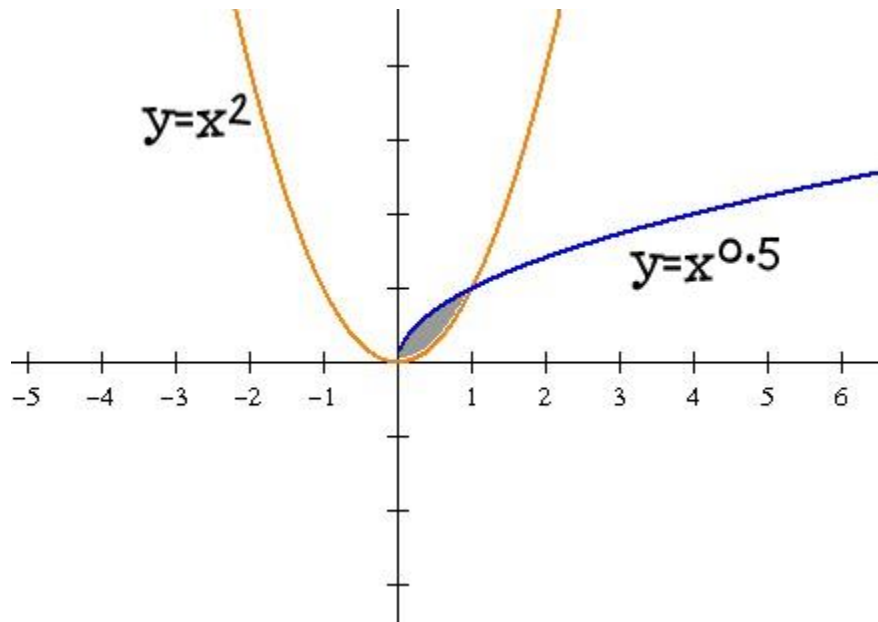


M8. $y = 2^x$, $y = 2^{-x}$ from -1 to 1

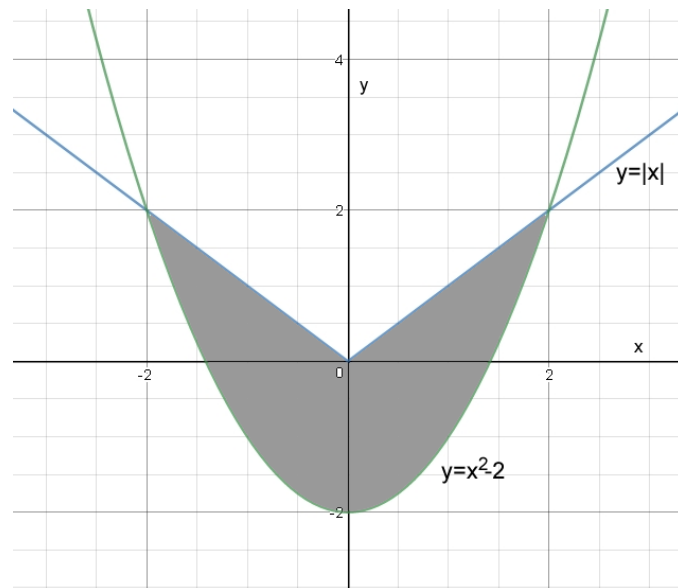
Set up only $A = 2 \int_0^1 (2^x - 2^{-x}) dx$ since it is symmetrical



$$M9. A = \int_0^1 (\sqrt{x} - x^2) dx = \left[\frac{2x^{3/2}}{3} - \frac{x^3}{3} \right]_0^1 = \frac{2}{3}(1)^{3/2} - \frac{1}{3} - (0) = \frac{1}{3}$$



M10.



$$y = |x|$$

x	y
0	0
1	1
-1	1
2	2

$$y = x^2 - 2$$

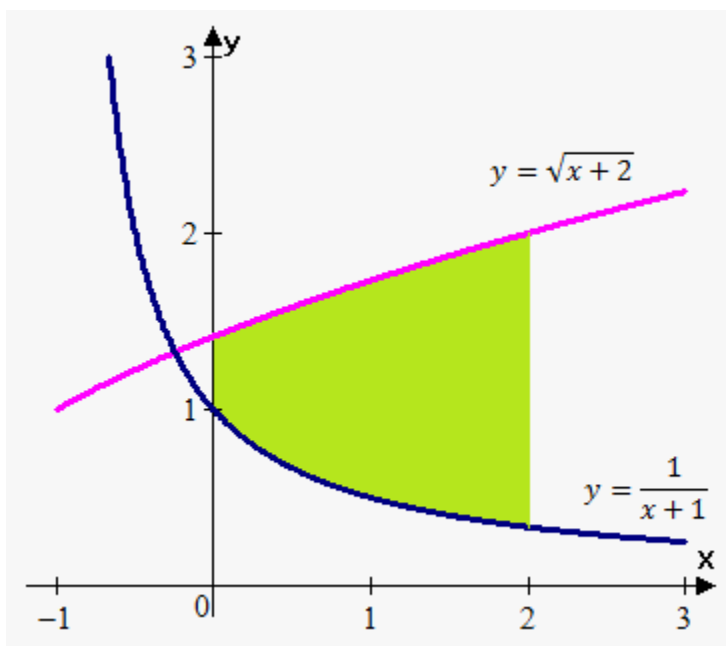
x	y
0	-2
1	-1
-1	-1
2	2

Absolute value of x is just x here since x is positive

$$\begin{aligned} A &= 2 \int_0^2 [x - (x^2 - 2)] dx \\ &= 2 \int_0^2 (x - x^2 + 2) dx \\ &= 2 \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x \right]_0^2 \\ &= 2 \left[\left(\frac{2^2}{2} - \frac{2^3}{3} + 2(2) \right) - 0 \right] \\ &= 2 \left(2 - \frac{8}{3} + 4 \right) \\ &= 2 \left(6 - \frac{8}{3} \right) = 2 \left(\frac{18}{3} - \frac{8}{3} \right) = 2 \left(\frac{10}{3} \right) = \frac{20}{3} \end{aligned}$$

M11.

$$\begin{aligned} A &= \int_0^2 \left(\sqrt{x+2} - \frac{1}{x+1} \right) dx \\ &= \left[\frac{2}{3} (x+2)^{3/2} - \ln|x+1| \right]_0^2 \\ &= \left[\frac{2}{3} (4^{3/2}) - \ln 3 \right] - \left[\frac{2}{3} (2)^{3/2} - \ln 1 \right] \\ &= \frac{2}{3} (8) - \ln 3 - \frac{2}{3} \sqrt{8} \\ &= \frac{16}{3} - \ln 3 - \frac{2}{3} \sqrt{8} \end{aligned}$$



M12.

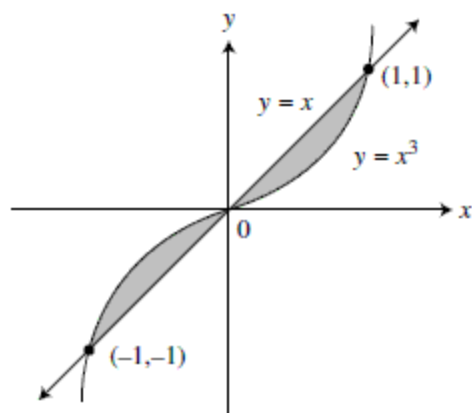


Figure 12.6-2

$$A = 2 \int_0^1 (x - x^3) dx \quad \text{using symmetry}$$

$$= 2 \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = 2 \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{1}{2}$$

M13. $y = \cos 2x$, $y = \sin x$ from $x = 0$, $\frac{\pi}{4}$

$$\begin{aligned}
 A &= \int_0^{\pi/6} (\cos 2x - \sin x) dx + \int_{\pi/6}^{\pi/4} (\sin x - \cos 2x) dx \\
 &= \left[\frac{\sin 2x}{2} + \cos x \right]_0^{\pi/6} + \left[-\cos x - \frac{\sin 2x}{2} \right]_{\pi/6}^{\pi/4} \\
 &= \left(\frac{\sin \frac{\pi}{3}}{2} + \cos \frac{\pi}{6} \right) - (\sin 0 + \cos 0) + \left(-\cos \frac{\pi}{4} - \frac{\sin \frac{\pi}{2}}{2} \right) - \left(-\cos \frac{\pi}{6} - \frac{\sin \frac{\pi}{3}}{2} \right) \\
 &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} - 1 - \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{4} \\
 &= \frac{2\sqrt{3}}{4} + \frac{\sqrt{3}}{2} - \frac{3}{2}
 \end{aligned}$$

$y = \cos 2x$

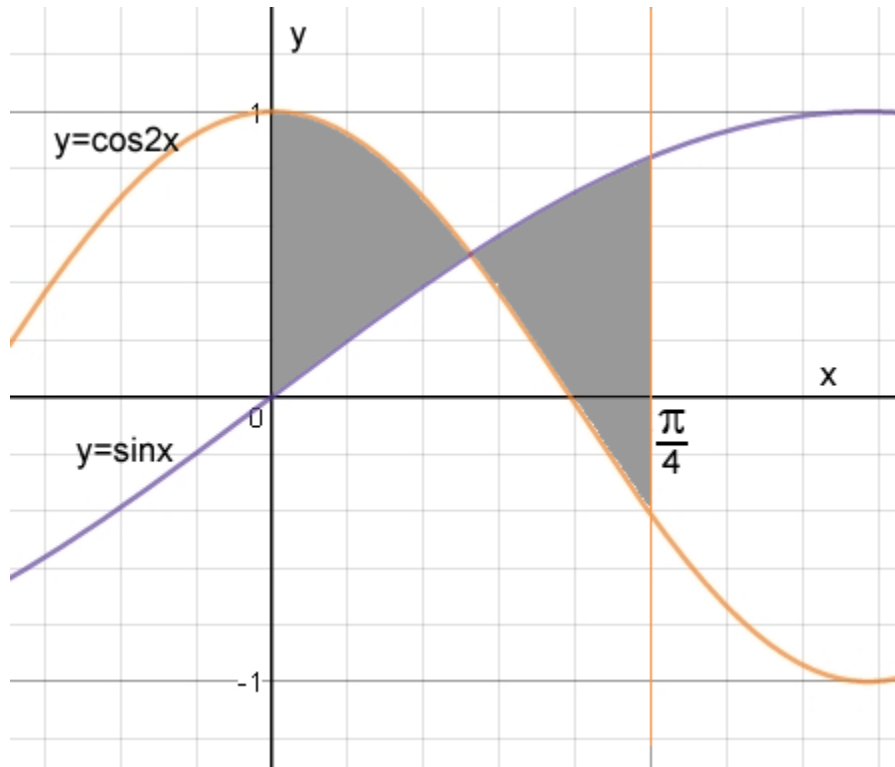
x	y
0	1
$\frac{\pi}{2}$	$\cos \pi = -1$
$\frac{\pi}{4}$	$\cos \frac{\pi}{2} = 0$
$\frac{\pi}{6}$	$\cos \frac{\pi}{3} = \frac{1}{2}$

↑

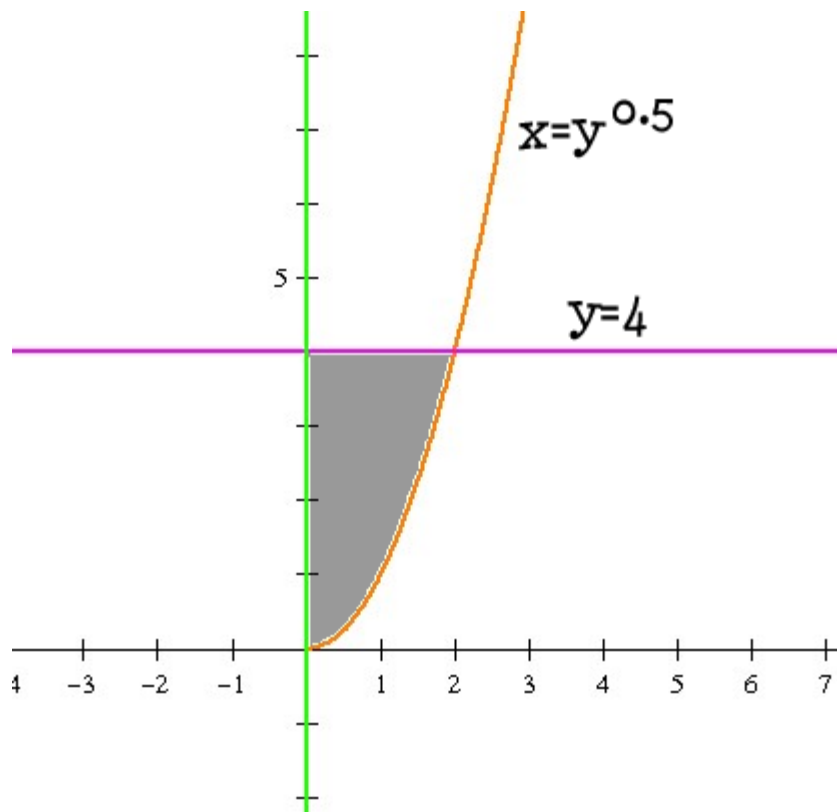
Point of intersection

$y = \sin x$

x	y
0	0
$\frac{\pi}{2}$	1
$\frac{\pi}{6}$	$\frac{1}{2}$



M14.



$$A = \int \text{top} - \text{bottom}$$

$$A = \int_0^2 (4 - x^2) dx$$

OR

$$A = \int \text{right} - \text{left}$$

$$A = \int_0^4 (\sqrt{y} - 0) dy$$

Point of intersection

$$x = \sqrt{y} \text{ and } y = 4$$

$$\therefore x = \sqrt{4}$$

$$A = \left[4x - \frac{x^3}{3} \right]_0^2 = 4(2) - \frac{2^3}{3} = \frac{8}{1} - \frac{8}{3} = \frac{24 - 8}{3} = \frac{16}{3}$$

M15. *Point of Intersection*

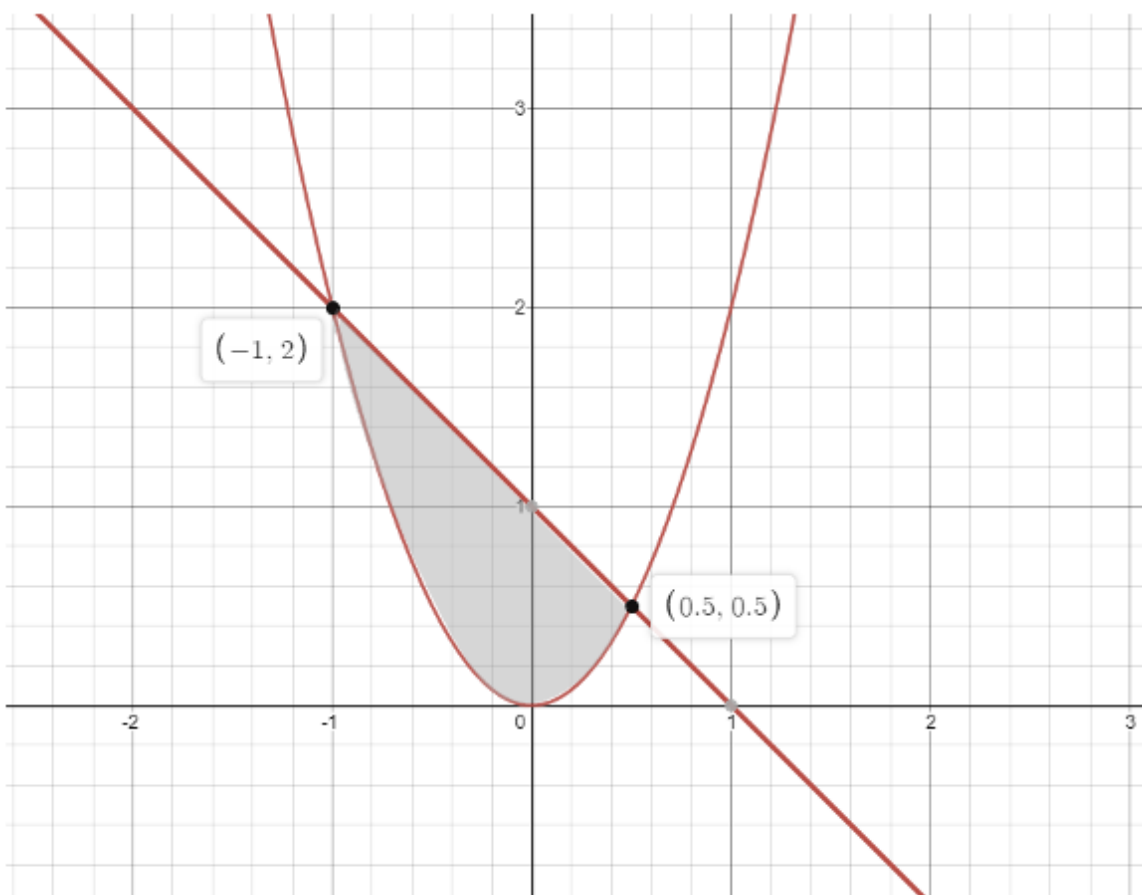
$$2x^2 = -x + 1$$

$$2x^2 + x - 1 = 0$$

$$(2x - 1)(x + 1) = 0$$

$$x = -1, \frac{1}{2}$$

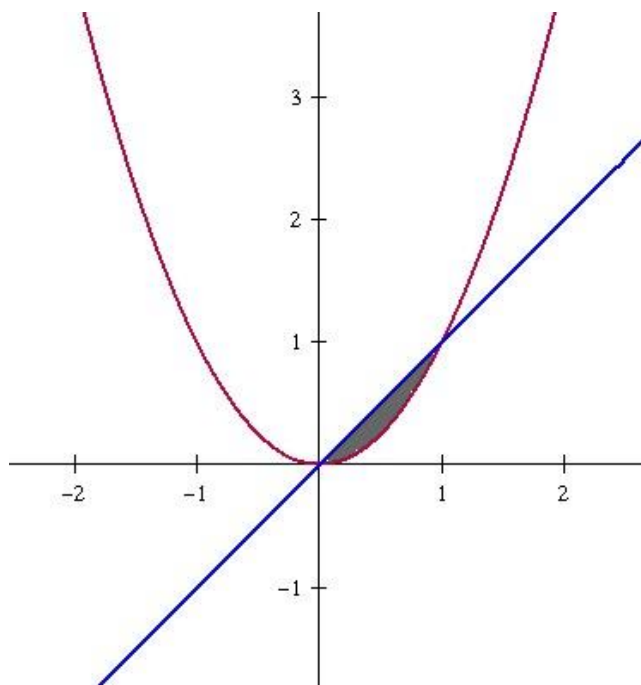
$$A = \int \text{top} - \text{bottom} = \int_{-1}^{1/2} [(-x + 1) - 2x^2] dx = \int_{-1}^{1/2} (-2x^2 - x + 1) dx$$



M16. Find the area of the region bounded by $y = x^2$ and $y = x$.

$$A = \int_0^1 (x - x^2) dx \quad x - \text{values}$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$



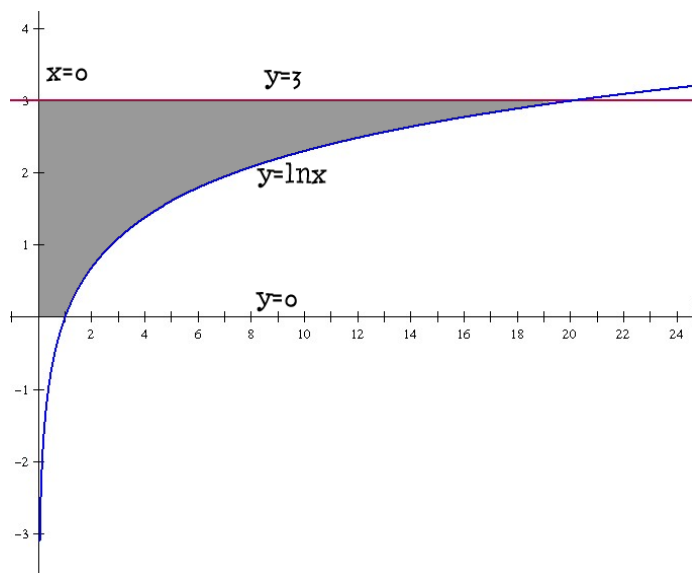
M17. Find the area of the region bounded by $y = \ln x$, $y = 3$, the x - axis and the y - axis.

$$\text{Point of Intersection } 3 = \ln x \quad x = e^3$$

$$A = \int_0^3 (e^y - 0) dy$$

$$= \int_0^3 e^y dy$$

$$A = [e^y]_0^3 = e^3 - e^0 = e^3 - 1$$



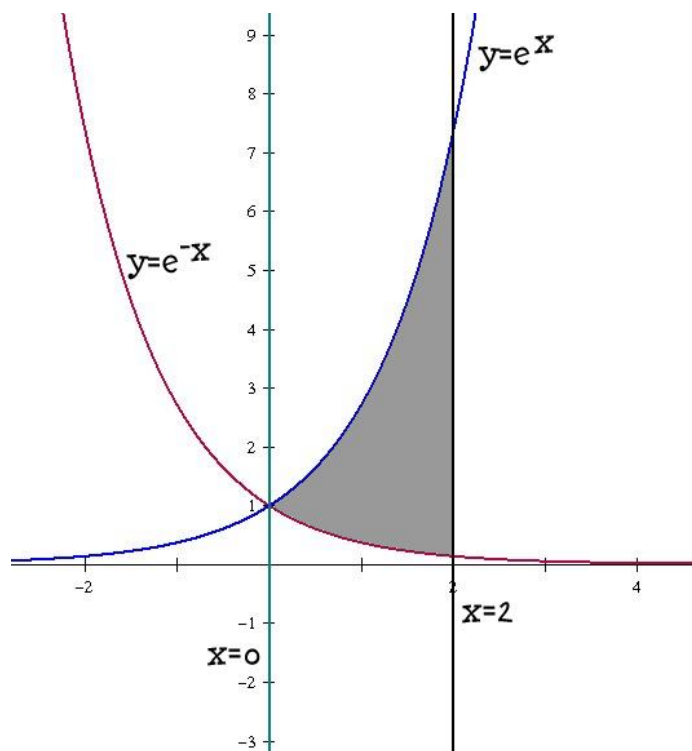
M18. Find the area of the region bounded by $y = e^x$ and $y = e^{-x}$ from $x = 0$ to $x = 2$.

$$A = \int_0^2 (e^x - e^{-x}) dx$$

$$= [e^x + e^{-x}]_0^2$$

$$= (e^2 + e^{-2}) - (e^0 + e^0)$$

$$= e^2 + \frac{1}{e^2} - 2$$



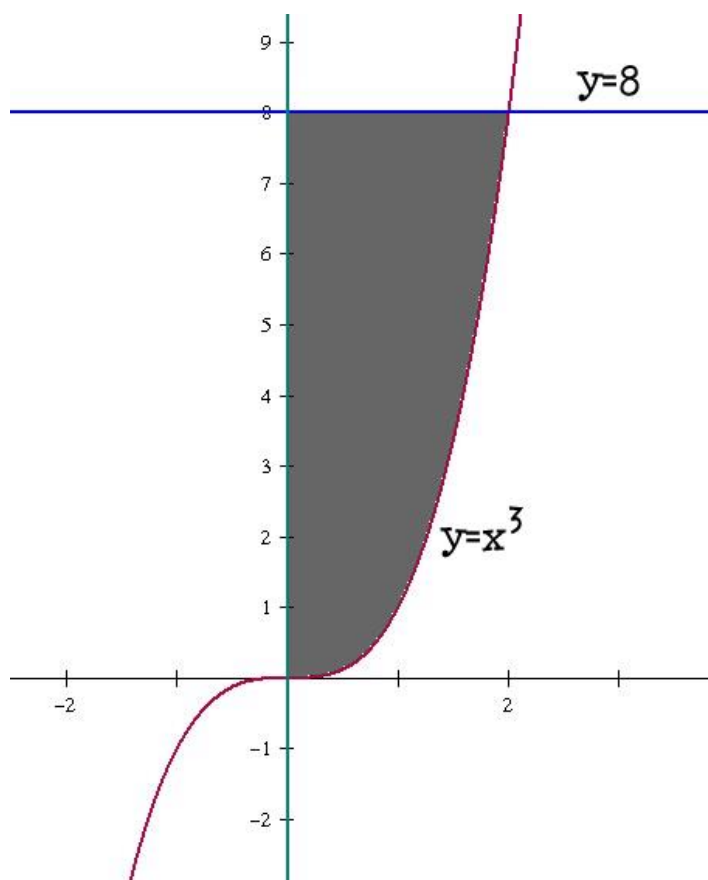
M19. Set up the integral representing the area of the region bounded by $y = x^3$, $y = 8$ and $x = 0$.

Point of Intersection $x^3 = 8$ $x = 2$

$$A = \int_0^8 \left(y^{\frac{1}{3}} - 0 \right) dy \quad \text{right} - \text{left}$$

$$= \left[\frac{3}{4} y^{\frac{4}{3}} \right]_0^8 = \frac{3}{4} (8)^{\frac{4}{3}} - 0 = \frac{3}{4} (16) = 12$$

$$\text{OR } \int_0^2 (8 - x^3) dx = \left[8x - \frac{x^4}{4} \right]_0^2 = 16 - 4 = 12 \quad \text{top} - \text{bottom}$$



M20. $y=x$ and $y^2=x+6$.

$$y^2 = x + 6 \quad \text{or} \quad x = y^2 - 6$$

x	y
-6	0
-2	2
-2	-2
3	3

$$A = \int_{-2}^3 (y) - (y^2 - 6) dy = \int_{-2}^3 (-y^2 + y + 6) dy$$

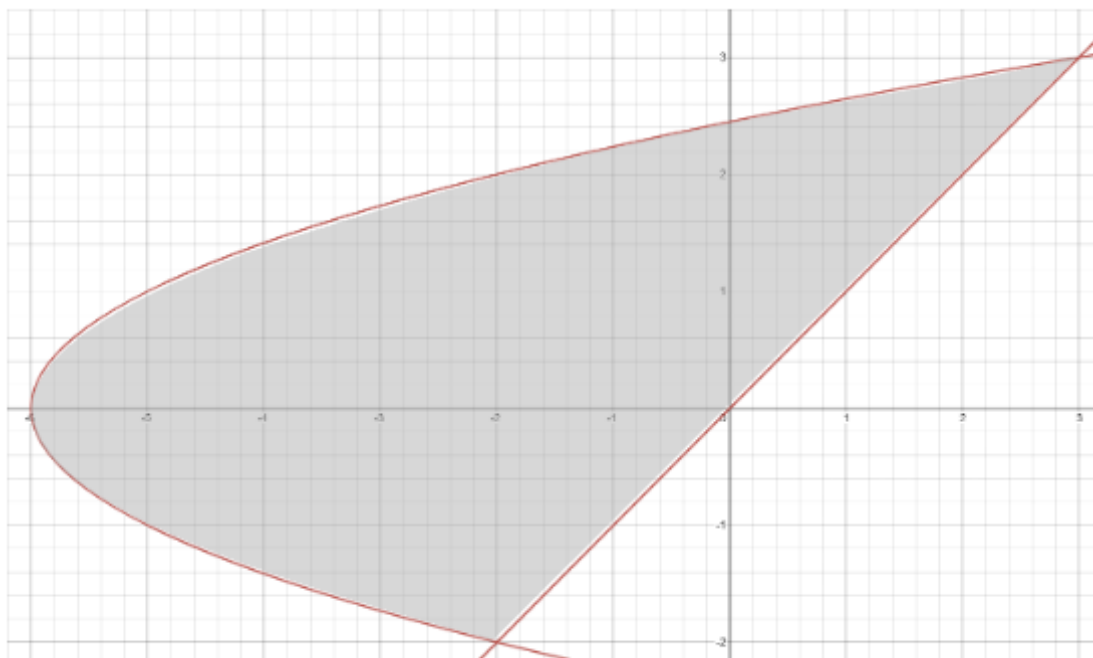
find point of intersection, set $x=x$

$$y = y^2 - 6$$

$$y^2 - y - 6 = 0$$

$$0 = (y - 3)(y + 2)$$

$$y = 3, -2$$



M21. Find the area bounded by the curves $x = y^2 + 3y$ and

$$y = x - 3$$

x	y
3	0
0	-3

$$x = y^2 + 3y = 0$$

$$y(y + 3) = 0$$

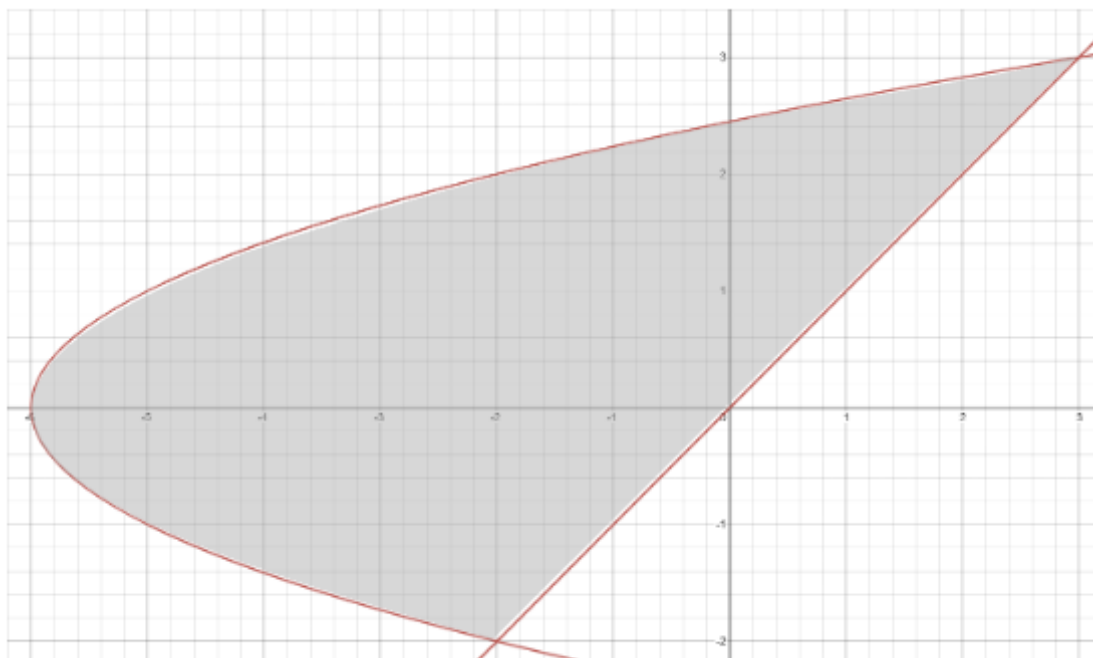
$$y = 0, -3 \quad A = \int_{-3}^1 [(y + 3) - (y^2 + 3y)] dy = \int_{-3}^1 (-y^2 - 2y + 3) dy$$

Point of intersection $y^2 + 3y = y + 3$

$$y^2 + 2y - 3 = 0$$

$$(y + 3)(y - 1) = 0$$

$$y = -3, 1$$



M22.

$$\sin x = \cos x$$

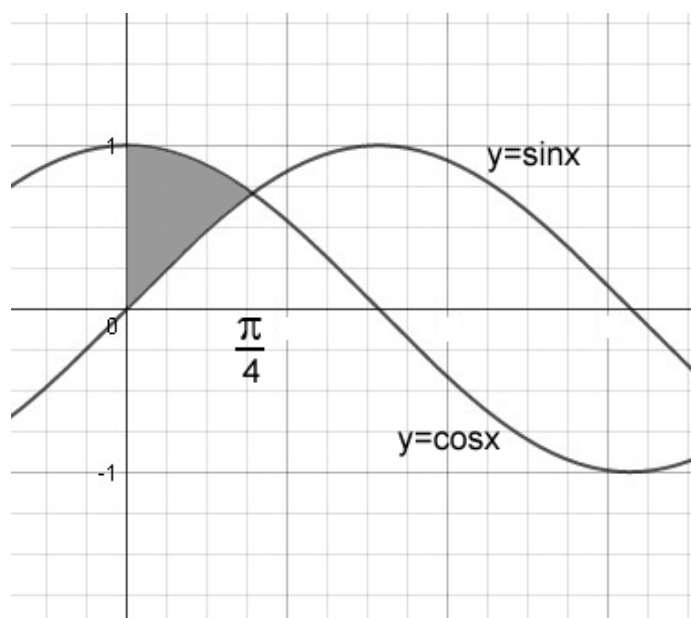
$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

$$x = \frac{\pi}{4} \text{ from special } \Delta's$$

$$A = \int_a^b (\text{top} - \text{bottom}) dx$$

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx$$



M23. It is bounded by the x-axis

$$y = x - 6 \text{ or } x = y + 6$$

$$y = \sqrt{x} \text{ or } x = y^2$$

Point of intersection $y = y$

$$(\sqrt{x})^2 = (x - 6)^2$$

$$x = x^2 - 12x + 36$$

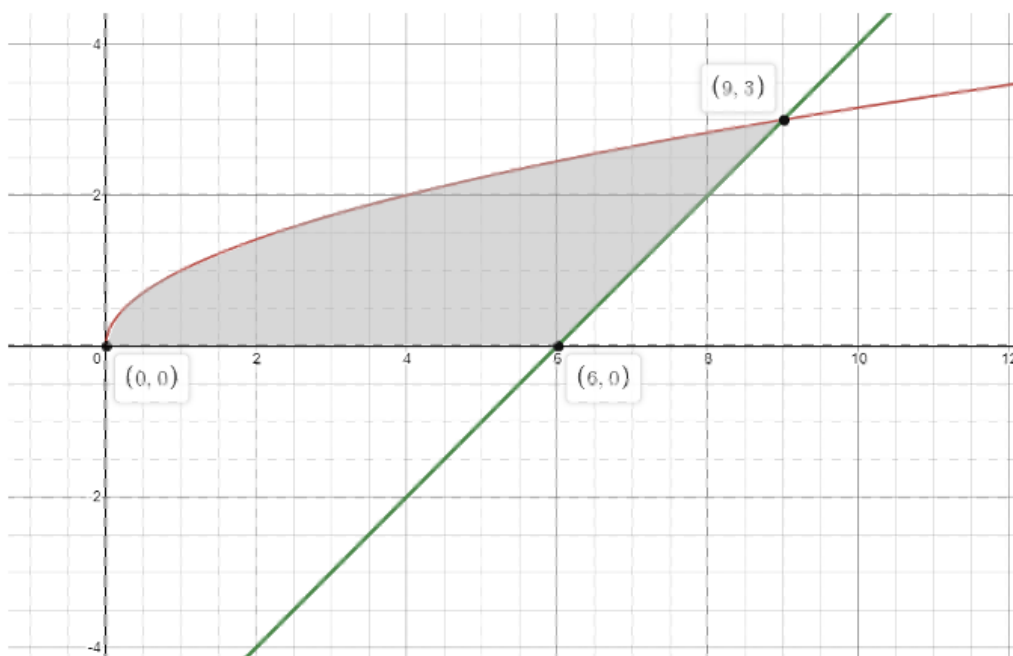
$$0 = x^2 - 13x + 36$$

$$0 = (x - 4)(x - 9)$$

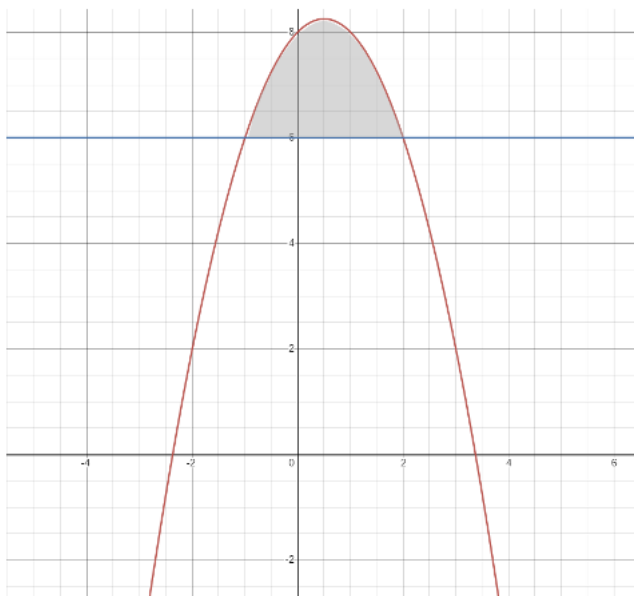
$$x = 4, 9 \quad y = \sqrt{4} = 2$$

$$A = \int_0^3 \left[\text{right} - \text{left} \right] dy \text{ or do } \int_0^9 \left[\text{top} - \text{bottom} \right] dx$$

$$= \int_0^3 (-y^2 + y + 6) dy$$



M24.



Point of intersection

$$-x^2 + x + 8 = 6$$

$$0 = x^2 - x - 2$$

$$0 = (x - 2)(x + 1)$$

$$x = 2, -1$$

$$A = \int_{-1}^2 [(-x^2 + x + 8) - 6] dx = \int_{-1}^2 (-x^2 + x + 2) dx$$

$$= \left[\frac{-x^3}{3} + \frac{x^2}{2} + 2x \right]_{-1}^2 = \left(\frac{-8}{3} + \frac{4}{2} + 4 \right) - \left(\frac{1}{3} + \frac{1}{2} - 2 \right)$$

$$= 6 - \frac{8}{3} - \frac{1}{3} - \frac{1}{2} + 2 = 8 - 3 - \frac{1}{2}$$

$$= 5 - \frac{1}{2} = \frac{9}{2}$$

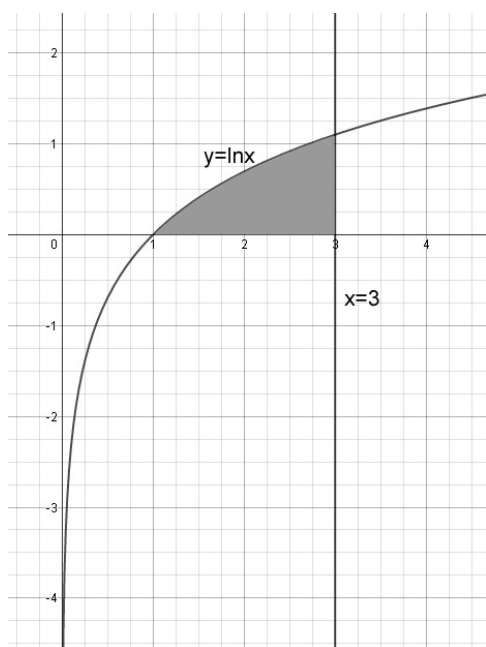
M25.

$y = \ln x$ intersects with $x = 3$

$$A = \int_a^b (\text{top} - \text{bottom}) dx \quad \text{at } \ln 3$$

$$A = \int_1^3 (\ln x - 0) dx = [x \ln x - x]_1^3 = (3 \ln 3 - 3) - (\ln 1 - 1) = 3 \ln 3 - 2$$

Integration by parts would be use $u = \ln x$ and $dv = dx$, so that $du = 1/x dx$ and $v = x$

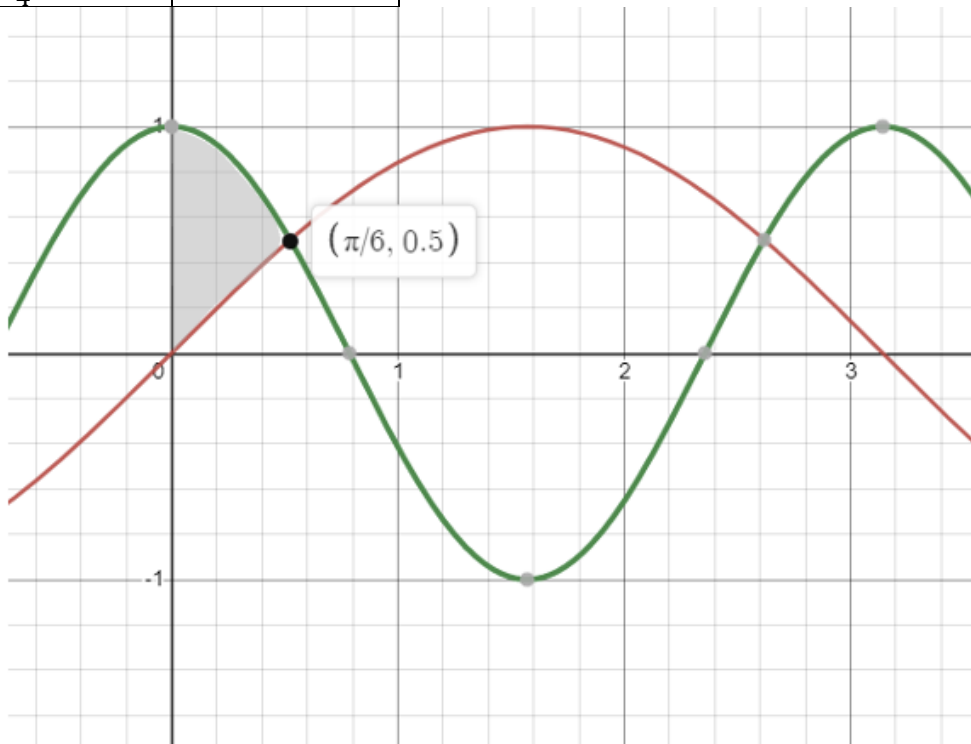


You can also use right/left and get $\int_0^{\ln 3} (3 - e^y) dy = [3y - e^y]_0^{\ln 3}$

M26.

$$y = \cos 2x$$

x	y
0	1
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	0



$$\begin{aligned}
 A &= \int_0^{\pi/6} (\cos 2x - \sin x) dx \\
 &= \left[\frac{\sin 2x}{2} + \cos x \right]_0^{\pi/6} \\
 &= \left(\frac{\sin(\pi/3)}{2} + \cos \pi/6 \right) - (0 + 1) \\
 &= \left(\frac{\sqrt{3}}{2} \left(\frac{1}{2} \right) + \frac{\sqrt{3}}{2} \right) - 1 = \frac{\sqrt{3} + 2\sqrt{3} - 4}{4} = \frac{3\sqrt{3} - 4}{4}
 \end{aligned}$$

N. Volume of Solids of Revolution

Example 1.

Interval= [1,4]

Thickness= Δx Top= $x^2 - 4x + 5$

Bottom=0

Volume= $\pi \int_1^4 ((x^2 - 4x + 5)^2 - (0)^2) dx$ *V of one disc = $\pi(x^2 - 4x + 5)^2 \Delta x$*

Example 2.

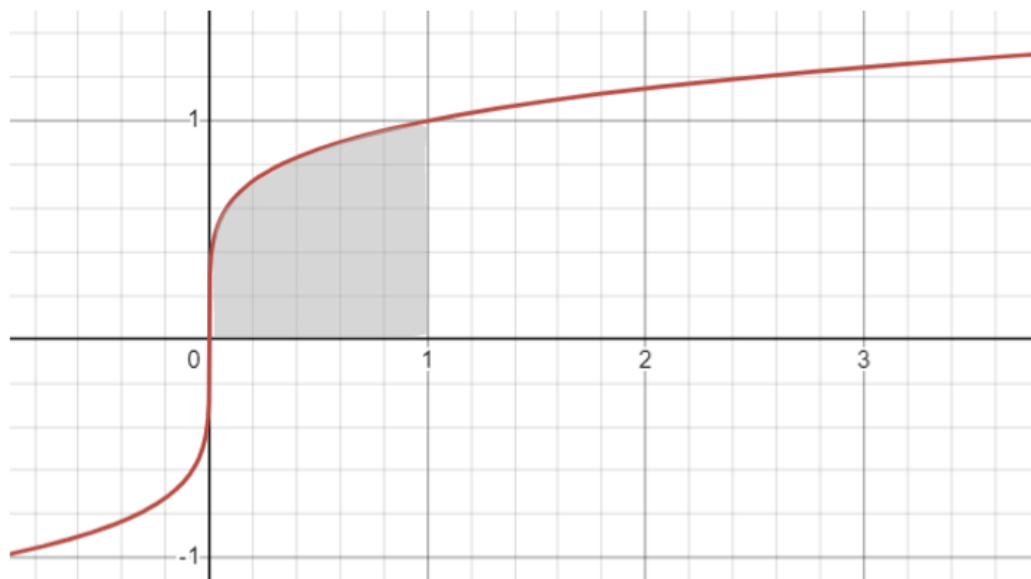
Interval=[0,1]

Thickness= Δx Top= $\sqrt[5]{x}$

Bottom= 0

Volume of one disc= $\pi(\sqrt[5]{x})^2 \Delta x$

$$\begin{aligned} V &= \pi \int_0^1 \left[(\sqrt[5]{x})^2 - 0^2 \right] dx = \pi \int_0^1 x^{\frac{2}{5}} dx = \pi \left[\frac{5}{7} x^{\frac{7}{5}} \right]_0^1 \\ &= \pi \left[\frac{5}{7} - 0 \right] = \frac{5\pi}{7} \end{aligned}$$



Example 3.

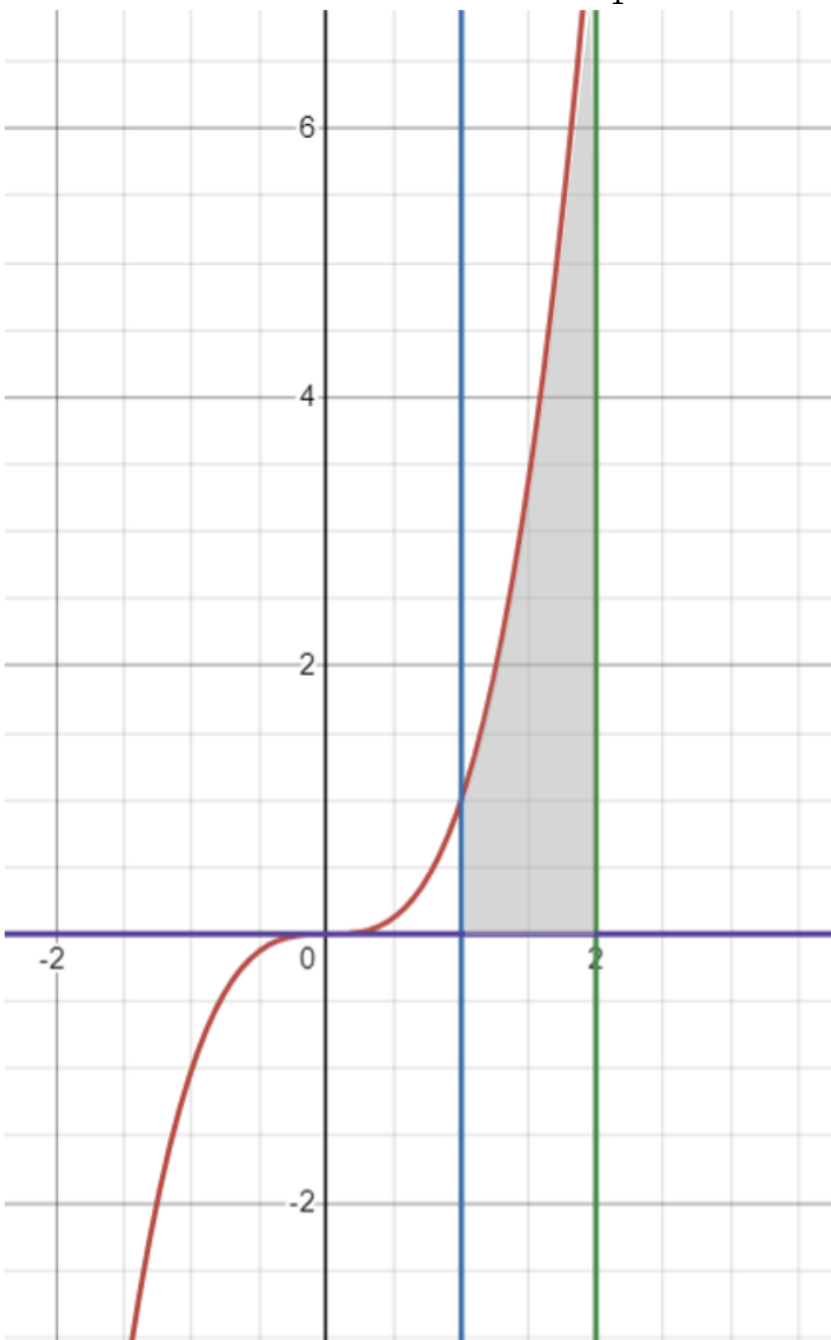
Interval=[1, 2]

Thickness= Δx Top= x^3

Bottom=0

Volume of one disc= $\pi(x^3)^2\Delta x$

$$V = \pi \int_1^2 (x^3)^2 dx = \pi \int_1^2 x^6 dx = \pi \left[\frac{x^7}{7} \right]_1^2 = \pi \left[\frac{2^7}{7} - \frac{1}{7} \right]$$



Example 4.About y -axis interval= $[0,4]$ ← y - valuesThickness= Δy Top= $y^{\frac{1}{3}}$

Bottom= 0

Volume of one disc= $\pi(y^{\frac{2}{3}})\Delta y$

$$V = \pi \int_0^4 (y^{\frac{1}{3}})^2 dy = \pi \int_0^4 y^{\frac{2}{3}} dy = \pi \left[\frac{3}{5} y^{\frac{5}{3}} \right]_0^4 = \pi \left[\frac{3}{5} (4)^{\frac{5}{3}} \right]$$

Example 5. Set up, but do not evaluate $y = \cos x$

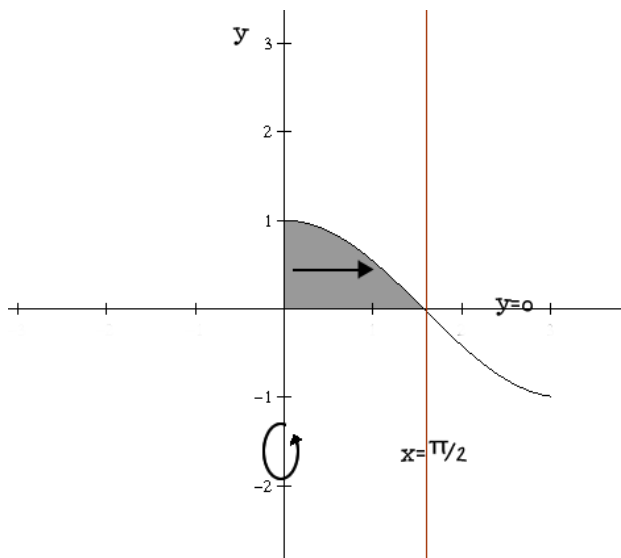
$$x = \cos^{-1} y$$

Interval= $[0,1]$ Thickness= Δy Top= $\cos^{-1} y$

Bottom = 0

Volume of one disc= $\pi(\cos^{-1} y)^2 \Delta y$

$$V = \pi \int_0^1 (\cos^{-1} y)^2 dy$$



Washer's Method

Example 1. $y = \sqrt[3]{x}$ $y = \frac{x}{4}$ $\sqrt[3]{x} = \frac{x}{4}$ $\frac{x^3}{64} = x$
 $x^3 - 64x = 0$
 $x(x^2 - 64) = 0$
 $x = 0, 8, -8$

$$y = \frac{x}{4} = \frac{8}{4} = 2$$

Interval=[0,2]

Thickness= Δy

Outer radius= $4y$

Inner radius= y^3

$$\text{Volume} = \pi \int_a^b A(y) dy = \pi \int_0^2 [(4y)^2 - (y^3)^2] dy = \pi \int_0^2 [16y^2 - y^6] dy$$

Example 2.

Point of intersection $x^2 + 1 = x + 3$
 $x^2 - x - 2 = 0$
 $(x - 2)(x + 1) = 0$ $x = -1, 2$

Interval=[-1,2]

Thickness= Δx

Outer radius= $x + 3$

Inner radius= $x^2 + 1$

$$V = \pi \int_a^b A(x) dx = \pi \int_{-1}^2 [(x + 3)^2 - (x^2 + 1)^2] dx$$

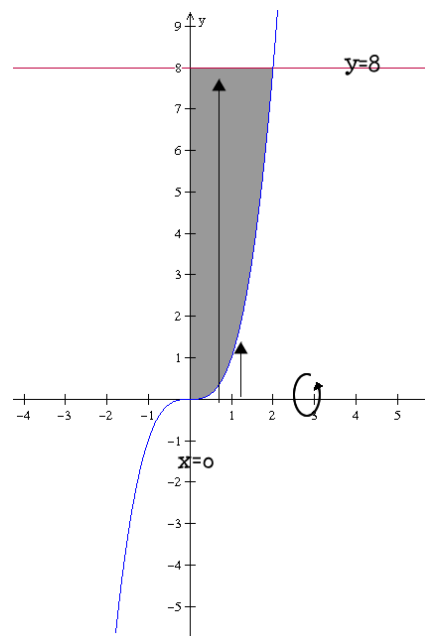
Example 3. Point of intersection $x^3 = 8 \quad x = 2$

Interval= $[0,2]$

Thickness= Δx

Outer radius= 8

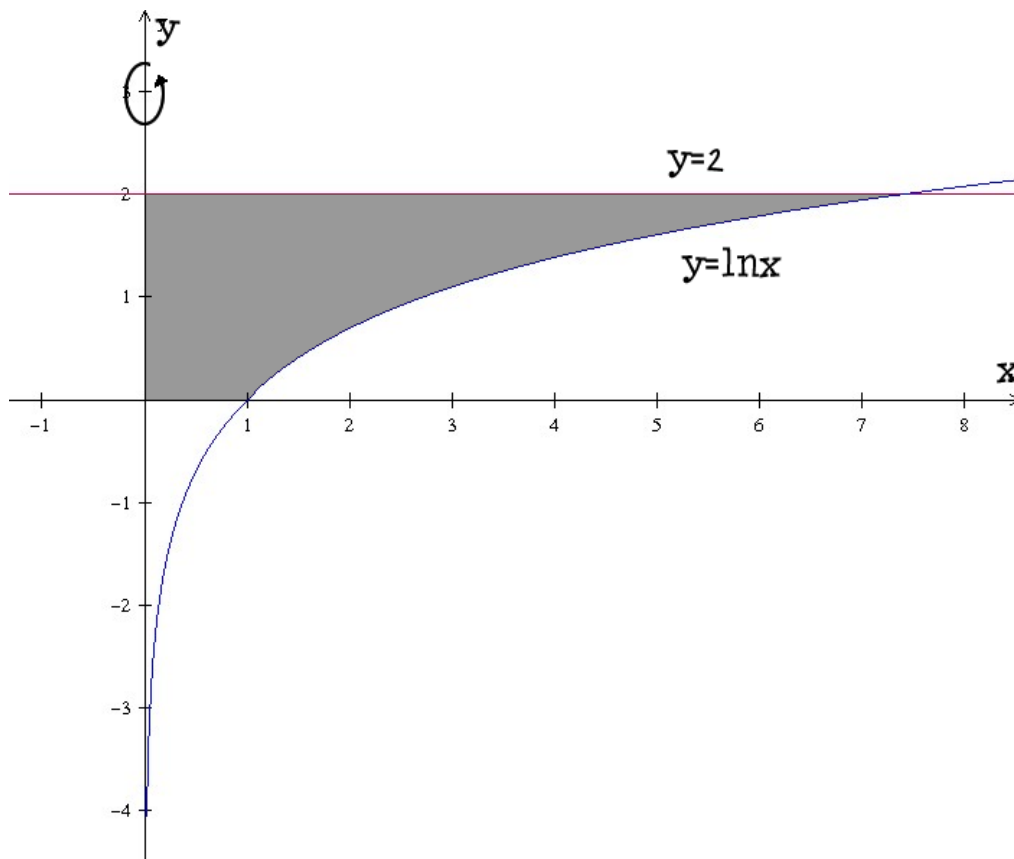
Inner radius= x^3



$$\begin{aligned}
 V &= \pi \int_0^2 [8^2 - (x^3)^2] dx = \pi \int_0^2 (64 - x^6) dx = \pi \left[64x - \frac{x^7}{7} \right]_0^2 \\
 &= \pi \left[64(2) - \frac{2^7}{7} - (0 - 0) \right] = \pi \left(128 - \frac{128}{7} \right) = \frac{768\pi}{7}
 \end{aligned}$$

Example 4.

$y = \ln x$ is the same graph as $x = e^y$



Example 4. a) $V = \pi \int_0^2 [(e^y)^2 - 0^2] dy$

$$= \pi \int_0^2 e^{2y} dy = \pi \left[\frac{e^{2y}}{2} \right]_0^2 = \pi \left[\frac{e^4}{2} - \frac{e^0}{2} \right] = \pi \left[\frac{e^4}{2} - \frac{1}{2} \right] = \pi \left(\frac{e^4 - 1}{2} \right)$$

b) $V = \pi \int_{-\infty}^2 [(e^y)^2 - 0^2] dy$

Revolving About other Lines**Example 5.**

Interval=[0,3]

Thickness= Δx Outer radius= $4 - (x^2 - 2x) = -x^2 + 2x + 4$ Inner radius= $4 - x$

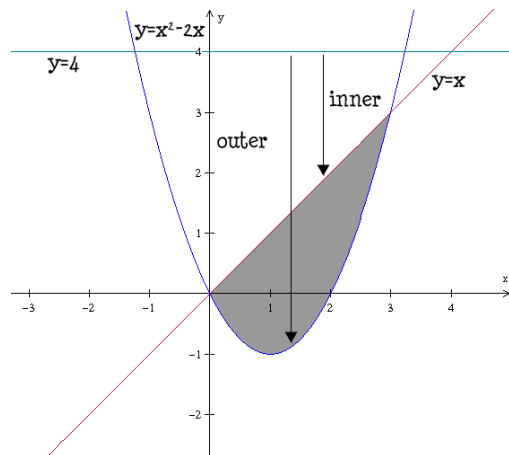
$$x^2 - 2x = x$$

$$x^2 - 3x = 0$$

$$x(x - 3) = 0$$

$$x = 0, 3$$

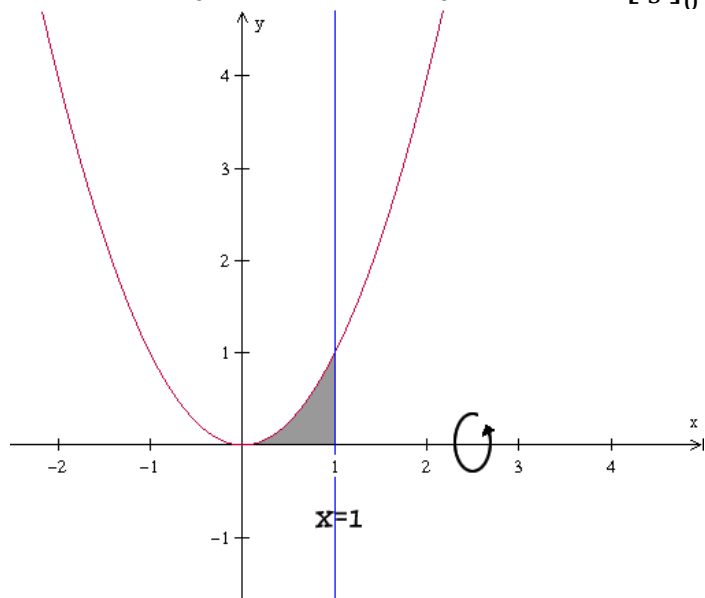
$$V = \pi \int_0^3 [(-x^2 + 2x + 4)^2 - (4 - x)^2] dx$$



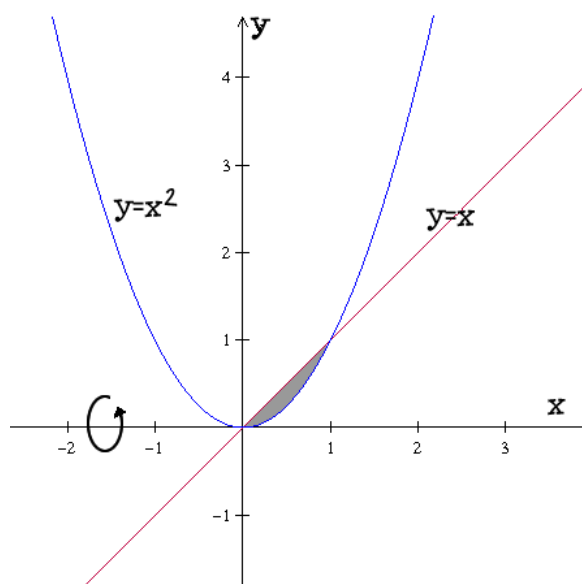
Practice Exam Questions on Volumes of Solids of Revolution

Find the Volume of each of the following using either the Disc method or the Washer method.

$$N1. V = \pi \int_0^1 (x^2)^2 dx = \pi \int_0^1 x^4 dx = \pi \left[\frac{x^5}{5} \right]_0^1 = \pi \left[\frac{1}{5} - 0 \right] = \frac{\pi}{5}$$



$$N2. V = \pi \int_0^1 [(x)^2 - (x^2)^2] dx = \pi \int_0^1 (x^2 - x^4) dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 \\ = \pi \left[\frac{1}{3} - \frac{1}{5} \right] = \frac{2\pi}{15}$$



N3. $x = -1$ parallel to y -axis $\therefore y, dy$ $x = \sqrt{y}$

Interval=[0,1]

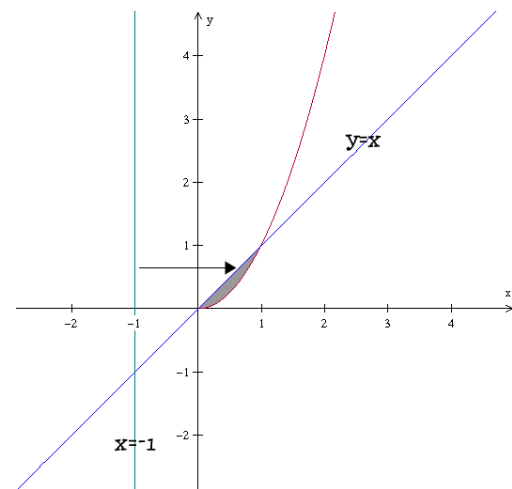
Thickness= Δy

Outer radius (right-left)= $\sqrt{y} + 1$

Inner radius= $y + 1$

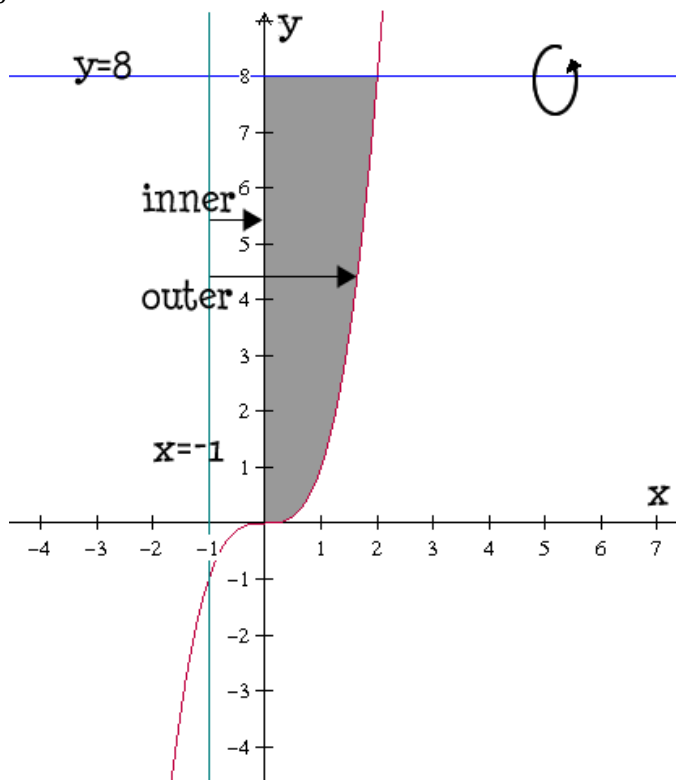
$$\begin{aligned} x^2 &= x \\ x^2 - x &= 0 \\ x(x - 1) &= 0 \\ x &= 0, 1 \quad \therefore y = 0, 1 \text{ since } y = x \end{aligned}$$

$$V = \pi \int [(\sqrt{y} + 1)^2 - (y + 1)^2] dy$$



N4. “right - left”

$$\begin{aligned} V &= \pi \int_0^8 \left[\left(y^{\frac{1}{3}} + 1 \right)^2 - (0 + 1)^2 \right] dy = \pi \int_0^8 \left[y^{\frac{2}{3}} + 2y^{\frac{1}{3}} \right] dy \\ &= \pi \left[\frac{3}{5} y^{\frac{5}{3}} + \frac{2y^{\frac{4}{3}}}{\frac{4}{3}} \right]_0^8 = \pi \left[\frac{3}{5} (8)^{\frac{5}{3}} + \frac{3^{\frac{4}{3}}}{2} \right] = \pi \left[\frac{96}{5} + \frac{48}{2} \right] \end{aligned}$$



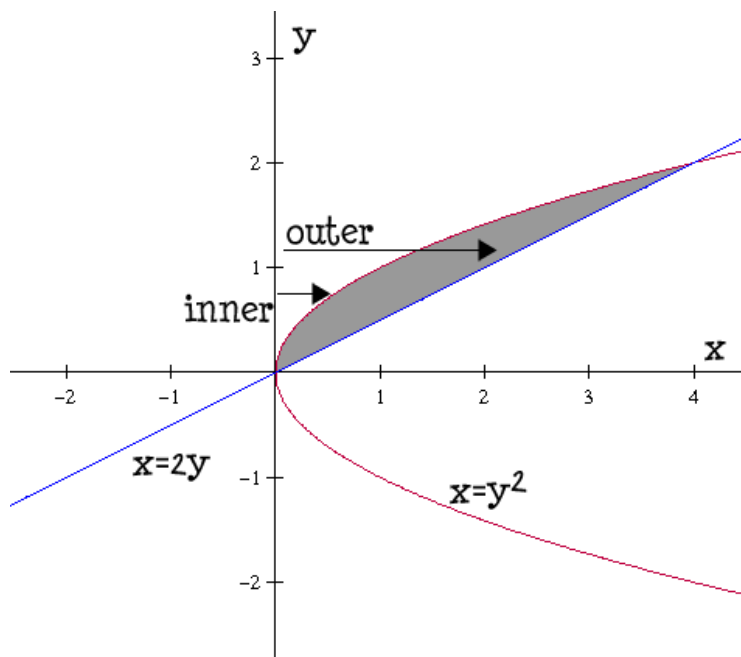
N5. $2y = y^2$

$$0 = y^2 - 2y$$

$$0 = y(y - 2) \quad \therefore y = 0, 2$$

$$V = \pi \int_0^2 [(2y)^2 - (y^2)^2] dy = \pi \int_0^2 (4y^2 - y^4) dy = \pi \left[\frac{4y^3}{3} - \frac{y^5}{5} \right]_0^2$$

$$= \pi \left[\frac{4(2)^3}{3} - \frac{2^5}{5} \right] = \frac{64\pi}{15}$$



N6. Parallel to y-axis (to $x = -1$) $\therefore y, dy$

$$x = 1 \quad y = 1^3 = 1$$

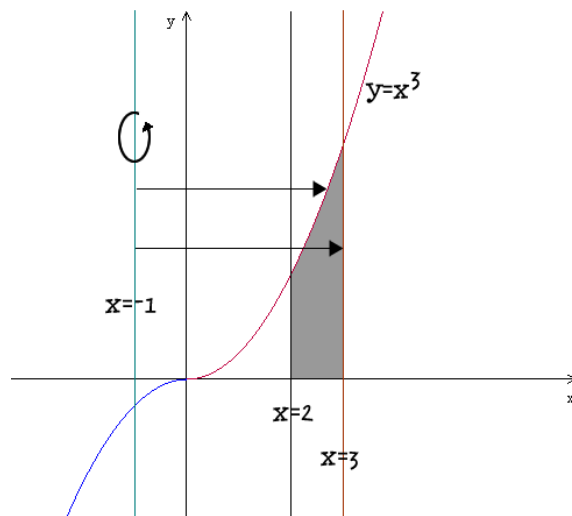
$$x = 2 \quad y = 2^3 = 8$$

Pt of intersection

$$y = x^3 \text{ and } x = 3 \text{ is } y = 3^3 = 27$$

$$V = \pi \int_0^{27} \left[(3 + 1)^2 - \left(y^{\frac{1}{3}} + 1 \right)^2 \right] dy$$

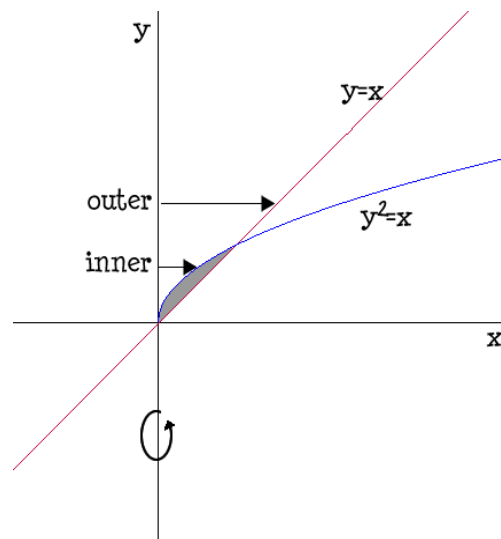
$$= \pi \int_0^{27} \left(4^2 - \left(y^{\frac{1}{3}} + 1 \right)^2 \right) dy$$



N7. Point of intersection $y^2 = y$
 $y^2 - y = 0$
 $y(y - 1) = 0 \quad y = 0, 1$

$$V = \pi \int_0^1 [(y)^2 - (y^2)^2] dy = \pi \int_0^1 (y^2 - y^4) dy = \pi \left[\frac{y^3}{3} - \frac{y^5}{5} \right]_0^1$$

$$= \pi \left[\frac{1}{3} - \frac{1}{5} \right] = \frac{2\pi}{15}$$



N8. Find the area of the region bounded by the curves $y = x + 2$ and $y = x^2$. Let R be the region. Find the volume formed by revolving the region R about the x-axis.

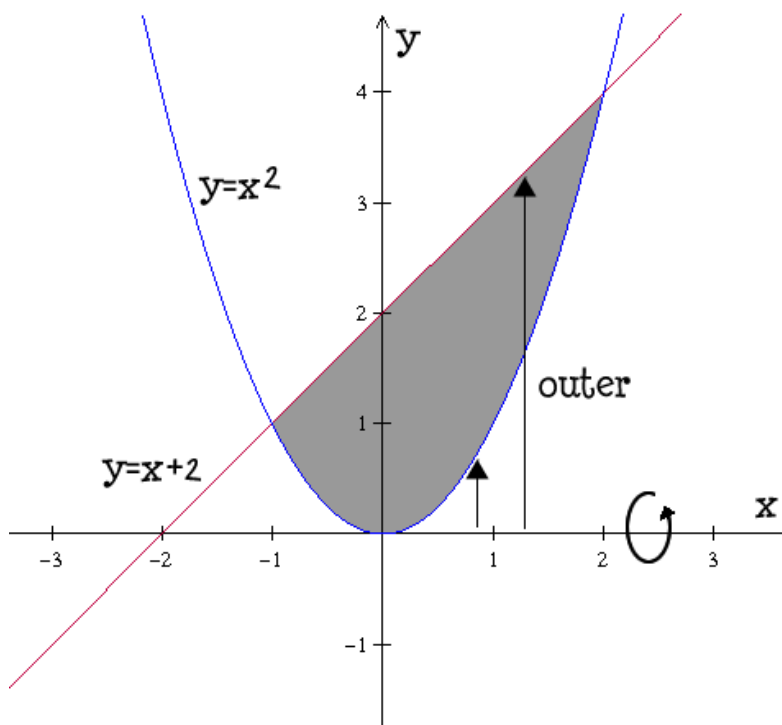
$$x + 2 = x^2$$

$$0 = x^2 - x - 2$$

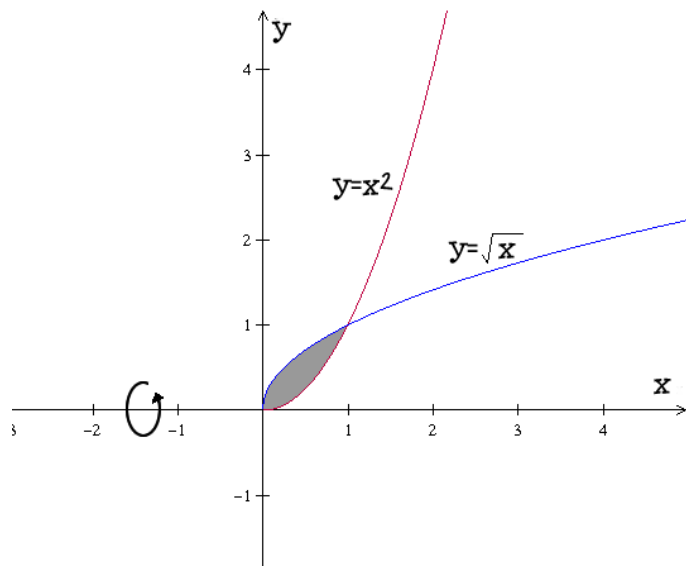
$$0 = (x - 2)(x + 1) \quad x = 2, -1$$

$$V = \pi \int_{-1}^2 [(x + 2)^2 - (x^2)^2] dx = \pi \int_{-1}^2 (x^2 + 4x + 4 - x^4) dx$$

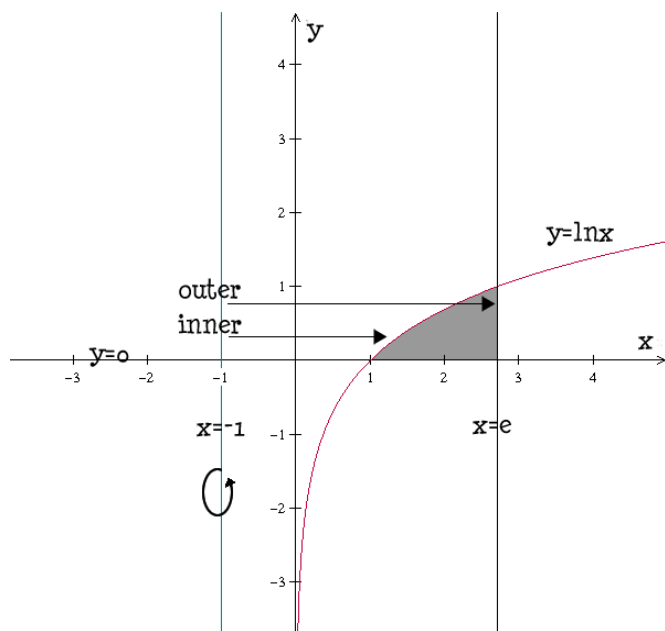
$$= \pi \left[\frac{x^3}{3} + 2x^2 + 4x - \frac{x^5}{5} \right]_{-1}^2 \quad \text{etc}$$



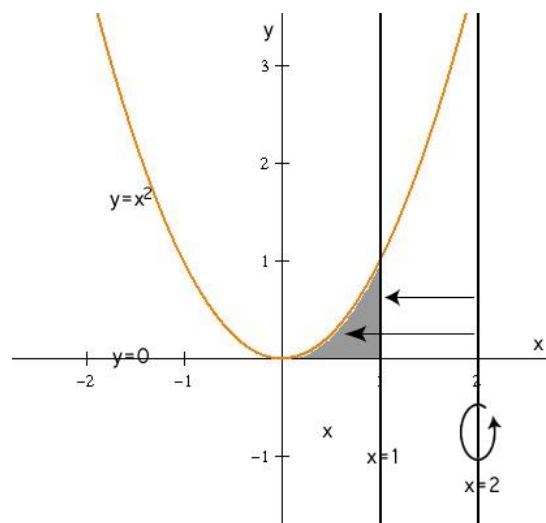
$$\begin{aligned} \text{N9. } V &= \pi \int_0^1 [(\sqrt{x})^2 - (x^2)^2] dx = \pi \int_0^1 (x - x^4) dx \\ &= \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} \right) = \frac{3\pi}{10} \end{aligned}$$



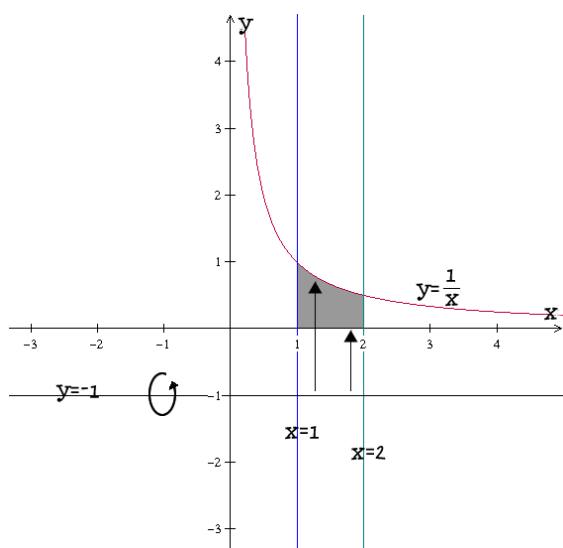
$$\text{N10. } V = \pi \int_0^1 [(e+1)^2 - (e^y + 1)^2] dy \dots \text{etc}$$



$$\begin{aligned}
 \text{N11. } V &= \pi \int_0^4 \left[(2 - \sqrt{y})^2 - (2 - 1)^2 \right] dy = \pi \int_0^4 (4 - 2\sqrt{y} + y - 1) dy \\
 &= \pi \int_0^4 (3 - 2\sqrt{y} + y) dy = \pi \left[3y - \frac{2y^{\frac{3}{2}}}{\frac{3}{2}} + \frac{y^2}{2} \right]_0^4 \text{ etc}
 \end{aligned}$$



$$\begin{aligned}
 \text{N12. } V &= \pi \int_1^2 \left[\left(\frac{1}{x} + 1 \right)^2 - (0 + 1)^2 \right] dx \\
 &= \pi \int_1^2 \left[\left(\frac{1}{x} + 1 \right) \left(\frac{1}{x} + 1 \right) - 1 \right] dx = \pi \int_1^2 [x^{-2} + 2x^{-1} + 1 - 1] dx \\
 &= \pi \left[\frac{x^{-1}}{-1} + 2 \ln x \right]_1^2 = \pi \left[\frac{-1}{x} + 2 \ln x \right]_1^2 \\
 &= \pi \left[\left(\frac{-1}{2} + 2 \ln 2 \right) - (1 + 2 \ln 1) \right] = \pi \left(\frac{1}{2} + \ln 4 \right)
 \end{aligned}$$



N13. Point of intersection $x^4 = x$

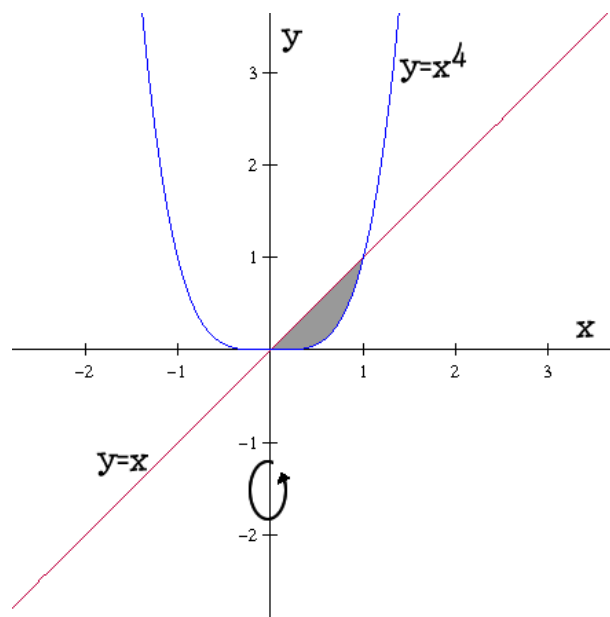
$$x^4 - x = 0$$

$$x(x^3 - 1) = 0 \quad x = 0, 1$$

$$y = x^4 \quad \therefore y = 0, 1$$

$$V = \pi \int_0^1 \left[\left(y^{\frac{1}{4}} \right)^2 - (y)^2 \right] dy = \pi \int_0^1 \left[y^{\frac{1}{2}} - y^2 \right] dy$$

$$= \pi \left[\frac{2}{3} y^{\frac{3}{2}} - \frac{y^3}{3} \right]_0^1 = \pi \left[\frac{2}{3} (1)^{\frac{3}{2}} - \frac{1}{3} \right] = \frac{\pi}{3}$$



N14. Point of intersection $x^2 + 1 = 2x + 1$

$$x^2 - 2x = 0$$

$$x(x - 2) = 0 \quad x = 0, 2$$

$$y = x^2 + 1 \quad \therefore y = 0^2 + 1 = 1$$

$$y = 2^2 + 1 = 5$$

$$V = \pi \int_1^5 \left[(\sqrt{y-1})^2 - \left(\frac{1}{2}y - \frac{1}{2} \right)^2 \right] dy$$

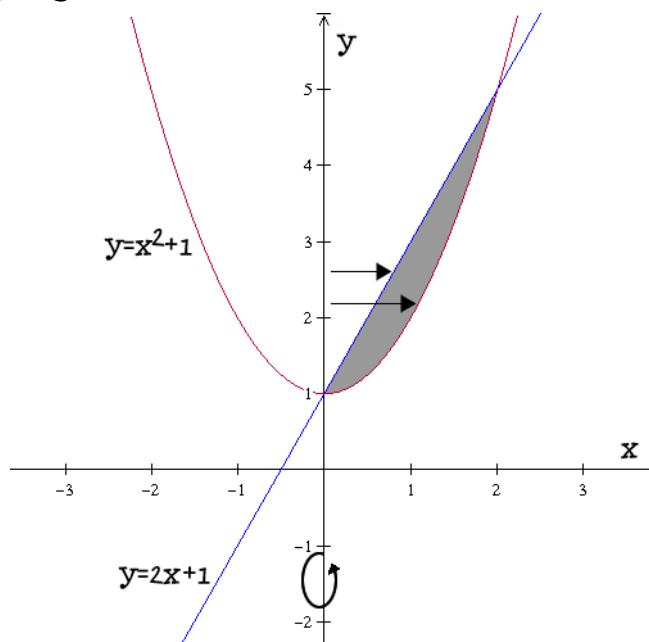
$$= \pi \int_1^5 \left[y - 1 - \left(\frac{1}{4}y - \frac{1}{2}y + \frac{1}{4} \right) \right] dy =$$

$$\pi \int_1^5 \left(\frac{5}{4}y - \frac{5}{4} \right) dy \quad \text{etc..}$$

Pt of intersection $x^2 + 1 = 2x + 1$

$$x^2 - 2x = 0$$

$x=0, 2$ substitute into either equation for y and get
 $y=1$ ($x=0$) and $y=5$ ($x=2$).



In the following two questions, the region R is the region shown below.

$$V = \pi \int_0^2 [(x^2)^2 - (0)^2] dx = \pi \int_0^2 x^4 dx$$

N15. The answer is A.

$$\begin{aligned} \text{N16. } V &= \pi \int_0^4 [(2)^2 - (\sqrt{y})^2] dy = \pi \int_0^4 (4 - y) dy = \pi \left[4y - \frac{y^2}{2} \right]_0^4 \\ &= \pi \left[4(4) - \frac{4^2}{2} \right] = \pi(16 - 8) = 8\pi \end{aligned}$$

N17.

Interval=[0,1]

Thickness= Δy

Outer radius (right - left)= $2 - \sqrt{y}$

Inner radius (right - left)= $2 - 1 = 1$

$$\begin{aligned} A(y) &= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \\ &= \pi(2 - \sqrt{y})^2 - \pi(1)^2 \end{aligned}$$

$$\begin{aligned} V &= \pi \int_0^1 [(2 - \sqrt{y})^2 - (1)^2] dy = \pi \int_0^1 (4 - 2\sqrt{y} + y - 1) dy \\ &= \pi \int_0^1 (-2\sqrt{y} + y + 3) dy \end{aligned}$$

*Best of luck on the
exam!!*